THE ENDOMORPHISMS OF THE TOTAL OPERATOR DOMAIN OF AN INFINITE MODULE

BY

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The total operator domain of a finite vector space of dimensionality n over a division ring is a total matrix algebra of order n^2 over a division ring anti-isomorphic with the given ring. The only nonzero endomorphisms of this total matrix algebra are the inner automorphisms, the automorphisms induced by automorphisms of the division ring, and products of these two([1]).

If we now start with an infinite vector space with Hamel basis over a division ring P, the total operator domain can be thought of as a matrix algebra of infinite order over a division algebra $\overline{P}$ anti-isomorphic with P. Since the choice of elements in the infinite matrices is restricted, the matrix algebra should perhaps not be called total, though it is a maximal ring contained in the set of all infinite matrices with elements in $\overline{P}$. The present paper is a study of the endomorphisms of this total operator domain.

To avoid the assumption of the well-ordering of any set, the infinite vector space is assumed to have a countable Hamel basis over P. Most of the methods introduced and the results carry over, however, for a basis of any cardinal number if the well-ordering assumption is used. As in the finite case, it is shown that the only nonzero endomorphisms are meromorphisms of the domain. However, the meromorphisms need not be automorphisms.

A formula is given for all meromorphisms of the operator ring. Simplifications occur in the cases of $\overline{P}$-meromorphisms and automorphisms. In the latter case the results are the same as those for the finite matrix algebras.

The total operator domain has no nonzero anti-endomorphisms.

1. **Introduction.** If Ξ is a module with nonzero submodules $\Xi_1, \Xi_2, \cdots$ and if every nonzero element ξ of Ξ has a unique form of representation $\xi=\eta_{r_1}+\eta_{r_2}+\cdots+\eta_{r_n}$ as the sum of nonzero elements η_{r_i} lying in Ξ_{r_i}, we shall say that Ξ is the direct sum of the submodules $\Xi_1, \Xi_2, \cdots$ and shall write

$$\Xi = \Xi_1 + \Xi_2 + \cdots = \sum_i \Xi_i.$$

Let P be a division ring and let Ξ be a regular P-module. Then $\alpha\xi=0$ for α in P and ξ in Ξ if and only if $\alpha=0$ or $\xi=0$. Further, the identity ϵ of P must be an identity operator for Ξ, for if $\epsilon\xi\neq\xi$, then $\epsilon(\xi-\epsilon\xi)=0$ while $\epsilon\neq 0$ and $\xi-\epsilon\xi\neq 0$.

Presented to the Society, April 27, 1946; received by the editors February 11, 1946, and, in revised form, November 8, 1946.

([1]) N. Jacobson, *The theory of rings*, Mathematical Surveys, vol. 2, New York, 1943, p. 23.

If $\xi_1, \xi_2, \cdots$ are elements in Ξ, $P(\xi_1, \xi_2, \cdots)$ will designate the least P-submodule of Ξ containing $\xi_1, \xi_2, \cdots$ and will consist of all finite sums $\alpha_1\xi_1+\alpha_2\xi_2+\cdots+\alpha_n\xi_n$, α_i in P. For ξ in Ξ, $P\xi$ denotes the P-module generated by ξ. The elements $\xi_1, \xi_2, \cdots$ are linearly independent if

$$P(\xi_1, \xi_2, \cdots) = P\xi_1 + P\xi_2 + \cdots.$$

In case there exist elements $\xi_1, \xi_2, \cdots$ in Ξ such that

$$\Xi = P\xi_1 + P\xi_2 + \cdots \tag{1.1}$$

then the set of elements $\xi_1, \xi_2, \cdots$ is called a proper P-basis of Ξ. We shall assume that Ξ has a countably infinite proper P-basis throughout this paper. Hence if ξ is in Ξ and Ξ has the basis (1.1) then $\xi=\sum_{i=1}^{n}\alpha_i\xi_i$, α_i in P, and if $\sum_{i=1}^{n}\alpha_i\xi_i=\sum_{i=1}^{n}\beta_i\xi_i$, then $\alpha_i=\beta_i$, $i=1, 2, \cdots, n$.

It is known that the P-submodules of Ξ will have certain properties(2).

(A) Every P-submodule of Ξ has a countable proper P-basis and the number of elements in the basis is an invariant of the submodule.

(B) If Ξ_1 is a P-submodule of Ξ, there exists a P-submodule Ξ_2 such that $\Xi=\Xi_1+\Xi_2$; that is, Ξ is completely reducible with respect to P.

A mapping of Ξ into itself or part of itself, $\xi\to\xi'$, is a P-endomorphism if and only if, for $\xi\to\xi'$, $\eta\to\eta'$, α in P,

$$\xi + \eta \to \xi' + \eta', \tag{1.2}$$

$$\alpha\xi \to \alpha\xi'. \tag{1.3}$$

This endomorphism will be represented by an operator a; $\xi\to a\xi$. With this convention (1.2) and (1.3) can be written

$$a(\xi + \eta) = a\xi + a\eta, \tag{1.2'}$$

$$\alpha(a\xi) = a(\alpha\xi). \tag{1.3'}$$

The set of all P-endomorphisms of Ξ is called the total operator domain of Ξ and is labelled Δ. Elements of Δ are denoted by $a, b, c, \cdots$; the elements a, b of Δ are equal, $a=b$, if and only if $a\xi=b\xi$ for all ξ in Ξ. As is well known, Δ is a ring under the operations of addition and multiplication defined by

$$(a + b)\xi = a\xi + b\xi, \tag{1.4}$$

$$(ab)\xi = a(b\xi). \tag{1.5}$$

The unity element of Δ is denoted by e, the zero element by 0.

Consider now the proper P-basis of Ξ given in (1.1). Any element a of Δ obviously determines a mapping $\xi_i\to a\xi_i$ of the basis elements into Ξ. A converse of this statement is now given.

(2) M. H. Ingraham, *A general theory of linear sets*, Trans. Amer. Math. Soc. vol. 27 (1925) p. 178.

THEOREM 1.1. *A mapping of the elements of a proper P-basis of Ξ into Ξ can be extended to a unique P-endomorphism of Ξ.*

Proof. Let $\xi_i \to m(\xi_i)$ be a mapping of the basis (1.1) into Ξ. If a is the mapping of Ξ defined by

$$a\left(\sum_i \alpha_i \xi_i\right) = \sum_i \alpha_i m(\xi_i),$$

then a is a P-endomorphism and a is an extension of m as $a\xi_i = m(\xi_i)$. Evidently a is unique.

The mapping $\xi \to \alpha\xi$, α in P, of Ξ is a P-endomorphism if P is a field. In this case, P can be considered a subring of Δ: it is easily shown that P is the center of Δ. However, if P is a division ring, the mapping $\xi \to \alpha\xi$, α in P, of Ξ is not in general a P-endomorphism. A situation somewhat analogous to the case of P a field is obtained by defining as in Theorem 1.1 the P-endomorphism $\bar{\alpha}$ for the basis (1.1) as

$$(1.6) \qquad \bar{\alpha}\xi_i = \alpha\xi_i, \qquad i = 1, 2, \cdots.$$

The ring $\overline{P} = \{\bar{\alpha}; \alpha \in P\}$ is a division subring of Δ and is anti-isomorphic to P. It is important to note that $\overline{P}$ is independent on the choice of basis of Ξ, that is, $\bar{\alpha}\xi$ and $\alpha\xi$ are in general unequal. The center C of P is a field as is the center $\overline{C}$ of $\overline{P}$. It is apparent that C and $\overline{C}$ are isomorphic. If ξ is any element of Ξ,

$$\xi = \sum_{i=1}^{n} \gamma_i \xi_i, \qquad \gamma_i \in P,$$

and if $\bar{\alpha}$ is in $\overline{C}$, then

$$\bar{\alpha}\xi = \sum_{i=1}^{n} \gamma_i \bar{\alpha}\xi_i = \sum_{i=1}^{n} \gamma_i \alpha \xi_i = \alpha\xi.$$

We may therefore identify $\overline{C}$ with C. This will be done in the future.

2. **Preliminary discussion of Ξ and Δ.** If a is in Δ and Ξ_1 is a P-submodule of Ξ, then the set $a\Xi_1$ of all elements of the form $a\xi$, ξ in Ξ_1, is a P-module.

DEFINITION 2.1. The order of a P-submodule Ξ_1 of Ξ is the number of elements in a proper P-basis of Ξ_1, and is either a positive integer or $\aleph_0$ depending on whether the proper P-basis has a finite or infinite number of elements. The order of Ξ_1 is designated by $o(\Xi_1)$. The order of an element a of Δ is the order of $a\Xi$, that is, $o(a) = o(a\Xi)$.

LEMMA 2.1. *If Ξ_1 and Ξ_2 are P-submodules of Ξ and if a and b are in Δ, then:*

(1) $o(\Xi_2) \leq o(\Xi_1)$ *if* $\Xi_2 \subset \Xi_1$,

(2) $o(\Xi_1, \Xi_2) \leq o(\Xi_1) + o(\Xi_2)$,

(3) $o(\Xi_1+\Xi_2)=o(\Xi_1)+o(\Xi_2)$ *if* $\Xi_1\cap\Xi_2=0$,
(4) $o(a\Xi_1)\leqq o(\Xi_1)$,
(5) $o(ab)\leqq o(a)$,
(6) $o(ab)\leqq o(b)$,
(7) $o(a+b)\leqq o(a)+o(b)$.

Proof. (1)–(3) are obvious. If $\eta_1, \eta_2, \cdots$ is a proper P-basis of Ξ_1, then the set $a\eta_1, a\eta_2, \cdots$ includes a proper P-basis of $a\Xi_1$. Hence (4) follows readily. As $(ab)\Xi$ is contained in $a\Xi$, (5) follows from (1). If we replace Ξ_1 by $b\Xi$ in (4) we obtain (6). As $(a+b)\Xi$ is contained in $(a\Xi, b\Xi)$, (7) follows from (2).

Corresponding to any element a in Δ there is a set H_a contained in Ξ made up of all elements of Ξ that are annihilated by a. If $a\xi=a\eta=0$, then $a(\alpha\xi+\beta\eta)=0$ for any α, β in P. Thus H_a is a P-module.

LEMMA 2.2. *To each element a in Δ there corresponds a decomposition of Ξ into a direct sum*:

$$\Xi = \sum_i P\delta_i + H_a, \tag{2.1}$$

where $a\Xi$ has the proper P-basis $a\delta_1, a\delta_2, \cdots$.

Proof. Since H_a is a P-submodule of Ξ, there exists a P-submodule Ξ_1 such that $\Xi=\Xi_1+H_a$. Then $a\Xi=a\Xi_1$. If $\Xi_1=\sum_i P\delta_i$, the elements $a\delta_1, a\delta_2, \cdots$ must be linearly independent since $a(\sum_i\alpha_i\delta_i)=0$ if and only if $\sum_i\alpha_i\delta_i$ is in H_a which in turn is true if and only if $\alpha_i=0$ for all i. Hence $a\Xi=\sum_i P(a\delta_i)$.

LEMMA 2.3. *To each element a in Δ and to each decomposition* (2.1) *of Ξ there corresponds on element $\bar a$ in Δ such that*:
(1) *$a\bar a$ and $\bar a a$ are idempotent.*
(2) *$a\bar a a=a$, $\bar a a\bar a=\bar a$.*
(3) *$o(a\bar a)=o(\bar a a)=o(a)$.*
(4) *$a\bar a\delta_i=\delta_i$, $i=1, 2, \cdots$.*

Proof. Select the P-submodule Ξ_1 so that $\Xi=a\Xi+\Xi_1$. We have, from Lemma 2.2, $\Xi=\sum_i P\delta_i+H_a$, $a\Xi=\sum_i P(a\delta_i)$. Define $\bar a$ to be the element of Δ for which

$$\bar a(a\delta_i) = \delta_i,\ i = 1, 2, \cdots;\ \bar a\Xi_1 = 0.$$

By Theorem 1.1, $\bar a$ exists. Immediately we have that

$$(\bar a a)\delta_i = \delta_i,\ i = 1, 2, \cdots;\ (\bar a a)H_a = 0,$$

and $\bar a a$ is idempotent. Further

$$(a\bar a)a\delta_i = a\delta_i,\ i = 1, 2, \cdots;\ (a\bar a)\Xi_1 = 0,$$

so that $a\bar a$ is also idempotent.

Now $o(a)$ is the number of elements in the set $a\delta_1, a\delta_2, \cdots$ as is $o(a\bar a)$, while $o(\bar{a}a)$ is the number of elements in the set $\delta_1, \delta_2, \cdots$. Thus $o(a) = o(a\bar{a}) = o(\bar{a}a)$. Also

$$\bar{a}(a\delta_i) = (\bar{a}a\bar{a})(a\delta_i) = \delta_i, \ i = 1, 2, \cdots; \ \bar{a}\Xi_1 = (\bar{a}a\bar{a})\Xi_1 = 0,$$
$$a\delta_i = (a\bar{a}a)\delta_i, \ i = 1, 2, \cdots; \qquad aH_a = (a\bar{a}a)H_a = 0,$$

from which we conclude $\bar{a} = \bar{a}a\bar{a}$, $a = a\bar{a}a$.

Conclusion (2) of this lemma shows that Δ is a regular ring in the sense of von Neuman[8].

We see in passing that if a is an idempotent element of Δ, then

$$\Xi = a\Xi + H_a.$$

This is a consequence of the Peirce decomposition $\xi = a\xi + (\xi - a\xi)$, where $a\xi$ lies in $a\Xi$, $\xi - a\xi$ lies in H_a.

LEMMA 2.4. *If e_1 and e_2 are idempotent elements of Δ for which $o(e_1) = o(e_2)$, then there exist elements d_1 and d_2 of Δ such that:*
(1) $o(d_1) = o(d_2) = o(e_1)$,
(2) $d_1 = d_1e_1 = e_2d_1$, $d_2 = d_2e_2 = e_1d_2$.
(3) $e_1 = d_2d_1$, $e_2 = d_1d_2$.
(4) $e_1 = d_2e_2d_1$, $e_2 = d_1e_1d_2$.

Proof. Let $\Xi = \sum_i P\delta_i + H_{e_1} = \sum_i P\gamma_i + H_{e_2}$ with $e_1\delta_i = \delta_i$, $e_2\gamma_i = \gamma_i$, $i = 1, 2, \cdots$. Define d_1 and d_2 to be the elements of Δ for which

$$d_1\delta_i = \gamma_i, \ i = 1, 2, \cdots; \ d_1H_{e_1} = 0,$$
$$d_2\gamma_i = \delta_i, \ i = 1, 2, \cdots; \ d_2H_{e_2} = 0.$$

This is possible since $o(e_1) = o(e_2)$. Then (1) and (2) follow immediately. Also

$$d_2d_1\delta_i = \delta_i, \ i = 1, 2, \cdots; \ d_2d_1H_{e_1} = 0,$$
$$d_1d_2\gamma_i = \gamma_i, \ i = 1, 2, \cdots; \ d_1d_2H_{e_2} = 0,$$

so that (3) follows. (4) is a consequence of (2) and (3).

In case one of the idempotent elements of this lemma is the unity e of Δ, we have the following corollary.

COROLLARY. *If a is an idempotent element of Δ with $o(a) = o(e)$, then there exist elements s and t in Δ such that:*
(1) $e = ts$, $a = st$.
(2) $s = as$, $t = ta$.
(3) $e = tas$.

LEMMA 2.5. *Let a and b be any two elements of Δ and suppose $o(a) \leq o(b)$. Then there exist elements c_1 and c_2 in Δ such that $a = c_1bc_2$.*

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(8) J. von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U. S. A. vol. 22 (1936) p. 707.

Proof. Select decompositions (2.1) corresponding to a and b: $\Xi = \sum_i P\delta_i + H_a$ with $a\,\Xi = \sum_i P(a\delta_i)$; $\Xi = \sum_i P\gamma_i + H_b$ with $b\Xi = \sum_i P(b\gamma_i)$. Define c to be the element of Δ such that

$$
\begin{aligned}
c\gamma_i &= \gamma_i, && i = 1, 2, \cdots, o(a),\\
c\gamma_i &= 0 && \text{for } i > o(a) \text{ in case } o(a) < o(b),\\
cH_b &= 0.
\end{aligned}
$$

Thus c is idempotent and $o(c) = o(a)$.

By Lemma 2.3 there exists $\bar{b}$ such that $\bar{b}b\gamma_i = \gamma_i$, $i = 1, 2, \cdots$, $\bar{b}bH_b = 0$. It follows that $c\bar{b}b = c$. Using the same lemma, an element $\bar{a}$ exists such that $\bar{a}a$ is idempotent, $a = a\bar{a}a$, and $o(\bar{a}a) = o(a) = o(c)$. By Lemma 2.4 there exist elements d_1 and d_2 such that $\bar{a}a = d_1cd_2$. Hence

$$a = a\bar{a}a = ad_1cd_2 = ad_1c\bar{b}bd_2,$$

and the lemma follows for $c_1 = ad_1c\bar{b}$ and $c_2 = d_2$.

Corollary. *If a is any element of Δ for which $o(a) = o(e)$ there exist elements p and q in Δ such that*

$$paq = e.$$

To any proper P-basis $\xi_1, \xi_2, \cdots$ of Ξ there corresponds a set of elements e_{ij}, $i, j = 1, 2, \cdots$, of Δ defined by

$$e_{ij}\xi_j = \xi_i, \qquad e_{ij}\xi_k = 0, \qquad j \neq k;\ i, j, k = 1, 2, \cdots. \tag{2.2}$$

These elements have the well known multiplication table

$$e_{ij}e_{jk} = e_{ik}, \qquad e_{ij}e_{nk} = 0, \qquad j \neq n;\ i, j, k, n = 1, 2, \cdots, \tag{2.3}$$

with the elements e_{ii}, $i = 1, 2, \cdots$, being idempotent. Obviously $o(e_{ij}) = 1$ for all i and j, and $o(e_{11} + e_{22} + \cdots + e_{nn}) = n$.

Lemma 2.6. *The set, Ω, of elements a in Δ for which $o(a) < o(e)$ is a proper subring of Δ.*

Proof. To say that $o(a) < o(e)$ is to say that $o(a)$ is finite. Lemma 2.1 guarantees that this set is a ring. Since it does not contain e and since it contains at least the elements e_{ij} defined above, it is a proper subring of Δ.

Theorem 2.1. *The ring Ω is the only proper ideal contained in Δ.*

Proof. If a is in Ω, then ab and ba are in Ω for any b in Δ. For, by Lemma 2.1, $o(ba) \leqq o(a)$, $o(ab) \leqq o(a)$. Hence Ω is an ideal.

If I is an ideal of Δ not contained in Ω, there exists an element a in I such that $o(a) = o(e)$. By the corollary to Lemma 2.5 there exist elements p and q in Δ such that $paq = e$ and I contains e: $I = \Delta$. If I is a proper ideal of Δ contained in Ω, I contains an element a of positive order $o(a)$. Then $o(e_{ii}) \leqq o(a)$

and, by Lemma 2.5, I must contain all e_{ii}, $i=1, 2, \cdots$. Hence I contains $e_{11}+e_{22}+\cdots+_{nn}$ for all integers n and as this element is of order n we may argue that I contains all elements of order less than or equal to n for arbitrary n, that is, $I=\Omega$.

DEFINITION 2.2. The countable set $a_1, a_2, \cdots$ of elements of Δ is said to be algebraically summable if and only if to every ξ in Ξ there corresponds an integer $n=N(\xi)$ such that a_n is in the set and $a_k\xi=0$ for every element a_k of the set for which $k>N(\xi)$. If $a_1, a_2, \cdots$ is algebraically summable, then $a_1+a_2+\cdots$ is given by

$$(a_1 + a_2 + \cdots)\xi = a_1\xi + a_2\xi + \cdots + a_n\xi, \qquad n = N(\xi).$$

It is evident from this definition that every finite set of elements of Δ is algebraically summable, and if $a_1, a_2, \cdots$ is algebraically summable, $a_1+a_2+\cdots$ is an element of Δ.

Consider now the basis of Ξ given in (1.1) and the corresponding elements e_{ij} of Δ given in (2.2) and (2.3). Corresponding to this basis of Ξ is a division ring $\bar{P}$ contained in Δ and anti-isomorphic to P. Now for $\bar{\alpha}$ in $\bar{P}$,

$$\bar{\alpha}e_{ij}\xi_j = \bar{\alpha}\xi_i = \alpha\xi_i = e_{ij}\alpha\xi_j = e_{ij}\bar{\alpha}\xi_j, \quad \bar{\alpha}e_{ij}\xi_k = e_{ij}\bar{\alpha}\xi_k = 0,$$
$$i, j, k = 1, 2, \cdots; j \neq k,$$

so that $\bar{\alpha}e_{ij}=e_{ij}\bar{\alpha}$ for all i, j. That is, the elements of $\bar{P}$ commute with all e_{ij}.

If a is an element of Δ, there exist elements α_{ij} in P such that

$$a\xi_j = \sum_{i=1}^{n_j} \alpha_{ij}\xi_i, \qquad j = 1, 2, \cdots.$$

Let a_j be defined by

$$a_j = \sum_{i=1}^{n_j} \bar{\alpha}_{ij}e_{ij}, \qquad j = 1, 2, \cdots.$$

Now $a_j\xi_k=0$ for $j\neq k$ so that the set $a_1, a_2, \cdots$ is algebraically summable. Further

$$a\xi_j = (a_1 + a_2 + \cdots)\xi_j = a_j\xi_j = \sum_{i=1}^{n_j} \alpha_{ij}\xi_i, \quad j = 1, 2, \cdots,$$

and it follows that $a=a_1+a_2+\cdots$. This can be expressed in a double summation as

$$a = \sum_{j=1}^{\infty} \sum_{i=1}^{n_j} \bar{\alpha}_{ij}e_{ij}. \tag{2.4}$$

We shall frequently abbreviate this to $a=\sum\bar{\alpha}_{ij}e_{ij}$.

THEOREM 2.2. *The set of elements e_{ij}, $i, j=1, 2, \cdots$, is a $\bar{P}$ basis for Δ*

under the operation of algebraic summation. Further, if a is given by (2.4), *then*

$$e_{rs}ae_{ij} = \bar{\alpha}_{si}e_{rj}, \qquad i, j, r = 1, 2, \cdots; s = 1, 2, \cdots, n_i. \tag{2.5}$$

From (2.5) we derive that the only elements of Δ that commute with all e_{ij} are the elements of $\overline{P}$. For if a is given by (2.4) and $ae_{ij}=e_{ij}a$, $i, j=1, 2, \cdots$, then, from (2.5), $\bar{\alpha}_{si}=0$ if $s\neq i$, and $\bar{\alpha}_{rr}e_{rk}=e_{rk}\bar{\alpha}_{kk}$. Hence $a=\bar{\alpha}_{rr}$, $r=1, 2, \cdots$, and a is in $\overline{P}$.

THEOREM 2.3. *If the sets $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ are algebraically summable and $c, c_1, c_2, \cdots$ are arbitrary elements of Δ, then the sets $a_1+b_1, a_2+b_2, \cdots, c_1a_1, c_2a_2, \cdots$ and $a_1c, a_2c, \cdots$ are algebraically summable and*

$$(a_1 + a_2 + \cdots) + (b_1 + b_2 + \cdots) = (a_1 + b_1) + (a_2 + b_2) + \cdots, \tag{2.6}$$

$$a_1c + a_2c + \cdots = (a_1 + a_2 + \cdots)c, \tag{2.7}$$

$$ca_1 + ca_2 + \cdots = c(a_1 + a_2 + \cdots). \tag{2.8}$$

3. **Endomorphisms of Δ.** The principal topic of this paper is the discussion of the endomorphisms of Δ. For completeness we define all terms.

DEFINITION 3.1. A mapping θ of Δ into a subring Δ^θ of Δ is an endomorphism if and only if for a, b in Δ,

(i) $$(a+b)^\theta=a^\theta+b^\theta,$$

(ii) $$(ab)^\theta=a^\theta b^\theta.$$

If in addition

(iii) $$a^\theta=b^\theta \qquad \text{if and only if } a=b$$

then θ is a meromorphism. An automorphism is a meromorphism θ for which $\Delta^\theta=\Delta$. An endomorphism θ is complete if and only if whenever the set $a_1, a_2, \cdots$ is algebraically summable,

(iv) the set $a_1^\theta, a_2^\theta, \cdots$ is algebraically summable,

(v) $$(a_1+a_2+\cdots)^\theta=a_1^\theta+a_2^\theta+\cdots.$$

If (ii) be replaced by

(ii′) $$(ab)^\theta=b^\theta a^\theta$$

then θ is an anti-endomorphism. An endomorphism θ is a $\overline{P}$-endomorphism (relative to the basis (1.1)) if and only if for $\bar{\alpha}$ in $\overline{P}$,

(vi) $$(\bar{\alpha}a)^\theta=\bar{\alpha}a^\theta,\ (a\bar{\alpha})^\theta=a^\theta\bar{\alpha}.$$

If u and v are elements of Δ such that $vu=e$, then the correspondence θ defined by

$$a^\theta = uav$$

is a meromorphism. We shall call θ an inner meromorphism. If and only if $uv=vu=e$ will θ be an inner automorphism.

A set of meromorphisms $\theta_1, \theta_2, \cdots$ of Δ is said to be orthogonal if and only if

$$e^{\theta_i}e^{\theta_j} = 0, \qquad i \neq j,\ i, j = 1, 2, \cdots.$$

If the set $e^{\theta_1}, e^{\theta_2}, \cdots$ is algebraically summable, then as $a^{\theta_i}=a^{\theta_i}e^{\theta_i}$ the set $a^{\theta_1}, a^{\theta_2}, \cdots$ is also algebraically summable as a consequence of Theorem 2.3.

DEFINITION 3.2. If the set of meromorphisms $\theta_1, \theta_2, \cdots$ is orthogonal and if the set $e^{\theta_1}, e^{\theta_2}, \cdots$ is algebraically summable, then the correspondence θ defined by

$$a^\theta = a^{\theta_1} + a^{\theta_2} + \cdots$$

is called the direct sum of the meromorphisms $\theta_1, \theta_2, \cdots$ and is written

$$\theta = \theta_1 \oplus \theta_2 \oplus \cdots.$$

THEOREM 3.1. *If θ is the direct sum of the meromorphisms $\theta_1, \theta_2, \cdots$ then θ is a meromorphism of Δ. If $\theta_1, \theta_2, \cdots$ are $\overline{P}$-meromorphisms, then θ is a $\overline{P}$-meromorphism.*

Proof. For any elements a, b in Δ,

$$a^{\theta_i}b^{\theta_j} = (a^{\theta_i}e^{\theta_i})(e^{\theta_j}b^{\theta_j}) = 0, \qquad i \neq j,\ i, j = 1, 2, \cdots.$$

Thus

$$a^\theta b^\theta = \left(\sum_i a^{\theta_i}\right)\left(\sum_j b^{\theta_j}\right) = \sum_i a^{\theta_i}b^{\theta_i} = (ab)^\theta.$$

Also

$$a^\theta + b^\theta = \sum_i a^{\theta_i} + \sum_j b^{\theta_j} = \sum_i (a+b)^{\theta_i} = (a+b)^\theta$$

in view of (2.6). If $a^\theta=b^\theta$, then $e^{\theta_i}b^\theta=e^{\theta_i}a^\theta$ so that $a^{\theta_i}=b^{\theta_i}$. As θ_i is a meromorphism, $a=b$. Hence θ is a meromorphism of Δ.

The second part of the theorem follows immediately from Theorem 2.3.

It was shown in Theorem 2.2 that the elements e_{ij}, $i, j=1, 2, \cdots$, constitute a $\overline{P}$-basis for Δ under the operation of algebraic summation. The completeness of an endomorphism θ allows us to select a similar basis for Δ^θ. For if a is any element of Δ expressed in the form (2.4), then

$$a^\theta = \sum_{j=1}^{\infty}\sum_{i=1}^{n_j} \bar{\alpha}_{ij}^\theta e_{ij}^\theta. \tag{3.1}$$

Hence we can assert that the elements e_{ij}^θ, $i, j=1, 2, \cdots$, form a $\overline{P}^\theta$-basis for Δ^θ under the operation of algebraic summation.

An endomorphism that is not a meromorphism is said to be proper. A trivial proper endomorphism is $a^\theta=0$ for all a in Δ. This is the zero endomorphism. The next few pages will be devoted to the task of showing that the zero endomorphism is the only proper endomorphism of Δ and also that every endomorphism of Δ is complete.

If θ is a proper endomorphism of Δ, there exist elements b_1 and b_2 in Δ, $b_1 \neq b_2$, such that $b_1^\theta = b_2^\theta$. An equivalent statement is that there exists a nonzero element b in Δ such that $b^\theta=0$, that is, $b=b_1-b_2$. Express b in the form $b=\sum \bar{\beta}_{ij}e_{ij}$. As b is unequal to zero, there exist integers r and s such that $\bar{\beta}_{rs}\neq 0$. From (2.5)

$$(\bar{\beta}_{rs})^{-1}e_{mr}be_{sn} = e_{mn}, \qquad m, n = 1, 2, \cdots .$$

An application of the endomorphism θ yields $e_{mn}^\theta=0$, for all m, n.

If in addition to being proper θ is also complete, then (3.1) shows that $a^\theta=0$ for every a in Δ, and we have established the following theorem.

THEOREM 3.2. *A nonzero complete endomorphism of Δ is necessarily a meromorphism.*

We temporarily divert our attention from the main topic of the paper to establish a lemma that is needed in the next theorem. Let the set of positive integers $(1, 2, 3, \cdots)$ be denoted by ρ. For any subset π of ρ, let the vector I_π be defined by

$$I_\pi = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots)$$

where

$$\epsilon_n = \epsilon, \quad n \in \pi; \qquad \epsilon_n = 0, \quad n \notin \pi,$$

ϵ being the identity of P.

Define the vector space Q as

$$Q = P[I_\pi; \pi \subset \rho]$$

so that Q is the least vector space over P that contains I_π for every subset π of ρ.

LEMMA 3.1. *The vector space Q does not have a countable P-basis.*

Proof. If Q has a countable proper P-basis: $Q=P\zeta_1+P\zeta_2+\cdots$ then define

$$\zeta_i' = \zeta_i - \sum_{j<i} \gamma_{ji}\zeta_i$$

where the γ_{ji} are selected so that ζ_i' has the maximum possible number of successive zeros as its first coordinates. The set $\zeta_1', \zeta_2', \cdots$ must also be a proper P-basis for Q, as no two of the ζ_i' can have the same number of successive zeros as their first coordinates. Now the vectors $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots)$ where $\epsilon_n = \epsilon$ and $\epsilon_k = 0$, $k \neq n$, $n = 1, 2, \cdots$, must be in Q. Hence it must be possible to reorder the new basis and possibly multiply by elements of P so that it becomes $\eta_1, \eta_2, \cdots$, where

$$\eta_k = (0, 0, \cdots, \epsilon, \alpha_{k\,k+1}, \alpha_{k\,k+2}, \cdots),$$

ϵ being the kth coordinate.

To show that $Q = P\eta_1 + P\eta_2 + \cdots$ is impossible, let $\mu_1 = 0$, $\nu_1 = \epsilon$ and recursively

$$\mu_n = \sum_{i=1}^{n-1} \nu_i \alpha_{in}, \qquad \begin{cases} \nu_n = \epsilon \text{ if } \mu_n = 0, \\ \nu_n = -\mu_n \text{ if } \mu_n \neq 0. \end{cases}$$

Select the vector

$$I = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots)$$

so that

$$\epsilon_n = \epsilon \quad \text{if} \quad \mu_n = 0, \qquad \epsilon_n = 0 \quad \text{if} \quad \mu_n \neq 0.$$

If, as we are assuming, $I = \sum_{i=1}^{N} \beta_i \eta_i$, a direct computation of $\beta_1, \beta_2, \cdots, \beta_N$ shows that $\beta_i = \nu_i$ so that $I = \sum_{i=1}^{N} \nu_i \eta_i$. However, then

$$\epsilon_{N+1} = \sum_{i=1}^{N} \nu_i \alpha_{i\,N+1} = \mu_{N+1},$$

which is not possible as $\epsilon_k \neq \mu_k$ for all values of k. Thus I is not in $P\eta_1 + P\eta_2 + \cdots$ and Q cannot have a countable P-basis.

Incidentally, we note that Q is not in general the set of all vectors $(\alpha_1, \alpha_2, \cdots, \alpha_n \cdots)$ over P. For example, the vector $(\epsilon, 2\epsilon, \cdots, n\epsilon, \cdots)$ is not in Q if P is not a finite field. Obviously Q is the set of all countably infinite vectors in case P is a finite field.

Consider now an endomorphism θ of Δ. We shall establish the fact that θ must be a complete endomorphism. To this end, let $\pi_1, \pi_2, \cdots$ be any countable partition of ρ, the set of positive integers. That is

$$\pi_1 \cup \pi_2 \cup \cdots = \rho, \qquad \pi_i \cap \pi_j \text{ void}, \qquad i \neq j,\ i, j = 1, 2, \cdots.$$

Select a proper P-basis for Ξ, say (1.1), and the corresponding set, e_{ij}, of elements of Δ defined in (2.2). Let

$$e(\pi_i) = \sum_{j \in \pi_i} e_{jj}, \qquad i = 1, 2, \cdots.$$

THEOREM 3.3. *The set* $e^\theta(\pi_1), e^\theta(\pi_2), \cdots$ *is algebraically summable.*

Proof. The theorem is obvious if the partition of ρ is finite, so let us assume an infinite partition, that is, an infinite number of π_i.

If the set $e^\theta(\pi_1)$, $e^\theta(\pi_2)$, $\cdots$ were not algebraically summable, there would exist a ξ_j of the basis (1.1), say ξ_1 for convenience, and an infinite set of positive integers, $i_1, i_2, \cdots$, such that

$$e^\theta(\pi_n)\xi_1 \neq 0, \qquad n = i_1, i_2, \cdots. \tag{1}$$

By Theorem 2.2, we can write

$$e^\theta(\pi_{i_n}) = \sum_{r,s} \bar{\alpha}_{nrs} e_{rs}, \qquad n = 1, 2, \cdots. \tag{2}$$

From (1), for every integer n there exists an integer k_n such that

$$\bar{\alpha}_{nk_n1} \neq 0, \qquad n = 1, 2, \cdots.$$

Let π be any subset of ρ, and define

$$\bar{e}(\pi) = \sum_{n \in \pi} e(\pi_{i_n}).$$

This is algebraically summable, as, for any integer k, $e(\pi_n)\xi_k$ is unequal to zero for exactly one value of n. Again $\bar{e}^\theta(\pi)$ can be written as

$$\bar{e}^\theta(\pi) = \sum_{r,s} \bar{\beta}_{\pi rs} e_{rs}. \tag{3}$$

The elements $e(\pi_i)$, $\bar{e}(\pi)$ are idempotent, and

$$\begin{aligned} e^\theta(\pi_{i_n})\bar{e}^\theta(\pi) &= e^\theta(\pi_{i_n}) && (n \in \pi) \\ &= 0 && (n \notin \pi). \end{aligned} \tag{4}$$

The substitution of (2) and (3) in (4) yields

$$\begin{aligned} \sum_j \bar{\alpha}_{nrj}\bar{\beta}_{\pi js} &= \bar{\alpha}_{nrs} && (n \in \pi) \\ &= 0 && (n \notin \pi), \end{aligned} \tag{5}$$

$r, s = 1, 2, \cdots$. The summation on j is finite for all values of r, s and π and is independent of n as we see from (3). In particular, if $r = k_n$ and $s = 1$, (5) takes on the form

$$\begin{aligned} \sum_{j=1}^{t_\pi} \bar{\alpha}_{nk_nj}\bar{\beta}_{\pi j1} &= \bar{\alpha}_{nk_n1} && (n \in \pi) \\ &= 0 && (n \notin \pi). \end{aligned} \tag{5'}$$

For convenience of notation we let

$$\begin{aligned} \bar{\alpha}_{ni} &= (\bar{\alpha}_{nk_n1})^{-1}\bar{\alpha}_{nk_ni}, && i = 1, 2, \cdots, \\ \bar{\beta}_{\pi i} &= \bar{\beta}_{\pi i1}, && i = 1, 2, \cdots, t_\pi. \end{aligned}$$

Then (5′) can be written as

$$\sum_{j=1}^{t_\pi} \bar{\alpha}_{nj}\bar{\beta}_{\pi j} = \epsilon_n, \qquad \begin{cases}\epsilon_n = \epsilon, & n \in \pi,\\ \epsilon_n = 0, & n \notin \pi.\end{cases} \tag{6}$$

Let the vectors ζ_i be defined by

$$\zeta_i = (\bar{\alpha}_{1i}, \bar{\alpha}_{2i}, \cdots, \bar{\alpha}_{ni}, \cdots), \qquad i = 1, 2, \cdots,$$

and the vector I_π by

$$I_\pi = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots).$$

Introduce the vector space Q as

$$Q = [\zeta_1, \zeta_2, \cdots]\bar{P}.$$

That is, it is composed of all finite sums $\sum_i \zeta_i\bar{\alpha}_i$, $\bar{\alpha}_i$ in $\bar{P}$. Now (6) can be written as

$$\sum_{j=1}^{t_\pi} \zeta_j\bar{\beta}_{\pi j} = I_\pi.$$

Thus I_π is in Q. Since π is arbitrary, all vectors of the form

$$I = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots), \qquad \epsilon_n = 0 \text{ or } \epsilon,$$

are in Q. That such a conclusion is impossible follows from Lemma 3.1. The assumption that $e^\theta(\pi_1)$, $e^\theta(\pi_2)$, $\cdots$ is not algebraically summable must be false, and the theorem is established.

A particular instance of this last theorem is $\pi_n = (n)$, $n = 1, 2, \cdots$. Hence the set e_{11}^θ, e_{22}^θ, $\cdots$ is algebraically summable. The endomorphism θ is not assumed to be complete, so even though we know $e_{11}^\theta + e_{22}^\theta + \cdots$ is in Δ and $e_{11} + e_{22} + \cdots = e$, we cannot say that $e^\theta = e_{11}^\theta + e_{22}^\theta + \cdots$ without further justification. Let

$$\bar{e} = \sum_{i=1}^{\infty} e_{ii}^\theta,$$

and let e^0 be defined by

$$e^\theta = \bar{e} + e^0.$$

We shall show that $e^0 = 0$.

From (2.7) and (2.8), $e^\theta\bar{e} = \bar{e}e^\theta = \bar{e}$. As $\bar{e}$ and e^θ are idempotent, we have

$$e^0e^0 = e^0, \qquad e^0e^\theta = e^\theta e^0 = e^0, \qquad e^0\bar{e} = \bar{e}e^0 = 0.$$

Also, as $e_{ij}^\theta = e_{ij}^\theta\bar{e} = \bar{e}e_{ij}^\theta$,

$$e_{ij}^\theta e^0 = e^0 e_{ij}^\theta = 0, \qquad i, j = 1, 2, \cdots.$$

Let $\pi_1, \pi_2, \cdots$ be any countably infinite partition of the set of integers

ρ such that all π_n are infinite sets. Let π_k be the set of integers $n_{k1}, n_{k2}, \cdots$ where $n_{k1} < n_{k2} < \cdots$, $k = 1, 2, \cdots$. Define

$$e(\pi_k) = \sum_{i \in \pi_r} e_{ii}, \qquad k = 1, 2, \cdots,$$

$$e_k = \sum_{i=k}^{\infty} e_{ii}, \qquad k = 1, 2, \cdots,$$

$$p_k = \sum_{i=1}^{\infty} e_{k+i-1\, n_{ki}}, \qquad k = 1, 2, \cdots,$$

$$q = \sum_{j=1}^{\infty} \sum_{i=1}^{j} e_{n_{ik}j}, \qquad \text{where } k = j - i + 1.$$

A simple computation shows that

$$p_k e(\pi_k) = p_k,\ p_k e(\pi_k) q = e_k, \qquad k = 1, 2, \cdots.$$

From Theorem 3.3, the set $e^\theta(\pi_1), e^\theta(\pi_2), \cdots$ is algebraically summable and hence, from Theorem 2.3, the set $p_1^\theta e^\theta(\pi_1) q^\theta, p_2^\theta e^\theta(\pi_2) q^\theta, \cdots$ is also algebraically summable. In view of the above equations, $e_1^\theta, e_2^\theta, \cdots$ is an algebraically summable set.

Now $e_1^\theta = e^\theta$, and generally

$$e_k^\theta = e^\theta - \sum_{i=1}^{k-1} e_{ii}^\theta, \qquad k = 2, 3, \cdots,$$

and therefore

$$e_k^\theta e^0 = e^0 e_k^\theta = e^0 e^\theta = e^\theta e^0 = e^0.$$

Select any element ξ of Ξ. As $e_1^\theta, e_2^\theta, \cdots$ is algebraically summable, there exists an integer $N(\xi)$ such that $e_k^\theta \xi = 0$, $k > N(\xi)$, and consequently

$$e^0 e_k^\theta \xi = e^0 \xi = 0.$$

It follows that $e^0 \xi = 0$ for all ξ in Ξ so that $e^0 = 0$. This establishes the following lemma.

LEMMA 3.2. *If θ is an endomorphism of Δ, then $e_{11}^\theta, e_{22}^\theta, \cdots$ is algebraically summable and*

$$e^\theta = e_{11}^\theta + e_{22}^\theta + \cdots.$$

Consider any algebraically summable set $a_1, a_2, \cdots$ of elements of Δ, and let $a = a_1 + a_2 + \cdots$. By definition, there exist integers $N_1, N_2, \cdots$ such that

$$a_n \xi_k = 0, \qquad n > N_k,\ k = 1, 2, \cdots.$$

An alternate way of writing this equation is

$$a_n e_{kk} = 0, \qquad n > N_k,\ k = 1, 2, \cdots.$$

In view of the preceding lemma, for any ξ in Ξ there exists an integer $N(\xi)$ such that

$$\overset{\theta}{e}_{nn}\xi = 0, \qquad n > N(\xi),$$

and therefore

$$\overset{\theta}{a}_k\xi = (\overset{\theta}{a}_k\overset{\theta}{e}_{11} + \overset{\theta}{a}_k\overset{\theta}{e}_{22} + \cdots + \overset{\theta}{a}_k\overset{\theta}{e}_{nn})\xi, \qquad n = N(\xi),\ k = 1, 2, \cdots.$$

Select the integer M as

$$M = \max(N_1, N_2, \cdots, N_n), \qquad n = N(\xi).$$

Then

$$\overset{\theta}{a}_k\overset{\theta}{e}_{ii} = 0, \qquad k > M,\ i = 1, 2, \cdots, N(\xi),$$

so that

$$\overset{\theta}{a}_k\xi = 0, \qquad k > M,$$

and $a_1, a_2, \cdots$ is algebraically summable. Further,

$$ae_{kk} = a_1e_{kk} + a_2e_{kk} + \cdots + a_ne_{kk}, \qquad n = N_k,\ k = 1, 2, \cdots,$$

so that

$$\overset{\theta}{a}\,\overset{\theta}{e}_{kk} = \overset{\theta}{a}_1\overset{\theta}{e}_{kk} + \overset{\theta}{a}_2\overset{\theta}{e}_{kk} + \cdots + \overset{\theta}{a}_n\overset{\theta}{e}_{kk} = (\overset{\theta}{a}_1 + \overset{\theta}{a}_2 + \cdots)\overset{\theta}{e}_{kk}, \qquad k = 1, 2, \cdots.$$

If we sum up both sides of this equation with respect to k,

$$\overset{\theta}{a} = \overset{\theta}{a}_1 + \overset{\theta}{a}_2 + \cdots.$$

We have established the following theorem.

THEOREM 3.4. *Every endomorphism of Δ is a complete endomorphism*[4].

If θ is a proper endomorphism, then θ is a proper complete endomorphism so that θ is the zero endomorphism because of Theorem 3.2. Hence we have the following corollary.

COROLLARY. *Every nonzero endomorphism of Δ is a complete meromorphism.*

4. Determination of all meromorphisms of Δ. Consider any meromorphism θ of Δ into Δ^θ. By the preceding section, θ is a complete meromorphism. Let

(4) If we introduce the weak topology in Δ we convert Δ into a topological ring and it is easy to see that algebraic summability coincides with convergence in this topology. Theorem 3.4 can thus be stated as the result that every endomorphism of Δ is continuous in the weak topology. (This remark is due to the referee.)

$$\Xi^\theta = e^\theta \Xi, \qquad H^\theta = H_{e^\theta},$$

that is, Ξ^θ is the set of all $e^\theta\xi$ while H^θ is the set of all ξ annihilated by e^θ, ξ in Ξ. From the Peirce decomposition,

$$\Xi = \Xi^\theta + H^\theta.$$

As $a^\theta\xi = e^\theta(a^\theta\xi)$, $a^\theta\xi$ is in Ξ^θ, while if $e^\theta\xi = 0$ then $a^\theta\xi = 0$, and therefore both Ξ^θ and H^θ are Δ^θ-modules. In the work to follow, we shall use the basis of Ξ given in (1.1): $\Xi = P\xi_1 + P\xi_2 + \cdots$, and the corresponding basis elements of Δ, the e_{ij} defined in (2.2).

Let ξ be any nonzero element of Ξ^θ. As $e^\theta\xi = \xi$, and in view of Lemma 3.2, there exists an integer n such that $\overset{\theta}{e}_{nn}\xi \neq 0$. Define

$$\eta_n = \overset{\theta}{e}_{nn}\xi; \qquad \eta_i = \overset{\theta}{e}_{in}\eta_n, \qquad i = 1, 2, \cdots.$$

From (2.3) it is evident that

$$\overset{\theta}{e}_{ij}\eta_j = \eta_i; \qquad \overset{\theta}{e}_{ij}\eta_k = 0, \qquad j \neq k;\ i, j, k = 1, 2, \cdots. \tag{4.1}$$

And as $\eta_n = \overset{\theta}{e}_{ni}\eta_i$, all η_i must be different from zero. If

$$\sum_{i=1}^{m} \alpha_i\eta_i = 0, \qquad \alpha_i \text{ in } P,$$

then

$$\overset{\theta}{e}_{rs}\sum_{i=1}^{m} \alpha_i\eta_i = \alpha_s\eta_r = 0, \qquad r, s = 1, 2, \cdots, m.$$

This is possible only if $\alpha_s = 0$, $s = 1, 2, \cdots, m$, so that the following lemma is established.

LEMMA 4.1. *A set of nonzero elements* $\eta_1, \eta_2, \cdots$ of Ξ^θ *satisfying* (4.1) *is necessarily linearly independent.*

The division rings P and $\bar{P}$ have a common center C. The set of all elements of Δ of the form

$$\sum_{j=1}^{\infty}\sum_{i=1}^{n_j} \alpha_{ij}\overset{\theta}{e}_{ij}, \qquad \alpha_{ij} \text{ in } C,$$

with the operations of Δ is a ring R. In case P is a field and θ is a P-endomorphism R is just Δ^θ.

If a subset Ξ_1 of Ξ is both a P-module and an R-module, Ξ_1 is called a (P, R)-module. Any nonzero (P, R)-module, Ξ_1, is a minimal (P, R)-module if no proper nonzero submodule of Ξ_1 is a (P, R)-module. As an example of a minimal (P, R)-module, let the nonzero elements $\eta_1, \eta_2, \cdots$ of Ξ have property (4.1). Then $\Xi_1 = P\eta_1 + P\eta_2 + \cdots$ is a minimal (P, R)-module. It is not in general true that Ξ_1 is a Δ^θ-module, for $\bar{\alpha}^\theta\eta_i$ need not be in Ξ_1. An example to be given later will illustrate this point. It is evident that every minimal (P, R)-module Ξ_1 has a proper P-basis $\eta_1, \eta_2, \cdots$ satisfying (4.1), as the element ξ in the paragraph preceding Lemma 4.1 could be selected from Ξ_1.

THEOREM 4.1. *The module Ξ^θ has a decomposition into a direct sum*

$$\Xi^\theta = \Xi_1 + \Xi_2 + \cdots \tag{4.2}$$

of minimal (P, R)-modules. Each Ξ_k has a representation

$$\Xi_k = P\eta_{1k} + P\eta_{2k} + \cdots \tag{4.3}$$

with the property

$$\overset{\theta}{e}_{ij}\eta_{jk} = \eta_{ik}; \qquad \overset{\theta}{e}_{ij}\eta_{nk} = 0, \qquad j \neq n;\ i, j, k, n = 1, 2, \cdots, \tag{4.4}$$

and $\eta_{jk} = \overset{\theta}{e}_{jm}\xi_n$ for some choice of m and n.

Proof. To any integer k there corresponds an integer N_k such that $\overset{\theta}{e}_{nn}\xi_k = 0$, $n > N_k$, because of Lemma 3.2. Thus $e^\theta\xi_k = \overset{\theta}{e}_{11}\xi_k + \overset{\theta}{e}_{22}\xi_k + \cdots + \overset{\theta}{e}_{mm}\xi_k$, $m = N_k$. Define

$$\eta_{ink} = e_{in}\xi_k, \qquad i, n, k = 1, 2, \cdots,$$
$$\Xi_{nk} = P\eta_{1nk} + P\eta_{2nk} + \cdots, \qquad n, k = 1, 2, \cdots.$$

As

$$\overset{\theta}{e}_{ij}\eta_{jnk} = \eta_{ink}; \qquad \overset{\theta}{e}_{ij}\eta_{mnk} = 0, \qquad j \neq m;\ i, j, k, m, n = 1, 2, \cdots,$$

each Ξ_{nk} is either zero or a minimal (P, R)-module. The element $e^\theta\xi_k$ is in $(\Xi_{1k}, \Xi_{2k}, \cdots, \Xi_{nk})$, $n = N_k$, and therefore $\Xi^\theta = \bigcup(\Xi_{nk})$, $n, k = 1, 2, \cdots$. Any one of the usual methods of eliminating the dependent Ξ_{nk} will yield a decomposition such as (4.2).

LEMMA 4.2. *The number of minimal (P, R)-modules in any decomposition (4.2) of Ξ^θ is an invariant of θ.*

Proof. If $\Xi^\theta = \Xi_1 + \Xi_2 + \cdots = \Xi_1' + \Xi_2' + \cdots$ with each Ξ_n, Ξ_m' a minimal (P, R)-module and

$$\Xi_k = P\eta_{1k} + P\eta_{2k} + \cdots, \qquad k = 1, 2, \cdots,$$
$$\Xi_k' = P\eta_{1k}' + P\eta_{2k}' + \cdots, \qquad k = 1, 2, \cdots,$$
$$\overset{\theta}{e}_{ij}\eta_{jk} = \eta_{ik}; \qquad \overset{\theta}{e}_{ij}\eta_{nk} = 0, \qquad j \neq n;\ i, j, k, n = 1, 2, \cdots,$$
$$\overset{\theta}{e}_{ij}\eta_{jk}' = \eta_{ik}'; \qquad \overset{\theta}{e}_{ij}\eta_{nk}' = 0, \qquad j \neq n;\ i, j, k, n = 1, 2, \cdots,$$

then

$$\overset{\theta}{e}_{11}\Xi^\theta = P\eta_{11} + P\eta_{12} + \cdots = P\eta_{11}' + P\eta_{22}' + \cdots,$$

the order of a P-module, that is, the number of elements in a proper P-basis is an invariant of the P-module. The above equation shows that the number of η_{1i} equals the number of η'_{1i}, and therefore the lemma follows.

DEFINITION 4.1. The number of minimal (P, R)-modules in a decomposition (such as (4.2)) of Ξ^θ is called the order of θ, $o(\theta)$, and is a positive integer or $\aleph_0$.

Let us now assume that Ξ^θ has the decomposition (4.2) into a direct sum of minimal (P, R)-modules Ξ_k, and each Ξ_k has the basis (4.3) with property (4.4). As $\Xi = \Xi^\theta + H^\theta$, and because of Theorem 1.1, there exist unique elements $u_1, u_2, \cdots, v_1, v_2, \cdots$ of Δ defined by

$$\begin{aligned} &u_k\eta_{ik} = \xi_i; \qquad u_k\eta_{ij} = 0, \qquad j \neq k;\ u_kH^\theta = 0,\ i, j, k = 1, 2, \cdots, \\ &v_k\xi_i = \eta_{ik}, \qquad\qquad i, k = 1, 2, \cdots. \end{aligned} \tag{4.5}$$

It is evident that u_k and v_k are semi-inverses of each other. In fact

$$u_kv_k = e; \qquad u_kv_j = 0, \qquad j \neq k;\ j, k = 1, 2, \cdots. \tag{4.6}$$

The set $u_1, u_2, \cdots$ is algebraically summable as $u_k\eta_{in} = 0$, $k > n$, and $u_kH^\theta = 0$. In view of Theorem 2.3, $v_1u_1, v_2u_2, \cdots$ is also an algebraically summable set, and

$$\begin{aligned} (v_1u_1 + v_2u_2 + \cdots)\eta_{nm} &= \eta_{nm}, \qquad n, m = 1, 2, \cdots, \\ (v_1u_1 + v_2u_2 + \cdots)H^\theta &= 0. \end{aligned}$$

However e^θ has exactly the same effect on this basis: $e^\theta\eta_{nm} = \eta_{nm}$, $e^\theta H^\theta = 0$; so as a consequence

$$e^\theta = v_1u_1 + v_2u_2 + \cdots. \tag{4.7}$$

From (2.2), (4.4), and (4.5) we derive

$$\begin{aligned} &\overset{\theta}{e_{ij}}v_k\xi_j = \eta_{ik} = v_k\xi_i = v_ke_{ij}\xi_j; \qquad i, j, k = 1, 2, \cdots, \\ &\overset{\theta}{e_{ij}}v_k\xi_n = v_ke_{ij}\xi_n = 0, \qquad n \neq j;\ i, j, k, n = 1, 2, \cdots. \end{aligned}$$

And therefore

$$\overset{\theta}{e_{ij}}v_k = v_ke_{ij}, \qquad i, j, k = 1, 2, \cdots. \tag{4.8}$$

From this we obtain $\overset{\theta}{e_{ij}}v_ku_k = v_ke_{ij}u_k$, so that the use of (4.7) yields

$$\overset{\theta}{e_{ij}} = \sum_k v_ke_{ij}u_k, \qquad i, j = 1, 2, \cdots, \tag{4.9}$$

the summation going from 1 to $o(\theta)$.

We recall that corresponding to the basis (1.1), $\Xi = P\xi_1 + P\xi_2 + \cdots$, there is a division subring $\overline{P}$ of Δ anti-isomorphic to P; $\bar{\alpha}$ in $\overline{P}$ is defined by $\bar{\alpha}\xi_i = \alpha\xi_i$, $i = 1, 2, \cdots$. Under the meromorphism θ, $\overline{P}$ is carried into $\overline{P}^\theta$.

For any $\bar{\alpha}^\theta$ in $\overline{P}^\theta$ and ξ in Ξ^θ, $\bar{\alpha}^\theta\xi$ is also in Ξ^θ. Hence

$$\bar{\alpha}^\theta \eta_{rs} = \sum_{i,j} \gamma_{ijrs}\eta_{ij}, \qquad r, s = 1, 2, \cdots .$$

Now $e^\theta_{ij}\bar{\alpha}^\theta = \bar{\alpha}^\theta e^\theta_{ij}$, so that, on multiplying this equation by e^θ_{kr}, we obtain

$$\bar{\alpha}^\theta \eta_{ks} = \sum_{j} \gamma_{rjrs}\eta_{kj}, \qquad r, s, k = 1, 2, \cdots .$$

Thus the γ's are independent of the r in η_{rs}. However, the γ's are functions of α, and to show this we rewrite the equation as

$$\bar{\alpha}^\theta \eta_{rs} = \sum_{i} \overset{\alpha}{\rho}_{is}\eta_{ri}, \qquad r, s = 1, 2, \cdots . \tag{4.10}$$

In this equation $\overset{\alpha}{\rho}_{is} = 0$ if i is larger than some integer which is a function of s and α.

Similarly, for any β in P,

$$\bar{\beta}^\theta \eta_{rs} = \sum_{i} \overset{\beta}{\rho}_{is}\eta_{ri}.$$

Then

$$\bar{\beta}^\theta \bar{\alpha}^\theta \eta_{rs} = \sum_{i} \overset{\alpha}{\rho}_{is} \sum_{j} \overset{\beta}{\rho}_{ji}\eta_{rj}.$$

On the other hand,

$$\overline{\alpha\beta}^\theta \eta_{rs} = \sum_{i} \overset{\alpha\beta}{\rho}_{is}\eta_{ri},$$

and as $\overline{\alpha\beta}^\theta = \bar{\beta}^\theta\bar{\alpha}^\theta$, we must have

$$\overset{\alpha\beta}{\rho}_{rs} = \sum_{i} \overset{\alpha}{\rho}_{is}\overset{\beta}{\rho}_{ri}.$$

The corresponding equation in the division ring $\bar{P}$ is

$$\overset{\alpha\beta}{\bar{\rho}}_{rs} = \sum_{i} \overset{\beta}{\bar{\rho}}_{ri}\overset{\alpha}{\bar{\rho}}_{is}. \tag{4.11}$$

Evidently

$$\overset{\alpha+\beta}{\bar{\rho}}_{rs} = \overset{\alpha}{\bar{\rho}}_{rs} + \overset{\beta}{\bar{\rho}}_{rs} \tag{4.12}$$

also holds.

The correspondence

$$\bar{\alpha} \to f_\alpha = \sum_{r,s} \overset{\alpha}{\bar{\rho}}_{rs}e_{rs}, \tag{4.13}$$

where r is finite for each s and s has $o(\theta)$ for its range, is an isomorphism between $\bar{P}$ and a subring of Δ. For from (4.11) and (4.12) we have

$$f_{\alpha\beta} = f_\beta f_\alpha, \qquad f_{\alpha+\beta} = f_\alpha + f_\beta,$$

and as $0^\theta = 0$, $f_\alpha = 0$ if and only if $\bar{\alpha} = 0$. Now P and $\bar{P}$ are anti-isomorphic under the correspondence $\alpha \leftrightarrow \bar{\alpha}$, so these equations show that $\bar{\alpha} \leftrightarrow f_\alpha$ is an isomorphism between $\bar{P}$ and a subring of Δ.

With the aid of (1.3′), (4.5), and (4.10), remembering that $\overset{\alpha}{\rho_{is}}\xi_r = \overset{\alpha}{\bar{\rho}_{is}}\xi_r$, we obtain

$$\overset{\theta}{\bar{\alpha}} v_s \xi_r = \overset{\theta}{\bar{\alpha}} \eta_{rs} = \sum_i \overset{\alpha}{\rho_{is}} v_i \xi_r = \sum_i v_i \overset{\alpha}{\bar{\rho}_{is}} \xi_r,$$

and therefore

$$\overset{\theta}{\bar{\alpha}} v_s = \sum_i v_i \overset{\alpha}{\bar{\rho}_{is}}.$$

Using the same methods employed in deriving (4.9) from (4.8), we obtain finally

$$\overset{\theta}{\bar{\alpha}} = \sum_{r,s} v_r \overset{\alpha}{\bar{\rho}_{rs}} u_s. \tag{4.14}$$

In this summation, r has a finite range for each value of s and s has $o(\theta)$ as its range.

We shall now combine (4.9) and (4.14) to get an explicit formulation of a^θ for any a in Δ. Let a have the representation

$$a = \sum_{i,j} \bar{\alpha}_{ij} e_{ij}.$$

As θ is complete,

$$\overset{\theta}{a} = \sum_{i,j} \overset{\theta}{\bar{\alpha}_{ij}} \overset{\theta}{e_{ij}}.$$

In view of (4.9) and (4.14),

$$\overset{\theta}{a} = \sum_{r,s} v_r \left(\sum_{i,j} \overset{\alpha_{ij}}{\bar{\rho}_{rs}} e_{ij} \right) u_s.$$

We have established the following theorem.

THEOREM 4.2. *If θ is a meromorphism of Δ, there exists an algebraically summable set of elements $u_1, u_2, \cdots, o(\theta)$ in number and a corresponding set $v_1, v_2, \cdots$ with the property*

$$u_k v_k = e, \qquad u_k v_j = 0, \qquad j \neq k;\ j, k = 1, 2, \cdots, o(\theta),$$

and an isomorphism between $\bar{P}$ and a subring of Δ, $\bar{\alpha} \leftrightarrow f_\alpha = \sum \overset{\alpha}{\bar{\rho}_{rs}} e_{rs}$, such that for any element a of Δ given by $a = \sum \bar{\alpha}_{ij} e_{ij}$,

$$\overset{\theta}{a} = \sum_{r,s} v_r \left(\sum_{i,j} \overset{\alpha_{ij}}{\bar{\rho}_{rs}} e_{ij} \right) u_s. \tag{4.15}$$

The converse of this theorem takes the following form:

THEOREM 4.3. *If the set $u_1, u_2, \cdots, u_N$, N a positive integer or $\aleph_0$, is algebraically summable, and if there exist elements $v_1, v_2, \cdots, v_N$ for which*

$$u_k v_k = e, \qquad u_k v_j = 0, \qquad j \neq k;\ j, k = 1, 2, \cdots, N,$$

and if

$$\bar{\alpha} \leftrightarrow f_\alpha = \sum_{r,s=1}^{N} \bar{\rho}_{rs}^{\alpha} e_{rs}$$

is an isomorphism between $\overline{P}$ and a subring of Δ, then for $a = \sum \bar{\alpha}_{ij} e_{ij}$ the correspondence θ defined by

$$a^\theta = \sum_{r,s=1}^{N} v_r \Big(\sum_{i,j} \bar{\rho}_{rs}^{\alpha_{ij}} e_{ij} \Big) u_s$$

is a meromorphism of Δ.

Proof. It is apparent that $a^\theta + b^\theta = (a+b)^\theta$. Also, if $a^\theta = 0$, then $u_r a^\theta v_s = 0$ so that

$$\bar{\rho}_{rs}^{\alpha_{ij}} = 0, \qquad r, s = 1, 2, \cdots, N;\ i, j = 1, 2, \cdots;$$

then

$$f_{\alpha_{ij}} = 0, \qquad i, j = 1, 2, \cdots,$$

and $\alpha_{ij} = 0$ for all i, j. Hence $a = 0$.

If $b = \sum \bar{\beta}_{ij} e_{ij}$,

$$a^\theta b^\theta = \sum_{r,s,t=1}^{N} v_r \Big(\sum_{i,j,k} \bar{\rho}_{rs}^{\alpha_{ij}} \bar{\rho}_{st}^{\beta_{jk}} e_{ik} \Big) u_t.$$

Calling $ab = \sum \bar{\gamma}_{ij} e_{ij}$ and remembering that

$$\sum_{s=1}^{N} \bar{\rho}_{rs}^{\alpha} \bar{\rho}_{st}^{\beta} = \bar{\rho}_{rt}^{\beta\alpha},$$

then we have

$$a^\theta b^\theta = \sum_{r,s=1}^{N} v_r \Big(\sum_{i,j,k} \bar{\rho}_{rs}^{\beta_{jk}\alpha_{ij}} e_{ik} \Big) u_s,$$

and as $\bar{\gamma}_{ij} = \sum_k \bar{\alpha}_{ik} \bar{\beta}_{kj} = \sum_k \beta_{kj} \alpha_{ik}$,

$$a^\theta b^\theta = \sum_{r,s=1}^{N} v_r \Big(\sum_{i,j} \bar{\rho}_{rs}^{\gamma_{ij}} e_{ij} \Big) u_s.$$

This establishes the theorem.

The above representation assumes a simple form in case θ is an automorphism. As u_1, v_1 are not zero, there exists a b in Δ, $b=\sum\bar{\beta}_{ij}e_{ij}$, such that $v_1u_1=b^\theta\neq 0$. By Theorem 4.2

$$v_1u_1 = \sum_{r,s} v_r\left(\sum_{i,j}\bar{\rho}_{rs}^{\beta_{ij}}e_{ij}\right)u_s.$$

From (4.6) we derive, if $o(\theta)>1$,

$$u_1(v_1u_1)v_1 = \sum_{i,j}\bar{\rho}_{11}^{\beta_{ij}}e_{ij} = e,$$

$$u_k(v_1u_1)v_1 = \sum_{i,j}\bar{\rho}_{k1}^{\beta_{ij}}e_{ij} = 0, \qquad k > 1,$$

$$u_1(v_1u_1)v_k = \sum_{i,j}\bar{\rho}_{1k}^{\beta_{ij}}e_{ij} = 0, \qquad k > 1.$$

Hence

$$\bar{\rho}_{11}^{\beta_{ij}} = \bar{\epsilon} \text{ if } i=j; \qquad \bar{\rho}_{11}^{\beta_{ij}} = 0 \text{ if } i\neq j.$$

$$\bar{\rho}_{rs}^{\beta_{ij}} = 0 \qquad \text{if } r+s>2.$$

A glance at (4.13) reveals that

$$\bar{\beta}_{ij}=0,\quad i\neq j;\qquad \bar{\beta}_{ii}=\bar{\epsilon}.$$

Thus $v_1u_1=e^\theta$ and consequently $v_ku_k=0$, $k>1$. The order of θ is one and Theorem 4.2 becomes[5]:

THEOREM 4.4. *If θ is an automorphism of Δ, there exists an automorphism $\bar{\alpha}\leftrightarrow\bar{p}^\alpha$ of $\overline{P}$ and a regular element u of Δ such that, if $a=\sum\alpha_{ij}e_{ij}$, then*

$$a^\theta = u^{-1}\left(\sum \bar{\rho}^{\alpha_{ij}}e_{ij}\right)u.$$

As an example of the above, take P as rational quaternions, $P=R_a(1, i, j, k)$. Let Ξ be the linear set of all vectors $(\alpha_1, \alpha_2, \cdots)$ over P of order type ω with only a finite number of nonzero coordinates. Then Δ is the ring of all $\omega\times\omega$ matrices $(\bar{\alpha}_{rs}: r, s=1, 2, \cdots)$ over $\overline{P}$, a ring anti-isomorphic to P, with finitely nonzero columns (that is, $\bar{\alpha}_{rs}=0$ for $r>N_s$, N_s an integer depending on s).

Select ξ_k as the vector with 1 as the kth coordinate and 0's elsewhere. Then

$$\Xi = P\xi_1+P\xi_2+\cdots.$$

For this basis of Ξ, e_{ij} is the matrix with 1 in the ith row and jth column and 0's elsewhere.

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(5) This theorem is a special case of a theorem given by N. Jacobson for dense rings. See N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 47 (1945) p. 318, Theorem 32.

For α in P, the correspondence

$$\bar{\alpha} \leftrightarrow -\bar{j}\bar{\alpha}\bar{j}e_{11} + (\bar{j}\bar{\alpha}\bar{i} - \bar{\alpha}\bar{k})e_{12} - \bar{k}\bar{\alpha}\bar{k}e_{22}$$

is an isomorphism between $\bar{P}$ and a subring of Δ. Let

$$u_1 = \sum_{i=1}^{\infty} e_{i\,2i-1}, \quad u_2 = \sum_{i=1}^{\infty} e_{i\,2i}, \quad v_1 = \sum_{i=1}^{\infty} e_{2i-1\,i}, \quad v_2 = \sum_{i=1}^{\infty} e_{2i\,i}.$$

Then

$$u_1v_1 = u_2v_2 = e; \qquad u_1v_2 = u_2v_1 = 0$$

in accordance with (4.6). For the matrix a in Δ, the correspondence θ defined by

$$a^\theta = -v_1\bar{j}a\bar{j}u_1 + v_1(\bar{j}a\bar{i} - a\bar{k})u_2 - v_2\bar{k}a\bar{k}v_2$$

is a meromorphism of Δ as a consequence of Theorem 4.3. The order of θ, $o(\theta)$, equals 2, as

$$\Xi_1 = P\xi_1 + P\xi_3 + \cdots, \qquad \Xi_2 = P\xi_2 + P\xi_4 + \cdots,$$

and

$$\Xi = \Xi^\theta = \Xi_1 + \Xi_2,$$

with Ξ_1 and Ξ_2 minimal (P, R)-modules.

5. **The $\bar{P}$-meromorphisms of Δ.** Theorem 2.3 indicates the importance of the division ring $\bar{P}$ in a given representation of Δ as a matrix algebra. The meromorphism θ of Δ is a $\bar{P}$-meromorphism if for a in Δ, $\bar{\alpha}$ in $\bar{P}$,

$$(\bar{\alpha}a)^\theta = \bar{\alpha}a^\theta, \qquad (a\bar{\alpha})^\theta = a^\theta\bar{\alpha}.$$

An equivalent condition is that

$$\bar{\alpha}^\theta = \bar{\alpha}e^\theta = e^\theta\bar{\alpha}.$$

If θ is a $\bar{P}$-meromorphism, then by Theorem 4.1 there exists a representation of Ξ^θ as a sum of minimal (P, R)-modules:

$$\Xi^\theta = \Xi_1 + \Xi_2 + \cdots$$

with

$$\Xi_k = P\eta_{1k} + P\eta_{2k} + \cdots$$

and

$$\overset{\theta}{e}_{ij}\eta_{jk} = \eta_{ik}; \qquad \overset{\theta}{e}_{ij}\eta_{nk} = 0, \qquad j \neq n.$$

Furthermore, $\eta_{jk} = \overset{\theta}{e}_{jn}\xi_m$ for some choice of m and n depending on j and k. Hence

$$\overset{\theta}{\bar{\alpha}}\eta_{jk} = \overset{\theta}{\bar{\alpha}}\overset{\theta}{e}_{jn}\xi_m = e_{jn}\overset{\theta}{\bar{\alpha}}\overset{\theta}{\xi}_m = e_{jn}\overset{\theta}{e}\overset{\theta}{\bar{\alpha}}\xi_m = \overset{\theta}{\alpha}_{jn}\alpha\xi_m = \alpha\overset{\theta}{e}_{jn}\xi_m = \alpha\eta_{jk}$$

for $j, k=1, 2, \cdots$ and any α in P. From (4.10),

$$\overset{\alpha}{\rho}_{jj} = \alpha; \qquad \overset{\alpha}{\rho}_{jk} = 0, \qquad j \neq k.$$

Theorem 4.2 then takes the following form:

THEOREM 5.1. *If θ is a $\overline{P}$-meromorphism of Δ of order $o(\theta)$, there exists an algebraically summable set of elements $u_1, u_2, \cdots, o(\theta)$ in number, and a corresponding set $v_1, v_2, \cdots$ with the property*

$$u_kv_k = e; \qquad u_kv_j = 0, \qquad j \neq k; j, k = 1, 2, \cdots, o(\theta)$$

such that

$$a^\theta = \sum_k v_kau_k. \tag{5.1}$$

Evidently

$$a^\theta v_k = v_ka; \qquad u_ka^\theta = au_k, \qquad k = 1, 2, \cdots,$$

and as

$$\bar{\alpha}^\theta v_k = \bar{\alpha}e^\theta v_k = \bar{\alpha}v_k, \qquad u_k\bar{\alpha}^\theta = u_ke^\theta\bar{\alpha} = u_k\bar{\alpha},$$

it follows that

$$\bar{\alpha}v_k = v_k\bar{\alpha}, \qquad \bar{\alpha}u_k = u_k\bar{\alpha}, \qquad k = 1, 2, \cdots. \tag{5.2}$$

In other words, the ring $\overline{P}$ commutes elementwise with the u_k and v_k.

Let θ_k be the inner meromorphism defined by

$$a^{\theta_k} = v_kau_k.$$

From (5.2), θ_k is a $\overline{P}$-meromorphism. The meromorphisms $\theta_1, \theta_2, \cdots$ are orthogonal and the set $e^{\theta_1}, e^{\theta_2}, \cdots$ is algebraically summable. We conclude that the following theorem holds.

THEOREM 5.2. *A meromorphism θ is a $\overline{P}$-meromorphism if and only if there exists a set of inner $\overline{P}$-meromorphisms $\theta_1, \theta_2, \cdots$ containing $o(\theta)$ elements such that*

$$\theta = \theta_1 \oplus \theta_2 \oplus \cdots.$$

COROLLARY. *If θ is a $\overline{P}$-automorphism of Δ, then θ is an inner automorphism.*

If θ and ϕ are meromorphisms of Δ, then the product of θ and ϕ, $\theta\phi$, defined by $a^{\theta\phi}=(a^\theta)^\phi$, is also a meromorphism of Δ. If θ and ϕ are $\overline{P}$-meromorphisms, $\theta\phi$ is also a $\overline{P}$-meromorphism.

THEOREM 5.3. *If θ is a $\overline{P}$-meromorphism of Δ and if θ has the two decom-*

positions

$$\theta = \theta_1 \oplus \theta_2 \oplus \cdots = \phi_1 \oplus \phi_2 \oplus \cdots$$

into direct sums of inner $\bar{P}$-meromorphisms, then the sets $\theta_1, \theta_2, \cdots$ and $\phi_1, \phi_2, \cdots$ have the same number of elements, $o(\theta)$, and there exists a $\bar{P}$-automorphism τ of Δ such that

$$\phi_i = \theta_i\tau, \qquad i = 1, 2, \cdots .$$

Proof. As θ_i and ϕ_j are inner $\bar{P}$-meromorphisms,

$$a^{\theta_i} = v_i a u_i, \qquad i = 1, 2, \cdots ,$$

$$a^{\phi_j} = t_j a s_j, \qquad j = 1, 2, \cdots ,$$

and

$$u_i v_i = e, \qquad i = 1, 2, \cdots ,$$

$$s_j t_j = e, \qquad j = 1, 2, \cdots .$$

Now $e^{\theta_i}e^{\theta_j} = 0$ if $i \neq j$ and similarly for ϕ_j, so that

$$u_i v_j = 0, \qquad i \neq j,$$

$$s_i t_j = 0, \qquad i \neq j.$$

Any element $\bar{\alpha}$ of $\bar{P}$ commutes with all u_i, v_i, s_j, t_j. Also

$$e^{\theta} = v_1 u_1 + v_2 u_2 + \cdots = t_1 s_1 + t_2 s_2 + \cdots .$$

Let ζ_{ik} and η_{ik} of Ξ be defined by $\zeta_{ik} = v_k \xi_i$; $\eta_{ik} = t_k \xi_i$, $i, k = 1, 2, \cdots$, from which it follows that

$$u_k \zeta_{ik} = \xi_i; \qquad u_k \zeta_{ij} = 0, \qquad j \neq k,$$

$$s_k \eta_{ik} = \xi_i; \qquad s_k \eta_{ij} = 0, \qquad j \neq k.$$

It is also seen that for any α in P,

$$\alpha\zeta_{ik} = \bar{\alpha}\zeta_{ik}; \qquad \alpha\eta_{ik} = \bar{\alpha}\eta_{ik}, \qquad i, k = 1, 2, \cdots .$$

If Ξ_k and Π_j be defined by

$$\Xi_k = e^{\theta_k}\Xi, \qquad \Pi_j = e^{\phi_j}\Xi,$$

then

$$\Xi^{\theta} = \Xi_1 + \Xi_2 + \cdots = \Pi_1 + \Pi_2 + \cdots .$$

Furthermore,

$$\Xi_k = P\zeta_{1k} + P\zeta_{2k} + \cdots , \qquad k = 1, 2, \cdots ,$$

$$\Pi_j = P\eta_{1j} + P\eta_{2j} + \cdots , \qquad j = 1, 2, \cdots ,$$

and

$$\overset{\theta}{e}_{ij}\zeta_{jk} = \zeta_{ik}; \qquad \overset{\theta}{e}_{ij}\zeta_{nk} = 0, \qquad j \neq n,$$
$$\overset{\theta}{e}_{ij}\eta_{ik} = \eta_{ik}; \qquad \overset{\theta}{e}_{ij}\eta_{nk} = 0, \qquad j \neq n.$$

As each Ξ_k and Π_j is a minimal (P, R)-module, we derive from Lemma 4.2 that the correspondence $\Xi_k \leftrightarrow \Pi_k$ is 1-1.

Now $\Xi = \Xi^\theta + H^\theta$, where H^θ is the set of all ξ annihilated by e^θ. As $H^\theta = P(\xi_1 - e^\theta\xi_1, \xi_2 - e^\theta\xi_2, \cdots)$,

$$H^\theta = P\delta_1 + P\delta_2 + \cdots$$

where $\delta_i = \xi_j - e^\theta\xi_j$, j being a function of i. There exists a regular element p of Δ defined by

$$p\eta_{ik} = \zeta_{ik}, \quad i, k = 1, 2, \cdots; \qquad p\delta_i = \delta_i, \quad i = 1, 2, \cdots.$$

For any α in P, $\bar{\alpha}\delta_i = \alpha\delta_i$ so that

$$\bar{\alpha}p = p\bar{\alpha}.$$

The inner automorphism τ defined by

$$a^\tau = p^{-1}ap$$

is a $\overline{P}$-automorphism.

From the definition of p, and as $s_k(\xi_i - e^\theta\xi_i) = u_k(\xi_i - e^\theta\xi_i) = 0$ for all k and i, it follows that

$$u_kp\eta_{ik} = \xi_i = s_k\eta_{ik}; \qquad u_kp\eta_{in} = 0 = s_k\eta_{in}, \qquad n \neq k;\ i, k, n = 1, 2, \cdots,$$
$$u_kp\delta_i = 0 = s_k\delta_i, \qquad i, k = 1, 2, \cdots.$$

Therefore

$$s_k = u_kp, \qquad k = 1, 2, \cdots.$$

Similarly, as

$$v_k\xi_i = \zeta_{ik} = pt_k\xi_i, \qquad i, k = 1, 2, \cdots,$$

it follows that

$$v_k = pt_k.$$

The above equations yield

$$a^{\phi_k} = p^{-1}v_kau_kp = a^{\theta_k\tau}, \qquad k = 1, 2, \cdots.$$

Thus $\phi_k = \theta_k\tau$ and the theorem follows.

6. **Anti-endomorphisms of Δ.** In the case of a finite total matrix algebra over a division ring, the product of an anti-automorphism of the division ring with the transformation that takes every element of the algebra into

its transpose is an anti-automorphism of the algebra. Moreover, every anti-automorphism of the algebra can be expressed as the product of such an anti-automorphism with an inner automorphism(6).

If $a = \sum \bar{\alpha}_{rs} e_{rs}$ is an element of Δ, it is not true in general that the transpose of a, $a^{\mathrm{T}} = \sum \bar{\alpha}_{rs} e_{sr}$, is also an element of Δ. Thus any anti-endomorphisms of Δ must be of a different type than those of a finite matrix algebra. Actually it will be shown that Δ has no nonzero anti-endomorphisms.

Let θ denote an anti-endomorphism of Δ. Consider any infinite partition $\pi_1, \pi_2, \cdots$ of the set of positive integers into mutually exclusive subsets such that all π_i contain an infinite number of elements. Define the elements e_i of Δ by

$$e_i = \sum_{j \in \pi_i} e_{jj}, \qquad i = 1, 2, \cdots .$$

In the proof of Theorem 3.3, the only products of elements of Δ used were commutative. The conclusions of the theorem are valid therefore for the anti-endomorphism θ. Thus the set $e_1^\theta, e_2^\theta, \cdots$ is algebraically summable.

Denote the integers of π_k by $n_{k1}, n_{k2}, \cdots$, and let

$$r = \sum_{i,j=1}^{\infty} e_{jn_{ij}}, \qquad s_k = \sum_{j=1}^{\infty} e_{n_{kj}j}, \qquad k = 1, 2, \cdots .$$

As $n_{ij} \neq n_{ik}$ for $j \neq k$, both r and s are in Δ. A simple multiplication yields

$$r e_k s_k = e, \qquad k = 1, 2, \cdots ,$$

and therefore

$$s_k^\theta e_k^\theta r^\theta = e^\theta, \qquad k = 1, 2, \cdots .$$

Now in view of Theorem 2.4, the set $s_1^\theta e_1^\theta r^\theta, s_2^\theta e_2^\theta r^\theta, \cdots$, that is, the set $e^\theta, e^\theta, \cdots$, is algebraically summable. However, the countably infinite set $e^\theta, e^\theta, \cdots$ is algebraically summable if and only if $e^\theta = 0$. The anti-endomorphism θ must be the zero endomorphism, and we have proved the following theorem.

THEOREM 6.1. *The ring Δ has no nonzero anti-endomorphisms.*

(6) N. Jacobson, *The theory of rings*, Mathematical Surveys vol. 2, New York, 1943, p. 23.

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