COHOMOLOGY THEORY OF LIE GROUPS AND LIE ALGEBRAS

BY

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INTRODUCTION

The present paper lays no claim to deep originality. Its main purpose is to give a systematic treatment of the methods by which topological questions concerning compact Lie groups may be reduced to algebraic questions concerning Lie algebras[1]. This reduction proceeds in three steps: (1) replacing questions on homology groups by questions on differential forms. This is accomplished by de Rham's theorems[2] (which, incidentally, seem to have been conjectured by Cartan for this very purpose); (2) replacing the consideration of arbitrary differential forms by that of invariant differential forms: this is accomplished by using invariant integration on the group manifold; (3) replacing the consideration of invariant differential forms by that of alternating multilinear forms on the Lie algebra of the group.

We study here the question not only of the topological nature of the whole group, but also of the manifolds on which the group operates. Chapter I is concerned essentially with step 2 of the list above (step 1 depending here, as in the case of the whole group, on de Rham's theorems). Besides considering invariant forms, we also introduce "equivariant" forms, defined in terms of a suitable linear representation of the group; Theorem 2.2 states that, when this representation does not contain the trivial representation, equivariant forms are of no use for topology; however, it states this negative result in the form of a positive property of equivariant forms which is of interest by itself, since it is the key to Levi's theorem (cf. later).

Chapter II is concerned with step 3 of the above list. It is then necessary to assume that the group operates transitively on the manifold under consideration, that is, that this manifold is a homogeneous space relative to the group. Theorem 13.1, in connection with Theorem 2.3, indicates a method by which the Betti numbers of any homogeneous space attached to a connected compact Lie group may be computed algebraically. However, applications of this theorem are still lacking. In particular, it is desirable to obtain an algebraic proof of Samelson's theorem[3] to the effect that, if a closed subgroup

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[1] These methods are due to E. Cartan, *Sur les invariants intégraux de certains espaces homogènes clos*, Annales Société Polonaise de Mathématique vol. 8 (1929) pp. 181–225. They were used by R. Brauer to determine the Betti numbers of the classical groups (C. R. Acad. Sci. Paris vol. 201 (1935) pp. 419–421).

[2] G. de Rham, *Sur l'analysis situs des variétés à n dimensions*, J. Math. Pures Appl. vol. 10 (1931) pp. 115–200.

[3] H. Samelson, *Beiträge zur Topologie der Gruppen-Mannigfaltigkeiten*, Ann. of Math. vol. 42 (1941) pp. 1091–1137; Satz VI, p. 1134.

H of a connected compact Lie group G is not homologous to 0, then the cohomology ring of G is the product of the cohomology rings of H and G/H.

The topological questions on compact Lie groups, once they have been reduced to algebraic questions on Lie algebras, suggest a certain number of purely algebraic objects, which may be constructed in relation with any Lie algebra over a field of characteristic zero. One arrives in this way to the notion of the cohomology groups of an arbitrary Lie algebra L, which is the object of Chapters III and IV (Chapter III is concerned with the cohomology groups which correspond to invariant forms, Chapter IV with those which correspond to equivariant forms). Properties of these cohomology groups may be derived either from transcendental properties of compact groups (such properties apply only to semi-simple Lie algebras and are then obtained by making use of H. Weyl's "unitary trick") or purely algebraically. Thus, Whitehead's algebraic lemma ([4]) (which is used to prove Levi's theorems) states in our terminology that the second cohomology groups of any semi-simple Lie algebra always reduce to $\{0\}$. The first cohomology groups also reduce to $\{0\}$, and this fact can be used to prove algebraically the full reducibility of representations of semi-simple Lie algebras (cf. the paper of Hochschild quoted above). In general, the second cohomology group of any Lie algebra L (with respect to the trivial representation) is the dual space of the full exterior center of L, a notion which was introduced by Ado([5]). The theorem proved by Ado in this connection can be restated by saying that if $L \neq \{0\}$ is nilpotent, then its second cohomology group is not equal to $\{0\}$. Finally, we show (following Cartan) that the third cohomology group of a semi-simple algebra $L \neq \{0\}$ never reduces to $\{0\}$, which proves that the third Betti number of a compact connected semi-simple Lie group is always not equal to 0.

We make constant use of the notions and theorems contained in the book by one of us (C. Chevalley, *Theory of Lie groups*, I, Princeton University Press, 1940); this book will be referred to as LG.

Chapter I. Manifolds with operators

1. **Differential forms on a manifold.** Let M be a manifold of dimension d and class C^2. At every point m of M we consider the space $V(m)$ of tangent vectors to M at m [LG, p. 76]([6]). $V(m)$ is a vector space of dimension d over

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[4] J. H. C. Whitehead, *On the decomposition of an infinitesimal group*, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 229–237. Cf. also G. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math. vol. 64 (1942) pp. 667–694.

[5] I. Ado, *Über die Structur der endlichen kontinuierlichen Gruppen*, Bull. Soc. Phys.-Math. Kazan (3) vol. 6 (1934) pp. 38–42. (Russian with German summary.)

[6] In LG, only analytic manifolds were considered. The definitions can be slightly modified in order to allow us to treat the case of manifolds of class $C^k (k \geq 1)$. These modifications are trivial except as regards the definition of tangent vectors. This definition should be formulated as follows in the case of manifolds M of class C^k. Let m be any point of M and let A be the class

the field R of real numbers. Given an arbitrary finite-dimensional vector space V over R, we denote by $C^q(m, V)$ the vector space of all q-linear alternating functions defined on $V(m)$ with values in V. If $V = R$ then $C^q(m) = C^q(m, R)$ is the set of homogenous elements of order q in the (contravariant) Grassmann algebra of $V(m)$. By definition $C^0(m, V) = V$.

A V-differential form (or shorter: V-form) of order q on M is a function ω which to each $m \in M$ assigns an element $\omega(m) \in C^q(m, V)$. If $V = R$ we omit the prefix V- and speak of differential forms on M.

The usual definition of the differential $d\omega$ of a form of class C^1 [LG, p. 148] can be carried over to V-forms in the following manner. We select a basis $v_1, \cdots, v_k$ for the vectors in V. The V-form ω can then be written as $\omega = \omega^{(1)}v_1 + \cdots + \omega^{(r)}v_k$ where $\omega^{(k)}$ are differential forms. Define $d\omega = d\omega^{(1)}v_1 + \cdots + d\omega^{(r)}v_k$. Clearly $d\omega$ is a V-form of order $q+1$ independent of the choice of the basis.

A V-form ω will be called *regular* if both ω and $d\omega$ are of class C^1. In the sequel all forms will be assumed regular without explicit statement. If ω is regular then (as a consequence of Stokes' formula) $dd\omega = 0$, hence $d\omega$ also is regular.

V-forms ω such that $d\omega = 0$ are called *closed*. Those of the form $\omega = d\theta$ where θ is a V-form of one lower order are called *exact*. Since $dd\omega = 0$, every exact V_r-form is closed. The quotient space of the linear set of closed V-forms of order q by the subspace of the exact V-forms of order q will be denoted by $D^q(M, V)$ and by $D^q(M)$ if $V = R$. By analogy with topology $D^q(M, V)$ will be called the q-dimensional *cohomology group of M* obtained using V-forms.

If $V = R$ then we also have the Grassmann multiplication of differential forms which to two forms ω^p and ω^q of order p and q respectively defines a form $\omega^p \square \omega^q$ of order $p+q$. This multiplication has the property [LG, p. 148] that

$$(1.1) \qquad d(\omega^p \square \omega^q) = d\omega^p \square \omega^q + (-1)^p \omega^p \square d\omega^q$$

which implies that the product of two closed forms is closed and that the product of a closed form and of an exact one (taken in any order) is exact. Thus the Grassmann multiplication defines a multiplication of elements in $D^p(M)$ and $D^q(M)$ with values in $D^{p+q}(M)$. The direct sum $D(M)$ of the groups $D^q(M)$ for all $q \geq 0$ thus becomes a ring (or rather an algebra over R) which we shall call the cohomology ring of M obtained using differential forms.

Let $P: V \to V$ be a linear transformation. For every $f \in C^q(m, V)$ we then

of functions of type C^k at m. A tangent vector L to M at m is a linear real-valued function defined on A satisfying the following condition: let $f_1, \cdots, f_r \in A$ and let $\phi(u_1 \cdots, u_r)$ be a real-valued function of r real arguments, which is of type C^k in the neighborhood of $(a_1, \cdots, a_r)$ where $a_i = f_i(m)$ $(1 \leq i \leq r)$. If $f \in A$ can be expressed in the neighborhood of m as $f = \phi(f_1, \cdots, f_r)$ then $Lf = \sum_{i=1}^r (\partial \phi / \partial u_i)(a_1, \cdots, a_r)Lf_i$.

have the composite function $Pf \in C^q(m, V)$ and the correspondence $f \to Pf$ is a linear transformation $P: C^q(m, V) \to C^q(m, V)$. Hence for each V-form ω on M we may define a V-form $P\omega$ by setting $(P\omega)m = P(\omega m)$. It is easy to see that $P\omega$ is regular if ω is and that

$$(1.2) \qquad\qquad d(P\omega) = Pd\omega,$$

$$(1.3) \qquad\qquad P_1(P_2\omega) = (P_1P_2)\omega,$$

$$(1.4) \qquad\qquad P(r_1\omega_1 + r_2\omega_2) = r_1P\omega_1 + r_2P\omega_2, \qquad\qquad r_1, r_2 \in R.$$

Consider two manifolds M_1, M_2, a transformation $T: M_1 \to M_2$ (all of class C^2) and a V-form ω on M_2. If $m \in M_1$ then T defines [LG, p. 78] a linear mapping of the tangent vector spaces $dT: V(m) \to V(Tm)$. If f is any q-linear function on $V(Tm)$ then fT defined by

$$(fT)(x_1, \cdots, x_q) = f(dTx_1, \cdots, dTx_q), \qquad\qquad x_1, \cdots, x_q \in V(m),$$

is a q-linear function on $V(m)$. Using this notation we define a V-form ωT on M_1 by setting

$$(\omega T)m = \omega(Tm)T .$$

If ω is regular then so is ωT. The following properties of ωT will be used [LG, p. 152]:

$$(1.5) \qquad\qquad d(\omega T) = (d\omega)T,$$

$$(1.6) \qquad\qquad (r_1\omega_1 + r_2\omega_2)T = r_1(\omega_1T) + r_2(\omega_2T), \qquad\qquad r_1, r_2 \in R,$$

$$(1.7) \qquad\qquad P(\omega T) = (P\omega)T,$$

$$(1.8) \qquad\qquad (\omega T_2)T_1 = \omega(T_2T_1),$$

where, in (1.8), T_1 is a mapping of class C^2 of M_1 into M_2 and T_2 a mapping of class C^2 of M_2 into some third manifold M_3 of class C^2; ω is a form on M_3.

2. **Equivariant forms.** We shall assume that a topological group G is acting as a group of transformations on M. By this we mean that for each $g \in G$ a transformation

$$T_g: M \to M$$

of class C^2 is given such that

$$(2.1) \qquad\qquad T_g(m) \text{ is continuous in } g \text{ and } m \text{ simultaneously,}$$

$$(2.2) \qquad\qquad T_{g_1g_2} = T_{g_1}T_{g_2},$$

$$(2.3) \qquad\qquad T_e(m) = m \qquad\qquad \text{for each } m \in M,$$

where e denotes the unit element of G.

We shall also assume that a representation P of G with a finite-dimensional vector space V over R as representation space is given. For each $g \in G$ a

linear transformation $P_g: V \to V$ is then defined satisfying conditions analogous to (2.1)–(2.3).

A (regular) V-form ω on M will be called *equivariant* provided

$$(2.4) \qquad\qquad P_g\omega = \omega T_g$$

for each $g \in G$. It follows from (1.2) and (1.5) that if ω is equivariant then $d\omega$ is equivariant. Propositions (1.4) and (1.6) imply that if ω_1 and ω_2 are equivariant, so is $r_1\omega_1 + r_2\omega_2$ for any $r_1, r_2 \in R$.

A V-form which is the differential of an equivariant form will be called equivariantly exact. The quotient space of the linear set of closed equivariant V-forms of order q on M by the subset of equivariantly exact V-forms will be denoted by $E^q(M, P)$. We shall refer to $E^q(M, P)$ as the q-dimensional cohomology group of M obtained using equivariant V-forms.

If $V = R$ and P is the trivial representation we shall write $E^q(M)$ instead of $E^q(M, P)$. Since the Grassmann product of two equivariant forms is equivariant, it follows as before that the direct sum of the spaces $E^q(M)$ forms a ring $E(M)$.

The equivariant V-forms form a linear subspace of the space of V-forms. This leads to a natural homomorphism of the cohomology groups

$$(2.5) \qquad\qquad \pi : E^q(M, P) \to D^q(M, V).$$

If $V = R$ and P is trivial, π is a ring homomorphism.

The following theorem will be proved in §6.

THEOREM 2.1. *If G is compact then (2.5) maps $E^q(M, P)$ isomorphically into a subspace of $D^q(M, V)$.*

A more detailed analysis of (2.5) follows from the decomposition of P into irreducible components. Such a decomposition always exists if G is compact. Let then $V = V_1 + \cdots + V_k$ be a direct decomposition of V into irreducible invariant subspaces and let P_i be the corresponding representations of G in V_i. Every V-form ω then decomposes uniquely as

$$\omega = \omega_1 + \cdots + \omega_k$$

with ω_i being a V_i-form. There result direct sum decompositions

$$E^q(M, P) = \sum_i E^q(M, P_i), \qquad D^q(M, V) = \sum_i D^q(M, V_i)$$

and an appropriate decomposition of (2.5). Hence we may concentrate our attention on irreducible representations.

In the following two theorems it is assumed that G is compact and connected. The proofs will be given in §6.

THEOREM 2.2. *If the representation P of G is irreducible and nontrivial*

then $E^q(M, P) = \{0\}$, that is, every closed equivariant V-form is equivariantly exact.

THEOREM 2.3. *If $V = R$ and P is trivial, the correspondence* (2.5) *is a ring isomorphism onto*

$$E(M) \approx D(M),$$

that is, the equivariant forms lead to the same cohomology ring as all the regular differential forms on M.

3. **The averaging process.** We assume that G is compact. This implies the existence of a Haar measure with the measure of G being 1. The definition and properties of integrals of real valued functions defined on G carry over in a trivial fashion to functions with values in a finite-dimensional vector space V over R

Given a continuous V-form ω of order q on M consider the family of V-forms

$$\omega^g = P_{g^{-1}}\omega T_g, \qquad\qquad g \in G.$$

For each $m \in M$, $\omega^g(m)$ is a continuous function on G with values in the vector space $C^q(m, V)$ (this will be established in the course of the proof of (3.1) and (3.2) below). Hence the integral

$$(I\omega)m = \int_G \omega^g(m)dg$$

is a well defined element of $C^q(m, V)$. The V-form $I\omega$ thus obtained has the following properties (for ω regular)

(3.1) $I\omega$ is regular,

(3.2) $d(I\omega) = I(d\omega)$,

(3.3) $I\omega$ is equivariant,

(3.4)   if ω is equivariant then $I\omega = \omega$.

Proof of (3.1) and (3.2). Having selected a base in the vector space V, we can represent ω by its components $\omega^{(k)}$ with respect to this base; each $\omega^{(k)}$ is then a differential form of class C^1 on M. Let $m_0 \in M$ and $g_0 \in G$; we select coordinate systems $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ on M at m_0 and $T_{g_0}(m_0)$ respectively. We can find a cubic neighbourhood $U = U(m_0, g_0)$ of m_0 with respect to $(x_1, \cdots, x_n)$, a cubic neighbourhood W of $T_{g_0}(m_0)$ with respect to $(y_1, \cdots, y_n)$ and a neighbourhood $N = N(m_0, g_0)$ of g_0 in G such that $T_g(U) \subset W$ for all $g \in N$. In W each $\omega^{(k)}$ has a certain analytic expression

$$\omega^{(k)} = \sum_{i_1, \cdots, i_q} A^{(k)}_{i_1, \cdots, i_q}(y_1, \cdots, y_n)dy_{i_1} \cdots dy_{i_q}$$

with respect to the coordinates $y_1, \cdots, y_n$; the functions $A^{(k)}_{i_1, \ldots, i_q}$ are of class C^1. On the other hand if $m \in U$ is a point with coordinates $(x_1, \cdots, x_n)$, and if $g \in N$, the y-coordinates of $T_g(m)$ are functions

$$y_i = \theta_i(x_1, \cdots, x_n; g)$$

of $x_1, \cdots, x_n; g$. For g fixed, θ_i are of class C^2 in $x_1, \cdots, x_n$. We have

$$(\omega^{(k)} T_g)(m) = \sum_{i_1, \ldots, i_q} A^{(k)}_{i_1, \ldots, i_q}(\theta_1, \cdots, \theta_n) d\theta_{i_1} \cdots d\theta_{i_q}.$$

On the other hand the mapping P_g can be represented (with respect to the base in V) by a matrix $\rho_{kl}(g)$, and we have

$$(\omega^g(m))^{(k)} = \sum_l \rho_{kl}(g^{-1}) \sum_{i_1, \ldots, i_q} A^{(l)}_{i_1, \ldots, i_q}(\theta_{i_1}, \cdots, \theta_n) d\theta_{i_1} \cdots d\theta_{i_q}.$$

Now, it follows from a theorem of Montgomery[7] that not only the functions θ_i but also their partial derivatives of order 1 and 2 are continuous functions of $(x_1, \cdots, x_n; g)$. This implies that

$$(\omega^g(m))^{(k)} = \sum_{i_1, \ldots, i_q} B^{(k)}_{i_1, \ldots, i_q}(x_1, \cdots, x_n; g) dx_{i_1} \cdots dx_{i_q}$$

where each $B^{(k)}_{i_1, \ldots, i_q}$ for g fixed is of class C^1 and is continuous, together with its partial derivatives in $(x_1, \cdots, x_n; g)$.

Let A be any measurable subset of $N(m_0, g_0)$. Then

$$\left(\int_A \omega^g dg \right)^{(k)}(m) = \sum_{i_1 \cdots i_q} \left(\int_A B^{(k)}_{i_1 \cdots i_q}(x_1, \cdots, x_n; g) dg \right) dx_{i_1} \cdots dx_{i_q}$$

and the functions $\int_A B^{(k)}_{i_1, \ldots, i_q}(x_1, \cdots, x_n; g) dg$ are of class C_1. Consequently $\int_A \omega^g dg$ is a V-form on M of class C^1 on $U(m_0, g_0)$.

Moreover, we have

$$\frac{\partial}{\partial x_{i_0}} \int_A B^{(k)}_{i_1, \ldots, i_q}(x_1, \cdots, x_n, g) dg = \int_A \frac{\partial}{\partial x_{i_0}} B^{(k)}_{i_1, \ldots, i_q}(x_1, \cdots, x_n, g) dg.$$

Therefore

$$\left(d \int_A \omega^g dg \right)(m) = \left(\int_A d\omega^g dg \right)(m) = \left(\int_A (d\omega)^g dg \right)(m)$$

for $m \in U(m_0, g_0)$.

Keeping m_0 fixed we now vary g_0. Since G is compact, there is a finite sequence $g_i, \cdots, g_s$ such that the neighbourhoods $N(m_0, g_i)$ cover G. Let $U(m_0) = \cap_i U(m_0, g_i)$. It follows that $\int_G \omega^g dg$ is of class C^1 on $U(m_0)$ and that $d(\int_G \omega^g dg)(m) = (\int_G (d\omega)^g dg)(m)$ for $m \in U(m_0)$. Hence $I\omega$ is of class C_1 and

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[7] D. Montgomery, *Topological groups of differentiable transformations*, Ann. of Math. vol. 46 (1945) pp. 382–387; Theorem 1, p. 383 and Corollary 1, p. 386.

$d(I\omega) = I(d\omega)$. Consequently $d(I\omega)$ also is of class C^1 and $I\omega$ is regular.

Proof of (3.3). For every $h \in G$ and $m \in M$, we have

$$[(I\omega)T_h](m) = I\omega(T_hm)T_h = \left(\int \omega^g(T_hm)dg \right) T_h$$

$$= \int \omega^g(T_hm)T_h dg = \int (\omega^g T_h)(m)dg = \int (P_{g^{-1}}\omega^g T_g T_h)(m)dg$$

$$= \int (P_h\omega^{gh})(m)dg = P_h \int \omega^{gh}(m)dg = P_h \int \omega^g(m)dg$$

$$= [P_h I(\omega)](m).$$

Hence $(I\omega)T_h = P_h I(\omega)$.

Proof of (3.4). If ω is equivariant then $\omega^g = \omega$ for all $g \in G$ and $(I\omega)(m)$ $= \int_G \omega(m)dg = \omega(m)$ since G has measure 1.

4. Integration of forms over cycles. We shall use the singular homology theory as developed by one of the authors ([8]). In particular $S(M)$ will denote the singular complex of the manifold M. The cells of $S(M)$ are equivalence classes of singular simplexes T which are continuous maps into M

$$T : s \to M$$

of euclidean simplexes s with ordered vertices.

Suppose now that the manifold M is of class C^k. If the mapping T can be extended to a neighborhood U_s of s (in the cartesian space containing s) in such a way that the extended map T' be of class C^k, then we shall say that T is a singular simplex of class C^k. The singular simplexes of class C^k form a closed subcomplex $S^k(M)$ of $S(M)$, and we have the identity chain transformation

$$\epsilon^k : S^k(M) \to S(M).$$

It has been proved([9]) that ϵ^k induces isomorphisms of the respective homology and cohomology groups.

We now return to our assumption that M is of class C^2 and we shall consider only chains and cycles in $S^2(M)$. Let

$$T : s \to M$$

be a singular q-dimensional simplex of class C^2. We shall assume that T has been extended to a map T' of class C^2 of some neighbourhood U_s of s. Given a continuous differential form ω of order q on M we then have the form $\omega T'$ on U_s and the integral $\int_s \omega T'$ is defined in the usual way. This integral is inde-

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[8] S. Eilenberg, *Singular homology theory*, Ann. of Math. vol. 45 (1944) pp. 407–447.

[9] S. Eilenberg, *Singular homology in differentiable manifolds*, Ann. of Math. vol. 48 (1947) pp. 670–681.

pendent of the choice of the extension T' and of the choice of T within its equivalence class.

We define

$$\int_T \omega = \int_s \omega T'.$$

Given a q-dimensional chain $c = \sum r_i T_i$ in $S^2(M)$ (coefficients in R) we define

$$\int_c \omega = \sum r_i \int_{T_i} \omega.$$

If ω is of class C^1 then $\omega T'$ is of class C^1 and the classical Stokes formula implies that

$$\int_c d\omega = \int_{\partial c} \omega$$

for every $(q+1)$-chain c. Hence if ω is closed then $\int_{\partial c}\omega = 0$. It follows that for every homology class z of dimension q and every closed form ω of order q and class C^1 the integral $\int_z \omega$ is defined without ambiguity.

Let ω be a closed V-form of class C^1. Taking a base in V, denote by $\omega^{(k)}$ the components of ω with respect to this base and by $\int_z \omega$ the element of V whose components are the real numbers $\int_z \omega^{(k)}$. Clearly $\int_z \omega$ is independent of the choice of base in V.

If $P: V \to V$ is a linear transformation then clearly

$$(4.1) \qquad \int_z P\omega = P\left(\int_z \omega\right).$$

Let two manifolds M_1, M_2 and a mapping $R: M_1 \to M_2$, all of class C^2, be given. Let ω be a V-form on M_2 of order q and class C^1 and let $T: s \to M_1$ be a singular q-simplex of class C^1. With T' defined as before, we have $(\omega R) T' = \omega(RT')$. Therefore

$$\int_T \omega R = \int_s \omega R T' = \int_{RT} \omega$$

where $RT: s \to M_2$ is a singular simplex of M_2. Consequently for every closed form ω on M_2

$$(4.2) \qquad \int_z \omega R = \int_{Rz} \omega$$

where Rz is the image of the homology class z under the homomorphism of the homology groups induced by R.

5. **Formulation of de Rham's theorems.** Let $H_q(M)$ and $H^q(M)$ denote

the q-dimensional homology and cohomology groups of M with real coefficients defined using the complex $S(M)$. The groups $H^q(M)$ and $H_q(M)$ are in duality, the product of a cohomology class f and a homology class z being the Kronecker index $KI(f, z)$.

For a fixed closed differential form ω of order q on M the expression $\int_z \omega$ is a linear function on $H_q(M)$, hence there is a unique cohomology class $f \in H^q(M)$ such that

$$\int_z \omega = KI(f, z)$$

for each $z \in H_q(M)$. De Rham's theorems(^{8}) imply that the correspondence $\omega \to f$ establishes an isomorphism.

$$D^q(M) \approx H^q(M).$$

Moreover, this isomorphism is a ring isomorphism of the cohomology ring $D(M)$ obtained using differential forms (with the Grassmann multiplication) with the cohomology ring $H(M)$ (with the cup product as multiplication).

In the next section we shall use the following part of the previously stated theorem:

(5.1) *If ω is a closed form such that $\int_z \omega = 0$ for every homology class, then ω is exact.*

There is no explicit proof of the above theorem in the literature. De Rham's original proof is valid for closed manifolds M carrying a simplicial decomposition of a rather special kind(^{10}).

Theorems 2.2 and 2.3 combined with the theorem of de Rham imply that if G is compact and connected then both $D^q(M, V)$ and $E^q(M, P)$ are isomorphic with certain multiples (in the sense of direct sum) of the cohomology group $H^q(M)$ (real coefficients). In the case of D^q the multiplicity is the dimension of V, in the case of E^q it is the number of times the trivial representation occurs in the irreducible decomposition of the representation P.

6. **Proofs of Theorems 2.1–2.3.** To prove Theorem 2.1 consider an equivariant closed V-form ω which is exact. Then $\omega = d\theta$ for some V-form θ. By (3.2) and (3.4) we have

$$d(I\theta) = I(d\theta) = I\omega = \omega.$$

Since $I\theta$ is equivariant by (3.3), it follows that ω is equivariantly exact.

Before we proceed with the proofs of Theorems 2.2 and 2.3 we prove the following proposition:

(6.1) *If G is connected then for every $g \in G$ and every homology class z in M, $T_g z = z$.*

(^{10}) Cf. footnote 3, p. 62.

Let M be given as some simplicial decomposition and let K be a sub-complex of M containing a cycle of the homology class z. Let K_1 be a complex containing K in its interior. We may then find a neighbourhood U of the identity e in G such that $T_g(K) \subset K_1$ for each $g \in U$. If we consider the family of mappings $T_g: K \to K_1$, $g \in U$, it follows that there is a neighbourhood $U_1 \subset U$ of the identity such that any map $T_g: K \to K_1$ with $g \in U_1$ is homotopic with the identity map $T_e: K \to K_1$. Hence $T_g z = z$ for $g \in U_1$. Since G is connected, this holds for any $g \in G$.

Proof of Theorem 2.2. Let ω be a closed equivariant V-form on M. Since $P_g \omega = \omega T_g$ for each $g \in G$ it follows from (4.1), (4.2) and (6.1) that for every homology class z

$$P_g \int_z \omega = \int_z P_g \omega = \int_z \omega T_g = \int_{T_g z} \omega = \int_z \omega.$$

Since this holds for every $g \in G$ and since the representation P is irreducible and nontrivial it follows that $\int_z \omega = 0$. Since this holds for every z, (5.1) implies that ω is exact and, by Theorem 2.1, ω is equivariantly exact.

Proof of Theorem 2.3. We have already shown that $\pi: E^q(M) \to D^q(M)$ is an isomorphism into. It is therefore sufficient to prove that $E^q(M)$ is mapped onto $D^q(M)$. Let ω be a closed form of order q on M. Consider the integral $\int_z I\omega$ over a q-dimensional homology class z. We have

$$\int_z I\omega = \int_z \int_G \omega^g dg = \int_z \int_G \omega T_g dg.$$

Since all the functions involved are continuous, Fubini's theorem can be applied; reversing the order of integration and using (4.2) and (6.1) we have

$$\int_z I\omega = \int_G \int_z \omega T_g dg = \int_G \int_{T_g z} \omega dg = \int_G \int_z \omega dg = \int_z \omega.$$

Hence $\int_z (\omega - I\omega) = 0$ and, by (5.1), $\omega - I\omega$ is exact. This completes the proof.

7. Double equivariance. Let G and H be two groups operating on M. We shall assume here that $V = R$. A regular differential form ω on M will be called doubly equivariant provided

$$\omega T_g = \omega = \omega T_h$$

for all $g \in G$ and $h \in H$. As before we may define cohomology groups $\tilde{E}^q(M)$ using doubly equivariant forms only, and the cohomology ring $\tilde{E}(M)$. As before we have a natural ring homomorphism

(7.1) $$\pi: E(M) \to D(M).$$

THEOREM 7.1. *If G and H are compact and connected and if the transformations T_g and T_h commute,*

$$T_g T_h = T_h T_g$$

for all $g \in G$, $h \in H$, then (7.1) is a ring isomorphism onto

$$\tilde{E}(M) \approx D(M).$$

Proof. Consider the direct product $G \times H$. For $(g, h) \in G \times H$ define $T_{(g,h)} = T_g T_h$. It follows from our assumptions that $G \times H$ is a compact and connected group operating on M. Let ω be a form equivariant relative to $G \times H$, then $\omega T_g T_h = \omega$. Taking $g = e_G$ we find $\omega T_h = \omega$ and similarly $\omega T_g = \omega$ so that ω is doubly equivariant. Conversely every doubly equivariant form is equivariant relative to $G \times H$. Thus Theorem 7.1 is a consequence of Theorem 2.3.

CHAPTER II. LOCALIZATION

8. **The transitive case.** We shall assume now that the compact group G operates on the connected manifold M transitively, that is, that for each pair m_1, $m_2 \in M$ there is an element g in G such that $T_g m_1 = m_2$. We further assume that G operates on M effectively, that is, that none of the transformations T_g, except T_e, leave all the points of M fixed.

Let m_0 be a point of M, and let H be the group of elements $h \in G$ such that $T_h m_0 = m_0$. Then for any $g \in G$, $T_g m_0$ depends only on the coset gH of g modulo H. The mapping $gH \rightarrow T_g m_0$ is then a 1-1 continuous mapping of G/H onto M, and since G/H is compact, it is a homeomorphism. On the other hand it follows from a theorem of Montgomery[11] that G is in this case a Lie group; therefore G/H admits the structure of an analytic manifold. Bochner and Montgomery[12] have also proved that the mapping $gH \rightarrow T_g m_0$ and its inverse are both of class C^2. Therefore, we may assume without loss of generality that M is identical with G/H and that $T_{g_1}(g_2 H) = (g_1 g_2)H$.

This being done, the assumption that G is compact is no longer needed. In the remainder of this chapter G will be an arbitrary Lie group and H a closed subgroup of G.

For the moment we shall study the simple case when H is the trivial subgroup and $M = G$. We shall return to the case of a nontrivial H at the end of this chapter.

Given any function f defined on G it will be convenient to denote by f_e the value of f at the unit element e of G.

Let L be the Lie algebra of the group G [LG, p. 101]. The elements of L are left invariant infinitesimal transformations x of G. Hence x_e is an element of the tangent vector space $V_e = V(e)$ to G at e. Because of the left invariance condition, x is entirely determined by x_e, and we may therefore regard V_e as the vector space of the Lie algebra L.

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[11] Cf. footnote 7, Theorem 2, p. 387.

[12] S. Bochner and D. Montgomery, *Groups of differentiable and real or complex analytic transformations*, Ann. of Math. vol. 46 (1945) pp. 685–694.

Given any V-form ω of order q on G, the element ω_e of $C^q(1, V)$ is a q-linear alternating function on V_e to V. We define

$$\{\omega\}(x_1, \cdots, x_q) = \omega_e(x_{1e}, \cdots, x_{qe}), \qquad x_i \in L.$$

This way to each V-form ω of order q on G there corresponds a q-linear alternating function $\{\omega\}$ with arguments in the Lie algebra L and values in the vector space V. The correspondence $\omega \to \{\omega\}$ is obviously linear. Moreover if ω is equivariant then $\{\omega\} = 0$ implies $\omega = 0$. The last fact follows from the remark that the condition $P_h\omega = \omega T_h$ implies that $\omega(g)$ can be obtained from ω_e by means of suitable linear transformations, and hence $\omega_e = 0$ implies $\omega = 0$.

The passage from ω to $\{\omega\}$ will be referred to as *localization*.

9. Localization of left invariant forms. We shall assume that $V = R$ and that the representation P of G in V is trivial. The equivariance condition on a differential form then becomes a condition of *left invariance*: $\omega = \omega T_h$ where $T_h g = hg$. We consider two Grassmann algebras: 1° the algebra of left invariant differential forms on G, 2° the (contravariant) Grassmann algebra of the vector space of the Lie algebra L. The correspondence $\omega \to \{\omega\}$ is then multiplicative:

$$(9.1) \qquad \{\omega_1 \square \omega_2\} = \{\omega_1\} \square \{\omega_2\}.$$

THEOREM 9.1. *Let G be a Lie group. The correspondence $\omega \to \{\omega\}$ establishes an isomorphism between the algebra of left invariant differential forms on G and the (contravariant) Grassmann algebra of the vector space of the Lie algebra L of G. Moreover*

$$(9.2) \quad \begin{aligned} &\{d\omega\}(x_1, \cdots, x_{q+1}) \\ &= \frac{1}{q+1} \sum_{k<l} (-1)^{k+l+1} \{\omega\}([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1}). \end{aligned}$$

The symbol $\frown$ over a variable indicates that the appropriate variable is to be omitted.

Proof. Let f be a q-linear alternating function on the Lie algebra L. Represent f as a finite sum of Grassmann products

$$f^k = f_1^k \square \cdots \square f_q^k$$

where f_i^k is a linear function on L. For each f_i^k there is [LG, p. 152] an analytic left invariant form ω_i^k of order 1 on G such that $\{\omega_i^k\} = f_i^k$. Define $\omega^k = \omega_1^k \square \cdots \square \omega_q^k$. Clearly ω^k is analytic and left invariant and by (9.1) $\{\omega^k\} = f^k$. This proves the first half of theorem, and in addition it proves that every left-invariant form is analytic.

Before we proceed with the proof of (9.2) it will be convenient to give this expression a slightly different form. We first notice that the right-hand side of (9.2) is an alternating function of the variables $x_1, \cdots, x_{q+1}$. Hence if

we apply the operation of alternation, the right-hand side will not change its value. On the other hand the right-hand side of (9.2) is the sum of $q(q+1)/2$ terms each of which is obtained from the term

$$(9.3) \qquad \{\omega\}([x_1, x_2], x_3, \cdots, x_{q+1})$$

by applying certain permutations to the variables and multiplying by ± 1 according to whether the permutation was even or odd. Hence applying the operation of alternation to the various summands in (9.2) will each time give the same result as alternating (9.3). Hence (9.2) is equivalent with the following formula,

$$(9.4) \qquad \{d\omega\}(x_1, \cdots, x_{q+1}) = \frac{q}{2} A[\{\omega\}([x_1, x_2], x_3, \cdots, x_{q+1})]$$

where A is the abbreviation for "alternation."

We prove (9.4) by induction on q. For $q=1$, (9.4) reads

$$\{d\omega\}(x_1, x_2) = \{\omega\}([x_1, x_2])/2.$$

This formula for left invariant differential forms ω of order 1 is known [LG, p. 153].

Assume that (9.4) holds for orders less than q. Let

$$\omega = \omega_1 \,\square\, \omega_2$$

be a left invariant form of order q which is a Grassmann product of a left invariant form ω_1 of order 1 and a left invariant form ω_2 of order $q-1$. Since

$$d\omega = d\omega_1 \,\square\, \omega_2 - \omega_1 \,\square\, d\omega_2$$

we have, by the inductive hypothesis,

$$\{d\omega\}(x_1, \cdots, x_{q+1}) = \frac{1}{2} A[\{\omega_1\}([x_1, x_2])\{\omega_2\}(x_3, \cdots, x_{q+1})]$$

$$- \frac{q-1}{2} A[\{\omega_1\}(x_1)\{\omega_2\}([x_2, x_3], x_4, \cdots, x_{q+1})]$$

$$= \frac{1}{2} A[\{\omega\}([x_1, x_2], x_3, \cdots, x_{q+1})]$$

$$- \frac{q-1}{2} A[\{\omega\}(x_1, [x_2, x_3], x_4, \cdots, x_{q+1})]$$

$$= \frac{q}{2} A[\{\omega\}([x_1, x_2], x_3, \cdots, x_{q+1})].$$

10. Localization of equivariant V-forms. Let P be a representation of the Lie group G by means of linear transformations of the n-dimensional vector space V. Having chosen basis vectors $v_1, \cdots, v_n$ in V each linear

transformation P_g is described by a matrix $|a_{ij}(g)|$. The coefficients $a_{ij}(g)$ are analytic functions of g, and their differentials da_{ij} are analytic forms of order 1 on G. For every x in the Lie algebra L of G one considers the matrix

$$P(x) = |\{da_{ij}\}(x)|$$

which again may be regarded as a linear transformation of V into itself. The mapping $x \rightarrow P(x)$ is a representation of the Lie algebra L.

THEOREM 10.1. *Let G be a Lie group and P a representation of G by means of linear transformations of a finite-dimensional vector space V over the field of real numbers. The correspondence $\omega \rightarrow \{\omega\}$ establishes an isomorphism of the space of equivariant V-forms ω of order q on G and the space of q-linear alternating functions defined on the Lie algebra L of G with values in V. Moreover*

$$\{d\omega\}(x_1, \cdots, x_{q+1})$$

(10.1)
$$= \frac{1}{q+1} \sum_{k<l} (-1)^{k+l+1} \{\omega\}([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1})$$

$$+ \frac{1}{q+1} \sum_k (-1)^{k+1} P(x_k) \{\omega\}(x_1, \cdots, \hat{x}_k, \cdots, x_{q+1}).$$

Proof. Let f be a q-linear alternating function on L to V. In terms of the basis vectors $v_1, \cdots, v_n$ in V, f can be written as

$$f = f_1 v_1 + \cdots + f_n v_n$$

where each f_i is a q-linear alternating function on L with real values. By Theorem 9.1 there exist left invariant analytic forms Ω_i of order q on G such that $\{\Omega_i\} = f_i$. Define the left invariant V-form Ω on G by $\Omega = \Omega_1 v_1 + \cdots + \Omega_n v_n$ then set

$$\omega(g) = P_g \Omega(g)$$

or more precisely

(10.2)
$$\omega(g) = \sum_{i,j} a_{ij}(g) \Omega_j(g) v_i.$$

Since the $a_{ij}(g)$'s are analytic it follows that ω is an analytic V-form of order q on G. Since $\omega_e = P_e \Omega_e = \Omega_e$ it follows that $\{\omega\} = \{\Omega\} = f$.

Let $g \in G$ and let $T_g : g' \rightarrow gg'$ be the corresponding left translation of G. Since Ω is left invariant we have

$$(\omega T_g)(g') = \omega(gg') T_g = P_{gg'} \Omega(gg') T_g$$
$$= P_{gg'} (\Omega T_g) g' = P_g P_{g'} \Omega(g') = P_g \omega(g')$$
$$= (P_g \omega)(g').$$

Hence $\omega T_g = P_g \omega$ and ω is equivariant. This completes the proof of the first

part of the theorem and incidentally shows that every equivariant V-form on G is analytic.

We proceed with the proof of (10.1). Differentiating (10.2) we have

$$d\omega = \sum_{i,j} (a_{ij} d\Omega_j + da_{ij} \square \Omega_j) v_i.$$

Hence

$$(d\omega)_e = \sum_{i,j} (a_{ij} d\Omega_j)_e v_i + \sum_{i,j} (da_{ij})_e \square (\Omega_j)_e v_i$$

$$= (d\Omega)_e + \sum_{i,j} (da_{ij})_e \square (\Omega_j)_e v_i.$$

Applying both sides to $x_1, \cdots, x_{q+1}$ in the Lie algebra L we have

$$\{d\omega\}(x_1, \cdots, x_{q+1}) = \{d\Omega\}(x_1, \cdots, x_{q+1})$$
$$+ \sum_{i,j} \{da_{ij}\} \square \{\Omega_j\}(x_1, \cdots, x_{q+1}) v_i.$$

The first term on the right-hand side is, by Theorem 9.1,

$$\{d\Omega\}(x_1, \cdots, x_{q+1})$$

$$= \frac{1}{q+1} \sum_{k<l} (-1)^{k+l+1} \{\Omega\}([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1})$$

$$= \frac{1}{q+1} \sum_{k<l} (-1)^{k+l+1} \{\omega\}([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1}).$$

The second term evaluates as follows

$$\sum_{i,j} \{da_{ij}\} \square \{\Omega_j\}(x_1, \cdots, x_{q+1}) v_i$$

$$= A\left[\sum_{i,j} \{da_{ij}\}(x_1) \{\Omega_j\}(x_2, \cdots, x_{q+1}) v_i \right]$$

$$= A[F(x_1)\{\Omega\}(x_2, \cdots, x_{q+1})]$$

$$= A[P(x_1)\{\omega\}(x_2, \cdots, x_{q+1})]$$

$$= \frac{1}{q+1} \sum_k (-1)^{k+1} P(x_k)\{\omega\}(x_1, \cdots, \hat{x}_k, \cdots, x_{q+1}).$$

This completes the proof of Theorem 10.1.

11. **Invariance under right translations.** Given an element $h \in G$ we consider the right translation $R_h : G \to G$ defined by $R_h g = gh$. Let P be a representation of G with representation space V and let ω be an equivariant V-form of order q on G. Consider the V-form ωR_h. We have

$$\omega R_h T_g = \omega T_g R_h = P_g(\omega R_h)$$

which shows that ωR_h is equivariant. If we further denote by C_h the inner automorphism $g \rightarrow hgh^{-1}$ induced by h then

$$\omega R_h = P_h \omega T_{h^{-1}} R_h = P_h \omega C_{h^{-1}}.$$

This proves that $\omega = \omega R_h$ if and only if $\omega C_h = P_h \omega$. Since both ω and ωR_h are equivariant, the relation $\omega = \omega R_h$ is equivalent with $\omega_e = (\omega R_h)_e$ which in turn is equivalent with $(\omega C_h)_e = (P_h \omega)_e$. Hence we see that a necessary and sufficient condition for $\omega = \omega R_h$ is that

(11.1) $$\{\omega\}(C_h x_1, \cdots, C_h x_q) = P_h \{\omega\}(x_1, \cdots, x_q)$$

where $x_1, \cdots, x_q \in L$ and C_h is the automorphism of L induced by the inner automorphism C_h.

THEOREM 11.1. *Let H be a closed and connected subgroup of G, and let L_H be the corresponding subalgebra of L. Let ω be an equivariant V-form of order q on G relative to a representation P. Then $\omega = \omega R_h$ for all $h \in H$ if and only if*

(11.2) $$P(x')\{\omega\}(x_1, \cdots, x_q)$$
$$= \{\omega\}([x_1, x'], x_2, \cdots, x_q) + \cdots + \{\omega\}(x_1, \cdots, x_{q-1}, [x_q, x'])$$

for all $x' \in L_H$ and $x_1, \cdots, x_q \in L$.

Proof. Let $\mathfrak{A}^q$ be the vector space of all q-linear alternating functions on L with values in V. Given $h \in H$ and $f \in \mathfrak{A}^q$, define the elements $P_h f$ and $Q_h f$ of $\mathfrak{A}^q$ by setting

$$(P_h f)(x_1, \cdots, x_q) = P_h(f(x_1, \cdots, x_q)),$$
$$(Q_h f)(x_1, \cdots, x_q) = f(C_h x_1, \cdots, C_h x_q).$$

Both correspondences $P_h : f \rightarrow P_h f$ and $Q_h : f \rightarrow Q_h f$ give linear representations P and Q of H with $\mathfrak{A}^q$ as representation space. We are interested in characterizing those $f \in \mathfrak{A}^q$ for which

(11.3) $$Q_h f = P_h f \qquad \qquad \text{for all } h \in H.$$

Let $h = \exp(tx')$ for $x' \in L_H$. Consider

$$Q_{x'} f = \lim_{t \to 0} \frac{Q_h f - f}{t}, \qquad P_{x'} f = \lim_{t \to 0} \frac{P_h f - f}{t}.$$

Clearly (11.3) implies

(11.4) $$Q_{x'} f = P_{x'} f \qquad \qquad \text{for all } x' \in L_H.$$

Conversely (11.4) implies (11.3) for all h in a neighbourhood of e in H [LG, p. 118] and since H is connected this implies (11.3) for all $h \in H$. We have

(11.5) $$(P_{x'} f)(x_1, \cdots, x_q) = P_{x'}(f(x_1, \cdots, x_q)).$$

Since [LG, p. 124]

$$[x, x'] = \lim_{t \to 0} \frac{C_h x - x}{t}$$

and

$$(Q_{x'}f)(x_1, \cdots, x_q) = \lim_{t \to 0} \frac{f(C_h x_1, \cdots, C_h x_q) - f(x_1, \cdots, x_q)}{t}$$

it follows from the linearity of f that

(11.6)
$$\begin{aligned}(Q_{x'}f)(x_1, \cdots, x_q) \\ = f([x_1, x'], x_2, \cdots, x_q) + \cdots + f(x_1, \cdots, x_{q-1}, [x_q, x']).\end{aligned}$$

Propositions (11.4)–(11.6) combine to give formula (11.2).

12. Invariant forms. A differential form ω on a Lie group G will be called invariant if it is both left and right invariant. It follows from Theorem 11.1 that

(12.1) *A left invariant form ω of order q on a connected Lie group G is invariant if and only if*

(12.2) $\{\omega\}([x_1, x], x_2, \cdots, x_q) + \cdots + \{\omega\}(x_1, \cdots, x_{q-1}, [x_q, x]) = 0$

for all $x, x_1, \cdots, x_q \in L$.

Assuming that ω is invariant we compute $\{d\omega\}$.

$$\begin{aligned}\{d\omega\}(x_1, \cdots, x_{q+1}) \\ = \frac{1}{q+1} \sum_{i<j} (-1)^{i+j+1} \{\omega\}([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1}) \\ = \frac{1}{2(q+1)} \sum_{i \neq j} \epsilon_{ij} \{\omega\}([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1})\end{aligned}$$

where $\epsilon_{ij} = (-1)^{i+j+1}$ if $i<j$, and $\epsilon_{ij} = (-1)^{i+j}$ if $i>j$. For each fixed $j = 1, \cdots$, $q+1$ the sum vanishes in virtue of (12.2). Hence $\{d\omega\} = 0$ and

(12.3) *Every invariant form is closed.*

Assume now that the Lie group G is compact and connected. Theorem 7.1 implies that the cohomology ring of G obtained using invariant forms only is isomorphic (under the natural inclusion homomorphism) with the cohomology ring of G obtained using all the regular differential forms. Combining this with (12.3) we find:

THEOREM 12.1. *Let G be a compact and connected Lie group. Every cohomology class of differential forms on G contains precisely one invariant form. The invariant forms span a ring isomorphic with the conomology ring of the manifold G.*

If in addition to being compact and connected G is also semi-simple then G possesses a Riemannian metric invariant with respect to both left and right translations. Moreover Hodge[13] has shown that the harmonic forms with respect to this Riemannian metric coincide with the forms that are invariant. Thus Theorem 12.1 gives a relatively simple proof of Hodge's theorems on harmonic forms on a Riemannian manifold in the case where this manifold is the group manifold of a semi-simple compact group.

13. The case of a homogeneous space. We now return to the consideration of a homogenous space $G/H = M$, H being a closed subgroup of the Lie group G. We denote by Π the mapping $g \to gH$ of G onto M. We use the symbols T_g and R_g to denote the left and right translations of G while $\bar{T}_g$ will denote the transformations $g'H \to gg'H$ of M onto itself. We have

(13.1) $$\Pi T_g = \bar{T}_g \Pi$$

(13.2) $$\Pi R_h = \Pi \qquad\qquad \text{for } h \in H.$$

Given any V-form ω on M we consider the V-form $\omega\Pi$ on G. We have

(13.3) $$d(\omega\Pi) = (d\omega)\Pi,$$

(13.4) $$\omega\Pi = 0 \quad \text{if and only if } \omega = 0.$$

Let P be a representation of G with V as representation space.

(13.5) $\omega\Pi$ *is equivariant if and only if ω is.*

In fact, $P_g(\omega\Pi) = (P_g\omega)\Pi$ and $(\omega\Pi)T_g = (\omega\bar{T}_g)\Pi$. Hence (13.4) implies that $P_g(\omega\Pi) = (\omega\Pi)T_g$ if and only if $P_g\omega = \omega\bar{T}_g$.

We shall now characterize the V-forms on G of the form $\omega\Pi$.

Let $g \in G$. Consider the tangent vector spaces $V(g)$ at g and $V(\Pi g)$ at Πg. The linear mapping $d\Pi : V(g) \to V(\Pi g)$ is then a mapping onto.

(13.6) *Given a V-form Ω of order q on G there is a V-form ω on M such that $\Omega = \omega\Pi$ if and only if the following two conditions hold:*

(13.7) $\Omega_g(x_1, \cdots, x_q) = 0$ *if $d\Pi x_i = 0$ for at least one of the vectors $x_i \in V(g)$.*

(13.8) $\Omega R_h = \Omega$ *for all $h \in H$.*

The necessity of (13.7) is obvious. If $\Omega = \omega\Pi$ then $\Omega R_h = \omega\Pi R_h = \omega\Pi = \Omega$ which proves (13.8).

Assume now that Ω is given satisfying (13.7) and (13.8). Given a point $m \in M$ select $g \in G$ so that $\Pi g = m$. Given vectors $x_1, \cdots, x_q \in V(m)$ select vectors $\bar{x}_1, \cdots, \bar{x}_q \in V(g)$ such that $d\Pi \bar{x}_i = x_i$. Define $\omega_m(x_1, \cdots, x_q) = \Omega_g(\bar{x}_1, \cdots, \bar{x}_q)$. It follows from the linearity of Ω_g and from (13.7) that $\omega_m(x_1, \cdots, x_q)$ does not depend upon the choice of $\bar{x}_1, \cdots, \bar{x}_q$. If we replace g by gh for some $h \in H$, then (13.7) implies that

$$\Omega_g(\bar{x}_1, \cdots, \bar{x}_q) = (\Omega R_h)_g(\bar{x}_1, \cdots, \bar{x}_q) = \Omega_{gh}(dR_h\bar{x}_1, \cdots, dR_h\bar{x}_q).$$

[13] W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambridge University Press, 1941.

Since $d\Pi dR_h \bar{x}_i = x_i$, this proves that $\omega_m(x_1, \cdots, x_q)$ is independent of the choice of g. It is clear that ω is a V-form such that $\Omega = \omega\Pi$. Moreover ω is analytic if Ω is.

THEOREM 13.1. *Let G be a connected Lie group and H a closed connected subgroup of G. Denote by L and L_H the Lie algebras of G and H respectively, by M the homogenous space G/H, and by Π the mapping $g \to gH$ of G onto M. Consider G as a group of operators for M by setting $\bar{T}_g(g'H) = gg'H$. Let P be a representation of G and V the representation space of V.*

The mapping $\omega \to \{\omega\Pi\}$ establishes a 1-1 correspondence between the equivariant V-forms ω of order q on M and those q-linear alternating functions $\{\Omega\}$ on L which satisfy the conditions

(13.9) $\{\Omega\}(x_1, \cdots, x_q) = 0$ *if at least one x_i is in L_H,*

(13.10) $P(x)\{\Omega\}(x_1, \cdots, x_q) = \{\Omega\}([x_1, x], x_2, \cdots, x_q) + \cdots$

$$+ \{\Omega\}(x_1, \cdots, x_{q-1}[x_q, x])$$

for $x \in L_H$, $x_1, \cdots, x_q \in L$.

To prove Theorem 13.1 we only need to remark that (13.9) and (13.10) are equivalent with (13.7) and (13.8). In fact, if Ω is equivariant, condition (13.7) need only be stated for $g = e$ and then it becomes condition (13.9) since L_H is the kernel of the linear mapping $L = V(e) \to V(\Pi e)$. Conditions (13.8) and (13.10) are equivalent in view of Theorem 11.1.

CHAPTER III. COHOMOLOGY GROUPS OF LIE ALGEBRAS

14. The cohomology ring of a Lie algebra. Let L be a Lie algebra over a field K of characteristic 0. We denote by $C(L)$ the contravariant Grassmann algebra over L, and by $C^q(L)$ the space of elements of order q of $C(L)$. The elements of $C^q(L)$ for $q > 0$ are the q-linear alternating functions in L with values in K and $C^0(L) = K$. The elements of $C^q(L)$ will also be called q-dimensional cochains in L. For two cochains $f_1 \in C^p(L)$, $f_2 \in C^q(L)$ the product $f_1 \square f_2 \in C^{p+q}(L)$ is the Grassmann or alternating product

$$(f_1 \square f_2)(x_1, \cdots, x_{p+q}) = A(f_1(x_1, \cdots, x_p)f_2(x_{p+1}, \cdots, x_{p+q}))$$

where A stands for the alternating operator applied to the variables $x_1, \cdots, x_{p+q} \in L$.

Guided by the formula (9.2), we define for each cochain $f \in C^q(L)$ a cochain $\delta f \in C^{q+1}(L)$ called the *coboundary* of f as follows

$$(\delta f)(x_1, \cdots, x_{q+1})$$

(14.1)
$$= \frac{1}{q+1} \sum_{k<l} (-1)^{k+l+1} f([x_k, x_l], x_1, \cdots, \hat{x}_k, \cdots, \hat{x}_l, \cdots, x_{q+1}).$$

If $q = 0$ we set $\delta f = 0$. As we saw in §9, δf can be defined alternatively as

$$\delta f(x_1, \cdots, x_{q+1}) = \frac{q}{2} A(f([x_1, x_2], x_3, \cdots, x_{q+1})).$$

The operation δ maps $C^q(L)$ linearly into $C^{q+1}(L)$. We shall show that for $f_1 \in C^p(L), f_2 \in C^q(L)$

(14.2) $$\delta(f_1 \square f_2) = (\delta f_1) \square f_2 + (-1)^p f_1 \square \delta f_2.$$

In the case $p = 0$, the formula is obvious. The case $p = 1$ follows from exactly the same computation as the one made in §9. From now on we proceed by induction. Assume that (14.2) is valid for $f_1' \in C^{p-1}(L)$. Let $f \in C^1(L)$ and $f_1 = f \square f_1'$. Then

$$\begin{aligned}
\delta(f_1 \square f_2) = \delta(f \square f_1' \square f_2) &= (\delta f) \square f_1' \square f_2 - f \square \delta(f_1' \square f_2) \\
&= (\delta f) \square f_1' \square f_2 - f \square (\delta f_1') \square f_2 + (-1)^p f \square f_2' \square \delta f_2 \\
&= \delta(f \square f_1') \square f_2 + (-1)^p (f \square f_1') \square \delta f_2 \\
&= (\delta f_1) \square f_2 + (-1)^p f_1 \square \delta f_2.
\end{aligned}$$

This proves (14.2) for any f_1 of the form $f \square f_1'$. It follows by linearity that (14.2) holds for all $f_1 \in C^p(L)$.

We next prove for $f \in C^q(L)$

(14.3) $$\delta\delta f = 0.$$

The formula is obvious for $q = 0$. For $q = 1$ we have

$$6(\delta\delta f)(x_1, x_2, x_3) = 2\delta f([x_1, x_2], x_3) + 2\delta f([x_2, x_3], x_1) - 2\delta f([x_1, x_3], x_2)$$

$$= f([[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2]) = 0.$$

For $q > 1$ we proceed by induction. Assume first that $f = f_1 \square f_2$ where $f_1 \in C^1(L)$, $f_2 \in C^{p-1}(L)$. Then by (14.2) $\delta f = (\delta f_1) \square f_2 - f_1 \square \delta f_2$ and $\delta\delta f = (\delta\delta f_1) \square f_2 + (\delta f_1) \square \delta f_2 - (\delta f_1) \square \delta f_2 + f_1 \square \delta\delta f_2 = 0$. It follows by linearity that (14.3) holds for every $f \in C^q(L)$.

Having established (14.3) we proceed with the familiar definitions of combinatorial topology. A cochain f is a *cocycle* provided $\delta f = 0$. The cocycles of dimension q form a subspace $Z^q(L)$ of $C^q(L)$. A cochain $f \in C^q(L)$ is a *coboundary* if it is of the form $\delta f'$ for some $f' \in C^{q-1}(L)$. The coboundaries form a subspace $B^q(L)$ of $Z^q(L)$. If $q = 0$ then $B^q(L) = 0$ by definition. The factor space $H^q(L) = Z^q(L)/B^q(L)$ is called the *qth cohomology group of the Lie algebra L*.

If $f_1 \in Z^p(L)$, $f_2 \in Z^q(L)$ it follows from (14.2) that $f_1 \square f_2 \in Z^{p+q}(L)$; if furthermore either $f_1 \square B^p(L)$ or $f_2 \square B^q(L)$ then $f_1 \square f_2 \in B^{p+q}(L)$. It follows that a multiplication between elements of $H^p(L)$ and $H^q(L)$ with values in $H^{p+q}(L)$ is defined and that, n being the dimension of L, the direct sum $\sum_{p=0}^{n} H^p(L)$ is thus made into a ring $H(L)$ called the *cohomology ring* of the Lie algebra L.

For $q = 0$ we have $Z^0 = C^0 = K$ and $B^0 = 0$ so that

(14.4) $$H^0(L) = K.$$

For $q=1$ we have $B^1=0$ so that $H^1=Z^1$. If $f\in C^1(L)$ then $(\delta f)(x_1, x_2)$ $=2^{-1}f([x_1, x_2])$. If we therefore denote by $[L, L]$ the subalgebra of L spanned by elements of the form $[x_1, x_2]$, then it appears that f is a cocycle if and only if it vanishes on $[L, L]$. Hence

(14.5) $H^1(L)$ *is the conjugate space of* $L/[L, L]$.

We observe that if n is the dimension of L and $q>n$ then every q-dimensional cochain is identically zero and $H^q(L)=0$.

It is worth observing that an identical cohomology ring is obtained by dropping the factor $1/(q+1)$ in (14.1). In this modified form the preceding definitions apply to Lie algebras over a field of any characteristic.

15. Connections with Lie groups. Theorem 9.1 implies the following theorem.

THEOREM 15.1. *If L is the Lie algebra of the Lie group G, then $H^q(L)$ is isomorphic with the cohomology group $E^q(G)$ obtained using the left invariant differential forms on G. The ring $H(L)$ is isomorphic with the ring $E(G)$.*

Applying Theorem 2.3 and de Rham's theorem we find:

THEOREM 15.2. *If L is the Lie algebra of the compact and connected Lie group G, then $H^q(L)$ is isomorphic with the qth cohomology group $H^q(G)$ with real coefficients and the ring $H(L)$ is isomorphic with the cohomology ring $H(G)$ of G.*

As a corollary we obtain the following generalization of a theorem of Pontrjagin[14].

THEOREM 15.3. *Two locally isomorphic compact connected Lie groups have isomorphic cohomology rings.*

16. Semi-simple Lie algebras. Let L be a Lie algebra over a field K of characteristic 0. A representation P of L with a vector space V as representation space will be called fully reducible if to every P-invariant subspace V_1 of V there is a P-invariant subspace V_2 of V such that V is the direct sum V_1+V_2. If every representation of L is fully reducible then L is called semi-simple.

In particular consider the adjoint representation defined by $\mathrm{ad}(x):y$ $\to[y, x]$ for which the linear space of L is the representation space. The invariant subspaces are then precisely the ideals of L. Hence

(16.1) *If L is semi-simple then for every ideal L_1 in L there is an ideal L_2 in L such that L is the direct sum L_1+L_2.*

Using (16.1) we prove

(14) L. Pontrjagin, *Homologies in compact Lie groups*, Rec. Math. (Mat. Sbornik) N.S. vol. 6 (1939) pp. 389–422.

(16.2) *If L is semi-simple and L_1 is an ideal in L then both L_1 and L/L_1 are semi-simple.*

Proof. Using the natural homomorphism $L \rightarrow L/L_1$ every representation of L/L_1 gives a representation of L. Since the representation of L thus obtained is fully reducible the same holds for the representation of L/L_1. To prove that L_1 is semi-simple we observe that, in virtue of (16.1), L_1 is isomorphic with L/L_2 where L_2 is an ideal of L.

(16.3) *A semi-simple Lie algebra L has center 0.*

Assume that L is semi-simple and has a nonzero center. Since the center is an ideal, we may assume in virtue of (16.2) that L is its own center. Let $x_1, \cdots, x_n$ be a base in L and let V be a 2-dimensional vector space with generators v_1, v_2. Define a representation P of L in V by setting

$$P_{x_1}v_1 = v_2, \qquad P_{x_1}v_2 = 0, \qquad P_{x_i}v_j = 0 \qquad \text{for } i = 2, \cdots, n; j = 1, 2.$$

In this representation the subspace V_2 generated by v_2 is invariant, but since $P_{x_1}(\alpha v_1 + \beta v_2) = \alpha v_2$ for $\alpha, \beta \in K$, no other 1-dimensional subspace is invariant. Hence L is not semi-simple.

An alternative way of formulating (16.3) is

(16.4) *The adjoint representation of a semi-simple Lie algebra is faithful.*

(16.5) *If L is semi-simple then $L = [L, L]$.*

Indeed, $[L, L]$ is an ideal. Therefore, by (16.2), $L/[L, L]$ is semi-simple. Hence, by (16.3), $L/[L, L] = \{0\}$.

From (16.5) and (14.5) we deduce

(16.6) *If L is semi-simple then $H^1(L) = \{0\}$.*

Using (16.6) we shall now prove the following theorem.

THEOREM 16.1. *A compact connected Lie group G is semi-simple if and only if its fundamental group is finite([16]).*

Proof. Let L be the Lie algebra of G. Assume that G is semi-simple. Then L is semi-simple and, by (16.6), $H^1(L) = \{0\}$. Hence, by Theorem 15.2, $H^1(G) = \{0\}$ and the first Betti number of G is zero. This implies that the 1-dimensional homology group of G with integral coefficients is finite. However, this group is isomorphic with the fundamental group of G, since this fundamental group is abelian.

Conversely assume that the fundamental group of G is finite. Then the universal covering group $\tilde{G}$ of G is compact. Since $\tilde{G}$ is simply connected, every representation of the Lie algebra L is induced by some representation of $\tilde{G}$ [LG, p. 113]. $\tilde{G}$ being compact, every representation of $\tilde{G}$ is fully reducible. Hence the same follows for every representation of L. Hence L is semi-simple and so is G.

([16]) Cf. H. Samelson, *A note on Lie groups*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 870–873.

17. **The unitary trick.** A semi-simple Lie algebra (over the field of reals) will be called compact if it is the Lie algebra of some compact connected Lie group. Any proposition concerning the cohomology rings of compact connected semi-simple Lie groups translates in virtue of Theorem 15.2 into a similar proposition concerning the cohomology rings of compact Lie algebras. The "unitary trick" is a general method which then allows us to extend the proposition to arbitrary semi-simple Lie algebras over any field of characteristic 0.

Let L be a Lie algebra over a field K (of characteristic 0) and let N be an extension of K. We shall denote by L_N the Lie algebra obtained from L by extending the ground field from K to N. L is semi-simple if and only if L_N is semi-simple.

A property P of Lie algebras will be called *linear* provided: 1° if a Lie algebra L has the property P, then so does L_N for any extension N of the ground field; 2° if L_N has the property P for some extension N of the ground field then L has property P.

The essence of the unitary trick is then embodied in the following theorem.

THEOREM 17.1. *Let P be a linear property. If all compact Lie algebras have property P then all semi-simple Lie algebras have property P.*

Proof. First assume that L is a semi-simple Lie algebra over the field C of complex numbers. H. Weyl has proved[16] that there is a compact Lie algebra L' such that L'_C is isomorphic with L. Hence L has the property P.

Next assume that L is a semi semi-simple Lie algebra over a field K which can be deduced from the field of rationals by the adjunction of a finite number of elements. Then K may be regarded as a subfield of C. Since L_C is semi-simple it has the property P and therefore L has the property P.

Finally let L be a semi-simple Lie algebra over an arbitrary field K of characteristic 0, and let $x_1, \cdots, x_n$ be a base of L. We then have

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k, \qquad\qquad c_{ijk} \in K.$$

Let K_0 be the smallest subfield of K containing the quantities c_{ijk}. Then the vector space $L_0 = K_0 x_1 + K_0 x_2 + \cdots + K_0 x_n$ has the structure of a Lie algebra over K_0, the brackets $[x_i, x_j]$ having the same meaning in L_0 as in L. It is easily seen that L_0 is semi-simple and that $(L_0)_K = L$. Since the field K_0 can be deduced from the field of rationals by the adjunction of a finite number of elements, it follows that L_0 has the property P and therefore L also has the property P.

18. **Hopf's theorem.** We shall apply the unitary trick to carry over cer-

(16) H. Weyl, *Theorie der Darstellung kontinuierlichen half-einfacher Gruppen durch lineare Transformationen* II, Math. Zeit. vol. 24 (1926) pp. 328–376; Satz 6, p. 375.

tain theorems proved by H. Hopf[17] for compact Lie groups to semi-simple Lie algebras. Hopf's theorem asserts that the cohomology ring $H(G)$ of a compact connected Lie group is isomorphic with the direct sum of cohomology rings of a finite number of odd-dimensional spheres.

THEOREM 18.1. *The cohomology ring $H(L)$ of a semi-simple Lie algebra L over a field K of characteristic 0 is isomorphic with the direct sum of the cohomology rings (over K) of a finite number of odd-dimensional spheres.*

Proof. Hopf's theorem combined with Theorem 15.2 gives a proof for compact Lie algebras. It therefore remains to verify that the property described in the conclusion of the theorem is linear. Let then N be an extension of the ground field K. A simple argument shows then that the cohomology ring $H(L_N)$ regarded as an algebra over N is obtained from $H(L)$ by extending the groundfield from K to N. Hence if the conclusion of the theorem holds for $H(L)$ it also holds for $H(L_N)$ and vice versa.

Let p_q denote the qth Betti number of L, that is, the dimension over K, of the vector space $H^q(L)$. The polynomial

$$(18.1) \qquad P_L(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n,$$

where n is the dimension of L, is then called the Poincaré polynomial of L. Theorem 18.1 implies that if L is semi-simple then

$$(18.2) \qquad P_L(t) = (1 + t^{m_1}) \cdots (1 + t^{m_l})$$

where $m_1, \cdots, m_l$ are odd integers. Substituting $t = -1$ in (18.1) and (18.2) shows that

$$\sum_{q=0}^{n} (-1)^q p_q = 0$$

which is the algebraic counterpart of the known fact that the Euler characteristic of a compact Lie group is 0. Substituting $t = 1$ we find

$$\sum_{q=0}^{n} p_q = 2^l.$$

The quantity l in the case of a compact Lie group is called the rank of G and is the maximal dimension of the abelian subgroups of G.

Since we already know that $H^1(L) = \{0\}$ for L semi-simple it follows that $m_i \geqq 3$ for $i = 1, \cdots, l$. Consequently

(18.3) If L is semi-simple then $H^q(L) = \{0\}$ for $q = 1, 2, 4$.

The case $q = 2$ will be given an algebraic proof later. We shall also show that $H^3(L) \neq \{0\}$.

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[17] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. of Math. vol. 42 (1941) pp. 22–52.

19. Invariant cochains. Guided by the results of (12.1) we define a cochain $f \in C^q(L)$ to be invariant if

$$(19.1) \qquad f([x_1, x], x_2, \cdots, x_q) + \cdots + f(x_1, \cdots, x_{q-1}, [x_q, x]) = 0$$

for all $x, x_1, \cdots, x_q \in L$. The same computation as in §12 shows that

(19.2) *Every invariant cochain is a cocycle.*

THEOREM 19.1. *Let L be a semi-simple Lie algebra over a field of characteristic* 0. *Every cohomology class of $H^q(L)$ contains exactly one invariant cocycle. The invariant cocycles constitute a ring isomorphic with the cohomology ring $H(L)$.*

This theorem could be derived from Theorem 12.1 by the application of the unitary trick. The following proof is purely algebraic.

In the following lemmas the semi-simplicity of L is not assumed.

For each $x \in L$ we define a linear mapping $d_x : C^q(L) \to C^q(L)$ as follows

$$(19.3) \qquad \begin{aligned} &(d_x f)(x_1, \cdots, x_q) \\ &\qquad = f([x_1, x], x_2, \cdots, x_q) + \cdots + f(x_1, \cdots, x_{q-1}, [x_q, x]) \end{aligned}$$

and $d_x f = f$ if $q = 0$.

Condition (19.1) can now be written as

$$(19.1') \qquad\qquad\qquad d_x f = 0.$$

We shall establish the following properties of d_x:

$$(19.4) \qquad\qquad d_x(f_1 \,\square\, f_2) = (d_x f_1) \,\square\, f_2 + f_1 \,\square\, d_x f_2;$$

$$(19.5) \qquad\qquad d_y d_x f - d_x d_y f = d_{[x,y]} f;$$

$$(19.6) \qquad\qquad d_x(\delta f) = \delta(d_x f);$$

(19.7) *If f is a cocycle then $d_x f$ is a coboundary.*

Ad(19.4). The formula is obvious if either f_1 or f_2 has degree zero. Let then $f_1 \in C^1(L)$ and $f_2 \in C^q(L)$, $q > 0$. From the definition of the Grassmann multiplication

$$(f_1 \,\square\, f_2)(x_1, \cdots, x_{q+1}) = \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} f_1(x_i) f_2(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1})$$

hence

$$\begin{aligned} &[d_x(f_1 \,\square\, f_2)](x_1, \cdots, x_{q+1}) \\ &\qquad = -\frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} f_1([x_i, x]) f_2(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}) \\ &\qquad\quad + \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} f_1(x_i)(d_x f_2)(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}) \end{aligned}$$

which proves (19.4) in this case. Assume now that (19.4) holds whenever $f_1 \in C^{p-1}(L)$. Let now $g_1 \in C^1(L)$, $g_2 \in C^{p-1}(L)$ and let $f_1 = g_1 \square g_2$. Then

$$d_x(f_1 \square f_2) = d_x(g_1 \square g_2 \square f_2) = (d_x g_1) \square g_2 \square f_2 + g_1 \square d_x(g_2 \square f_2)$$
$$= (d_x g_1) \square g_2 \square f_2 + g_1 \square (d_x g_2) \square f_2 + g_1 \square g_2 \square d_x f_2$$
$$= (d_x f_1) \square f_2 + f_1 \square (d_x f_2).$$

This implies (19.4) for every f_1 of the form $g_1 \square g_2$, and by linearity (19.4) holds for all f_1, f_2.

Ad(19.5). The formula is trivial for $f \in C^0(L)$. Assume $f \in C^1(L)$. Then (19.5) becomes $-f([[z, x], y]) + f([[z, y], x]) = -f([z, [x, y]])$ which is a consequence of the Jacobi identity. Assume now that (19.5) holds for $f \in C^{q-1}(L)$. Let $f \in C^q(L)$ be of the form $f = f_1 \square f_2$ with $f_1 \in C^1(L)$. Then by (19.4)

$$d_{[x,y]}(f_1 \square f_2) = (d_{[x,y]} f_1) \square f_2 + f_1 \square d_{[x,y]} f_2$$
$$= (d_y d_x f_1) \square f_2 - (d_x d_y f_1) \square f_2 + f_1 \square d_y d_x f_2 - f_1 \square d_x d_y f_2$$
$$= d_y d_x(f_1 \square f_2) - d_x d_y(f_1 \square f_2).$$

q.e.d.

Ad(19.6). The formula is trivial for $f \in C^0(L)$. Let $f \in C^1(L)$ then

$$- (\delta d_x f)(x_1, x_2) = - 2^{-1}(d_x f)([x_1, x_2]) = 2^{-1}f([[x_1, x_2], x])$$

while

$$- (d_x \delta f)(x_1, x_2) = \delta f([x_1, x], x_2) + \delta f(x_1, [x_2, x])$$
$$= 2^{-1}f([[x_1, x], x_2]) + 2^{-1}f([x_1, [x_2, x]])$$

and (19.6) follows from the Jacobi identity. Assume now that (19.6) holds for $f \in C^{q-1}(L)$. Let $f = f_1 \square f_2 \in C^q(L)$ with $f_1 \in C^1(L)$. Then by (14.2) and (19.4)

$$(d_x \delta)(f_1 \square f_2) = d_x[(\delta f_1) \square f_2 - f_1 \square \delta f_2]$$
$$= (d_x \delta f_1) \square f_2 + (\delta f_1) \square d_x f_2 - (d_x f_1) \square \delta f_2 - f_1 \square d_x \delta f_2$$
$$= (\delta d_x f_1) \square f_2 + (\delta f_1) \square d_x f_2 - (d_x f_1) \square \delta f_2 - f_1 \square \delta d_x f_2$$
$$= \delta[(d_x f_1) \square f_2] + \delta[f_1 \square d_x f_2] = \delta d_x(f_1 \square f_2).$$

q.e.d.

Ad(19.7). Since f is a cocycle we have

$$0 = \delta f(x, x_1, \cdots, x_q)$$
$$= \frac{1}{q+1} \sum_{i=1}^{q} (-1)^{i+1} f([x, x_i], x_1, \cdots, \hat{x}_i, \cdots, x_q)$$
$$+ \frac{1}{q+1} \sum_{i<j} (-1)^{i+j+1} f([x_i, x_j], x, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_q)$$

$$= \frac{1}{q+1} (d_x f)(x_1, \cdots, x_q)$$

$$- \frac{1}{q+1} \sum_{i<j} (-1)^{i+j+1} f(x, [x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_q).$$

If we define the cochain $g \in C^{q-1}(L)$ by

$$g(x_1, \cdots, x_{q-1}) = q f(x, x_1, \cdots, x_{q-1})$$

the last term above becomes $(\delta g)(x_1, \cdots, x_q)$. Therefore $\delta g = d_x f$.

With these preliminaries we are ready to prove Theorem 19.1. (19.4) implies

(19.8) *If f_1 and f_2 are invariant then $f_1 \square f_2$ is invariant.*

Proposition (19.5) expresses the fact that d_x is a representation of the Lie algebra L with $C^q(L)$ as representation space. Proposition (19.6) implies

$$d_x Z^q(L) \subset B^q(L) \subset Z^q(L), \qquad d_x B^q(L) \subset B^q(L).$$

Since L is semi-simple $Z^q(L)$ is the direct sum of $B^q(L)$ and some space T^q invariant under the representation $x \to d_x$. The proof will be complete if we show that T^q consists precisely of all the invariant cocycles. For each $f \in T^q$ we have $d_x f \in T^q$ and $d_x f \in B^q$. Hence $d_x f = 0$ and f is invariant. In order to show that all the invariant cocycles are in T^q it is sufficient to show that B^q contains no invariant cocycles.

Consider now the representation $x \to d_x$ with $C^{q-1}(L)$ as representation space. Since $Z^{q-1}(L)$ is an invariant subspace and L is semi-simple, C^{q-1} is the direct sum of Z^{q-1} and some invariant subspace D^{q-1}. Since $\delta : C^{q-1} \to B^q$ maps C^{q-1} onto B^q with Z^{q-1} as kernel, it follows that δ maps D^{q-1} onto B^q isomorphically.

Let now $f \in B^q(L)$ be an invariant cocycle. Then $f = \delta g$ for some $g \in D^{q-1}$. By (19.6) we have $0 = d_x f = d_x \delta g = \delta d_x g$. Hence $d_x g \in Z^{q-1}$. But $d_x g \in D^{q-1}$, whence $d_x g = 0$. Consequently g is invariant and by (19.2) $f = \delta g = 0$. This completes the proof.

20. **The bilinear form of a representation.** Let P be a representation of a Lie algebra L with V as representation space. For $x, y \in L$ define

$$B(x, y) = \text{Trace} (P_x P_y).$$

Then B is a bilinear symmetric form on L and

(20.1) $$B([x, z], y) = B(x, [z, y])$$

since

$$B([x, z], y) = \text{Trace} (P_{[x,z]} P_y) = \text{Trace} (P_x P_z P_y - P_z P_x P_y)$$

$$= \text{Trace} (P_x (P_y P_z - P_z P_y)) = B(x, [z, y]).$$

The following lemma will be used in the sequel.

(20.2) *If L is semi-simple and the representation P is faithful, the bilinear form associated with P is nonsingular.*

This lemma is equivalent to Theorem 1.4 in the paper of Hochschild quoted in footnote 4. This Theorem 1.4 follows easily from Theorem 1.3 of the same paper, which is a well known theorem of Cartan. The matrix which represents the bilinear form of a representation has been considered by Casimir and is sometimes called Casimir's matrix. The fact that it is regular was essential in the algebraic proof given by van der Waerden of the full reducility of representations of semi-simple Lie algebras.

21. **The groups $H^2(L)$ and $H^3(L)$.** Using Theorem 19.1 we shall now prove the following theorem.

THEOREM 21.1. *If L is a semi-simple Lie algebra over a field of characteristic 0 then $H^1(L) = \{0\}$, $H^2(L) = \{0\}$ and $H^3(L) \neq \{0\}$.*

Proof. The first part of the theorem was proved earlier. In order to prove $H^2(L) = 0$ it suffices to show that every invariant 2-cocycle f in L is zero. Since f is a cocycle we have

$$f([x_1, x_2], x_3) - f([x_1, x_3], x_2) + f([x_2, x_3], x_1) = 0.$$

The first two terms cancel out since f is invariant. Hence $f([x_2, x_3], x_1) = 0$. But $[L, L] = L$, whence $f = 0$.

In order to prove that $H^3(L) \neq 0$ it suffices to exhibit a nonvanishing invariant 3-cochain in L. Consider the adjoint representation $\mathrm{ad}(x): y \to [y, x]$. Define $B(x, y) = \mathrm{Trace}\,(\mathrm{ad}(x)\mathrm{ad}(y))$. Then B is a bilinear symmetric form on L. Furthermore by (20.1)

(21.1) $B([x, z], y) = B(x, [z, y]).$

Now define

$$f(x_1, x_2, x_3) = B([x_1, x_2], x_3).$$

It follows from (21.1) that f is alternating and therefore $f \in C^3(L)$. Further

$$f([x_1, x], x_2, x_3) + f(x_1, [x_2, x], x_3) + f(x_1, x_2, [x_3, x])$$
$$= B([[x_1, x], x_2], x_3) + B([x_1, [x_2, x]], x_3) + B([x_1, x_2], [x_3, x])$$
$$= B([[x_1, x_2], x], x_3) - B([x_1, x_2], [x, x_3]) = 0$$

and f is invariant.

If f is identically zero then $B([x, y], z)$ is identically zero. Since $[L, L] = L$ this implies that $B(x, y) = 0$ for all $x, y \in L$. This contradicts (20.2) since the adjoint representation is faithful.

As a corollary we find:

THEOREM 21.2. *The first and second Betti numbers of a compact semi-simple Lie group are* 0. *The third is different from* 0.

As a further corollary we prove:

THEOREM 21.3. *The n-dimensional sphere S^n is a group manifold for the values $n = 1, 3$ only.*

Proof. Suppose that S^n is a group manifold and $n > 1$. Since S^n is compact it can be represented as a Lie group G. Since $n > 1$, S^n is simply-connected and therefore, by Theorem 16.1, G is semi-simple. The preceding theorem then implies that $n = 3$.

22. Relative cohomology groups. Let L' be a subalgebra of a Lie algebra L over a field of characteristic 0. A cochain $f \in C^q(L)$ will be called *orthogonal* to L' provided the following two conditions hold

$$(22.1) \qquad\qquad f(x_1, \cdots, x_q) = 0 \qquad\qquad \text{if } x_1 \in L',$$

$$(22.2) \qquad\qquad d_x f = 0 \qquad\qquad \text{for } x \in L'.$$

These cochains form a subspace $C^q(L, L')$ of $C^q(L)$.

(22.3) If $f_1 \in C^p(L, L')$ and $f_2 \in C^q(L, L')$ then $f_1 \square f_2 \in C^{p+q}(L, L')$.

Clearly $f_1 \square f_2$ satisfies condition (22.1). Condition (22.2) follows directly from (19.4).

(22.4) If $f \in C^q(L, L')$ then $\delta f \in C^{q+1}(L, L')$.

Indeed for $x \in L'$ we have $(\delta f)(x, x_1, \cdots, x_q) = (d_x f)(x_1, \cdots, x_q) = 0$ if $x \in L'$ and $d_x \delta f = \delta d_x f = 0$.

We define $Z^q(L, L') = Z^q(L) \cap C^q(L, L')$ and $B^q(L, L') = \delta C^{q-1}(L, L')$ for $q > 0$ and $B^0(L, L') = 0$. The relative cohomology group of L mod L' is then defined as the quotient space $H^q(L, L') = Z^q(L, L')/B^q(L, L')$. The cohomology ring $H(L, L')$ is defined similarly as in §14.

Theorem 13.1 combined with Theorem 2.3 and de Rham's theorem then imply the following theorem.

THEOREM 22.1 *Let G be a compact connected Lie group, H a closed connected subgroup of G and M the homogenous space G/H. Let L be the Lie algebra of G and L_H the subalgebra corresponding to H. Then the cohomology groups $H^q(L, L_H)$ and $H^q(M)$ are isomorphic and the rings $H(L, L_H)$ and $H(M)$ are ring isomorphic.*

We remark that if L' is an ideal in L then condition (22.2) is a consequence of (22.1). Every q-cochain of L orthogonal to L' may therefore be regarded as a q-cochain of the quotient algebra L/L' and vice versa. This correspondence commutes with δ and thus gives $H^q(L, L') \approx H^q(L/L')$.

CHAPTER IV. COHOMOLOGY GROUPS ASSOCIATED WITH A REPRESENTATION

23. Definition of the cohomology groups. Let L be a Lie algebra over a field K of characteristic 0, and let P be a representation of L by linear transformations of a vector space V of finite dimension over K. A q-linear alternating mapping of L into V will be called a q-dimensional V-cochain or shorter q-V-cochain. The q-V-cochains form a space $C^q(L, V)$. By definition $C^0(L, V) = V$.

We define a linear mapping $f \to \delta f$ of $C^q(L, V)$ into $C^{q+1}(L, V)$ by the formula

(23.1)
$$
\begin{aligned}
(\delta f)&(x_1, \cdots, x_{q+1}) \\
&= \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} P(x_i) f(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}) \\
&+ \frac{1}{q+1} \sum_{i<j} (-1)^{i+j+1} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{q+1}).
\end{aligned}
$$

If $q = 0$ then $f \in V$ and δf is defined by

$$(\delta f)(x) = P(x)f.$$

We propose to prove that

(23.2)
$$\delta \delta f = 0.$$

A direct proof would be quite cumbersome. We shall give a short proof based on some auxiliary concepts that will be needed in the sequel.

For each $x \in L$ we define a linear mapping $d_x : C^q(L, V) \to C^q(L, V)$ by setting

(23.3)
$$
\begin{aligned}
(d_x f)(x_1, \cdots, x_q) = P_x[f(x_1, \cdots, x_q)] &- f([x_1, x], x_2, \cdots, x_q) \\
&- \cdots - f(x_1, \cdots, x_{q-1}[x_q, x]).
\end{aligned}
$$

If $q = 0$, d_x is defined by

$$d_x f = P_x f.$$

For $x \in L$ we also define a linear mapping $f \to f_x$ of $C^{q+1}(L, V)$ into $C^q(L, V)$ by setting

(23.4)
$$f_x(x_1, \cdots, x_q) = (q+1)f(x, x_1, \cdots, x_q).$$

A direction substitution into the definitions yields the following two formulae, for $x, y \in L$ and $f \in C^q(L, V)$

(23.5)
$$(d_x f)_y = d_x(f_y) - f_{[y, x]},$$

(23.6)
$$(\delta f)_x = d_x f - \delta(f_x).$$

We shall now prove that

(23.7)
$$d_y d_z f - d_x d_y f = d_{[x,y]} f$$

which expresses the fact that $x \to d_x$ is a representation of L. If $f \in C^0(L, V)$ then (23.7) becomes $P_y P_x f - P_x P_y f = P_{[x,y]} f$ which is true since P is a representation. Assume that (23.7) has been proved for all $f \in C^{q-1}(L, V)$ and let $f \in C^q(L, V)$, $q > 0$. Then for every $z \in L$ we have by (23.5)

$$(d_y d_z f)_z = d_y[(d_z f)_z] - (d_z f)_{[z,y]}$$
$$= d_y d_x(f_z) - d_y(f_{[z,x]}) - (d_z f)_{[z,y]}$$
$$= d_y d_x(f_z) - (d_y f)_{[z,x]} - f_{[[z,x],y]} - (d_z f)_{[z,y]}.$$

Therefore

$$(d_y d_z f)_z - (d_x d_y f)_z = d_y d_x(f_z) + f_{[[z,y],x]} - d_x d_y(f_z) - f_{[[z,x],y]}$$
$$= d_{[x,y]}(f_z) - f_{[z,[x,y]]} = (d_{[x,y]} f)_z.$$

Since this holds for each $z \in L$, (23.7) follows.

Next we prove

(23.8)
$$\delta d_x f = d_x \delta f.$$

If $f \in C^0(L, V)$ then

$$(d_x \delta f)(x_1) = P_x[\delta f(x_1)] - \delta f([x_1, x]) = P_x P_{x_1} f - P_{[x_1, x]} f$$
$$= P_{x_1} P_x f = P_{x_1}(d_x f) = (\delta d_x f)(x_1)$$

and (23.8) is proved in this case. Assume now that (23.8) has been proved for all $f \in C^{q-1}(L, V)$ and let $f \in C^q(L, V)$, $q > 0$. Applying (23.5), (23.6) and (23.7) gives

$$(\delta d_x f)_y - (d_x \delta f)_y = d_y d_x f - \delta[(d_x f)_y] - d_x[(\delta f)_y] + (\delta f)_{[y,x]}$$
$$= d_y d_x f - \delta d_x f_y - \delta(f_{[y,x]}) - d_x[(\delta f)_y] + d_{[y,x]} f - \delta(f_{[y,x]})$$
$$= d_y d_x f - \delta d_x(f_y) - d_x d_y f + d_x \delta(f_y) + d_{[y,x]} f$$
$$= d_x \delta(f_y) - \delta d_x(f_y) = 0.$$

Since this holds for every $y \in L$, (23.8) follows.

We can now prove (23.2). If $f \in C^0(L, V)$ then $2(\delta \delta f)(x_1, x_2) = P_{x_1}[(\delta f)(x_2)] - P_{x_2}[(\delta f)(x_1)] + \delta f([x_1, x_2]) = P_{x_1} P_{x_2} f - P_{x_2} P_{x_1} f + P_{[x_1, x_2]} f = 0$ which proves (23.2) in this case. Assume now that (23.2) holds for all $f \in C^{q-1}(L, V)$ and let $f \in C^q(L, V)$, $q > 0$. Then by (23.6) and (23.8)

$$(\delta \delta f)_x = d_x \delta f - \delta[(\delta f)_x] = d_x \delta f - \delta d_x f + \delta \delta(f_x) = 0.$$

Since this holds for every $x \in L$, (23.2) follows.

Having proved that $\delta \delta f = 0$ we define the space $Z^q(L, P)$ of q-P-cocycles as the kernel of the transformation $\delta : C^q \to C^{q+1}$, and the space $B^q(L, P)$ of q-P-coboundaries as the image δC^{q-1}. By definition $B^0(L, P) = 0$. The

cohomology groups of L over P are then defined as the quotient space $H^q(L, P) = Z^q(L, P)/B^q(L, P)$.

Since $H^0(L, P) = Z^0(L, P)$ and $(\delta f)(x) = P_x f$ for every $x \in L$ and $f \in C^0(L, V)$, we conclude that:

(23.9) $H^0(L, P)$ *is the subspace of the invariant elements of* V.

For $q > \dim L$ we have $H^q(L, P) = 0$.

24. The case of semi-simple algebras.

THEOREM 24.1. *If L is a semi-simple Lie algebra over a field of characteristic* 0, *and P is an irreducible and nontrivial representation of L then* $H^q(L, P) = 0$ *for all dimensions q.*

This theorem shows that in the semi-simple case nothing is gained by studying cohomology groups over representations. In fact any representation P decomposes into irreducible representations and this carries with it a direct decomposition of the cohomology groups. Hence $H^q(L, P)$ is isomorphic with the direct sum of several copies of $H^q(L)$.

A proof of Theorem 24.1 could be derived from Theorems 2.2 and 10.1 by the application of the unitary trick. The following proof is algebraic[18].

Since the representation P is not trivial, the kernel L_0 of P is not all of L, since L is semi-simple there is an ideal L_1 in L such that L is the direct sum of L_0 and L_1. Let $y_1, \cdots, y_n$ be a base in L_1. Since the bilinear form $B(y, z)$ associated with the representation P of L_1 is nonsingular we can select a dual base $z_1, \cdots, z_n$ for L_1 such that

$$B(y_i, z_j) = \delta_j^i.$$

For each $x \in L$ we have $[y_i, x] \in L_1$ and therefore $[y_i, x] = \sum_{j=1}^n c_{ij} y_j$. Similarly $[x, z_i] = \sum_{i=1}^n d_{ij} z_j$. We shall prove that $c_{ij} = d_{ij}$. Indeed we have $B([y_i, x], z_j) = B(\sum_k c_{ik} y_k, z_j) = c_{ij}$ and similarly $B(y_i, [x, z_j]) = d_{ij}$. Hence $c_{ij} = d_{ij}$ by formula (20.1). This implies the following two propositions for any $x \in L$ and any linear function f of L to V.

(24.1) $$\sum_{i=1}^n P_{[y_i, x]} f(z_i) = \sum_{i=1}^n P_{y_i} f([x, z_i]),$$

(24.2) $$\sum_{i=1}^n (P_{y_i} P_{[x, z_i]} - P_{[y_i, x]} P_{z_i}) = 0.$$

Consider the linear transformation of V into itself

$$\Gamma = \sum_{i=1}^n P_{y_i} P_{z_i}.$$

Since the trace of Γ is n it follows that $\Gamma \neq 0$. Further

[18] Cf. J. H. C. Whitehead, *Certain equations in the algebra of a semi-simple infinitesimal group*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 220–237.

$$\Gamma P_x = \sum_{i=1}^{n} P_{y_i} P_{z_i} P_x = \sum_{i=1}^{n} P_{y_i}(P_{[z,z_i]} - P_{y_i} P_x P_{z_i})$$

$$= \sum_{i=1}^{n} (P_{y_i} P_{[z,z_i]} - P_{[y_i,z]} P_{z_i} + P_x P_y P_{z_i}) = P_x \Gamma.$$

Since Γ commutes with each P_x the space $\Gamma(V)$ is an invariant subspace of V. But the representation P of L is irreducible hence $\Gamma(V) = V$ and Γ has an inverse Γ^{-1}. Clearly $\Gamma^{-1} P_x = P_x \Gamma^{-1}$.

Let $f \in C^q(L, V)$ and let Λ be a linear transformation $V \to V$. Then $\Lambda f \in C^q(L, V)$. A direct computation shows that

(24.3)
$$(\delta \Lambda f - \Lambda \delta f)(x_1, \cdots, x_{q+1})$$
$$= \frac{1}{q+1} \sum_{i=1}^{q+1} (-1)^{i+1} (P_{x_j} \Lambda - \Lambda P_{x_j}) f(x_1, \cdots, \hat{x}_i, \cdots, x_{q+1}).$$

Suppose now that f is a cocycle. Consider the cochains $f_{z_i} \in C^{q-1}(L, V)$ and let $g = \sum_{i=1}^{n} P_{y_i} f_{z_i}$. Then

$$\delta g = \sum_{i=1}^{n} (\delta P_{y_i} f_{z_i} - P_{y_i} \delta f_{z_i}) + \sum_{i=1}^{n} P_{y_i} \delta f_{z_i}$$

and by (23.6)

$$\delta g = \sum_{i=1}^{n} (\delta P_{y_i} f_{z_i} - P_{y_i} \delta f_{z_i}) + \sum_{i=1}^{n} P_{y_i} d_{z_i} f.$$

Hence by (24.3)

$$(\delta g)(x_1, \cdots, x_q)$$
$$= \sum_{i=1}^{n} \frac{1}{q} \sum_{j=1}^{q} (-1)^{i+1} (P_{x_j} P_{y_i} - P_{y_i} P_{x_j}) f_{z_i}(x_1, \cdots, \hat{x}_j, \cdots, x_q)$$
$$+ \sum_{i=1}^{n} P_{y_i}(d_{z_i} f)(x_1, \cdots, x_q)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{q} (-1)^{i+1} P_{[y_i, z_j]} f(z_i, x_1, \cdots, \hat{x}_j, \cdots, x_q)$$
$$+ \sum_{i=1}^{n} P_{y_i} P_{z_i} f(x_1, \cdots, x_q)$$
$$- \sum_{i=1}^{n} \sum_{j=1}^{q} (-1)^{i+1} P_{y_i} f([x_j, z_i], x_1, \cdots, \hat{x}_j, \cdots, x_q).$$

The two double sums cancel in view of (24.1) and therefore $\delta g = \Gamma f$. Since Γ^{-1}

and P_x commute, (24.3) implies that $\delta\Gamma^{-1}g = \Gamma^{-1}\delta g = \Gamma^{-1}\Gamma f = f$. Hence f is a coboundary, q.e.d.

25. The group $H^1(L, P)$. A detailed discussion of the connections between the first cohomology groups and derivations in a Lie algebra is given by Hochschild[4].

THEOREM 25.1. *A Lie algebra L over a field of characteristic 0 is semi-simple if and only if $H^1(L, P) = \{0\}$ for every representation P of L.*

Proof. It follows from (16.6) and Theorem 24.1 that if L is semi-simple then $H^1(L, P) = 0$. To prove the converse, consider any representation P of L with representation space V and let U be an invariant subspace of V. Choose a subspace W of B (not necessarily invariant) such that V is the direct sum of U and W. For each $x \in L$ denote by $\tilde{P}_x$ the transformation P_x treated as a linear transformation $W \to V$. Clearly

$$\tilde{P}_x = Q_x + R_x$$

where $Q_x : W \to U$ and $R_x : W \to W$ are linear in both $x \in L$ and $w \in W$. Let S be the space of linear mappings $W \to U$. Then $Q_x \in S$. For $s \in S$ and $x \in L$ define $T_x s \in S$ by

$$T_x s = \tilde{P}_x s - s R_x.$$

Since

$$\tilde{P}_{x_2}\tilde{P}_{x_1} = \tilde{P}_{x_2}Q_{x_1} + \tilde{P}_{x_2}R_{x_1} = \tilde{P}_{x_2}Q_{x_1} + Q_{x_2}R_{x_1} + R_{x_2}R_{x_1}$$

and $\tilde{P}_{x_2}\tilde{P}_{x1} - \tilde{P}_{x1}\tilde{P}_{x_2} = \tilde{P}_{[x_1,x_2]}$ it follows that

$$R_{[x_1, x_2]} = R_{x_2}R_{x_1} - R_{x_1}R_{x_2},$$

(25.1)
$$Q_{[x_1, x_2]} = T_{x_2}Q_{x_1} - T_{x_1}Q_{x_2}.$$

Since

$$T_{x_2}T_{x_1}s = \tilde{P}_{x_2}T_{x_1}s - (T_{x_1}s)R_x.$$
$$= \tilde{P}_{x_2}\tilde{P}_x.s - \tilde{P}_{x_2}sR_{x_1} - \tilde{P}_{x_1}sR_{x_2} + sR_{x_2}R_x$$

therefore

$$T_{x_2}T_{x_1}s - T_{x_1}T_{x_2}s = \tilde{P}_{x_2}\tilde{P}_{x_1}s - \tilde{P}_{x_1}\tilde{P}_{x_2}s + sR_{x_1}R_{x_2} - sR_{x_2}R_x$$
$$= \tilde{P}_{[x_1, x_2]}s - sR_{[x_1, x_2]} = T_{[x_1, x_2]}s.$$

This shows that T is a representation of L with S as representation space. Then $Q \in C^1(L, S)$ and condition (25.1) means that $\delta Q = 0$. Since $H^1(L, T) = 0$, Q is a coboundary. Hence there is a linear transformation $s_0 \in S$ such that

$$Q_x = T_x s_0 = \tilde{P}_x s_0 - s_0 R_x.$$

Hence

$$\tilde{P}_x = Q_x + R_x = \tilde{P}_x s_0 - s_0 R_x + R_x$$

which implies

(25.2) $$P_x(w - s_0(w)) = (R_x - s_0 R_x)w$$

for each $x \in L$, $w \in W$.

Define

$$t(w) = w - s_0(w).$$

Since $s_0(w) \in U$, t maps W isomorphically onto some subspace $\overline{W}$ of V and V is the direct sum $U + \overline{W}$. Condition (25.2) then becomes $P_x(tw) = t(R_x w)$ which shows that $\overline{W}$ is invariant under P.

26. Extensions of Lie algebras. In this and the following section it is not assumed that the groundfield is of characteristic 0. Consequently the factor $1/(q+1)$ in the formula for the W coboundary will be omitted.

Let L^* and L be two Lie algebras and $\phi : L^* \to L$ a homomorphism of L^* onto L. The pair (L^*, ϕ) is called an *extension*. The kernel V of ϕ is called the kernel of the extension. If $[V, V] = 0$ the extension is said to have an *abelian kernel*. An extension is called *inessential* if there exists a subalgebra L' of L^* which is mapped by ϕ isomorphically onto L. As a vector space L^* is then the direct sum $L' + V$.

The proof of the following theorem is identical with the proof of an analogous theorem established by Hochschild([4]) for associative algebras.

THEOREM 26.1. *If every extension of L with an abelian kernel is inessential, then every extension of L is inessential.*

The extensions of L with an abelian kernel will be now studied in greater detail. Let (L^*, ϕ) be such an extension with kernel V. We select a linear mapping $u : L \to L^*$ such that $\phi u(x) = x$ for each $x \in L$. For each $x \in L$ we then have a mapping $P_x v = [v, u(x)]$ of V into itself. Since $[V, V] = 0$ it can be easily verified that P_x is independent of the choice of u. Further $P_y P_x v - P_x P_y v = [[v, u(x)], u(y)] - [[v, u(y)], u(x)] = [v, [u(x), u(y)]] = [v, u([x, y])] = P_{[x,y]}v$, and therefore P is a representation of L with V a representation space.

Suppose now that the space V and the representation P of L in V is given. Any pair (L^*, ϕ) where ϕ is a homomorphism of L^* onto L with kernel V, such that $[V, V] = 0$ and which leads to the given representation P, will be called an extension of L by P. Two such extensions (L_1^*, ϕ_1) and (L_2^*, ϕ_2) are isomorphic if there is an isomorphism $\theta : L_1^* \to L_2^*$ such that $\theta(v) = v$ for $v \in V$ and that $\phi_1 = \phi_2 \theta$.

Given an extension (L^*, ϕ) of L by P we select a linear mapping $u : L \to L^*$ with $\phi u(x) = x$. Since $\phi([u(x), u(y)]) = [x, y] = \phi u([x, y])$ therefore there is an element $f(x, y) \in V$ such that

$$[u(x), u(y)] = f(x, y) + u([x, y]).$$

The function f is called the *factor set* corresponding to the function u. Clearly f is a 2-V-cochain.

For x, y, $z \in L$ we have

$$[[u(x), u(y)], u(z)] = [f(x, y), u(z)] + [u([x, y]), u(z)]$$
$$= P_z f(x, y) + f([x, y], z) + u([[x, y], z]).$$

Therefore Jacobi's identity gives

$$P_z f(x, y) + P_y f(z, x) + P_x f(y, z) + f([x, y], z) + f([z, x], y) + f([y, z], x) = 0$$

or $(\delta f)(x, y, z) = 0$. It follows that the factor set f is a 2-P-cocycle.

If $\bar{u}$ is a different linear mapping $\bar{u}: L \rightarrow L^*$ such that $\phi\bar{u}(x) = x$, then $h(x) = \bar{u}(x) - u(x) \in V$ and h is a 1-V-cochain. If $\bar{f}$ is the factor set corresponding to $\bar{u}$ then

$$\bar{f}(x, y) = [\bar{u}(x), \bar{u}(y)] - \bar{u}([x, y]) = [u(x), u(y)] + [h(x), u(y)]$$
$$+ [u(x), h(y)] + [h(x), h(y)] - u([x, y]) - h([x, y])$$
$$= f(x, y) - [P_x h(y) - P_y h(x) + h([x, y])]$$

so that

$$\bar{f} = f - \delta h.$$

This proves that the cohomology class of f is independent of the choice of u. Thus to each extension (L^*, ϕ) corresponds a definite element of $H^2(L, P)$. Clearly two isomorphic extensions determine the same element of $H^2(L, P)$. Given an element of $H^2(L, P)$ a corresponding extension can be constructed as follows. Let f be a cocycle belonging to the given element of $H^2(L, P)$. As a vector space L^* is the direct sum $L + V$, with $\phi(x, v) = x$ and $u(x) = (x, 0)$. Commutation in L is defined by the formula

$$[(x_1, v_1), (x_2, v_2)] = ([x_1, x_2], P_{x_2} v_1 - P_{x_2} v_2 + f(x_1, x_2)).$$

It is further easy to see that an extension is inessential if and only if the corresponding cohomology class is zero.

This implies the following theorem.

THEOREM 26.2. *The elements of $H^2(L, P)$ are in a 1-1 correspondence with isomorphism classes of extensions of L by P.*

COROLLARY 26.1. *In order that every extension of L over P be inessential it is necessary and sufficient that $H^2(L, P) = 0$.*

Combining this with Theorem 26.1, we find:

THEOREM 26.3. *In order that every extension of a Lie algebra L be inessential it is necessary and sufficient that $H^2(L, P) = 0$ for every representation P.*

In view of Theorems 21.1 and 24.1 this condition is satisfied when L is a

semi-simple Lie algebra over a field of characteristic 0. Thus we obtain Levi's theorem:

THEOREM 26.4. *If L is a semi-simple Lie algebra over a field of characteristic 0, then every extension of L is inessential.*

27. **Irreducible extensions. Exterior center.** An extension (L^*, ϕ) of L by a representation P will be called *irreducible* if there is no proper subalgebra of L^* which is mapped by ϕ onto L. Let V_1 be a proper invariant subspace of V. This implies that V_1 is an ideal in L^* and L^*/V_1 may also be regarded as an extension of L. It is easy to see that (L^*, ϕ) is irreducible if and only if the extensions L^*/V_1 are essential for every proper invariant subspace V_1 of V.

We shall now consider extensions (L^*, ϕ) corresponding to the trivial representation of L in V. Such extensions are called *central*, because V is then in the center of the extension. These extensions are in a 1-1-correspondence with the element of the cohomology group $H^2(L, V)$ (we write V instead of the trivial representation of L in V).

Any linear map $V \to V_1$ induces in the obvious way a homomorphism $H^2(L, V) \to H^2(L, V_1)$. In particular every linear functional $\hat{v}$ on L (with values in the ground field) induces a homomorphism $\hat{v}_2$ of $H^2(L, V)$ into $H^2(L)$. This way every element $e \in H^2(L, V)$ determines a homomorphism

$$\Phi_e: \quad \widehat{V} \to H^2(L).$$

LEMMA 27.1. *The correspondence $e \to \Phi_e$ establishes an isomorphism of $H^2(L, V)$ with the space of all linear maps $\widehat{V} \to H^2(L)$ of the conjugate space of V into $H^2(L)$.*

The proof is simple and is left as an exercise to the reader.

Each central extension $E = (L^*, \phi)$ of L with kernel V determines uniquely an element of $H^2(L, V)$ and this in turn determines a linear map $\widehat{V} \to H^2(L)$ that we shall denote by ϕ_E. Thus $E \to \phi_E$ establishes a 1-1 correspondence between the equivalence classes of extensions E and the linear maps $\widehat{V} \to H^2(L)$.

THEOREM 27.1. *A central extension $E = (L^*, \phi)$ with kernel V is irreducible if and only if the corresponding linear transformation $\phi_E: \widehat{V} \to H^2(L)$ is an isomorphism into.*

Proof. Suppose that E is not irreducible. There is then a proper subalgebra L' of L^* which is mapped by ϕ onto L. Then $V' = L' \cap V$ is a proper subspace of V and there exists a nonzero element $\hat{v} \in \widehat{V}$ which annihilates V'. Let $u: L \to L'$ be a linear map such that $\phi u = $ identity. The factor set corresponding to this u will have values in V' and therefore will be annihilated by $\hat{v}$. Hence $\phi_E \hat{v} = 0$, which shows that ϕ_E is not an isomorphism.

Conversely, let $\hat{v} \in \widehat{V}$, $\hat{v} \neq 0$, and $\phi_E \hat{v} = 0$. Let V' be the space of zeros of

$\hat{v}$ and let V'' be a subspace of V such that V is the direct sum $V' + V''$. Let $u: L \to L^*$ be a linear map as before and let $f \in Z^2(L, V)$ be the corresponding factor set. The cocycle f determines cocycles $f' \in Z^2(L, V')$ and $f'' \in Z^2(L, V'')$ such that $f = f' + f''$. Since $\hat{v}f$ is a coboundary it follows that f'' is a coboundary. Let then $f'' = \delta h$ where $h \in \delta C^1(L, V'')$. Replacing the function $u: L \to L^*$ by

$$\bar{u}(x) = u(x) + h(x)$$

we find that the factor set is

$$\jmath = f - \delta h = f'.$$

Then $L' = \bar{u}(L) + V'$ is a proper subalgebra of L^* and ϕ maps L' onto L. Thus E is not irreducible.

Consider now the case when $V =$ conjugate space of $H^2(L)$. Let ϕ be the natural isomorphism of $\hat{V}$ with $H^2(L)$. Then there is a central extension $E = (L^*, \phi)$ with kernel V such that $\phi_E = \phi$. It follows from the previous theorem that E is irreducible. The kernel of ϕ is then called the full exterior center of L. We can therefore state the following theorem.

THEOREM 27.2. *$H^2(L)$ is the conjugate space of the full exterior center of the Lie algebra L.*

28. **The relative case.** The definition of §22 will now be carried over to the case of cohomology groups over a representation. Let L be a Lie algebra over a field of characteristic 0, L' a subalgebra of L and P a representation of L with the vector space V as representation space.

A q-V-cochain f on L will be called *orthogonal* to L' if, as in §22,

(28.1) $\qquad f(x_1, \cdots, x_q) = 0 \qquad$ if $x_1 \in L'$,

(28.2) $\qquad d_x f = 0 \qquad$ for $x \in L'$.

Condition (28.1) can also be equivalently written as

(28.1') $\qquad f_x = 0 \qquad$ if $x \in L'$.

It follows from (23.6) and (23.8) that if f is orthogonal to L' then δf also is. Consequently if we denote by $C^q(L, L', V)$ the q-V-cochains orthogonal to L' and define

$$Z^q(L, L', P) = Z^q(L, P) \cap C^q(L, L', V),$$
$$B^q(L, L', P) = \delta C^{q-1}(L, L', V), \qquad B^0(L, L', P) = 0,$$
$$H^q(L, L', P) = Z^q(L, L', P)/B^q(L, L', P),$$

we arrive at the definition of the cohomology group $H^q(L, L', P)$. As in §22 if L' is an ideal in L and the representation P is trivial on L' then $H^q(L, L', P) \approx H^q(L/L', P)$.

The definition of $H^q(L, L', P)$ clearly allows for a reformulation of Theorem 13.1.

As a generalization of Theorem 24.1 we prove the following theorem.

THEOREM 28.1. *If L is a semi-simple Lie algebra over a field of characteristic 0, L' is a subalgebra of L and P is an irreducible and nontrivial representation of L then $H^q(L, L', P) = 0$ for all dimensions q.*

Proof. We utilize the notation of the proof of Theorem 24.1. For every q-P-cocycle f we constructed a $(q-1)$-V-cochain

$$g = \sum_{i=1}^{n} P_{y_i} f_{z_i}$$

such that

$$\delta \Gamma^{-1} g = f.$$

Assume now that f is orthogonal to L'. We shall then show that $g' = \Gamma^{-1} g$ is orthogonal to L'.

Let $x \in L'$. Then

$$g_x = \sum_{i=1}^{n} (P_{y_i} f_{z_i})_x = \sum_{i=1}^{n} P_{y_i} (f_{z_i})_x = - \sum_{i=1}^{n} P_{y_i} (f_x)_{z_i} = 0.$$

Since Γ^{-1} is an automorphism of V it follows that

(28.3) $g'_x = 0$ for $x \in L'$.

From (23.6) we deduce

$$f_x = d_x g' - \delta(g'_x).$$

Hence for $x \in L'$, $d_x g' = 0$. This combined with (28.3) proves that g' is orthogonal to L', q.e.d.

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