# SOME GENERALIZATIONS OF QUASI-FROBENIUS ALGEBRAS

BY

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**Introduction.** Let $\mathfrak{A}$ be an algebra over a field $\mathfrak{k}$. $\mathfrak{A}$ is called quasi-Frobenius (QF) if it has a unit element and if every primitive right ideal is dual to a primitive left ideal; or, equivalently, if $\mathfrak{A}$ has a unit element and if every indecomposable direct constituent of the right regular representation is equivalent to an indecomposable direct constituent of the left regular representation. Properties of QF algebras and rings have been treated by Nakayama, Nesbitt, and the author [2, 3, 5, 6](1). Some of the most important properties of QF algebras do not characterize these algebras, but occur in more extensive classes. This leads us to the definitions that follow. Throughout this paper $\mathfrak{A}$ is a $\mathfrak{k}$-algebra with unity element.

QF-1 Algebra. $\mathfrak{A}$ is said to be a QF-1 algebra if every faithful representation of $\mathfrak{A}$ is its own second commutator.

QF-2 Algebra. $\mathfrak{A}$ is said to be a QF-2 algebra if every primitive left ideal, and every primitive right ideal, of $\mathfrak{A}$ has a unique minimal subideal.

A faithful representation $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is said to be a *minimal faithful representation* if deletion of any direct constituent of $\mathfrak{B}$ leaves a nonfaithful representation, that is, if the corresponding space $V$ is the direct sum of $V_1$ and $V_2$ with $V_2 \neq 0$ then $\mathfrak{B}_1$ is not faithful.

QF-3 Algebra. $\mathfrak{A}$ is said to be a QF-3 algebra if it has a unique minimal faithful representation.

We shall use the notation QF-12 to describe an algebra which is both QF-1 and QF-2, and so on.

Every QF algebra is QF-123. It is the purpose of the present paper to initiate the study of the above classes of algebras.

§1 contains definitions and notations for the paper. §2 gives an example to show that the class QF-1 is more general than the class QF. §3 contains a theorem which gives an equivalent definition for QF-2 algebras, §§4 and 5 discuss conditions under which a QF-2 algebra is QF, and give some properties of the Cartan invariants of QF-2 algebras. §6 contains the proof that every QF-2 algebra is also a QF-3 algebra. §§7 and 8 treat QF-3 algebras and include examples which illustrate the essentially more general character of QF-3 algebras as compared with QF-2 algebras. §9 treats a necessary condition on the class QF-13.

Some of the above definitions are given in the language of ideal theory

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(1) Numbers in brackets refer to bibliography.

and some in the language of representation theory. It would be desirable to be able to state each definition in both languages. I know of no way to express a definition for QF-1 algebras in the language of ideal theory. The discussion of §4, especially formulas (2), (2′), (3) and (3′), shows how to characterize QF-2 algebras in the language of representation theory. Theorem 5 gives an equivalent definition of QF-3 algebras in the language of ideal theory.

1. **Definitions and notation.** In what follows $\mathfrak{k}=\{a, b, c, \cdots\}$ shall denote a field; $\mathfrak{A}=\{\alpha, \beta, \gamma, \cdots\}$ shall denote a $\mathfrak{k}$-algebra with unit element; $V=\{v, v_1, \cdots\}$ shall denote a (finite-dimensional) right $\mathfrak{k}$-space. We shall denote by $\mathfrak{B}$ a homomorphism of $\mathfrak{A}$ into the algebra of all linear transformations of $V$ into itself, and shall call $\mathfrak{B}$ a *representation* of $\mathfrak{A}$ with corresponding *representation space* $V$. We shall regard $V$ as a left $\mathfrak{A}$-space and shall write $\alpha v$ for the image of $v$ under the linear transformation assigned to $\alpha$ by $\mathfrak{B}$. Thus we have $V$ as a right $\mathfrak{k}$-space and left $\mathfrak{A}$-space with the associativity condition $\alpha(va)=(\alpha v)a$ for all $\alpha\in\mathfrak{A}$, $v\in V$, and $a\in\mathfrak{k}$. Corresponding to each basis $v_1, \cdots, v_n$ of $V$ we can obtain a matrix representation $\alpha\rightarrow V(\alpha)$ by means of the equation $\alpha(v_1, \cdots, v_n)=(v_1, \cdots, v_n)V(\alpha)$. (Cf. [1, chap. III, p. 20]).

We construct the dual space $W$ for $V$ (cf. [6, p. 558]). $W$ is a left $\mathfrak{k}$-space and right $\mathfrak{A}$-space related to $V$ by an inner product $(w, v)$ which is a bilinear function from $W$ and $V$ to $\mathfrak{k}$ such that $(w\alpha, v)=(w, \alpha v)$ for all $w\in W$, $\alpha\in\mathfrak{A}$, and $v\in V$.

We may regard any left ideal of $\mathfrak{A}$ as a left representation space for $\mathfrak{A}$. A primitive left ideal $\mathfrak{l}$ of $\mathfrak{A}$ is said to be *dominant* if its dual space is $\mathfrak{A}$-isomorphic (as a right $\mathfrak{A}$-space) to some primitive right ideal $\mathfrak{r}$ of $\mathfrak{A}$. A primitive left ideal is said to be *subordinate* if it is $\mathfrak{A}$-isomorphic to a proper subideal of some dominant left ideal. A primitive left ideal is said to be *weakly subordinate* if its left annihilator in $\mathfrak{A}$ contains the left annihilator of a sum of proper subideals of dominant left ideals. The analogous definitions are made for right ideals. Obviously, every subordinate ideal is weakly subordinate and we shall see below (§8) that the converse is not true.

If two left $\mathfrak{A}$-spaces $V_1$ and $V_2$ have the property that the annihilator of $V_1$ is contained in the annihilator of $V_2$ we say that $V_2$ *is faithfully represented by* $V_1$. In this terminology a primitive left ideal is weakly subordinate if it is faithfully represented by a sum of proper subideals of dominant left ideals.

2. QF-1 **algebras.** Let $\alpha_1, \cdots, \alpha_5$ be basis elements for an algebra $\mathfrak{A}$ in which multiplication is defined by requiring that the mapping $\alpha=\alpha_1 a_1+\cdots+\alpha_5 a_5\rightarrow V(\alpha)$ where

$$(1)\qquad V(\alpha)=\begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_4 & a_2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & a_5 & a_3 \end{bmatrix}, \qquad a_i\in\mathfrak{k},$$

shall give a faithful matrix representation of $\mathfrak{A}$. The primitive left ideals of $\mathfrak{A}$ are $\mathfrak{l}_1=\alpha_1\mathfrak{k}+\alpha_4\mathfrak{k}$, $\mathfrak{l}_2=\alpha_2\mathfrak{k}+\alpha_5\mathfrak{k}$, and $\mathfrak{l}_3=\alpha_3\mathfrak{k}$. The primitive right ideals are $\mathfrak{r}_1=\mathfrak{k}\alpha_1$, $\mathfrak{r}_2=\mathfrak{k}\alpha_2+\mathfrak{k}\alpha_4$, and $\mathfrak{r}_3=\mathfrak{k}\alpha_3+\mathfrak{k}\alpha_5$. The ideals $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are dominant being dual respectively to $\mathfrak{r}_2$ and $\mathfrak{r}_3$. The ideals $\mathfrak{l}_3$ and $\mathfrak{r}_1$ are subordinate (to $\mathfrak{l}_2$ and $\mathfrak{r}_2$ respectively).

The sum $V=\mathfrak{l}_1+\mathfrak{l}_2$ is representation space for the faithful representation $\mathfrak{B}$ which was used above to define $\mathfrak{A}$. It is not difficult to prove that every faithful representation of $\mathfrak{A}$ has $\mathfrak{B}$ as a direct constituent (that is, that $\mathfrak{A}$ is a QF-3 algebra). Lemma II-C and Theorem II-E of [6, pp. 559–560] can be used to determine all faithful representations of $\mathfrak{A}$ and to show that $\mathfrak{A}$ is a QF-1 algebra. Since $V(\alpha)$ is not the reduced regular representation of $\mathfrak{A}$, $\mathfrak{A}$ is not a QF algebra (cf. [6, p. 559, Theorem II-D]). Hence, this example shows that *the class of* QF-1 *algebras properly contains the class of* QF *algebras.*

Direct examination of the primitive ideals shows that $\mathfrak{A}$ is a QF-2 algebra (see also Theorem 1 below). I know of no examples of QF-1 algebras which are not QF-12 algebras, and raise the question: *is every* QF-1 *algebra also a* QF-2 *algebra* (*or also a* QF-3 *algebra*)?

3. QF-2 **algebras.** The following theorem gives a second definition for QF-2 algebras:

THEOREM 1. *An algebra $\mathfrak{A}$ is a* QF-2 *algebra if and only if every primitive left ideal and every primitive right ideal is either dominant or subordinate.*

**Proof.** "*If.*" If a space $V$ has a unique minimal subspace, then its dual space $W$ has a unique maximal subspace. Every primitive left or right ideal has a unique maximal ideal (subspace), hence if two are dual each has also a unique minimal ideal. Thus every dominant left or right ideal has a unique minimal subideal. From that it follows that every subideal of a dominant left or right ideal has a unique minimal subideal and hence that every subordinate left or right ideal has a unique minimal subideal; this completes the proof that $\mathfrak{A}$ is QF-2.

"*Only if.*" We first remark that *any $\mathfrak{A}$-space $V$ which has a unique maximal subspace $V'$ is the homomorphic image of a primitive left ideal $\mathfrak{l}$ of $\mathfrak{A}$.* The ideal $\mathfrak{l}$ is one for which the factor space $\mathfrak{l}/\mathfrak{N}\mathfrak{l}$ is $\mathfrak{A}$-isomorphic to $V/V'$. ($\mathfrak{N}$ is the radical of $\mathfrak{A}$.) For let $e$ be a generating idempotent for $\mathfrak{l}$, that is, $\mathfrak{l}=\mathfrak{A}e$. Denote by $v$ any element of $V$ which maps into the residue class $e+\mathfrak{N}e$ of $\mathfrak{l}/\mathfrak{N}\mathfrak{l}$ under the $\mathfrak{A}$-homomorphism of $\mathfrak{B}$ onto $\mathfrak{l}/\mathfrak{N}\mathfrak{l}$. Then $ev$ has the same image as $v$ and so $v''=ev=v+v'$ where $v'\in V'$, and $V''=\mathfrak{A}v''$ is a subspace of $V$ which is not contained in $V'$ (since $v''$ has the same image as $v$) and therefore $V''=V$. Now the mapping $\lambda\rightarrow\lambda v''$ for each $\lambda\in\mathfrak{l}$ is clearly a homomorphism of $\mathfrak{l}$ onto $V$. The analogous result for right spaces is also true.

We now complete the proof of Theorem 1 by establishing the following lemma:

LEMMA 1. *If $\mathfrak{A}$ is a QF-2 algebra then every primitive left (right) ideal of $\mathfrak{A}$ is dual to a factor space of a dominant right (left) ideal of $\mathfrak{A}$.*

Let $\mathfrak{l}$ be a primitive left ideal of $\mathfrak{A}$. Then since $\mathfrak{l}$ has a unique minimal subideal its dual space $W$ has a unique maximal subspace and hence is homomorphic image of some primitive right ideal $\mathfrak{r}$ of $A$. It remains to show that $\mathfrak{r}$ is dominant. Let $V$ denote the dual space of $\mathfrak{r}$. Then $V$ is the homomorphic image of some primitive left ideal $\mathfrak{l}'$ under a homomorphism $\sigma$ with kernel $\mathfrak{t}'$ and $\mathfrak{l}$ is isomorphic to some subspace $V'$ of $V$. The lemma will follow if we show that $\mathfrak{t}'=0$. Let $\mathfrak{q}'$ be the counterimage of $V'$ under the homomorphism $\sigma$, and let $\mathfrak{m}$ be the unique minimal subideal of $\mathfrak{l}$. If $\mathfrak{m}\mathfrak{q}'\subseteq\mathfrak{t}'$ we would have $0=\sigma(\mathfrak{m}\mathfrak{q}')=\mathfrak{m}\sigma(\mathfrak{q}')=\mathfrak{m}V'$. But this is impossible since $V'$ is isomorphic to $\mathfrak{l}=\mathfrak{A}e$, whence $\mathfrak{m}=\mathfrak{m}e\neq 0$. Hence $\mathfrak{m}\mathfrak{q}'\not\subseteq\mathfrak{t}'$, and we can find $\lambda_1'$ in $\mathfrak{q}'$ such that $\mathfrak{m}\lambda_1'\not\subseteq\mathfrak{t}'$. Now the mapping $\lambda\rightarrow\lambda\lambda_1'$ is an isomorphism of $\mathfrak{l}$ onto the subideal $\mathfrak{q}_1'=\mathfrak{l}\lambda_1'$ of $\mathfrak{l}'$(an isomorphism since $\mathfrak{m}\lambda_1'\neq 0$ and $\mathfrak{m}$ is the unique minimal subideal of $\mathfrak{l}$). The set of $\lambda$ in $\mathfrak{l}$ such that $\lambda\lambda_1'\in\mathfrak{t}'$ is clearly a subideal $\mathfrak{q}$ of $\mathfrak{l}$, and since $\mathfrak{q}$ does not contain $\mathfrak{m}$ we have $\mathfrak{q}=0$; hence the intersection of $\mathfrak{q}_1'$ and $\mathfrak{t}'$ is 0. But in an ideal with unique minimal subideal any two nonzero ideals have nonzero intersection, and so from $\mathfrak{q}_1'\not\subseteq\mathfrak{t}'$ and therefore $\mathfrak{q}_1'\neq 0$ we conclude that $\mathfrak{t}'=0$. (Incidentally, the isomorphism between $\mathfrak{l}$ and $\mathfrak{q}_1'$ shows that $\mathfrak{l}$ is either subordinate or isomorphic to $\mathfrak{l}'$.)

**4. The mappings $\sigma$ and $\pi$.** The set of all primitive left ideals can be partitioned into equivalence classes under the relation of $\mathfrak{A}$-isomorphism, and we suppose that $\mathfrak{l}_1, \cdots, \mathfrak{l}_m$ is a set of representatives one from each equivalence class. Then we may write the unit element of $\mathfrak{A}$ as a sum of primitive orthogonal idempotents, $1=e_1+\cdots+e_m+\cdots+e_{m'}$ where $\mathfrak{l}_1=\mathfrak{A}e_1, \cdots, \mathfrak{l}_m=\mathfrak{A}e_m$. Now let $\mathfrak{A}$ be a QF-2 algebra and suppose the $\mathfrak{l}_i$, $i\leqq m$, ordered so that $\mathfrak{l}_1, \cdots, \mathfrak{l}_k$ are dominant, and the remaining ones subordinate. The primitive right ideals $\mathfrak{r}_1=e_1\mathfrak{A}, \cdots, \mathfrak{r}_m=e_m\mathfrak{A}$ then form a set of representatives for the equivalence classes of primitive right ideals.

We denote by $\mathfrak{L}_i$: $\alpha\rightarrow L_i(\alpha)$ the representation of $\mathfrak{A}$ with space $\mathfrak{l}_i$, and by $\mathfrak{R}_i$: $\alpha\rightarrow\mathfrak{R}_i(\alpha)$ the representation of $\mathfrak{A}$ with space $\mathfrak{r}_i$. The space $\mathfrak{A}e_i/\mathfrak{N}e_i$ is irreducible and we denote by $\mathfrak{F}_i$: $\alpha\rightarrow F_i(\alpha)$ the irreducible representation of $\mathfrak{A}$ which it provides. It is well known that every irreducible representation of $\mathfrak{A}$ is equivalent to one of $\mathfrak{F}_1, \cdots, \mathfrak{F}_m$, that $\mathfrak{F}_i$ is the unique top irreducible constituent of $\mathfrak{L}_i$, and that $\mathfrak{F}_i$ is the unique bottom irreducible constituent of $\mathfrak{R}_i$. The unique minimal subideal $\mathfrak{m}_i$ of $\mathfrak{l}_i$ is space for an irreducible representation of $\mathfrak{A}$; this must be one of the $\mathfrak{F}_j$ and we designate it by $\mathfrak{F}_{\sigma(i)}$. Then $\mathfrak{F}_{\sigma(i)}$ is the unique bottom constituent of $\mathfrak{L}_i$ and we have

$$L_i(\alpha) = \begin{bmatrix} F_i(\alpha) & & 0 \\ & \ddots & \\ * & & F_{\sigma(i)}(\alpha) \end{bmatrix}, \qquad i = 1, \cdots, m. \tag{2}$$

Similarly, the unique minimal subideal $\mathfrak{s}_i$ of $\mathfrak{r}_i$ is space for an irreducible representation $\mathfrak{F}_{\pi(i)}$ which is the unique top constituent of $\mathfrak{R}_i$, that is,

$$R_i(\alpha) = \begin{bmatrix} F_{\pi(i)}(\alpha) & & 0 \\ & \cdot & \\ * & & F_i(\alpha) \end{bmatrix}, \qquad i = 1, \cdots, m. \tag{2'}$$

It was seen in the proof of Lemma 1 that every primitive left ideal was dual to a factor space of some primitive right ideal; we can now make this result more specific, namely $\mathfrak{l}_i$ is dual to a factor space of $\mathfrak{r}_{\sigma(i)}$ and $\mathfrak{r}_i$ is dual to a factor space of $\mathfrak{l}_{\pi(i)}$, $i=1, \cdots, m$. Now, restating Lemma 1 in the language of representations, we have

$$L_{\pi\sigma(i)}(\alpha) = R_{\sigma(i)}(\alpha) = \begin{bmatrix} * & 0 \\ * & L_i(\alpha) \end{bmatrix}, \qquad i = 1, \cdots, m, \tag{3}$$

and dually

$$R_{\sigma\pi(i)}(\alpha) = L_{\pi(i)}(\alpha) = \begin{bmatrix} R_i(\alpha) & 0 \\ * & * \end{bmatrix}, \qquad i = 1, \cdots, m. \tag{3'}$$

The functions $\sigma$ and $\pi$ are mappings of the set $M = \{1, \cdots, m\}$ into itself. We denote by $\Sigma$ and $\Pi$ the ranges of $\sigma$ and $\pi$ respectively. Then Lemma 1 is equivalent to the following lemma:

LEMMA 2. *$\sigma\pi(i) = i$ if and only if $i \in \Sigma$, and $\pi\sigma(i) = i$ if and only if $i \in \Pi$. Moreover, $\mathfrak{l}_i$ is dominant if and only if $i \in \Pi$, and $\mathfrak{r}_i$ is dominant if and only if $i \in \Sigma$.*

It is clear that both $\Sigma$ and $\Pi$ contain exactly $k$ elements and if the domain of $\sigma$ is contracted from $M$ to $\Pi$, then $\sigma$ is 1-1 onto, its inverse being $\pi$ with domain cut down to $\Sigma$.

One characterization of QF-algebras is that *every primitive left ideal is dominant.* Stated in terms of the mappings $\sigma$ and $\pi$ this characterization yields the following theorem:

THEOREM 2. *A* QF-2 *algebra is* QF *if and only if*
(i) *No primitive left (right) ideal is subordinate, or*
(ii) $\Sigma = M$ (*or* $\Pi = M$), *or*
(iii) *$\sigma$ is a permutation on $M$ (or $\pi$ is a permutation on $M$).*

We note that $\Sigma = \Pi$ is not a sufficient condition for a QF-2 algebra to be QF. For consider the QF-2 algebra $\mathfrak{A}$ consisting of all matrices of the form

$$\begin{bmatrix} x_1 & 0 & 0 \\ x_3 & x_2 & 0 \\ x_5 & x_4 & x_1 \end{bmatrix}. \tag{4}$$

It is not QF and yet has $\Sigma = \Pi = \{1\}$, whereas $M = \{1, 2\}$.

5. **Cartan invariants of QF-2 algebras.** The left Cartan invariant $c_{ij}$ of any algebra $\mathfrak{A}$ is the multiplicity with which the irreducible constituent $\mathfrak{F}_j$ appears in $\mathfrak{L}_i$. Dually, the right Cartan invariant $c'_{ij}$ of $\mathfrak{A}$ is the multiplicity with which $\mathfrak{F}_j$ appears in $\mathfrak{R}_i$. It is easy to show that $c_{ij} = c'_{ji}$ = composition length of $e_j\mathfrak{A}e_i$ as a space over the completely primary ring $e_j\mathfrak{A}e_j$; if $\mathfrak{k}$ is algebraically closed, $c_{ij} = \mathfrak{k}$-dimension of $e_j\mathfrak{A}e_i$ (cf. [1, p. 106, Corollary 9.5c]).

If $\mathfrak{A}$ is a QF-2 algebra we have from formula (3) above that

$$c_{ij} \leqq c'_{\sigma(i)j} = c_{j\sigma(i)} \qquad \text{for all } i, j \text{ in } M \tag{5}$$

with equality holding if $i \in \Pi$, and dually from formula (3′)

$$c_{ij} = c'_{ji} \leqq c_{\pi(j)i} \qquad \text{for all } i, j \text{ in } M \tag{5'}$$

with equality holding if $j \in \Sigma$. We state this result in the following theorem:

Theorem 3. *Let $\mathfrak{A}$ be a* QF-2 *algebra with Cartan invariants $c_{ij}$, $i, j = 1, \cdots, m$. Then $c_{ij} \leqq c_{j\sigma(i)}$ with equality if $i \in \Pi$, and $c_{ij} \leqq c_{\pi(j)i}$ with equality if $j \in \Sigma$. Furthermore, if $i \notin \Pi$ then for at least one $j$ the inequality holds in the first relation and dually if $j \notin \Sigma$ then for at least one $i$ the inequality holds in the second relation.*

Note that for QF-algebras equality holds throughout in the above relations.

6. **Minimal faithful representations of QF-2 algebras.**

Lemma 3. *Let $\mathfrak{A}$ be any $\mathfrak{k}$-algebra with unity element, let $\mathfrak{l}_1, \cdots, \mathfrak{l}_k$ be a representative set of dominant primitive left ideals of $\mathfrak{A}$, and let $\mathfrak{S}$ be the representation of $\mathfrak{A}$ whose space is the (direct) sum $\mathfrak{l}_1, \cdots, \mathfrak{l}_k$. Then any faithful representation of $\mathfrak{A}$ can be written as a direct sum with $\mathfrak{S}$ as one of the summands.*

(This lemma is a restatement of Lemmas II-A and II-C [6, pp. 558, 559].)

Theorem 4. *If $\mathfrak{A}$ is a* QF-2 *algebra then $\mathfrak{S}$ is its unique minimal faithful representation, or, in other words, every* QF-2 *algebra is also a* QF-3 *algebra.*

**Proof.** Reference to Lemma 3 shows that we need only prove that $\mathfrak{S}$ is a faithful representation of $\mathfrak{A}$. Now the representation $\mathfrak{S}' = \mathfrak{L}_1 + \cdots + \mathfrak{L}_m$ afforded by the space $\mathfrak{l}_1 + \cdots + \mathfrak{l}_k + \cdots + \mathfrak{l}_m$ is the reduced left regular representation and as such is surely faithful. If $\mathfrak{l}_i$ is not dominant it follows from formula (3) that $\mathfrak{L}_i$ can be dropped from the sum $\mathfrak{L}_1 + \cdots + \mathfrak{L}_m$ without destroying its property of being faithful. When all $\mathfrak{L}_i$ for which $\mathfrak{l}_i$ is not dominant are dropped from $\mathfrak{S}'$ what remains is $\mathfrak{S}$; hence $\mathfrak{S}$ is faithful and the theorem is proved.

7. QF-3 **algebras.** The following theorem plays the same role for QF-3

algebras as Theorem 1 does for QF-2 algebras.

THEOREM 5. *An algebra $\mathfrak{A}$ is a* QF-3 *algebra if and only if every primitive left ideal and every primitive right ideal is either dominant or weakly subordinate.*

PROOF. "*If.*" Define $\mathfrak{S}$ and $\mathfrak{S}'$ as in the preceding section. Then the argument follows just as the proof of Theorem 4 except that the argument which justifies dropping a non-dominant summand must be slightly altered. But, if $\mathfrak{l}_i$ is weakly subordinate it is faithfully represented by $\mathfrak{S}$ (although not a bottom constituent of one indecomposable summand of $\mathfrak{S}$ as in the QF-2 case) and so can be dropped from $\mathfrak{S}'$ without destroying faithfulness.

"*Only if.*" We first show that $\mathfrak{S}$ is the unique minimal faithful representation, and then show that the non-dominant primitive ideals are weakly subordinate. Suppose that $\mathfrak{T}$ is the unique minimal faithful representation of $\mathfrak{A}$. Then every indecomposable direct constituent of $\mathfrak{T}$ must appear as an indecomposable direct constituent of every faithful representation of $\mathfrak{A}$. Let $\mathfrak{S}''$ be the reduced right regular representation. Both $\mathfrak{S}'$ and $\mathfrak{S}''$ are faithful, hence $\mathfrak{T}$ must be contained in the sum of all indecomposable direct constituents which are common to $\mathfrak{S}'$ and $\mathfrak{S}''$, that is, $\mathfrak{T}$ is contained in the sum of all indecomposable direct constituents of $\mathfrak{S}'$ whose spaces are dominant left ideals, or, in other words, $\mathfrak{T}$ is a constituent of $\mathfrak{S}$. On the other hand, we know by Lemma 3 that $\mathfrak{S}$ is a constituent of $\mathfrak{T}$, hence $\mathfrak{S}=\mathfrak{T}$.

Now consider a non-dominant primitive left ideal $\mathfrak{l}_i$ of $\mathfrak{A}$. Let $\mathfrak{m}_i$ be the sum of all minimal subideals of $\mathfrak{l}_i$ (that is, $\mathfrak{m}_i$ is the "sockel" of $\mathfrak{l}_i$). It follows from the general theory of rings [1, p. 8, Theorem 1.6B] that $\mathfrak{m}_i$ can be written as a direct sum of minimal subideals of $\mathfrak{l}_i$, say

$$\mathfrak{m}_i = \mathfrak{m}_{i1} + \cdots + \mathfrak{m}_{ir} \text{ (direct).} \tag{6}$$

As a minimal ideal, $\mathfrak{m}_{ij}$ is space for an irreducible representation $\mathfrak{F}_{\rho(ij)}$ of $\mathfrak{A}$, $j=1, \cdots, r$. Since $\mathfrak{S}$ is a faithful representation of $\mathfrak{A}$ we must have some dominant ideal $\mathfrak{l}_{\mu(ij)}$ which is not annihilated by $\mathfrak{m}_{ij}, j=1, \cdots, r$. This means that we can find an element $\lambda=\lambda_{\mu(ij)}$ in $\mathfrak{l}_{\mu(ij)}$ such that $\mathfrak{m}_{ij}\lambda\neq 0$ and such that $e_i\lambda=\lambda$ (where $e_i$ is a generating idempotent for $\mathfrak{l}_i$). Then $\mathfrak{l}_i\lambda$ is a subideal of $\mathfrak{l}_{\mu(ij)}$ isomorphic (as an $\mathfrak{A}$-space) to a factor space $\mathfrak{l}_i/\mathfrak{p}_{ij}$ of $\mathfrak{l}_i$, where $\mathfrak{p}_{ij}$ is a subideal of $\mathfrak{l}_i$ which does not contain $\mathfrak{m}_{ij}$. As a subideal of a dominant left ideal, $\mathfrak{l}_i\lambda$ has the unique minimal subideal $\mathfrak{m}_{ij}\lambda$ and hence $\mathfrak{p}_{ij}+\mathfrak{m}_{ij}\supseteq\mathfrak{m}_i$.

Since $\mathfrak{m}_i$ is the sum of all minimal subideals of $\mathfrak{l}_i$, any representation of $\mathfrak{A}$ which is annihilated by no subideal of $\mathfrak{m}_i$ must be faithful for $\mathfrak{l}_i$. This shows that $\mathfrak{l}_i$ is faithfully represented by the sum $\mathfrak{l}_i\lambda_{\mu(i1)}+ \cdots +\mathfrak{l}_i\lambda_{\mu(ir)}$. The theorem now follows from the remark that $\mathfrak{l}_i\lambda_{\mu(ij)}$ as factor space of the primitive left ideal $\mathfrak{l}_i$ must be a proper subideal of $\mathfrak{l}_{\mu(ij)}$ (for otherwise the dominant ideal $\mathfrak{l}_{\mu(ij)}$ would be isomorphic to the nondominant ideal $\mathfrak{l}_i$, cf. [1, p. 99, Theorem 9.2G]).

8. **Examples of QF-3 algebras.** The following examples of QF-3 algebras which are not QF-2 algebras show not only that the class QF-3 properly includes the class QF-2 but also illustrate various kinds of weakly subordinate ideals.

The first example is the algebra $\mathfrak{A}$, with $\mathfrak{k}$-basis $\alpha_1, \cdots, \alpha_7$, and general element $\alpha = x_1\alpha_1 + \cdots + x_7\alpha_7$, $x_i \in \mathfrak{k}$, defined by requiring that the mapping $\alpha \rightarrow V(\alpha)$ shall be a faithful matrix representation of $\mathfrak{A}$, where

$$V(\alpha) = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_3 & x_2 & 0 & 0 \\ x_4 & 0 & x_2 & 0 \\ x_7 & x_5 & x_6 & x_1 \end{bmatrix}. \tag{7}$$

$\mathfrak{A}$ is a QF-3 algebra with one dominant primitive left ideal $\mathfrak{l}_1$ with $\mathfrak{k}$-basis $\alpha_1, \alpha_3, \alpha_4, \alpha_7$ and one weakly subordinate primitive left ideal $\mathfrak{l}_2$ with $\mathfrak{k}$-basis $\alpha_2, \alpha_5, \alpha_6$. $L_1(\alpha) = V(\alpha)$ and

$$L_2(\alpha) = \begin{bmatrix} x_2 & 0 & 0 \\ x_5 & x_1 & 0 \\ x_6 & 0 & x_1 \end{bmatrix}.$$

No subideal of $\mathfrak{l}_1$ is $\mathfrak{A}$-isomorphic to $\mathfrak{l}_2$ although $\mathfrak{N} \cap \mathfrak{l}_1$ gives a faithful representation of $L_2(\alpha)$ (as required by Theorem 5). Each nonzero element $x_5\alpha_5 + x_6\alpha_6$ of $\mathfrak{l}_2 \cap \mathfrak{N}$ is basis element for a minimal subideal of $\mathfrak{l}_2$ and all of these minimal subideals are isomorphic as $\mathfrak{A}$-spaces.

The second example is the algebra $\mathfrak{A}$ of order 9 over $\mathfrak{k}$ defined by requiring that the representation

$$V(\alpha) = \begin{bmatrix} x_1 & & & & & \\ x_4 & x_3 & & & & \\ x_6 & x_5 & x_1 & & & \\ & & & x_2 & & \\ & & & x_7 & x_3 & \\ & & & x_9 & x_8 & x_2 \end{bmatrix} \tag{8}$$

be faithful. There are two dominant primitive left ideals $\mathfrak{l}_1 = \mathfrak{k}\alpha_1 + \mathfrak{k}\alpha_4 + \mathfrak{k}\alpha_6$, $\mathfrak{l}_2 = \mathfrak{k}\alpha_2 + \mathfrak{k}\alpha_7 + \mathfrak{k}\alpha_9$, and one weakly subordinate primitive left ideal $\mathfrak{l}_3 = \mathfrak{k}\alpha_3 + \mathfrak{k}\alpha_5 + \mathfrak{k}\alpha_8$. $\mathfrak{l}_3$ has only the two minimal subideals $\mathfrak{k}\alpha_5$ and $\mathfrak{k}\alpha_8$ which are not isomorphic as $\mathfrak{A}$-spaces. $\mathfrak{l}_3$ is $\mathfrak{A}$-isomorphic to the subideal $\mathfrak{k}(\alpha_4 + \alpha_7) + \mathfrak{k}\alpha_6 + \mathfrak{k}\alpha_9$ of $(\mathfrak{l}_1 + \mathfrak{l}_2) \cap \mathfrak{N}$.

Although the mappings $\sigma$ and $\pi$ no longer exist in the QF-3 case, we may still define $\Pi$ as the set of indices $i \in M$ for which $\mathfrak{l}_i$ is dominant and $\Sigma$ as the set of indices $i \in M$ for which $\mathfrak{r}_i$ is dominant. In the first two examples we have $\Sigma+\Pi$ a proper subset of $M$. The following example shows that it is possible to have $\Sigma+\Pi=M$ in QF-3 algebras which are not QF-2 algebras. We take for $\mathfrak{A}$ the algebra of order 15 over $\mathfrak{k}$ defined by its faithful representation

$$V(\alpha) = \begin{bmatrix} L_1(\alpha) & & \\ & L_2(\alpha) & \\ & & L_3(\alpha) \end{bmatrix} \tag{9}$$

where

$$L_1(\alpha) = \begin{bmatrix} x_1 & & & \\ x_7 & x_2 & & \\ x_8 & 0 & x_3 & \\ x_9 & x_{10} & x_{11} & x_4 \end{bmatrix},$$

$$L_2(\alpha) = \begin{bmatrix} x_2 & & \\ x_{10} & x_4 & \\ x_{12} & x_{13} & x_5 \end{bmatrix}, \qquad L_3(\alpha) = \begin{bmatrix} x_3 & & \\ x_{11} & x_4 & \\ x_{14} & x_{15} & x_6 \end{bmatrix}.$$

It is of some interest to discuss conditions under which a weakly subordinate ideal is isomorphic as $\mathfrak{A}$-space to a subideal of $(\mathfrak{l}_1+\cdots+\mathfrak{l}_k)\cap\mathfrak{N}$. The following lemma is useful in that connection.

LEMMA 4. *Let $\mathfrak{l}_1, \cdots, \mathfrak{l}_k$ be a representative set of the dominant left ideals of any $\mathfrak{k}$-algebra $\mathfrak{A}$ with unit element. Then any completely reducible subspace $\mathfrak{l}'$ of $\mathfrak{l}=\mathfrak{l}_1+\cdots+\mathfrak{l}_k$ is the direct sum of its irreducible subspaces, no two of which are $\mathfrak{A}$-isomorphic.*

**Proof.** It follows from the general theory of Loewy series [1, pp. 102–104] that $\mathfrak{l}'$ is a subspace (subideal of $\mathfrak{m}=\mathfrak{m}_1+\cdots+\mathfrak{m}_k$ where $\mathfrak{m}_i$ is the unique minimal subideal of $\mathfrak{l}_i$). No two of the $\mathfrak{m}_i$ can be isomorphic lest the corresponding dominant ideals be dual to the same primitive right ideal and therefore isomorphic to each other. The lemma now follows immediately from the general theory of fully reducible vector spaces [1, pp. 7–9].

THEOREM 6. *Let $\mathfrak{A}$ be a QF-3 algebra for which $\mathfrak{l}_1, \cdots, \mathfrak{l}_k$ are a representative set of dominant primitive left ideals, set $\mathfrak{l}=\mathfrak{l}_1+\cdots+\mathfrak{l}_k$ and let $\mathfrak{l}_i$ $(i>k)$ be a nondominant primitive left ideal of $\mathfrak{A}$. Then $\mathfrak{l}_i$ is $\mathfrak{A}$-isomorphic to a subideal of $\mathfrak{l}\cap\mathfrak{N}$ if and only if no two minimal subideals of $\mathfrak{l}_i$ are isomorphic as $\mathfrak{A}$-spaces.*

**Proof.** "*Only if.*" Suppose $\mathfrak{l}_i$ has two $\mathfrak{A}$-isomorphic minimal subideals $\mathfrak{m}_{i1}$ and $\mathfrak{m}_{i2}$ whose isomorphism is established by the mapping $\sigma: \lambda_1\rightarrow\lambda_2=\sigma(\lambda_1)$ for $\lambda_1\in\mathfrak{m}_{i1}$. Then the set of elements $\lambda_1+\sigma(\lambda_1)$ for $\lambda_1\in\mathfrak{m}_{i1}$ constitute a minimal ideal of $\mathfrak{l}_i$ distinct from either $\mathfrak{m}_{i1}$ or $\mathfrak{m}_{i2}$ but contained in their sum.

Hence, the sum of all minimal subideals of $\mathfrak{l}_i$ is not direct. Then according to Lemma 4, $\mathfrak{l}_i$ cannot be $\mathfrak{A}$-isomorphic to any subideal of $\mathfrak{l}\cap\mathfrak{N}$.

*"If."* Set $\lambda_0=\lambda_{\mu(i1)}+\cdots+\lambda_{\mu(ir)}$ where $\lambda_{\mu(ij)}$ is the element defined in the proof of Theorem 5, and consider the mapping

$$\sigma: \quad \lambda \to \sigma(\lambda) = \lambda\lambda_0$$

of $\mathfrak{l}_i$ into $\mathfrak{l}\cap\mathfrak{N}$. $\sigma$ is clearly a homomorphism into, and is an isomorphism unless some minimal subideal $\mathfrak{m}_{ij}$ of $\mathfrak{l}_i$ is mapped into zero. Suppose that $\sigma(\mathfrak{m}_{i1})=0$. Then we have

$$-\lambda_1\lambda_{\mu(i1)} = \lambda_1\lambda_{\mu(i2)} + \cdots + \lambda_1\lambda_{\mu(ir)} \tag{10}$$

for all $\lambda_1\in\mathfrak{m}_{i1}$. Now, $\mathfrak{A}\lambda_{\mu(ij)}$, as a subideal of a dominant ideal, has a unique minimal subideal, which is by our hypothesis not $\mathfrak{A}$-isomorphic to $\mathfrak{m}_{i1}$ if $j>1$. Again, since $\mathfrak{m}_{i1}$ is minimal the ideal $\mathfrak{m}_{i1}\lambda_{\mu(ij)}$ is either zero or isomorphic to $\mathfrak{m}_{i1}$. Hence, we have $\mathfrak{m}_{i1}\lambda_{\mu(ij)}=0$ for $j>1$, and therefore, according to equation (10), $\mathfrak{m}_{i1}\lambda_{\mu(i1)}=0$ contrary to our hypothesis that $\mathfrak{m}_{i1}$ does not annihilate $\mathfrak{l}_{\mu(i1)}$. Hence, $\sigma$ is an isomorphism and our theorem is proved.

## 9. QF-13 algebras.

THEOREM 7. *If $\mathfrak{A}$ is a* QF-13 *algebra, then $\Sigma+\Pi=M$, that is every irreducible representation appears either as top or as bottom constituent of the representation afforded by some dominant ideal.*

**Proof.** Suppose that $\mathfrak{A}$ is a QF-3 algebra for which $\Sigma+\Pi\neq M$ and let $i\notin\Sigma+\Pi$. Then consider the faithful representation

$$\mathfrak{B} = \begin{bmatrix} \mathfrak{F}_i & 0 \\ 0 & \mathfrak{S} \end{bmatrix}$$

where $\mathfrak{S}$ is (as above) the unique minimal faithful representation of $\mathfrak{A}$ and $\mathfrak{F}_i$ is the irreducible representation on the space $\mathfrak{l}_i/\mathfrak{N}\mathfrak{l}_i$. Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

(subdivision of rows and columns the same in $B$ and $\mathfrak{B}$) be any commutator of $\mathfrak{B}$. Then we have $B_{12}=0$ since $\mathfrak{F}_i$ is not a top constituent of any indecomposable constituent of $\mathfrak{S}$ and $B_{21}=0$ since $\mathfrak{F}_i$ is not a bottom constituent of any indecomposable constituent of $\mathfrak{S}$. Thus the commutator $\mathfrak{B}'$ has the form

$$\mathfrak{B}' = \begin{bmatrix} \mathfrak{B}_1' & 0 \\ 0 & \mathfrak{B}_2' \end{bmatrix}$$

where $\mathfrak{B}_1'$ is the commutator of $\mathfrak{F}_1$ and $\mathfrak{B}_2'$ is the commutator of $\mathfrak{S}$. Then the second commutator $\mathfrak{B}''$ has the form

$$\mathfrak{B}'' = \begin{bmatrix} \mathfrak{B}_1'' & 0 \\ 0 & \mathfrak{B}_2'' \end{bmatrix}$$

(zeros outside the diagonal blocks since

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

belongs to $\mathfrak{B}'$) where $\mathfrak{B}_1''$ is the commutator of $\mathfrak{B}_1'$ and $\mathfrak{B}_2''$ is the commutator of $\mathfrak{B}_2'$. Moreover, since $\mathfrak{B}_{12}' = \mathfrak{B}_{21}' = 0$ the constituents $\mathfrak{B}_1''$ and $\mathfrak{B}_2''$ of $\mathfrak{B}''$ are completely independent, that is, if

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \in \mathfrak{B}''$$

so do

$$\begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & C_2 \end{bmatrix}.$$

On the other hand, since $\mathfrak{S}$ is a faithful representation $\mathfrak{A}$, $\mathfrak{F}_i$ and $\mathfrak{S}$ are not completely independent. Hence, from the universal relation $\mathfrak{B}'' \supseteq \mathfrak{B}$ we conclude $\mathfrak{B}'' \supset \mathfrak{B}$, that is, $\mathfrak{A}$ is not a QF-1 algebra.

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