

INTERPOLATION SERIES

BY

R. CREIGHTON BUCK

1. Introduction. There is an extensive literature devoted to the study of the expansion of analytic functions into the interpolation series defined by a sequence $\{T_n\}$ of linear functionals. Some of these deal with specific series, and the problem is to determine the largest class of functions for which a given series is convergent [6; 11; 15; 21; 23]⁽¹⁾. Others treat a more or less general class of interpolation series; the results obtainable here are seldom as sharp as those obtained for a single series [1; 5; 10; 19; 20; 24]. A closely related problem is that of determining uniqueness classes. If C is a class of functions, $\{T_n\}$ is said to be total over C if $f \in C$ and $T_n(f) = 0$ for all n imply that $f = 0$; C is also said to be a uniqueness class for $\{T_n\}$, since if $T_n(f) = T_n(g)$ for all n , and for f and g in C , then $f = g$. If C is an expansion class for $\{T_n\}$ then it is also a uniqueness class. This remark has been used to obtain uniqueness classes for certain special sequences of functionals.

In the present paper, certain aspects of the convergence and summability of interpolation series will be discussed, following a method outlined in [4]. §§2 and 3 are devoted to the general treatment; succeeding sections take up in more detail the results which follow for special series. These contain and extend many of the known results. In addition, the theorems dealing with Mittag-Leffler summability are applied in §10 to obtain uniqueness theorems which in particular include all those of Gelfond [10]. Some of these results have been announced in a previous paper [4].

2. Integral formulae. We are concerned with the class K of entire functions of exponential type. If $f \in K$, then the growth function of f is defined by

$$(2.1) \quad h(\theta, f) = \limsup r^{-1} \log |f(re^{i\theta})|.$$

If H is any real function with period 2π , then $K[H(\theta)]$ is the class of all f in K such that $h(\theta, f) \leq H(\theta)$ for all θ ; infinite values of H are admissible. Thus, $K[A]$ is the class of entire functions of at most order 1 and type A . If ψ is a positive real function, then $K[\psi(r), H(\theta)]$ will be used for the class of all f in K such that the inequality

$$(2.2) \quad |f(re^{i\theta})| \leq \psi(r) \exp rH(\theta)$$

holds for all θ , and all sufficiently large r . The class $K[H(\theta)]$ is the intersection

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⁽¹⁾ Number in brackets refer to the bibliography at the end of the paper.

of the classes $K[e^{\lambda r}, H(\theta)]$ for all $\lambda > 0$. As in previous papers, $K(a, c)$ will be used in place of $K[H(\theta)]$ when H is chosen as the special function which takes the value c at $\pm\pi/2$, and the value a at 0 and π . (This class arises whenever f is studied from its behavior on the real axis.)

If G is a closed set of the plane, a supporting function for it is defined by

$$(2.3) \quad k(\theta, G) = \sup_{w \in G} \Re(we^{i\theta})$$

where $w = \rho e^{i\phi}$. (This differs from the usual definition in the sign of θ .) The function $k(\theta, G)$ is continuous in θ , has period 2π , has left and right derivatives everywhere, and is unchanged if G is replaced by its convex hull. If G_1 and G_2 are subsets of the plane then $G_1 \cdot G_2$ is the set of all points of the form $z_1 z_2$ where $z_1 \in G_1$, $z_2 \in G_2$. G is a star set (with respect to the origin) if $\lambda z \in G$ for all λ , $0 \leq \lambda \leq 1$, whenever $z \in G$. The set $(E' \cdot G)'$ will play a special role in succeeding sections, and will be referred to as the E -star of G . If E is chosen as the whole plane with the interval $[1, +\infty]$ of the reals deleted, then the E -star of G becomes simply the star of G —that is, the maximal star set contained in G .

A classical result of Pólya relates the growth function of f to the supporting function of a certain convex set. Let $f \in K$ and suppose that $f(z) = \sum_0^\infty a_n z^n / n!$; let $\phi(w)$ be the function regular at infinity having the development $\sum_0^\infty a_n / w^{n+1}$. ϕ is usually called the Borel transform of f . Let $D(f)$ be the closed convex hull of the set of singularities of ϕ , a bounded set. Then, the fundamental theorem of Pólya may be stated as follows [16, p. 585]:

If $f \in K$, then $h(\theta, f) = k(\theta, D(f))$ for all θ . Moreover, if Γ is a simple contour enclosing $D(f)$, then for all z ,

$$(2.4) \quad f(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{zw} \phi(w) dw.$$

Recently, S. Schmidli [20] has obtained a stronger form of this representation which may be stated in the following form⁽²⁾:

If $f \in K[\psi(r), h(\theta, f)]$ where

$$(2.5) \quad \int_0^\infty \psi(r) dr < \infty,$$

then

⁽²⁾ Not to be confused with the intersection of G_1 and G_2 which will be denoted by $G_1 \cap G_2$. The union will be denoted by $G_1 \cup G_2$, the complement of G by G' , and the boundary of G by ∂G .

⁽³⁾ Schmidli obtained this for $\psi(r) = 1/r^{1+\epsilon}$ by showing that ϕ is then continuous on $\partial D(f)$. The more general case requires no alteration in his proof. R. P. Boas has observed that if (2.5) holds, then $\phi(w)$ is uniformly bounded outside $D(f)$, and the formula (2.6) may be obtained from (2.4) by shrinking Γ .

$$(2.6) \quad f(z) = \frac{1}{2\pi i} \int_{\partial D(f)} e^{zw} \phi(w) dw.$$

We turn now to complex-valued functionals T defined on K . Let g be any entire function, and define T by

$$(2.7) \quad T(f) = \frac{1}{2\pi i} \int_{\Gamma} g(w) \phi(w) dw$$

where ϕ is the Borel transform of f , and Γ encloses $D(f)$. Since the Borel transform is linear, so is T . In the present paper, only functionals T of the above form will be discussed⁽⁴⁾. The function g is referred to as a generating function for T . If f is chosen so that $f(z) = e^{wz}$, then $T(f) = g(w)$; thus, $T(f)$ can be formally described as $g(d/dz)f(z)|_{z=0}$ [3].

3. Interpolation series. Let $\{T_n\}$ be a sequence of linear functionals, defined on all or part of K . A class $C \subset K$ is said to be a uniqueness class for $\{T_n\}$ if $T_n(f) = 0$ for all n implies $f = 0$ for any $f \in C$. Thus, a function of C is determined uniquely by the sequence of complex numbers $\{T_n(f)\}$. Suppose that it is possible to find a sequence of functions $\{u_n(z)\}$ which are orthogonal to $\{T_n\}$ in the sense that $T_n(u_m) = 0$ for all $n \neq m$, while $T_n(u_n) = 1$. Then any function f may be represented by the formal series

$$(3.1) \quad \sum_0^{\infty} u_n T_n(f).$$

Such a series will be called an interpolation series for $\{T_n\}$.

We will discuss classes of functions f for which the interpolation series is convergent, or is summable to f . Such classes must clearly be subclasses of the corresponding uniqueness class; this enables us to prove uniqueness theorems, which, however, can not be expected to be best possible, since there is reason to believe that a uniqueness class need not be an expansion class.

The argument for convergence is familiar [1; 10]. If $f(z) = e^{zw}$, then $T_n(f) = g_n(w)$, the generating function of T_n , and formally

$$(3.2) \quad e^{zw} = \sum_0^{\infty} u_n(z) g_n(w).$$

If this converges uniformly for all w on a simple contour Γ and for all z , and f is such that $D(f)$ lies in the interior of Γ , then using (2.4) and integrating termwise,

$$f(z) = \sum_0^{\infty} u_n(z) \frac{1}{2\pi i} \int_{\Gamma} g_n(w) \phi(w) dw = \sum_0^{\infty} u_n(z) T_n(f),$$

⁽⁴⁾ If K is given the weak topology in which $f_n \rightarrow f$ whenever $\lim f_n(z) = f(z)$ for all z , then any continuous linear functional T has the representation (2.7). See also [3].

convergent for all z .

If f satisfies the stronger growth conditions required to apply the Schmidli representation, we can say more: if (3.2) holds uniformly on Γ , and if f is such that $f \in K[\psi(r), h(\theta, f)]$ where $\int^\infty \psi(r) dr < \infty$, and $D(f)$ lies in the closed interior of Γ , then the interpolation series again converges to $f(z)$.

Turning to summability, we digress to describe the familiar notion of E -summability, of which Borel exponential means and Mittag-Leffler are special cases. Let $E(z) = \sum_0^\infty d_n z^n$ be an entire function, not a polynomial, with $d_n \geq 0$. If $\{s_n\}$ is a sequence of complex numbers with $|s_n|^{1/n} = O(1)$, then the series $\sum_0^\infty d_n s_n \lambda^n$ converges for all λ , $0 \leq \lambda < \infty$, to a function $H(\lambda)$. Consider the "mean," $\sigma(\lambda) = H(\lambda)/E(\lambda)$. If $\lim_{\lambda \rightarrow \infty} \sigma(\lambda) = S$ exists, then we write $(E)\text{-}\lim s_n = S$. This defines a totally regular method of summability. We use $(E)\text{-}\sum_0^\infty a_n$ for $(E)\text{-}\lim s_n$ where $s_n = \sum_0^\infty a_k$. These methods are of particular importance when applied to power series. Let E denote an open set associated with $E(z)$ such that $\lim_{\lambda \rightarrow \infty} E(z\lambda)/E(\lambda) = 0$, uniformly in any compact subset of E . The importance of the set E is that $(E)\text{-}\sum_0^\infty z^n = 1/(1-z)$ for all z in E , and the summability is uniform in any compact subset of E . This can be extended to a general function f and its Taylor series. The following proof differs slightly from that usually given for Borel summability [7].

THEOREM 3.1. *Let $f(z)$ be regular in a region R containing the origin. Then,*

$$f(z) = (E)\text{-}\sum_0^\infty f^{(n)}(0)z^n/n!$$

uniformly in any closed compact subset of $(R' \cdot E')'$, the E -star of R .

If $z \in E$ and $0 \leq \mu \leq 1$, then $|E(\mu z \lambda)/E(\lambda)| \leq |E(\mu z \lambda)/E(\lambda \mu)| = o(1)$ as λ increases, so that E may be assumed to be a star set. It follows that $G = (R' \cdot E)'$ is also a star set. Let G_0 be any compact subset of G . We can then choose an open set E_0 whose closure is a compact subset of E , and a region $R_0 \subset R$ such that $G_0 \subset (R'_0 \cdot E'_0)'$. In fact, since G is a star set, we can take R_0 to be simply connected, and its boundary $\Gamma = \partial R_0$ as a simple contour. Since 1 does not belong to E , and therefore not to E_0 , $(R'_0 \cdot E'_0)' \subset R_0$ and $G_0 \subset R_0$. For any z in G_0 , we have

$$s_n(z) = \sum_0^n f^{(k)}(0)z^k/k! = f(z) - \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \left(\frac{z}{t}\right)^{n+1} dt,$$

$$f(z) - \sigma(z, \lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \frac{z}{t} \frac{E(\lambda z/t)}{E(\lambda)} dt.$$

Since $G_0 \subset (R'_0 \cdot E'_0)'$ and $\Gamma \subset R'_0$, z/t lies in E_0 for all z in G_0 and t on Γ . Thus

$$\lim_{\lambda \rightarrow \infty} \frac{E(\lambda z/t)}{E(\lambda)} = 0$$

uniformly in z and t . Moreover, $f(t)$ is bounded on Γ , and $|t-z|$ is bounded away from zero, since G_0 is a compact subset of R_0 . Hence, $\lim_{\lambda \rightarrow \infty} |f(z) - \sigma(z, \lambda)| = 0$ uniformly in G_0 .

If we choose $E(z)$ as e^z , then E is the set of all z , $\Re(z) < 1$, and $(R' \cdot E)'$ is easily seen to be the Borel polygon of R . If $E(z)$ is taken as $\sum (z/\log(n+2))^n$, then E is the whole plane with the half line $y=0, x \geq 1$ deleted, and the E -star of R becomes simply the (Mittag-Leffler) star of R . Since the interval $[1, +\infty]$ always lies outside an E set, the E -star of R is a star set lying in the star of R .

Returning to the discussion of interpolation series, we can replace convergence by summability. If, for all z ,

$$(3.3) \quad e^{zw} = (E)\text{-}\sum_0^{\infty} u_n(z) g_n(w)$$

uniformly on Γ , then for all f such that $D(f)$ lies in the interior of Γ , and for all z ,

$$(3.4) \quad f(z) = (E)\text{-}\sum_0^{\infty} u_n(z) T_n(f).$$

If f is such that $f \in K[\psi(r), h(\theta, f)]$ where $\int^{\infty} \psi(r) dr < \infty$, and $D(f)$ lies in the closed interior of Γ , then (3.4) still holds.

We now assume that the generating functions of $\{T_n\}$ are of a special form. We suppose that $g_n(w) = [\zeta(w)]^n$ where $\zeta(w)$ is an entire function. Let Ω_w be a set containing the origin in which $\zeta(w)$ is univalent, and let Ω_{ζ} be the image of Ω_w under $\zeta = \zeta(w)$; let Δ_{ζ} be the largest open circle $|\zeta| < R$ in Ω_{ζ} , and let Δ_w be its image under $w = w(\zeta)$ in Ω_w ; let Ω_{ζ}^* be the E -star of Ω_{ζ} , and let Ω_w^* be its image in Ω_w . The central theorem of [4], proved there for a general kernel $\gamma(z)$ and stated below for the special kernel $\gamma(z) = e^z$ becomes:

THEOREM 3.2. *The formal interpolation series $\sum_0^{\infty} u_n(z) T_n(f)$ converges to $f(z)$ for all f such that $D(f) \subset \Delta_w$, and is E summable to $f(z)$ for all f with $D(f) \subset \Omega_w^*$.*

To obtain the sharper convergence forms, we must have $\sum_0^{\infty} u_n(z) [\zeta(w)]^n$ converging uniformly to e^{zw} on Γ . This usually requires a separate treatment; however, since e^{zw} is a regular function of ζ for ζ in Ω_{ζ} , we can state that if $\lim u_n(z) R^n = 0$, $\sum u_n(z) \zeta^n$ converges uniformly on all arcs of $|\zeta| = R$ where it is regular, considered as a function of z . In the succeeding sections, we take up special interpolation series and apply the results of this section in detail.

4. Newton series. To obtain this, we define the functionals $\{T_n\}$ by $T_n(f) = \Delta^n f(0) = (-1)^n \sum_0^{\infty} C_{n,k} (-1)^k f(k)$. The generating functions $g_n(w)$ have the form $[\zeta(w)]^n$ where $\zeta(w) = e^w - 1$, and the functions u_n are the polynomials $u_n(z) = C_{z,n} = z(z-1)(z-2) \cdots (z-n+1)/n!$. For Ω_w , we take the

strip $|v| < \pi$; Ω_ζ is then the ζ -plane with $[-\infty, -1]$ deleted. The circle Δ_ζ is $|\zeta| < 1$, and Δ_w is the set lying in Ω_w and satisfying $|e^w - 1| < 1$; the boundary of Δ_w has the equation $u = \log(2 \cos v)$. Applying Theorem 3.2 for convergence, we at once have:

THEOREM 4.1. *If $D(f) \subset \Delta_w$, the subset of Ω_w containing zero and bounded by $u = \log(2 \cos v)$, then the Newton series*

$$(4.1) \quad \sum_0^\infty C_{z,n} \Delta^n f(0)$$

converges to $f(z)$ for all z .

The condition $D(f) \subset \Delta_w$ may also be expressed in the form: $k(\theta, D(f)) < k(\theta, \Delta_w)$ for all θ . Since $k(\theta, D(f)) = h(\theta, f)$ and since [18, vol. 1, part III, no. 114]

$$k(\theta; \Delta_w) = \cos \theta \log(2 \cos \theta) + \theta \sin \theta$$

for all θ , $|\theta| < \pi/2$, we can rephrase this theorem [6].

COROLLARY 1. *If $h(\theta, f) < \cos \theta \log(2 \cos \theta) + \theta \cos \theta$ for all $|\theta| < \pi/2$, then the Newton series (4.1) converges to $f(z)$ for all z .*

Since the least value of $k(\theta, \Delta_w)$ is $\log 2$, the condition $f \in K[A]$, $A < \log 2$, is sufficient to insure the convergence of the Newton series [24, p. 55].

We can also obtain sharper results on convergence. Since $\lim_{n \rightarrow \infty} |C_{z,n}| = 0$ for all z such that $\Re(z) > 1$, the series $\sum C_{z,n} \zeta^n$ converges uniformly in any closed subset of the closed circle $|\zeta| \leq 1$ not containing the point -1 . If $f \in K$, the set $D(f)$ is bounded; thus, if $D(f)$ lies in the closure of Δ_w , its image under $\zeta = e^w - 1$ is a compact subset of the closed unit circle and does not contain -1 . Hence, if $D(f)$ lies in the closure of Δ_w and f obeys condition (2.5), we can again infer the convergence of the Newton series; the resulting theorem is not as strong as has been obtained by Nörlund who assumed only that f was regular in a half plane [15].

THEOREM 4.2. *If $f \in K[\psi(r), \theta \sin \theta + \cos \theta \log(2 \cos \theta)]$ where $\int^\infty \psi(r) dr < \infty$, then the Newton series (4.1) converges to $f(z)$ for all z , $\Re(z) > -1$.*

We now consider summability of the Newton series. In applying Theorem 3.2, we first observe that the E -star of Ω_ζ is simply $\{-1\} \cdot E$, which we write as $-E$; applying the general theorem, we have:

THEOREM 4.3. *If $D(f)$ lies in the image of the set $-E$ under $w = \log(1 + \zeta)$, then*

$$f(z) = (E) - \sum_0^\infty C_{z,n} \Delta^n f(0).$$

If we specialize this to Borel summability by choosing $E(z) = e^z$, then $-E$ is the half plane $\Re(z) > -1$, and its image is the strip $|v| < \pi/2^{(5)}$.

COROLLARY 1. If $f \in K(a, c)$ with $c < \pi/2$, then

$$(4.2) \quad f(z) = (B) - \sum_0^{\infty} C_{z,n} \Delta^n f(0).$$

For Mittag-Leffler summability, $-E$ is exactly Ω_{ζ} , and Ω_w^* is therefore Ω_w .

COROLLARY 2. If $f \in K(a, c)$ with $c < \pi$, then

$$(4.3) \quad f(z) = (ML) - \sum_0^{\infty} C_{z,n} \Delta^n f(0).$$

Convergence may also be established by the addition of Tauberian conditions. Two examples of this will be sufficient illustration of the method.

COROLLARY 3. If $f \in K(a, c)$, $c < \pi/2$, and $\Delta^n f(0) = o(1/n^{1/2})$ then the Newton series (4.1) converges to $f(z)$ for all z , $\Re(z) > -1$.

COROLLARY 4. If $f \in K(a, c)$, $c < \pi$, and $(-1)^n \Delta^n f(0) \geq 0$ for $n = 0, 1, \dots$, then the Newton series (4.1) converges to $f(z)$ for all z .

The first of these follows from Corollary 1; the Newton series is Borel summable for $z = -1$, and $C_{-1,n} = (-1)^n$ so that the terms of this series satisfy the Tauberian condition $o(1/n^{1/2})$. Applying a theorem of Hardy [14], the series converges; it then follows that since the Newton series converges for $z = -1$, it also converges for all z with $\Re(z) > -1$.

The second arises in a similar manner from Corollary 2; the series is ML summable for $z = -k$, and $C_{-k,n} \Delta^n f(0)$ is by hypothesis positive for each n . Since ML summability is totally regular, the series must in fact converge for $z = -k$, and since this holds for any k , we have convergence for all z .

5. Stirling series. To obtain one form of this interpolation series, we take $T_n(f) = \Delta^n f(-n/2) = (-1)^n \sum_0^n C_{n,k} (-1)^k f(k - n/2)$. The generating function $g_n(w)$ is $(e^w - 1)^n e^{-nw/2}$ which is $(\zeta(w))^n$ for $\zeta(w) = 2 \sinh(w/2)$. The functions $u_n(z)$ are

$$u_0(z) = 1, \quad u_n(z) = \frac{z}{n} C_{z-1+n/2, n-1}.$$

⁽⁵⁾ Using results of Dienes [8], it may be shown that $e^{zw} = (B) - \sum C_{z,n} [\zeta(w)]^n$, uniformly in compact subsets of $\Re(z) = 1$, if $\Re(z) > 0$. We can then obtain a stronger form of Corollary 1. If $|f(z)| \leq \psi(r) \exp \{a|x| + (\pi/2)|y|\}$ where $\int \psi(r) dr < \infty$, then $f(z) = (B) - \sum C_{z,n} \Delta^n f(0)$ for all z with $\Re(z) > 0$. This method will not work for ML summability, since we cannot expect the expansion of e^{zw} to be summable on the boundary of Ω_{ζ} ; for this reason no attempt will be made in the present paper to obtain sharper forms of the summability theorems.

$\zeta(w)$ is univalent in the strip $|v| < \pi$, which we may therefore take to be the set Ω_w ; Ω_ζ becomes the ζ -plane with cuts from $2i$ to $+\infty i$, and from $-2i$ to $-\infty i$ deleted. The circle Δ_ζ is $|\zeta| < 2$, and the set Δ_w is the subset of Ω_w described by $|\sinh(w/2)| < 1$. The supporting function of Ω_w is [18, III, no 115]:

$$(5.1) \quad k(\theta, \Delta_w) = \begin{cases} 2 \cos \theta \log [2^{1/2} \cos \theta + (\cos 2\theta)^{1/2}] \\ \quad + 2 \sin \theta \sin^{-1} (2^{1/2} \sin \theta), & |\theta| \leq \pi/4, \\ \pi |\sin \theta|, & \pi/4 \leq |\theta| \leq \pi/2, \\ k(\theta \pm \pi, \Delta_w), & \pi/2 \leq |\theta| \leq \pi. \end{cases}$$

Applying Theorem 3.2, we obtain the following result first proved by Nörlund using different methods [15].

THEOREM 5.1. *If $h(\theta, f) < k(\theta, \Delta_w)$, given by (5.1), then the Stirling series*

$$(5.2) \quad f(0) + \sum_1^\infty \frac{z}{n} C_{z-1+n/2, n-1} \Delta^n f(-n/2)$$

converges to $f(z)$ for all z .

Since $|u_n(z)| \leq O(2^{-n} n^{-3/2})$, the expansion of e^{zw} in powers of ζ converges for all ζ , $|\zeta| \leq 2$. This gives at once the following extension of a result of Schmidli [20].

THEOREM 5.2. *If $f \in K[\psi(r), k(\theta, \Delta_w)]$ where $\int^\infty \psi(r) dr < \infty$, then the Stirling series (5.2) converges to $f(z)$ for all z .*

The least value of $k(\theta, \Delta_w)$ is $\log(3+2^{3/2})$; thus, the condition that $f \in K[A]$ with $A < \log(3+2^{3/2})$ or $f \in K[\psi(r), \log(3+2^{3/2})]$ is sufficient to insure convergence. (The first of these is due to Pólya [17].)

For Borel summability we consider the corresponding E -star of Ω_ζ which in this case is the strip $|\Im(\zeta)| < 2$, the Borel polygon of Ω_ζ . Its image Ω_w^* is the set bounded by the curves $\cosh(u/2) \sin(v/2) = 1$. We therefore have Borel summability whenever $D(f)$ lies in the interior of this set. We cannot make use of the supporting function of this set as a bound on $h(\theta, f)$ since this set is not convex. However, since $\cos t \cosh t \leq 1$ for all t , $0 \leq t \leq \pi/2$, the square S with vertices $\pi, -\pi, \pi i, -\pi i$ lies in Ω_w^* , and is convex. Its supporting function is seen to be

$$k(\theta, S) = \begin{cases} \pi |\cos \theta| & \text{for } |\theta| < \pi/4 \text{ or } |\theta - \pi| < \pi/4, \\ \pi |\sin \theta| & \text{for } \pi/4 \leq |\theta| \leq 3\pi/4. \end{cases}$$

THEOREM 5.3. *If $D(f)$ lies in the region bounded by the curves $\cosh(u/2) \sin(v/2) = 1$, or if $h(\theta, f) < k(\theta, S)$, then*

$$f(z) = (B) \cdot \{f(0) + z/n C_{z-1+n/2, n-1} \Delta^n f(-n/2)\}.$$

COROLLARY. If $f \in K[\pi/2^{1/2}]$, then the Stirling series (5.2) is Borel summable to $f(z)$ for all z .

For the circle $|w| = \pi/2^{1/2}$ lies in S .

THEOREM 5.4. If $f \in K(a, c)$, $c < \pi$, then the Stirling series (5.2) is ML summable to $f(z)$ for all z .

6. Generalized difference series. If we take $T_n(f) = \Delta^n f(\beta n) = (-1)^n \cdot \sum_0^n C_{n,k} (-1)^k f(\beta n + k)$, we obtain a general class of interpolation series of which the Newton and Stirling discussed in the preceding sections are special cases. The generating functions are $g_n(w) = (e^w - 1)^n e^{\beta n w}$, and $\zeta(w) = e^{\beta w} (e^w - 1)$. The functions $u_n(z)$ are defined by

$$(6.1) \quad u_0(z) = 1, \quad u_n(z) = (z/n) C_{z-\beta n-1, n-1}.$$

We shall study these series for $\beta > 0$. The special case arising from $\beta = 1$ has been used by Selberg, and studied briefly by Gelfond [10; 22].

LEMMA. $\zeta(w) = e^{\beta w} (e^w - 1)$ is univalent in the region Ω_w containing the origin and bounded by the curve Γ whose equation is $u = \log \sin \beta v - \log \sin (\beta + 1)v$.

Computation shows that $\cos(\arg \zeta) = -1$ on Γ ; angles at $w = \log(\beta/(1+\beta))$ are doubled.

This gives us the set Ω_w . Its boundary Γ is convex, opens to the right, cuts the v axis at $w = \pm \pi i/(2\beta + 1)$, the u axis at $w = \log(\beta/(1+\beta))$, and has the lines $v = \pm \pi/(\beta + 1)$ for asymptotes. The image of Ω_w under $\zeta = \zeta(w)$ is the ζ -plane, with a cut from $\zeta_0 = \zeta(\log(\beta/(1+\beta)))$ to $-\infty$ deleted, which is therefore the set Ω_ζ .

Applying Theorem 3.2 for ML summability, we have:

THEOREM 6.1. If $D(f)$ lies in the region Ω_w bounded by the curve Γ whose equation is $u = \log \sin \beta v - \log \sin (\beta + 1)v$, then

$$(6.2) \quad f(z) = (ML) \cdot \left\{ f(0) + \sum_1^\infty \frac{z}{n} C_{z-\beta n-1, n-1} \Delta^n f(\beta n) \right\}.$$

It can be shown that the circle $|w| < \log(1+\beta)/\beta$ lies inside Ω_w if β is greater than $\beta_0 (\sim 1/12)$, the positive root of $\log(1+\beta)/\beta = 3/(2\beta + 1)$. This is the value for which the circle described becomes the circle of curvature. We then conclude that the expansion (6.2) holds for all $f \in K[A]$ with $A < \log(1+\beta)/\beta$.

For special choices of β , we can obtain more detailed results. For example, in the case studied by Selberg and Gelfond, $\beta = 1$, Γ has the equation: $u = -\log(2 \cos v)$.

COROLLARY. If $h(\theta, f) < (\pi - \theta) \sin \theta + \cos \theta \log(-2 \cos \theta)$ for $\pi/2 \leq |\theta| \leq \pi$, then

$$f(z) = (ML) - \left\{ f(0) + \sum_1^{\infty} \frac{z}{n} C_{z-n-1, n-1} \Delta^n f(n) \right\}.$$

We can also discuss convergence. For the special case that we are treating, $\zeta_0 = \zeta (\log 1/2) = -1/4$, and Δ_r is $|\zeta| < 1/4$ and Δ_w is the subset of Ω_w satisfying $|e^w(e^w - 1)| < 1/4$.

THEOREM 6.2. *If $D(f)$ lies in the region contained in Ω_w and bounded by $|e^w(e^w - 1)| = 1/4$ —for example, if $f \in K[A]$ with $A < \log(1 + 2^{1/2})/2$ —then*

$$(6.3) \quad f(0) + \sum_1^{\infty} \frac{z}{n} C_{z-n-1, n-1} \Delta^n f(n)$$

converges to $f(z)$ for all z .

In general, $u_n(z)$, given by (6.1), satisfies

$$|u_n(z)| = O(1)n^{-3/2}[\beta^{-\beta}(\beta + 1)^{1+\beta}]^n$$

so that $\sum u_n(z)\zeta^n$ converges uniformly for $|\zeta| = |\zeta_0|$.

THEOREM 6.3. *If $D(f)$ lies in the closure of Δ_w , and if $f \in K[\psi(r), h(\theta), f]$ where $\int \psi(r)dr < \infty$, then (6.3) converges to $f(z)$.*

7. Abel series. For this, we take $T_n(f) = f^{(n)}(n)$, and $g_n(w) = w^n e^{nw} = [\zeta(w)]^n$ with $\zeta(w) = we^w$. The functions u_n are given by

$$u_0(z) = 1, \quad u_n(z) = z(z - n)^{n-1}/n!.$$

LEMMA. $\zeta(w) = we^w$ is univalent in the region containing the origin and bounded by the curve whose equation is

$$(7.1) \quad \rho = \frac{\pi - |\phi|}{\sin |\phi|}$$

where $w = \rho e^{i\phi}$.

Computation shows that $\arg(we^w) = \pi$ when $\phi > 0$ and $\rho = (\pi - \phi)/\sin \phi$, and $-\pi$ when $\phi < 0$ and $\rho = -(\pi + \phi)/\sin \phi$. Angles at $w = -1$, $\zeta = -e^{-1}$ are doubled.

This gives us the set Ω_w ; its boundary, Γ , opens to the right, cuts the real axis at -1 , and has the lines $v = \pm\pi$ for asymptotes. Under $\zeta = we^w$, Ω_w is mapped onto the ζ -plane with the interval $[-e^{-1}, -\infty]$ deleted; this set is then Ω_r . The circle Δ_r is $|\zeta| < 1/e$, and Δ_w is the subset of Ω_w defined by $|we^{1+w}| < 1$, a convex set. Computing the supporting function of Δ_w , we have

$$(7.2) \quad k(\theta, \Delta_w) = \begin{cases} \sin 2\psi/2 \sin \theta & \text{for } 0 \leq |\theta| \leq 3\pi/4, \\ |\cos \theta| & \text{for } 3\pi/4 < |\theta| \leq \pi, \end{cases}$$

where

$$0 = \log \{ \cos \psi / \sin \theta \} + \sin \psi (\sin \psi + \cot \theta \cos \psi).$$

Applying Theorem 3.2, we obtain the following theorem on convergence.

THEOREM 7.1. *If $D(f) \subset \Delta_w$, or if $h(\theta, f) < k(\theta, \Delta_w)$, given by (7.2), then the Abel series*

$$(7.3) \quad f(0) + \sum_1^{\infty} \frac{z(z-n)^{n-1}}{n!} f^{(n)}(n)$$

converges to $f(z)$ for all z .

This result is not new [9; 11]. The least value of $k(\theta, \Delta_w)$ is assumed for $\theta = 0$, and is the positive root of $xe^{1+x} = 1$ which is approximately .278. Thus, we have convergence of the Abel series if $f \in K[A]$ where $A < .278$ [9; 12].

Stronger results follow as before; since $|u_n(z)| \leq O(1)e^n n^{-3/2}$ the series $\sum u_n(z)\zeta^n$ converges uniformly to e^{zw} for all ζ with $|\zeta| \leq e^{-1}$. This leads to the following slight extension of a result due to Schmidli [20].

THEOREM 7.2. *If $f \in K[\psi(r), k(\theta, \Delta_w)]$ where $\int^\infty \psi(r) dr < \infty$, then the Abel series (7.3) converges to $f(z)$ for all z .*

For Mittag-Leffler summability, Ω_f^* is Ω_f since this is already a star set, and Ω_w^* is the convex set Ω_w , bounded by the curve (7.1). Its supporting function is

$$(7.4) \quad k(\theta, \Omega_w) = \beta(\sin \theta - \cos \theta \cot \beta), \quad \pi/2 \leq |\theta| \leq \pi,$$

where

$$\tan \theta = \cot \beta - \beta \csc^2 \beta.$$

THEOREM 7.3. *If $h(\theta, f) < k(\theta, \Omega_w)$, then the Abel series (7.3) is ML summable to $f(z)$ for all z .*

The minimum value of $k(\theta, \Omega_w)$ is 1, achieved for $\theta = \pi$. Hence, the condition $f \in K[A]$, with $A < 1$, is sufficient for ML summability.

Borel summability, or more generally E summability, can also be discussed as before. The E -star of Ω_f is simply $\{-e^{-1}\} \cdot E$ and Ω_w^* is the image of this set under $w = w(\zeta)$. Hence, if $D(f)$ lies in this set, the Abel series (7.3) is E summable to $f(z)$. However, the region Ω_w^* will in general be quite complicated in description. Thus, for Borel, its boundary is the curve whose equation is $\log \rho + \rho \cos \phi + \log \cos(\rho \sin \phi + \phi) = -1$.

8. Generalized Newton series. Formally, the following expansion is valid:

$$(8.1) \quad e^{zw} = \beta^z \sum_0^{\infty} C_{z,n} \beta^{-n} (e^w - \beta)^n.$$

This converges for all z when $|e^w - \beta| < |\beta|$, and converges for all z such that $\Re(z) > -1$ when $|e^w - \beta| \leq |\beta|$ and w is finite. This expansion yields an interpolation series which reduces to the Newton series when $\beta = 1$. We take β real and positive. If $g_n(w) = (e^w - \beta)^n$ are chosen as generating functions for the functionals $\{T_n\}$, then

$$T_n(f) = (-\beta)^n \sum_0^n C_{n,k} (-1/\beta)^k f(k).$$

Since $\zeta(w) = e^w - \beta$, Ω_w is again the strip $|v| < \pi$ and Ω_ζ is the ζ -plane, with the interval $[-\beta, -\infty]$ deleted; the functions $u_n(z)$, from (8.1), are seen to be $\beta^z C_{z,n} \beta^{-n}$. The results for Borel and Mittag-Leffler summability are the same for this series as for the Newton series, since the corresponding sets Ω_w^* are the same.

THEOREM 8.1. *The series*

$$(8.2) \quad \beta^z \sum_0^\infty C_{z,n} \beta^{-n} T_n(f)$$

is Borel summable to $f(z)$ for all z if $f \in K(a, c)$ with $c < \pi/2$, and is Mittag-Leffler summable to $f(z)$ if $f \in K(a, c)$, $c < \pi$.

The interpolation series is convergent for a wider class of functions than the Newton series, if $\beta > 1$. The circle Δ_ζ is $|\zeta| < \beta$, and Δ_w is the set containing the origin and bounded by the curve $u = \log(2\beta \cos v)$. The supporting function of Δ_w is given by

$$k(\theta, \Delta_w) = \theta \sin \theta + \cos \theta \log(2\beta \cos \theta).$$

THEOREM 8.2. *If $h(\theta, f) < k(\theta, \Delta_w)$, then the generalized Newton series (8.2) converges to $f(z)$ for all z .*

As β increases, the set Δ_w approaches the strip $|v| < \pi/2$. If $f \in K(a, c)$ with $c < \pi/2$, then $D(f)$ is a compact subset of this open strip, and thus a subset of Δ_w for sufficiently large β .

COROLLARY. *If $f \in K(a, c)$, with $c < \pi/2$, then for sufficiently large β , the series (8.2) converges to $f(z)$ for all z .*

Making use of the convergence of (8.1) on the boundary of Δ_ζ , as observed above, we can also obtain the corresponding stronger convergence form.

THEOREM 8.3. *If $f \in K[\psi(r), \theta \sin \theta + \cos \theta \log(2\beta \cos \theta)]$ where $\int \psi(r) dr < \infty$, then the series (8.2) converges to $f(z)$ for all z with $\Re(z) > -1$.*

9. Lidstone series. This well known series arises from the choice of $g_{2n}(w) = w^{2n}$, $g_{2n+1}(w) = e^w w^{2n}$, which gives $T_{2n}(f) = f^{(2n)}(0)$, $T_{2n+1}(f) = f^{(2n)}(1)$. The generating functions are not of the form $[\zeta(w)]^n$ so that Theorem 3.2 does

not apply; however, the general method of §3 may still be used. We seek an expansion of the form

$$(9.1) \quad e^{zw} = \sum_0^{\infty} u_n(z) g_n(w).$$

Requiring that the g_n have the prescribed form, we write

$$(9.2) \quad \begin{aligned} e^{zw} &= F_1(w^2) + e^w F_2(w^2) \\ &= \frac{\sinh(1-z)w}{\sinh w} + e^w \frac{\sinh zw}{\sinh w} \end{aligned}$$

so that

$$u_{2n}(z) = \Lambda_n(1-z), \quad u_{2n+1}(z) = \Lambda_n(z)$$

where Λ_n is a polynomial of degree $2n+1$, determined by the formal identity [25]:

$$\frac{\sinh zw}{\sinh w} = \sum_0^{\infty} \Lambda_n(z) w^{2n}.$$

Accordingly, the formal Lidstone two-point expansion is

$$(9.3) \quad f(z) = \sum_0^{\infty} \Lambda_n(1-z) f^{(2n)}(0) + \sum_0^{\infty} \Lambda_n(z) f^{(2n)}(1).$$

In (9.2), the only singularities of F_1 and F_2 occur at the points $\pm \pi ni$, $n \geq 1$; the series (9.1) will then converge for $|w| < \pi$, be Borel summable for $|v| < \pi$, and be *ML* summable for $u \neq 0$, or $|v| < \pi$.

THEOREM 9.1. *If $f \in K[A]$, $A < \pi$, the Lidstone series (9.3) converge to $f(z)$ for all z . If $f \in K(a, c)$, $c < \pi$, then they are Borel summable to $f(z)$ for all z . If $D(f)$ does not contain any points of the lines $u=0$, $|v| \geq \pi$, then the series are Mittag-Leffler summable to $f(z)$ for all z .*

The convergence result is not as strong as can be proved; it is known that if $f \in K[\eta(r)/r^{1/2}, \pi]$ where $\eta(r) = o(1)$ as r increases, then the Lidstone expansion of f converges [2]. This illustrates again the limitations of the present method, for it can be easily seen that the expansion of e^{zw} in (9.1) does not converge on $|w| = \pi$.

The same approach can be made to the generalized n -point expansion and other essentially periodic type functionals [19; 20]. If $a_1, \dots, a_r$ are distinct complex numbers, we define $\{T_n\}$ by $T_{rn+j}(f) = f^{(rn)}(a_j)$ for $j=1, 2, \dots, r$, and $n=0, 1, 2, \dots$. The generating functions are given by

$$g_{rn+j}(w) = w^{rn} \exp(a_j w).$$

We first seek an expansion of the form

$$\begin{aligned} e^{zw} &= \sum_0^\infty u_n(z) g_n(w) = \sum_{j=1}^r \sum_{n=0}^\infty u_{rn+j}(z) w^{rn} \exp(a_j w) \\ &= \sum_{j=1}^r F_j(w^r) \exp(a_j w). \end{aligned}$$

Let σ be a primitive r th root of unity; then

$$\exp \sigma^k z w = \sum_{j=1}^r F_j(w^r) \exp(\sigma^k a_j w)$$

for $k=1, 2, \dots, r$. Solving these for the functions F_j , they are obtained as quotients of exponential polynomials. If

$$B(w) = B(w; a_1, \dots, a_r) = \begin{vmatrix} \exp(\sigma a_1 w) & \cdots & \exp(\sigma a_r w) \\ \vdots & & \vdots \\ \exp(\sigma^r a_1 w) & \cdots & \exp(\sigma^r a_r w) \end{vmatrix}$$

then $F_j(w^r) = B(w; a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_r) / B(w)$, which serves to define the polynomials $u_n(z)$.

It is readily seen that F_j is regular at $w=0$, and its only singularities occur at the other zeros of $B(w)$. If Ω_w^* is the star with vertices at these zeros, then the expansion

$$f(z) = \sum_0^\infty u_n(z) T_n(f) = \sum_{j=1}^r \sum_{n=0}^\infty u_{rn+j}(z) f^{(rn)}(a_j)$$

is Mittag-Leffler summable for all f such that $D(f) \subset \Omega_w^*$.

10. Uniqueness theorems. In this section, we shall use the preceding results to obtain uniqueness classes for certain functionals $\{T_n\}$. Clearly, if there is a formal interpolation series $f(z) = \sum u_n(z) T_n(f)$, then a convergence class is also a uniqueness class. However, in most cases, the functionals $\{T_n\}$ are total over a much larger class of functions than that for which the interpolation series converges. Gelfond, in an effort to treat these two problems in a unified manner, was able in certain cases to determine a uniqueness class for $\{T_n\}$ by constructing a new sequence of functionals $\{T_n^*\}$, total over the same class and therefore having the same uniqueness class, but having a larger expansion class for the corresponding interpolation series [10].

If we observe that in order to conclude that $f=0$ if it is known that $T_n(f)=0$ for $n=0, 1, \dots$, it is sufficient to have the corresponding interpolation series merely summable by some regular method; then we obtain at once all of Gelfond's results by applying Theorem 3.2, using Mittag-Leffler summability. This gives the following general theorem.

THEOREM 10.1. *If $g_n(w) = [\zeta(w)]^n$ are the generating functions of $\{T_n\}$, if*

Ω_w is a region of univalence of $\zeta(w)$, if Ω_ζ is the image of Ω_w under $\zeta = \zeta(w)$, if Ω_ζ^* is the star of Ω_ζ , and if Ω_w^* is its image in Ω_w under $w = w(\zeta)$, then the class of all f such that $D(f) \subset \Omega_w^*$ is a uniqueness class for $\{T_n\}$.

Specialization of $\zeta(w)$ at once yields the results of Gelfond, as well as new results and extensions. We give only a few of these.

COROLLARY 1. If $f \in K(a, c)$, $c < \pi$, and $\Delta^n f(0) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

The function $f(z) = \sin \pi z$ shows that the critical value of c cannot be increased.

COROLLARY 2. If $f \in K(a, c)$, $c < \pi$, and $\Delta^n f(-n/2) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

The example $f(z) = z \sin \pi z$ shows that π is again best.

COROLLARY 3. If $D(f)$ lies in the region containing zero, and bounded by the curve $u = \log \sin \beta v - \log \sin (1 + \beta)v$ for $\beta > 0$, and if $\Delta^n f(\beta n) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

Gelfond studied the special case $\beta = 1$; his result is included in the following corollary.

COROLLARY 4. If $h(\theta, f) < (\pi - \theta) \sin \theta + \cos \theta \log (-2 \cos \theta)$ for $\pi/2 \leq |\theta| \leq \pi$, and $\Delta^n f(n) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

COROLLARY 5. If $h(\theta, f) < k(\theta, \Omega_w)$ given by (7.4), and if $f^{(n)}(n) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

This was also obtained by Gelfond, although the function $k(\theta, \Omega_w)$ is incorrectly given. In particular, the class $K[A]$, with $A < 1$, is a uniqueness class for these Abel functionals. The example $f(z) = ze^{-z}$ shows that this value of A cannot be improved.

COROLLARY 6. If $f \in K(a, c)$, $c < \pi$, and for $n = 0, 1, \dots$ and some positive β ,

$$\sum_{k=0}^n C_{n,k} (-1/\beta)^k f(k) = 0,$$

then $f = 0$.

COROLLARY 7. If $f \in K$ and $D(f)$ does not contain any of the points $w = iv$, $|v| \geq \pi$, and if $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for $n = 0, 1, \dots$, then $f = 0$.

Although more general than that usually stated, this result also follows from the general characterization theorem of Schoenberg for Lidstone series [21].

It is often possible to obtain a uniqueness class for one sequence of functionals from that for another. If, for example, $\{T_n^*\}$ is a second sequence so

related to $\{T_n\}$ that if $T_n(f) = 0$ for all n implies that $T_n^*(f) = 0$ for all n , then a uniqueness class for $\{T_n^*\}$ is also one for $\{T_n\}$. This is the case with $T_n(f) = f(n)$, and $T_n^*(f) = \Delta^n f(0)$. Thus, from Corollary 1 above, we obtain the following form of Carlson's theorem [13].

COROLLARY 8. *If $f \in K(a, c)$, $c < \pi$, and if $f(n) = 0$ for $n = 0, 1, \dots$, then $f = 0$.*

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BROWN UNIVERSITY
PROVIDENCE, R.I.