THE REPRESENTATION OF ABSTRACT MEASURE FUNCTIONS

BY

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1. Introduction.

1.1. Several authors have discussed abstract "measures" and "integrals," the values of which need not be numerical(1). Our object here is to show that, in a sense, such abstract measures and integrals are often equivalent (under fairly general hypotheses) to combinations of ordinary numerical ones. We shall here consider only the theory of measure, to which the theory of abstract-valued integration can in most cases be reduced. The main theorem (3.1 (f) and §19) can be stated, roughly, as follows: a field of sets with an abstract measure on it is isomorphic to an ideal in the direct product of two such fields, on one of which the measure is an ordinary numerical one, while on the other factor the measure is trivial. (Precise definitions of the concepts involved will be given later.) As a consequence of this, the abstract measure can be replaced, to within isomorphism, by one whose values are (real-valued) numerical functions on a certain set, modulo a certain class of null functions.

We conclude the paper with a few deductions from this theorem, reserving however the application to abstract-valued integration for a subsequent paper.

1.2. *Formulation of the problem.* Instead of dealing directly with a field F of sets, on which a suitable abstract-valued measure λ is defined, it is more convenient to work with the Boolean algebra E of the sets of F *modulo null sets*—that is, with the algebra formed by the equivalence classes $\{x\}$ of sets belonging to F, where $y \in \{x\}$ means $\lambda(x+_2 y)=0$(2), the "measure" on E being given by setting $\lambda(\{x\})=\lambda(x)$. (That this relation *is* an equivalence relation, that $\lambda(x)=\lambda(y)$ whenever $y\in\{x\}$, and that the equivalence classes $\{x\}$ do form a Boolean σ-algebra(3) are in effect included in our postulates on λ.) Thus we *provisionally* define an abstract measure algebra to be an ordered pair (E, λ), where E is a Boolean σ-algebra and λ is an abstract measure-function on E, satisfying "reasonable" postulates (as yet unspecified). Two abstract measure algebras (E_1, λ_1) and (E_2, λ_2) are defined to be *isomorphic* if there exists an isomorphism of the algebras E_1 and E_2, say $\phi(x_1)=x_2$, such

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(1) We shall not be directly concerned here with abstract-valued integrals or with the closely related real-valued integrals of abstract-valued functions. For abstract-valued measures, see for example [4], [5], [10], and [14, §8] in the bibliography at the end of the paper.

(2) See 1.3 for the notations.

(3) See, for example [2, pp. 29, 88].

that $\lambda_2(\phi(x_1))=\lambda_2(\phi(y_1))$ if and only if $\lambda_1(x_1)=\lambda_1(y_1)$. If we further have $\lambda_2(\phi(x_1))=\lambda_1(x_1)$ for a suitable isomorphism ϕ, then (E_1, λ_1) and (E_2, λ_2) are said to be *isometric*.

We shall here be interested in abstract measure algebras only to within isomorphism; and this permits a further reduction of the problem. In (E, λ) we define the "*induced*" equivalence relation, $\sim$, by writing $x\sim y$ to mean $\lambda(x)=\lambda(y)$. The properties of (E, λ) to within isomorphism are evidently determined completely by this equivalence relation. Accordingly we define an *abstract measure algebra* to be either an ordered pair (E, λ) as before, or an ordered pair $(E, \sim)$, where E is a Boolean σ-algebra and $\sim$ is an equivalence relation on E satisfying the postulates to be given below. If $\sim$ is the equivalence relation induced by λ, we say that (E, λ) and $(E, \sim)$ are *naturally isomorphic*. The definition of isomorphism between two abstract measure algebras is now clear; for example, $(E_1, \sim)$ and $(E_2, \approx)$ will be isomorphic if and only if there exists a mapping ϕ of E_1 on E_2 such that (i) ϕ is an algebraic isomorphism and (ii) $\phi(x_1)\approx\phi(y_1)$ if and only if $x_1\sim y_1$.

1.3. *Notations*. As far as practicable, we adhere to the following notations. E denotes a Boolean σ-algebra[3] having o as zero element and e as unit element. Small letters a, b, x, y, and so on, usually denote elements of E, except that m, n, i, j usually denote positive integers; capital letters X, Y, and so on, usually denote subsets of E (or of some other set). Phrases like "$x\in E$, $X\subset E$" are generally omitted when the meaning is clear. The partial ordering relation in E is written $\leqq$, as usual; thus $x<y$ means $x\leqq y$ and $x\neq y$. (But $X\subset Y$ does not exclude the equality of X and Y.) The supremum of two elements of E is written $x\vee y$, and of an infinite collection is written $\bigvee x_i$; infima are written xy, $\bigwedge x_i$. We use aX to denote the set of all elements of the form ax, where $x\in X$.

If $xy=o$, we say that x and y are *disjoint*. A subset of E is said to be *disjoint* if every two of its elements are disjoint.

The *complement* of x is denoted by $-x$, and $x-y$ is used as an abbreviation for $x(-y)$. The *symmetric difference* of x and y is the element $(x-y)\vee(y-x)$; we shall usually denote this by $x+_2y$ ("addition mod 2").

E will generally be assumed to satisfy the *countable chain condition*, namely, there does not exist an uncountable disjoint subset of E. As is well known, this will imply that E is *complete* (that is, that for *every* $X\subset E$ both $\bigvee\{x \mid x\in X\}$ and $\bigwedge\{x\mid x\in X\}$ exist).

The words "algebra," "sub-algebra," used unqualified always refer to Boolean σ-algebras. A Boolean algebra in which only finite operations are to be considered will be called a "finitely additive algebra."

For any $a\in E$, the *principal ideal* $E(a)$ of a in E is the set of all $x\in E$ satisfying $x\leqq a$. $E(a)$ is evidently itself an algebra with $\leqq$, $\bigvee$, $\bigwedge$, having the same meaning as in E, but having a as unit element instead of e. A sub-algebra of $E(a)$ will be called a sub-algebra of E *relative to* a. Similarly, if F is a subalgebra of E, and $b \in F$, $F(b)$ denotes the principal ideal of b in F; it is clear that $F(b) = bF$, but the notation $F(b)$ is used to emphasize that $b \in F$.

Given $X \subset E(a)$, the sub-algebra relative to a which it generates is called the *Borel field* of X *relative to* a, and is written $\mathcal{B}_a(X)$; it is thus the smallest set $B \subset E(a)$ which satisfies (i) $B \supset X$, (ii) if $x \in B$, then $a - x \in B$, (iii) if $x_n \in B$ $(n = 1, 2, \cdots)$ then $\bigvee x_n \in B$. If in particular $a = e$, we omit the phrase "relative to a," and write the Borel field of X as $\mathcal{B}(X)$.

The usual notations of set theory are employed for subsets of E (or of other sets), the union of X and Y being written $X \cup Y$, and their intersection $X \cap Y$, and so on. The empty set is denoted by 0. The cardinal number of a set X is written $|X|$.

Greek letters are generally used as follows: α, β, and so on, are ordinal numbers; λ is an abstract measure, and μ is an ordinary numerical one; θ, ϕ, ψ, are mappings; ρ, σ, τ, are non-negative real numbers not exceeding 1, ρ usually being rational; and ϵ is a positive number less than 1.

Sequences (indicated by the notation $\{x_n\}$) can usually be either finite or (countably) infinite, except where the distinction is emphasized (for example, by writing $n = 1, 2, \cdots$, to ∞).

Where the abstract measure or equivalence relation intended is clear, we often write (E, λ) or $(E, \sim)$ simply as E.

2. **Postulates.**

2.1. Since the equivalence relation $\sim$, rather than the abstract measure λ, will be used in what follows, it is simpler to state our postulates in terms of $\sim$. (It would be easy but tedious to deduce them from "reasonable" postulates on λ[4]; and conversely the main theorem of this paper, stated in 3.1(f), will show that every equivalence relation satisfying our postulates is in fact induced by such a "reasonable" λ.) The postulates for the abstract measure algebra $(E, \sim)$ are:

(0) E is a Boolean σ-algebra satisfying the countable chain condition, and $\sim$ is an equivalence relation on E.

(I) If $x = \bigvee x_n$, $y = \bigvee y_n$, where $x_m x_n = o = y_m y_n$ $(m \neq n;\ m, n = 1, 2, \cdots)$, and if, for each n, $x_n \sim y_n$, then $x \sim y$.

(II) If $x \sim x'$, and if $y \leqq x$ is given, there exists $y' \leqq x'$ such that $y \sim y'$.

Before stating the last postulate, we need the following definition. An element x is *bounded* if, whenever $y \leqq x$ and $y \sim x$, we have $y = x$.

(III) If $x \sim y$ there exist *bounded* elements x_n, y_n $(n = 1, 2, \cdots)$ such that $x = \bigvee x_n$, $y = \bigvee y_n$, $x_m x_n = o = y_m y_n (m \neq n)$, and $x_n \sim y_n$.

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(4) The set of values taken by λ would be, for example, a lattice-ordered Abelian semigroup. Postulate I, for example, would be replaced by the requirement that λ be countably additive. One could deduce from [6, Theorem 10] that there is no loss in assuming the values of λ to be equivalence classes of numerical functions; but we shall derive independently a more precise result than this. For the device of replacing λ by $\sim$, cf. [10] and (though with different objectives) [4] and [12, Ch. VI].

2.2. *Numerical case.* To illustrate the meaning of these postulates, consider a "numerical measure algebra" (E, μ), in which $\sim$ is induced by an ordinary countably additive numerical function μ, taking non-negative real values and possibly the value ∞. Suppose that there exists $x \neq o$ for which $\mu(x) < \infty$. If the postulates 0–III are satisfied it readily follows that the bounded elements are precisely those of finite measure, and thence that μ is σ-finite (that is, $e = \bigvee a_n$ where $\mu(a_n) < \infty$). Further, μ must be "reduced" (that is, vanishes only for o). Conversely, if μ is a numerical countably additive measure on E, reduced and σ-finite, and if II holds, then elements of finite measure are certainly bounded, and postulates 0, I and III readily follow. Postulate II, though not automatically satisfied, can easily be seen to hold if either (i) E is non-atomic (5), or (ii) E is generated by a number of atoms all of equal measure (6). One would thus not expect II to be unduly restrictive in practice (7). There are, of course, important numerical measures which satisfy neither 0 nor III—for example, Hausdorff m-dimensional measure in Euclidean n-space $(n > m)$(8). In what follows, the phrase "*numerical measure algebra*" will mean (except where the contrary is explicitly stated) one in which the postulates are satisfied, and in which the measure takes at least one finite nonzero value.

We note that, for numerical measure algebras, isomorphism implies isometry, to within a nonzero constant numerical factor.

REMARK. It might seem preferable, in the interests of simplicity, to replace postulate III by the stronger requirement that *all* elements be bounded —corresponding in the numerical case to the restriction that $\mu(e) < \infty$. But this would be undesirable, apart from the loss of generality, since it turns out that even then it would be necessary to introduce numerical measure algebras in which some elements have infinite measure and are therefore unbounded.

2.3. *Elementary properties.* The following properties of an abstract measure algebra are easy deductions from the postulates.

(1) $e = \bigvee a_n$, where the elements a_n are bounded and disjoint.

(2) If x is bounded and $y \leqq x$, then y is bounded.

(3) o is bounded; conversely, if $x \sim o$ and x is bounded, then $x = o$.

(4) *If $x \sim o$, then $x = o$.* (From III, (3) and I.)

(5) An *atom* is an element $x \in E$ such that (i) $x \neq o$, (ii) if $o < y \leqq x$, then $y = x$.

(6) Conversely, it is easy to see that a numerical measure algebra which satisfies the postulates is of one of the following four types: (i) non-atomic; (ii) generated by $\aleph_0$ atoms of equal measure; (iii) the direct sum of a non-atomic algebra of finite total measure and of an algebra generated by a finite number of atoms; (iv) entirely generated by atoms and of finite total measure—the possibilities in this case can be further analyzed.

(7) Postulate II can be omitted if the values of the abstract measure λ are assumed to lie in a vector σ-lattice, for it can then be shown that the abstract measure algebra can be imbedded in one which satisfies all the postulates.

(8) Examples of abstract measures not satisfying the postulates are: Choquet's [4; 5], and Löwner's measure in Hilbert space (Ann. of Math. vol. 40 (1939) pp. 816–833).

(5) *Two given elements have maximal equivalent sub-elements.* More precisely; given $a, b \in E$, there exist $x \leqq a$, $y \leqq b$ such that (i) $x \sim y$, (ii) if $h \leqq a-x$, $k \leqq b-y$, and $h \sim k$, then $h=k=o$.

(The elements x and y are constructed by a transfinite induction argument —or equivalently by using Zorn's lemma—the transfinite induction necessarily terminating countably because of the countable chain condition. This type of argument is often called the "method of exhaustion.")

3. **Fundamental examples.**

3.1. We shall here define some particular abstract measure algebras which will play a fundamental part in the sequel.

(a) A numerical measure algebra (J, μ) (cf. 2.2). In particular, important measure algebras satisfying the postulates are: the measure algebra I^1 $(=(I^1, \mu))$ of Lebesgue measurable sets, modulo null sets, in the unit interval (0, 1); and the direct product $I^{\mathfrak{m}}$ of $\mathfrak{m}$ copies of I^1 (cf. [8, p. 419]), $\mathfrak{m}$ being any cardinal number. If $\mathfrak{m} \leqq \aleph_0$, I^1 and $I^{\mathfrak{m}}$ are well known to be isometric. Other measure algebras which will arise in the sequel are: K, the algebra generated by $\aleph_0$ atoms all of measure 1, and L, consisting of just two elements of measures 0 and 1 respectively.

(b) A "*trivial*" abstract measure algebra, U. By definition this is one in which *no* two distinct elements are equivalent(9). Obviously any algebra U satisfying the countable chain condition can be regarded as a trivial abstract measure algebra.

(c) If $(E, \sim)$ is any abstract measure algebra, so is the *relative algebra* (principal ideal) $E(a)$ (cf. 1.3), in which the equivalence relation is defined in the "natural" way; $x \sim y$ in $E(a)$ (where $x, y \leqq a$) if and only if $x \sim y$ in E. The same is true, more generally, of any *relative sub-algebra*, provided that it satisfies postulate II.

(d) If $(E_1, \sim)$ and $(E_2, \approx)$ are abstract measure algebras (satisfying our postulates, as always), then so is their "*free direct sum.*" This is defined as follows. The algebra is the "direct sum" $E_1 \oplus E_2$—that is, the set of ordered pairs (x_1, x_2) with $x_1 \in E_1$, $x_2 \in E_2$, the ordering relation being given by: $(x_1, x_2) \leqq (y_1, y_2)$ if and only if we have both $x_1 \leqq y_1$ and $x_2 \leqq y_2$. The equivalence relation in the direct sum is then given by: (x_1, x_2) and (y_1, y_2) are equivalent if and only if we have both $x_1 \sim y_1$ and $x_2 \approx y_2$.

Remark. This "free direct sum" is to be distinguished from the ordinary "direct sum" of two numerical measure algebras. There the algebra is the same $(E_1 \oplus E_2)$, but the equivalence relation is different, being induced by the measure μ, where $\mu(x_1, x_2) = \mu_1(x_1) + \mu_2(x_2)$, μ_1 and μ_2 being the measures in E_1 and E_2.

(e) A more complicated example is the "*direct product*" $J \otimes U$ of a numerical measure algebra $J = (J, \mu)$ (of type (a) above) and a trivial algebra U (of

(9) A numerical measure algebra can be trivial in this sense, though only if it is generated by atoms. Examples: L, and the algebra generated by a sequence of atoms of measures 3^{-n}.

type (b)). This may be characterized as follows. Suppose (E, M) is an abstract measure algebra in which the values of the abstract measure M are non-negative numerical *functions* on some set, modulo some class of null functions (for example, continuous functions on some space modulo functions which vanish except for sets of first category). Suppose E contains two sub-algebras J, U, such that:

(i) $E=\mathcal{B}(J\cup U)$.

(ii) If $x\in J$, $M(x)$ is a constant function, $=\mu(x)$ say.

(iii) If $x\in U$, $M(x)=\mu(e)$ times the characteristic function of some set (with the convention that $\infty\cdot 0=0$).

(It is further supposed, of course, that M is countably additive in the obvious sense, and that postulates 0–III are satisfied.) Then the sub-algebra (J, M) is clearly isomorphic with a numerical measure algebra (J,μ) (10); and it is not hard to show that the sub-algebra (U, M) is trivial. It further follows that, for any $x\in J$ and $u\in U$, $M(xu)=\mu(x)M(u)$. We say that (E, M) is the *direct product* of (J, μ) and U.

It will be proved in the next section that two *given* algebras (J, μ) and U always have a direct product. This product is moreover unique to within isomorphism; we shall not prove this explicitly, but it follows by essentially the same arguments as in §17.

(f) Finally, it is evident that a principal ideal (that is, relative algebra) in the free direct sum of two direct products of type (e) will be an abstract measure algebra. Our main theorem is that the converse is true; *every abstract measure algebra is isomorphic to one of this type.* (In fact, every abstract measure algebra is isomorphic to a principal ideal in a *single* direct product $J\otimes U$ of type (e) except that J and $J\otimes U$ need not satisfy postulates II and III above.) Thus *the four concepts* (E, λ), $(E, \sim)$, *abstract measure algebra of type* (f), *and* (E, λ) *in which the values of* λ *are numerical functions* (*modulo null functions*), *are all equivalent to within isomorphism.*

We shall also be able to deduce some simpler characterizations in some important special cases (§20).

4. Construction of the direct product $J\otimes U$.

4.1. In this section we show that a given numerical measure algebra $J=(J, \mu)$ and a given trivial abstract measure algebra U (satisfying our postulates) always have a direct product in the sense of 3.1(e). To save the trouble of proving explicitly that the direct product in the sense of 3.1(e) is unique (to within isomorphism), we shall use the unqualified term "direct product" in the rest of the paper to refer to the particular direct product to be constructed in this section. As the construction will depend on the notion of the "representation space" of an algebra, we begin by outlining the relevant properties.

4.2. *Representation spaces.* Let E be *any* Boolean σ-algebra, with or with-

(10) Provided μ is not always 0 or ∞.

out a measure. Then(11) there exists a compact (=bicompact) totally disconnected Hausdorff space R, the *representation space* of E, such that E is isomorphic to the Boolean algebra formed by the "open-closed" (=simultaneously open and closed) subsets of R. Let x^* denote the open-closed subset of R which corresponds to $x \in E$ in this isomorphism. Then we have

(1) The sets x^* form a basis for the open sets of R.

We note that for *finite* operations the "expected" relations hold; that is, $o^* = 0$, $e^* = R$, $(-x)^* = R - x^*$, $x \leqq y \rightleftarrows x^* \subset y^*$, $(x \vee y)^* = x^* \cup y^*$, and $(xy)^* = x^* \cap y^*$. For infinite operations the correspondence is more complicated, but it is easily seen from the laws for finite operations that

(2) $(\bigvee x_n)^* = \bigcup x_n^* \cup z$, $(\bigwedge x_n)^* = \bigcap x_n^* - t$,

where z and t are nowhere dense closed subsets of R (and are thus of the first category).

The *Borel field* $\mathcal{B}(R)$ of R is defined (cf. 1.3) to be the smallest family $\mathcal{B}'$ of subsets of R such that (i) $\mathcal{B}'$ contains all *open-closed* sets, (ii) if $X \in \mathcal{B}'$, $R - X \in \mathcal{B}'$, (iii) if $X_n \in \mathcal{B}'$ $(n = 1, 2, \cdots)$, $\bigcup X_n \in \mathcal{B}'$. We have:

(3) Any given $X \in \mathcal{B}(R)$ can be written in one and only one way as $X = x^* +_2 z$, where x^* is open-closed and z is of the first category(12).

For from (2) it is clear that the family $\mathcal{B}'$ of all sets so expressible as $x^* +_2 z$ satisfies (i), (ii) and (iii); hence it contains $\mathcal{B}(R)$ .To see that the expression is unique, we note that if $x^* +_2 z = y^* +_2 t$ (where x^*, y^* are open-closed and z, t are of the first category) then $z +_2 t = x^* +_2 y^*$, an open set; and since a non-empty open set is not of the first category we must have $z +_2 t = 0 = x^* +_2 y^*$, that is, $z = t$ and $x^* = y^*$.

Now assume that E satisfies the countable chain condition, and is thus complete. From (1) and (2) it now follows that

(4) If G is an open subset of R, so is its closure $\overline{G}$. As a corollary,

(5) If G_1, G_2 are disjoint open subsets of R, so are $\overline{G}_1$ and $\overline{G}_2$.

We shall be interested in numerical functions f on R whose values are between 0 and ∞ inclusive; for brevity we refer to them simply as "functions." The continuity of functions is defined in the obvious way(13), and a function f is said to be continuous *on* $X \subset R$ if f, considered only for the points of X, is continuous relative to the subspace X.

(6) LEMMA. *Given a function f on R which is continuous on $R - H$, where H is of the first category, there exists one and only one function g, continuous on R, such that $g(p) = f(p)$ for each $p \in R - H$.*

Given $p \in H$, we assert lim sup $f(q)$ = lim inf $f(q)$ as $q \rightarrow p$ through points

(11) See [15, Theorem 67].

(12) Thus E is σ-isomorphic, in a natural way, to the algebra formed by $\mathcal{B}(R)$ modulo sets of first category—proving the known theorem (cf. [9]) that the notions "Boolean σ-algebra" and "σ-field of sets modulo a σ-ideal" are equivalent.

(13) If $f(p) = \infty$, continuity of f at p means that, given any n, there exists a neighborhood of p throughout which $f > n$.

of $R-H$[14]. For if not, there exist (finite) real numbers ρ, σ such that $\limsup f(q)>\rho>\sigma>\liminf f(q)$. Define $K_1=\{q \mid q\in R-H \text{ and } f(q)>\rho\}$, $K_2=[q \mid q\in R-H \text{ and } f(q)<\sigma\}$. From the continuity of f, K_1 and K_2 are open relative to $R-H$; hence $K_i=G_i\cap(R-H)$ where G_i is open in R $(i=1, 2)$. Since $K_1\cap K_2=0$, we have $G_1\cap G_2\subset H$; and since a non-empty open set is not of the first category in R, it follows that $G_1\cap G_2=0$. Hence (by (5)) $\overline{G}_1\cap\overline{G}_2=0$, contradicting $p\in\overline{K}_1\cap\overline{K}_2\subset\overline{G}_1\cap\overline{G}_2$.

Define g by: $g(p)=f(p)$ for $p\in R-H$, and $g(p)=\lim f(q)$ for $p\in H$ (where $q\rightarrow p$ in $R-H$). It is easy to see that g is continuous on R. Thus the required function exists; and its uniqueness is immediate.

(7) LEMMA. *Let $\{f_n\}$ be a given sequence of continuous functions on R such that, for each $p\in R$, $f_1(p)\leqq f_2(p)\leqq \cdots$. Then there exists one and only one continuous function f on R such that $f(p)=\lim f_n(p)$ for each $p\in R-H$, H being a set of the first category.*

Let $f'(p)=\lim f_n(p)$; from the previous lemma it is enough to show that f' is continuous on $R-H$ where H is of the first category. We can suppose that f_n and f' are all uniformly bounded (on replacing f_n by $f_n/(1+f_n)$).

For each positive integer k, let $\{x_{ki}^*\}$ $(i=1, 2, \cdots)$ be a maximal disjoint collection (necessarily countable) of nonempty open-closed subsets of R, each having the property that there exists an integer $n=n(k, i)$ such that $f'(p)\leqq f_n(p)+1/k$ whenever $p\in x_{ki}^*$. Write $G_k=\bigcup_i x_{ki}^*$; clearly G_k is open. We assert $\overline{G}_k=R$. For suppose not. Define F_{nk} to be the set of all points $p\in R$ such that $f_m(p)\leqq f_n(p)+1/k$ whenever $m\geqq n$; evidently F_{nk} is closed, and $\bigcup_n F_{nk}=R$. Hence one of the sets $(R-\overline{G}_k)\cap F_{nk}$ $(n=1, 2, \cdots)$ must be somewhere dense, and so (from (4)) must contain a nonempty open-closed set y_k^*. But y_k^* is disjoint from every x_{ki}^* $(i=1, 2, \cdots)$, and yet satisfies the requirement which was imposed on the sets x_{ki}^*; and this contradicts the maximality of the collection $\{x_{ki}^*\}$.

Since $\overline{G}_k=R$, the set $H=\bigcup_k(R-G_k)$ is of the first category. Suppose $p\notin H$; then, for every k, p belongs to some x_{ki}^*. Writing n for $n(k, i)$, we see from the continuity of f_n that there is an open-closed set y^* such that $p\in y^*\subset x_{ki}^*$ and, whenever $q\in y^*$, $|f_n(q)-f_n(p)|<1/k$. Thus $|f'(q)-f'(p)|<3/k$ for all $q\in y^*$. That is, f' is continuous at every point of $R-H$, and is a fortiori continuous on $R-H$. Q.E.D.

REMARK. Arguments similar to the foregoing—which is evidently analogous to the proof of Egoroff's theorem—will occur again; we shall refer to them as "Egoroff arguments."

4.3. Now let (J, μ) be a numerical measure algebra, and let U be a trivial abstract measure algebra. (It is assumed that neither J nor U consists of o only.) Let R and S be the representation spaces of J and U respectively, and

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(14) Convergence is in the sense of Moore and Smith; cf. [3].

let $T=R\times S$ (the space formed by the ordered pairs (r, s), $r\in R$, $s\in S$, with the obvious definition of the topology). Define the *restricted Borel field* $\mathcal{B}_r(T)$ to be the Borel field generated (in the algebra of subsets of T) by the "rectangles" $x^*\times u^*$, x^* and u^* being open-closed subsets of R and S respectively. For each restricted Borel set $H\in\mathcal{B}_r(T)$ and for each $s\in S$ define $K(H, s)$ to be the set of all points $r\in R$ such that $(r, s)\in H$. It is easily seen that $K(H, s)\in\mathcal{B}(R)$ (as follows, for example, by the same reasoning as in 8.5 below). Hence (from (3) above) each $K(H, s)$ can be uniquely expressed as x^*+_2z, where x^* is open-closed in R and z is of the first category in R. The set x^* corresponds to a unique element $x\in J$, with measure $\mu(x)$. Thus we can define a function M'_H on S by setting

(8) $M'_H(s)=\mu(x)$, where $x^*+_2z=K(H, s)$.

We next show that

(9) There exists a function M_H, differing from M'_H on a set of first category (at most) in S, and continuous on S. (Obviously M_H will then be unique.)

Let $\{a_n\}$ be a disjoint sequence of elements of J such that $\mu(a_n)<\infty$ and $\bigvee a_n=j$, the unit-element in J. (The existence of such a sequence follows from the fact that J satisfies our postulates; cf. 2.3(1).) For each $H\in\mathcal{B}_r(T)$, define $H_n=H\cap(a_n^*\times S)$; thus the sets H_n belong to $\mathcal{B}_r(T)$, are disjoint (for fixed H), and their union is H—a set of first category (from (2)). Hence $M'_H=\sum_n M'_{H_n}$. By (7) it will thus suffice to prove that, for each n, M'_{H_n} differs from a continuous function only on a set of first category in S. Consider the family $\mathcal{B}'$ of those sets $H\in\mathcal{B}_r(T)$ for which this is true. Clearly $\mathcal{B}'$ contains the "rectangles" $x^*\times u^*$; also if $H'\supset H''$ and $H', H''\in\mathcal{B}'$ then $H'-H''\in\mathcal{B}'$ (for $M'_{(H'-H'')_n}=M'_{H'_n}-M'_{H''_n}$ since these functions are *finite*); and finally if $\{H^m\}$ is a disjoint sequence of sets in $\mathcal{B}'$ it readily follows from (7) again that $\bigcup H^m\in\mathcal{B}'$. Hence $\mathcal{B}'\supset\mathcal{B}_r(T)$, and (9) is established.

We now define $J\otimes U$ to consist of the algebra $\mathcal{B}_r(T)$ modulo "null sets"—those for which $M_H\equiv 0$. Formally, we write $H\equiv H'$ to mean that H, $H'\in\mathcal{B}_r(T)$ and $M_{H+_2H'}=0$ for every $s\in S$—that is, that, except for a first category set of s's, each set $K(H, s)+_2K(H', s)$ is of the first category in R—and write $\{H\}$ = set of all sets H' such that $H'\equiv H$. Algebraic operations on the equivalence classes $\{H\}$ $(H\in\mathcal{B}_r(T))$ are defined in the obvious way; it is easily seen that they form a Boolean σ-algebra. On this we define an abstract measure by setting $M\{H\}=M_H$ (which also $=M_{H'}$ if $H'\equiv H$); thus the values of the "measure" are continuous functions on S. This measure is evidently countably additive, provided that we define $\sum_{n=1}^{\infty}M\{H_n\}$ to mean not the ordinary sum of these functions, but the continuous function which differs from this sum on a set of the first category (at most).

An alternative procedure would be to redefine the "measure" of $\{H\}$ to be the set of all functions which differ from $M\{H\}$ on an at most first category subset of S; addition would have then have its natural meaning. This is in fact the procedure which was used in the formulation in 3.1(e); however, the choice of $M\{H\}$ to be a single continuous function is more convenient in proving that $J\otimes U$ has the desired properties.

4.4. It is easy to see that $J\otimes U$ (assuming it for the moment to satisfy our postulates) is in fact (to within isomorphism) a "direct product" of (J, μ) and U in the sense of 3.1(e). For $J\otimes U$ contains a sub-algebra J' consisting of all elements of the form $\{x^*\times S\}$ (x^* open-closed in R), and a sub-algebra U' consisting of the elements $\{R\times u^*\}$ (u^* open-closed in S). The function $M\{x^*\times S\}$ has evidently the constant value $\mu(x)$, and similarly $M\{R\times u^*\}$ has the value $\mu(j)$ for $s\in u^*$ (where j denotes the unit element of J) and the value 0 for $s\in S-u^*$. Thus (J', M) is isomorphic to the given measure algebra (J, μ), and (U', M) is isomorphic to the given trivial algebra U. All that remains to be verified is that $J\otimes U=\mathfrak{B}(J'\cup U')$. To see this, consider the family $\mathfrak{B}'$ of all *sets* $H\in\mathfrak{B}_r(T)$ for which the *elements* $\{H\}$ belong to $\mathfrak{B}(J'\cup U')$. Clearly $\mathfrak{B}'$ contains the "rectangles" $x^*\times u^*$, and contains $T-H$ and $\bigcup H_n$ if it contains H and H_n ($n=1, 2, \cdots$). Hence $\mathfrak{B}'\supset\mathfrak{B}_r(T)$, and so $\mathfrak{B}'=\mathfrak{B}_r(T)$, from which it follows that $\mathfrak{B}(J'\cup U')=J\otimes U$.

4.5. We must now verify that $J\otimes U$ satisfies our postulates (2.1), "$\sim$" being, of course, the equivalence relation induced by the abstract measure M.

(0) All that has to be proved here is that *$J\otimes U$ satisfies the countable chain condition*. Suppose there exists an uncountable disjoint collection of elements $\{H_\alpha\}$, all different from the zero element. As in the proof of (9) (4.2), we choose a disjoint sequence $\{a_n\}$ in J with $\mu(a_n)<\infty$ and $\bigvee a_n=j$. For some n, there will be uncountably many nonzero elements $\{H_\alpha\cap(a_n^*\times S)\}$, and from these we obtain a transfinite sequence of strictly increasing elements $\{L_\beta\}$, $\beta<\omega_1$, $\{L_1\}<\{L_2\}<\cdots<\{(a_n^*\times S)\}$. For each $\beta<\omega_1$, we have $M\{L_\beta\}<M(L_{\beta+1}\}$ for at least one $s\in S$; and since these functions are continuous there exists a nonempty open-closed set $u_\beta^*\subset S$ such that $M\{L_\beta\}<\rho_\beta<\sigma_\beta<M\{L_{\beta+1}\}$, where ρ_β, σ_β are fixed positive numbers, for all $s\in u_\beta^*$. For some positive integer m there will be uncountably many values (say) γ of β for which $\sigma_\beta-\rho_\beta>\mu(a_n)/m$. Thus no $s\in\bigcup u_\gamma^*$ can belong to more than m sets u_γ^*. Let $v^*(s)$ denote the intersection of all sets u_γ^* containing s; thus $v^*(s)$ is open-closed in S, and distinct sets $v^*(s)$ are disjoint. Since U satisfies the countable chain condition, the distinct sets $v^*(s)$ must be countable in number; and since each $v^*(s)$ is contained in u_γ^* for at most m values of γ, the γ's are countable—a contradiction.

(I) This is trivial here, since M is countably additive.

(II) Given $\{H\}$, $\{H'\}\in J\otimes U$ such that $M\{H\}=M\{H'\}$, and given $L'\in\mathfrak{B}_r(T)$ such that $\{L'\}\leqq\{H'\}$, we must show that there exists $L\in\mathfrak{B}_r(T)$ such that $L\subset H$ and $M\{L\}=M\{L'\}$. The proof is built up by first considering two special cases.

(i) *Suppose J has no atoms.* We need the following lemma:

(10) Given $H\in\mathfrak{B}_r(T)$, and given any continuous function ϕ on S, not identically zero, such that $0\leqq\phi\leqq M\{H\}$ (for all $s\in S$), there exists $L_1\in\mathfrak{B}_r(T)$

such that $L_1 \subset H$, $\{L_1\} \neq o$, and $M\{L_1\} \leqq \phi$.

For, from the continuity of ϕ, there exists a nonempty open-closed set $u^* \subset S$ and an $\epsilon > 0$ such that $\phi(s) \geqq \epsilon$ whenever $s \in u^*$. Let $\{c_n\}$ be a maximal disjoint collection of nonzero elements of J each of measure less than ϵ; since J is non-atomic, it is easily seen that $\bigvee c_n = j$. Write $L^n = H \cap (c_n^* \times u^*)$. Since $\sum_n M\{L^n\} = M\{H\} \geqq \phi$ (except on a first category set), there exists an n for which $M\{L^n\}$ is not identically 0, and we have only to take $L_1 = L^n$.

The usual "exhaustion" argument now shows that, under the hypotheses of the lemma, there exists $L \in \mathfrak{B}_r(T)$ such that $L \subset H$ and $M\{L\} = \phi$. (Take L to be a maximal L_1, modulo null sets; if the function $\phi' = \phi - M\{L\}$ were not identically 0, with the convention $\infty - \infty = 0$, the lemma applied to the set $H - L$ and function ϕ' would enable L to be enlarged.) In particular, on taking $\phi = M\{L'\}$, we see that II is established in this case.

(ii) *Suppose J is generated by a number of atoms, all of equal measure.* The proof proceeds as in case (i), the only difference being that we now restrict the continuous function ϕ to have only integral multiples of the measure of each atom (or ∞) as values.

(iii) *In the general case*, there are at most $\aleph_0$ atoms in J; let the distinct values of their measures be enumerated as μ_1, μ_2, and so on. Write b_n = supremum of all atoms in J of measure μ_n, and $b_0 = j - \bigvee b_n$. For each $X \in \mathfrak{B}_r(T)$ write $X^n = X \cap (b_n^* \times S)$. It is not hard to see, from the fact that J itself satisfies II, that if $M\{H\} = M\{H'\}$ then $M\{H^n\} = M\{H'^n\}$ for each n ($n = 0, 1, \cdots$)[15]. Now, given $\{L'\} \leqq \{H'\}$, we apply the argument in case (i) to obtain $L^0 \subset H^0$ such that $M\{L^0\} = M\{L'^0\}$, and similarly apply the argument in case (ii) to obtain $L^n \subset H^n$ such that $M\{L^n\} = M\{L'^n\}$ ($n \geqq 1$). Write $L = \bigcup L^n$ ($n \geqq 0$); evidently II follows.

(III) Given H, $H' \in \mathfrak{B}_r(T)$ such that $M\{H\} = M\{H'\}$, we decompose them into equivalent bounded elements as follows. As before, let $\{a_n\}$ be a disjoint sequence of elements of J such that $\mu(a_n) < \infty$ and $\bigvee a_n = j$. Write $X_n = H \cap (a_n^* \times S)$, $Y_n = H' \cap (a_n^* \times S)$. Define $H_1 = X_1$. From postulate II, there exists $H_1' \subset H'$ such that $M\{H_1'\} = M\{H_1\} \leqq \mu(a_1)$; hence $M\{H - H_1\} = M\{H' - H_1'\}$. Define $H_2' = Y_1 \cap (T - H_1')$; from II again there exists $H_2 \subset H - H_1$ such that $M\{H_2\} = M\{H_2'\} < \infty$, so that $M\{H - (H_1 \cup H_2)\} = M\{H' - (H_1' \cup H_2')\}$. Define $H_3 = X_2 \cap (T - (H_1 \cup H_2))$; and so on. We thus obtain disjoint sequences $\{H_n\}$, $\{H_n'\}$ such that $H = \bigcup H_n$, $H' = \bigcup H_n'$; and $M\{H_n\} = M\{H_n'\} < \infty$, so that the elements $\{H_n\}$, $\{H_n'\}$ are bounded, as required.

REMARK. The construction of $J \otimes U$ will apply even if $J = (J, \mu)$ is an *arbitrary* numerical measure algebra in which μ is σ-finite. Naturally, if J does not satisfy the postulates, $J \otimes U$ need not either.

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(15) Let $K(H, s) = x^* +_2 z$, $K(H', s) = x'^* +_2 z'$; then (ignoring a first category set of s's) $\mu(x) = \mu(x')$. If $x \geqq$ an atom of measure μ_1, $x' \geqq$ an element of J of measure μ_1, which must also be an atom. Iteration of this argument gives $\mu(b_n x) = \mu(b_n x')$ ($n = 0, 1, \cdots$), and the result follows.

4.6. We conclude this section by noting, for future use, that the elements of $J\otimes U$ are (roughly speaking) approximable by countable sums of rectangles. More precisely:

(11) LEMMA. *Suppose* $\mu(j)<\infty$. *Then, given* $H\in\mathcal{B}_r(T)$ *and* $\epsilon>0$, *there exist sequences* x_n^*, u_n^* *of open-closed subsets of* R *and* S *respectively, such that* (i) *the sets* u_n^* *are disjoint and cover* S *except for a set of first category, and* (ii) *we have*

$$M\{H +_2 \bigcup(x_n^* \times u_n^*)\} < \epsilon \qquad (\text{for all } s \in S)(^{16}).$$

Consider the family $\mathcal{B}'$ of sets $H\in\mathcal{B}_r(T)$ for which this is true (for every ϵ). Clearly $\mathcal{B}'$ contains the "rectangles" $x^*\times u^*$. It is also easy to see that if H, $H'\in\mathcal{B}'$ then so do $T-H$ and $H\cup H'$; thus $\mathcal{B}'$ is a finitely additive field, and to prove $\mathcal{B}'=\mathcal{B}_r(T)$ it will suffice to prove that if $H_n\in\mathcal{B}'$ and $H_1\subset H_2\subset\cdots$, then $\bigcup H_n\in\mathcal{B}'$. This follows by an "Egoroff argument" (cf. the proof of 4.2(7)).

5. Invariant elements.

5.1. DEFINITION. Returning now to an abstract measure algebra $(E, \sim)$ satisfying the postulates of 2.1, we define an element u to be *invariant* if, whenever $x\sim u$, we have $x\leqq u$. (Thus if u is invariant and *bounded*, we have further $x=u$.) Clearly no two distinct invariant elements can be equivalent. (The term "invariant" will be justified in 20.4(1).)

The invariant elements will play a fundamental part in the analysis of the structure of $(E, \sim)$. In fact, if E were a non-atomic numerical measure algebra, the only invariant elements would evidently be o and e; and this suggests, as is in fact the case, that the "trivial" factors in the desired "product" representation of E (cf. 3.1(f)) in the general case, will be formed from invariant elements.

The set of all invariant elements of E will be written U.

5.2. *If* u *is invariant and* $x\sim x'\leqq u$, *then* $x\leqq u$.

Write $y=x-u$; by II(2.1) there exists $y'\leqq x'$ such that $y\sim y'$. Now let $z=(u-y')\vee y$. From I(2.1) $z\sim(u-y')\vee y'=u$; hence, from the definition of invariance, $z\leqq u$. A fortiori $y\leqq u$, and so $y=o$ and $x\leqq u$.

5.3. If u is *not* invariant, there exist nonzero x, y such that $x\leqq u$, $y\leqq -u$, and $x\sim y$.

For there exists $x_1\sim u$ such that $x_1-u\neq o$. Define $y=x_1-u$. By II(2.1) there exists $x\leqq u$ such that $x\sim y$ (for $y\leqq x_1\sim u$).

5.4. *The invariant elements form a complete Boolean sub-algebra of* E.

It is trivial that o and e are invariant (2.3(4)). Thus we have only to prove (a) if u is invariant so is $-u$, and (in view of the countable chain condition) (b) if u_n is invariant ($n=1, 2, \cdots$) then so is $\bigvee u_n$. Now (a) is an immediate consequence of 5.3 and 5.2. To prove (b), let $u=\bigvee u_n$, and suppose

(16) This lemma can easily be modified so as to apply even if $\mu(j)=\infty$. Both lemma and proof can also be reformulated in terms of the properties given in 3.1 (e).

that u is not invariant. By 5.3, there exist x, y such that $x \leqq u$, $y \leqq -u$, and $x \sim y \neq o$. For some n we must have $xu_n \neq o$, and from postulate II we then have that there exists $y' \leqq y$ such that $y' \sim xu_n$. From 5.2, $y' \leqq u_n \leqq u$; but $y' \leqq y \leqq -u$, and so $y' = o$—that is, $xu_n = o$ (from 2.3(4)), a contradiction.

5.5. Definition. For each $x \in E$ we can (from 5.4) define the *invariant closure* $\bar{x}$ of x to be the *smallest* invariant element containing x. To simplify the printing of complicated expressions, we shall sometimes also write $\mathrm{Cl}(x)$ instead of $\bar{x}$. This closure operation will then have the usual properties familiar in topology ($\bar{o} = o$, $\bar{e} = e$, $\mathrm{Cl}(\mathrm{Cl}(x)) = \mathrm{Cl}(x) \geqq x$, $\mathrm{Cl}(x) \vee \mathrm{Cl}(y) = \mathrm{Cl}(x \vee y)$, and x is invariant if and only if $\bar{x} = x$), as well as some others, notably:

5.6. For any collection of elements $\{x_\alpha\}$, $\mathrm{Cl}(\bigvee x_\alpha) = \bigvee \bar{x}_\alpha$.

For it is trivial that $\mathrm{Cl}(\bigvee x_\alpha) \geqq \bigvee \bar{x}_\alpha$, while conversely $\bigvee \bar{x}_\alpha$ is invariant (5.4) and greater than or equal to $\bigvee x_\alpha$, and so greater than or equal to $\mathrm{Cl}(\bigvee x_\alpha)$.

5.7. *For any* x, $\bar{x} = \bigvee \{y \mid y \sim y' \leqq x\}$.

Let x^* denote the element on the right. Any invariant $u \geqq x$ will satisfy $u \geqq x^*$ (from 5.2), and so $\bar{x} \geqq x^*$. All that remains to be proved is that x^* is invariant; for clearly $x^* \geqq x$. If $x^* \notin U$, then (5.3) there exist z, t such that $z \leqq x^*$, $t \leqq -x^*$, $z \sim t$, and $z \neq o$. Hence for some $y \sim y' \leqq x$ we have $zy \neq o$; and postulate II(2.1) then gives the existence of $p \leqq t$ and $q \leqq y'$ such that $p \sim zy \sim q$. Since $q \leqq x$, this shows that $p \leqq x^*$; but $p \leqq t \leqq -x^*$, and so $p = o$. That is (2.3(4)), $zy = o$, a contradiction.

5.8. *If* $x \sim x'$, *then* $\mathrm{Cl}(x) = \mathrm{Cl}(x')$.

This is immediate from 5.7 and postulate II(2.1).

5.9. $\bar{x}\bar{y} = o \rightleftarrows$ whenever h, k are such that $h \leqq x$, $k \leqq y$, and $h \sim k$, then $h = k = o$.

The implication $\rightarrow$ is trivial, since $h \leqq \bar{x}$ and also $h \leqq \bar{y}$ (5.7). Conversely, if nonzero h, k exist such that $h \leqq x$, $k \leqq y$, and $h \sim k$, we have, on writing $u = \bar{h} = \bar{k}$ (from 5.8), $u \leqq \bar{x}\bar{y}$.

5.10. *If u is invariant*, $\mathrm{Cl}(xu) = \mathrm{Cl}(x)u$.

Trivially $\mathrm{Cl}(xu) \leqq \mathrm{Cl}(x)u$. Write $p = \mathrm{Cl}(x)u - \mathrm{Cl}(xu)$, and suppose $p \neq o$. Since $p \leqq \bar{x}$, 5.7 shows that $py \neq o$ for some $y \sim y' \leqq x$. Write $q = py$; then, by postulate II, there exists $q' \leqq x$ such that $q \sim q'$. Since $q' \sim q \leqq p \leqq u$, 5.2 shows that $q' \leqq u$; thus $q' \leqq xu$, and so (5.7) $q \leqq \mathrm{Cl}(xu)$. But $q \leqq p \leqq -\mathrm{Cl}(xu)$, so that $q = o$, a contradiction.

6. **Bounded elements.** In this section we develop further fundamental properties of the invariant and other elements of $(E, \sim)$ which depend on the notion of boundedness. (One example of the way in which this property is of service has already been furnished by the proof of 2.3(4).)

6.1. ("*Cancellation law*"). *If a, a' are both bounded and equivalent, and if $b \leqq a$, $b' \leqq a'$, and $b \sim b'$, then $a - b \sim a' - b'$.*

Remark. As will be proved later, it suffices that a be bounded; for the boundedness of a' will then follow from $a' \sim a$ (6.4).

From 2.3(5) and 5.9, there exist $x \leqq a-b$, $y \leqq a'-b'$, such that (i) $x \sim y$, (ii) on writing $p=(a-b)-x$, $q=(a'-b')-y$, we have $\bar{p}\bar{q}=o$. We shall prove $p=o$; a similar argument will then show that $q=o$, whence the result will follow.

Write $h=\bar{p}a'$; thus $hq=o$, and accordingly (since the elements b', y, q are disjoint and $b' \vee y \vee q=a'$) $h \leqq b' \vee y \sim b \vee x$. Thus (postulate II, 2.1) there exists $k \leqq b \vee x$ such that $k \sim h$. Since $p \vee k \leqq a \sim a'$, there exists $m \leqq a'$ such that $m \sim p \vee k$. Now, $k \leqq \bar{p}$ (from 5.2, since $k \sim h \leqq \bar{p}$), and therefore $h \vee k \leqq \bar{p}$; and since $\bar{p}$ is invariant, it follows (from 5.2 again) that $m \leqq \bar{p}$. Thus $m \leqq h \sim k$, so (from II again) there exists $n \leqq k$ such that $n \sim m \sim k \vee p$. But $k \vee p$ is bounded (from 2.3(2), for a is bounded). Hence $n=k \vee p$, so that $k \vee p \leqq k$; and since $kp=o$ (for $k \leqq b \vee x$) it follows that $p=o$; Q.E.D.

6.2. *Given $y \sim x = \bigvee x_n$, where the sequence $\{x_n\}$ is disjoint, there exists a disjoint sequence $\{y_n\}$ such that $y=\bigvee y_n$ and $y_n \sim x_n$ $(n=1, 2, \cdots)$.*

First assume that *x and y are both bounded.* In this case, postulate II gives the existence of $y_1 \leqq y$ such that $y_1 \sim x_1$; and from 6.1 we have $x-x_1 \sim y-y_1$. A straightforward induction, repeating this argument, gives a disjoint sequence of elements $y_n \leqq (y-(y_1 \vee y_2 \vee \cdots \vee y_{n-1}))$ such that $y_n \sim x_n$. Write $y'=\bigvee y_n$; thus $y' \leqq y$ and (from postulate I) $y' \sim x \sim y$. Since y is bounded, we have $y'=y$, and the result is established in this case.

In the general case, we apply postulate III (2.1) to write $x=\bigvee f_n$, $y=\bigvee g_n$, where $f_n \sim g_n$ and the sequences $\{f_n\}$, $\{g_n\}$ are each disjoint and bounded. Write $h_{mn}=x_m f_n$. The case already established enables us to write $g_n=\bigvee_m k_{mn}$ where $k_{mn} \sim h_{mn}$ and $k_{mn}k_{m'n}=o$ $(m \neq m')$. Define $y_m=\bigvee_n k_{mn}$; from postulate I we have $y_m \sim \bigvee_n h_{mn}=x_m$; the elements y_m are clearly disjoint; and finally $\bigvee y_m=\bigvee_{m,n} k_{mn}=\bigvee g_n=y$.

6.3. A necessary and sufficient condition that x be unbounded is that there exist an infinite disjoint sequence $\{z_n\}$, with $z_n \leqq x$, $z_n \neq o$, and $z_1 \sim z_2 \sim \cdots \sim z_n \sim \cdots$.

The sufficiency is trivial, since $\bigvee\{z_n \mid n \geqq 1\}$ and $\bigvee\{z_n \mid n \geqq 2\}$ are (from postulate I) equivalent, so that $\bigvee z_n$ (and so, a fortiori, x) is not bounded.

Conversely, suppose x is unbounded; then there exists $z_1 \leqq x$ such that $z_1 \neq o$ and $x \sim x-z_1$. The elements z_n are now defined by induction. When z_n has been defined, in such a way that $z_n \sim z_1$, $z_n \leqq x-(z_1 \vee \cdots \vee z_{n-1})=y_{n-1}$, say, and $y_n \sim x$, we apply 6.2 to the decomposition $x=z_1 \vee y_1$ to obtain an "equivalent" decomposition $y_n=z_{n+1} \vee y_{n+1}$ such that $z_{n+1}y_{n+1}=o$, $z_{n+1} \sim z_1$, and $y_{n+1} \sim y_1=x-z_1 \sim x$. Thus the inductive hypotheses are maintained, and the desired sequence $\{z_n\}$ is thereby constructed.

6.4. *If $x \sim x'$ and x is bounded, then so is x'.*

This follows from 6.3 and 6.2, since if x' were unbounded, the decomposition $x'=\bigvee z_n \vee (x'-\bigvee z_n)$ given by 6.3 could (6.2) be imitated in x, contradicting the boundedness of x.

REMARK. From these results one can show that if x, y are both bounded, then so is $x \vee y$; we omit the proof, since the result will not be needed.

6.5. Let $\{x_n\}$ be a sequence of bounded elements whose invariant closures are disjoint (that is, $\bar{x}_m \bar{x}_n = o$ if $m \neq n$). Then $\bigvee x_n$ is bounded.

Suppose $\bigvee x_n = x \sim x - y$ for some $y \leqq x$; we must prove $y = o$. Now, for each n we have $x_n \leqq x \sim x - y$; hence (postulate II) there exists $x_n' \leqq x - y$ such that $x_n' \sim x_n$. Now if $m \neq n$ we have $x_n' x_m \leqq \bar{x}_n$ (from 5.7), and also $x_n' x_m \leqq x_m \leqq \bar{x}_m$; hence $x_n' x_m = o$ $(m \neq n)$. It follows that $x_n' \leqq x_n$, and so (since x_n is bounded) $x_n' = x_n$; that is, $x_n y = o$ for each n, so that $y = xy = o$.

6.6. *If u is invariant and $x \sim y$, then $ux \sim uy$.*

First suppose x and y are both bounded. Then (postulate II) there exists $z \leqq y$ such that $z \sim ux$. From 5.2, $z \leqq u$, so $z \leqq uy$. Similarly there exists $t \leqq ux$ such that $t \sim uy$. By II again, there exists $t' \leqq z$ such that $t' \sim t \sim uy$. But uy is bounded (2.3(2)), so $t' \sim uy$; and consequently $z = uy$, whence the result.

The extension to the general case now follows from postulate III; we have $x = \bigvee x_n$, $y = \bigvee y_n$, where each of the sequences $\{x_n\}$, $\{y_n\}$ is disjoint, and where x_n, y_n are bounded and equivalent. The preceding case gives $ux_n \sim uy_n$; whence (from postulate I) $ux \sim uy$.

7. Equivalence classes.

7.1. *Notation.* We write $[x]$ for the "equivalence class" $\{x' | x' \sim x\}$, and write $[x] \leqq [y]$ (or $[y] \geqq [x]$) to mean that there exist $x' \in [x]$ and $y' \in [y]$ such that $x' \leqq y'$. Thus (from II, 2.1) there will then exist, for *each* $y' \in [y]$, an $x' \in [x]$ such that $x' \leqq y'$(17). It follows immediately that if $[x] \leqq [y]$ and $[y] \leqq [z]$ then $[x] \leqq [z]$ (so that $\leqq$ is in fact a partial ordering of the equivalence classes), and that if $[x] \leqq [y] \leqq [x]$ and x is *bounded* then $[x] = [y]$. (This holds even if x is unbounded; but this fact will not be needed.) We further write $[x] < [y]$ (or $[y] > [x]$) to mean that $[x] \leqq [y]$ and $[x] \neq [y]$(18). Of course, two given equivalence classes will in general be incomparable.

Note that $[x] \leqq [y]$ implies $\bar{x} \leqq \bar{y}$ (from 5.8). (The converse is false, in general.)

7.2. *If $x_1 \geqq x_2 \geqq \cdots$ is a decreasing sequence of bounded elements such that $[x_n] \geqq [y]$ for all n, then $[\bigwedge x_n] \geqq [y]$.*

Since $[x_1] \geqq [y]$, there exist $x_1' \sim x_1$, $y' \sim y$, such that $x_1' \geqq y'$. Now elements x_2', x_3', $\cdots$, such that $x_1' \geqq x_2' \geqq \cdots \geqq y'$ and $x_n' \sim x_n$, are defined by induction, as follows. When x_n' has been defined, postulate II gives the existence of $y^{n+1} \leqq x_{n+1}$ such that $y^{n+1} \sim y \sim y'$; and (from 6.4 and 6.1) $x_n' - y' \sim x_n - y^{n+1} \geqq x_n - x_{n+1}$. Hence (II) there exists $z_{n+1} \leqq x_n' - y'$ such that

(17) There will also exist, for each $x' \in [x]$, a $y' \in [y]$ such that $x' \leqq y'$; but the proof is not easy, except in the case in which y is bounded.

(18) One could go on to make $[x]$ play the role of an "abstract measure" $\lambda(x)$, by defining addition, and so on, suitably, along the lines of [12]; cf. also §12. We shall not use this procedure, since a much more specific abstract measure will be constructed later.

$z_{n+1} \sim x_n - x_{n+1}$. Define $x'_{n+1} = x'_n - z_{n+1}$; thus $x'_n \geqq x'_{n+1} \geqq y'$, and (from 6.1) $x'_{n+1} \sim x_n - (x_n - x_{n+1}) = x_{n+1}$.

Now $x'_1 - \bigwedge x'_n = \bigvee(x'_n - x'_{n+1}) \sim \bigvee(x_n - x_{n+1})$ (from I) $= x_1 - \bigwedge x_n$; hence (6.1 and 6.4) $\bigwedge x_n \sim \bigwedge x'_n \geqq y' \sim y$; and the result is proved.

7.3. *If* $x_1 \leqq x_2 \leqq \cdots$ *is an increasing sequence such that* $[x_n] \leqq [y]$ *for all* n, *and if* y *is bounded, then* $[\bigvee x_n] \leqq [y]$.

This is proved by an argument similar to the preceding(19).

7.4. *If* $[x] \leqq [y]$, *and* $u \in U$ (*the set of invariant elements*), *then* $[ux] \leqq [uy]$.

This is an immediate consequence of 6.6.

7.5. *Given* $a, b \in E$, *there exist "maximal" equivalent elements* $x \leqq a$, $y \leqq b$, *such that* (i) $\text{Cl}(a-x), \text{Cl}(b-y) = o$, (ii) *given any other equivalent elements* $x' \leqq a$, $y' \leqq b$, *we have* $[x'] \leqq [x]$ *and* $[y'] \leqq [y]$.

The existence of $x \leqq a$ and $y \leqq b$ such that $x \sim y$ and (i) holds is given by 2.3(5), in view of 5.9. We prove that they satisfy (ii). Write $z = x\,\text{Cl}(x'-x)$ and $t = x'z \vee (x'-x)$; thus $t \leqq x'\,\text{Cl}(x'-x)$. Since $t \leqq x' \sim y'$, there exists $s \sim t$ such that $s \leqq y' \leqq b$. But $s \leqq \text{Cl}t \leqq \text{Cl}(x'-x) \leqq \text{Cl}(a-x)$, so $s(b-y) = o$, and therefore $s \leqq y$. Thus $s \leqq y\text{Cl}(x'-x) \sim x\text{Cl}(x'-x)$ (from 6.6) $= z$, so that there exists $s' \leqq z$ such that $s' \sim s \sim t$. From postulate I we now have $x' = t \vee x'(x-z) \sim s' \vee x'(x-z) \leqq x$, and so $[x'] \leqq [x]$. Similarly $[y'] \leqq [y]$.

7.6. *Given elements* $a, b \in E$, *of which at least* a *is bounded, there exists* $u \in U$ *such that* (i) $[au] \leqq [bu]$, (ii) *if* v *is a nonzero invariant element disjoint from* u, *then* $[av] > [bv]$.

Let x, y be the elements given by 7.5. Write $u = -\text{Cl}(a-x)$; thus u is invariant (5.4). Further, $ua = ux \sim uy$ (from 6.6) $\leqq ub$, establishing (i). Finally, if v is invariant and $uv = o$, we have $v \leqq \text{Cl}(a-x)$ and therefore $v\,\text{Cl}\,(b-y) = o$; hence $bv = yv \sim xv \leqq av$, which shows that $[bv] \leqq [av]$, and that if $[bv] = [av]$ then $xv \sim av$—that is, $v(a-x) = o$, since a is bounded. But this would imply $o = \text{Cl}(v(a-x)) = v\text{Cl}(a-x)$ (from 5.10) $= v$. Hence if $v \neq o$ we must have $[bv] < [av]$.

8. **Relativization.**

8.1. Given any $a \in E$, it was pointed out in 3.1(c) that the "relative algebra" $(E(a), \sim)$, where $E(a)$ consists of all elements $x \leqq a$, satisfies our postulates. We shall here derive, for later use, some relations between this relative algebra $E(a)$ and E.

An element $v \leqq a$ which is invariant in $E(a)$ (that is, in $(E(a), \sim)$) will be said to be *invariant relative to a*, and the set of elements invariant relative to a will be denoted by U_a. For the other notations used in this section, we refer to 1.3.

8.2. A necessary and sufficient condition that an element $v \leqq a$ be invariant relative to a is that there exist $u \in U$ such that $v = au$. Further, u can be

(19) The restriction of boundedness can be removed in 7.3, but not in 7.2—just as in the familiar numerical case.

chosen so that $u \leqq \bar{a}$; and if this is done, the correspondence between u and v is 1-1.

The condition is sufficient; for if $v = au$, where $u \in U$, and if $v \sim v' \leqq a$, then $v' \leqq u$ (from the invariance of u), and so $v' \leqq v$—that is, $v \in U_a$.

The condition is necessary. For if $v \in U_a$, define $u = \bar{v}$. Trivially $v \leqq au$; let $w = au - v$, and suppose $w \neq o$. Then, since $w \leqq \bar{v}$, 5.7 shows that, for some $x \sim x' \leqq v$, we have $wx = y$ (say) $\neq o$. From II, there exists $y' \leqq x'$ such that $y \sim y'$; and since $y' \leqq v$ and $y \leqq w \leqq a$, 5.2 applied to $E(a)$ shows that $y \leqq v$. But $y \leqq w \leqq -v$; hence $y = o$, a contradiction. Thus $v = au$, where $u = \bar{v}$, so that $u \in U$ and $u \leqq \bar{a}$.

To show that the correspondence is 1-1, we must show that if $au_1 = au_2$ (where $u_1, u_2 \in U(\bar{a})$) then $u_1 = u_2$; and, writing $u = u_1 +_2 u_2$, it will suffice to show that if $au = o$ (where, from 5.4, $u \in U(\bar{a})$) then $u = o$. But we have $o = \mathrm{Cl}(au) = \mathrm{Cl}(a)u$ (5.10) $= u$.

The result just established can be restated as:

8.3 Corollary. *The algebra $U_a = aU$, and is isomorphic to the principal ideal $U(\bar{a})$ (of all elements of U which are $\leqq \bar{a}$) under the correspondence $v \rightarrow u = \bar{v}$, $u \rightarrow v = au$.*

8.4. An element of $E(a)$ is bounded relative to a (that is, in $E(a)$) if and only if it is bounded.

This is trivial from the definition of boundedness.

8.5. *For any set $P \subset E$, the Borel field relative to a generated by aP (in symbols, $\mathcal{B}_a(aP)$) is $a\mathcal{B}(P)$.*

Write $\mathcal{B}' = a\mathcal{B}(P) =$ set of all elements ax, where $x \in \mathcal{B}(P)$. Clearly $\mathcal{B}' \supset aP$; and if y and y_n belong to $\mathcal{B}'$, then so do $a - y$ and $\bigvee y_n$; hence, $\mathcal{B}' \supset \mathcal{B}_a(aP)$. Now write $\mathcal{B}'' = \mathcal{B}_a(aP) \vee E(-a) =$ set of all elements expressible as $y \vee z$ where $y \in \mathcal{B}_a(aP)$ and $z \in E(-a)$. It is easy to see that $\mathcal{B}'' \supset P$, and that if x and $x_n \in \mathcal{B}''$ then so do $-x$ and $\bigvee x_n$. Hence $\mathcal{B}'' \supset \mathcal{B}(P)$; so $a\mathcal{B}(P) \subset a\mathcal{B}'' = \mathcal{B}_a(aP)$, completing the proof.

9. Indecomposable elements.

9.1. *Definitions.* An element x is *decomposable* if there exist y_1, y_2 such that $o < y_i \leqq x$ $(i = 1, 2)$, $y_1 \sim y_2$, and $y_1 y_2 = o$. Otherwise x is said to be *indecomposable.* Clearly o is indecomposable, and so is each atom of E; there may, of course, be other indecomposable elements. The indecomposable elements are the easiest to deal with, and we shall dispose of them in this section; this will in effect reduce the problem of representing E to the case in which all nonzero elements are decomposable.

9.2. The following properties are easy consequences of the above definition.

(1) If x is indecomposable, and $y \leqq x$, then y is indecomposable.

(2) Indecomposable elements are bounded. (From 6.3.)

(3) If $x \sim x'$ and x is indecomposable, then so is x'. (From 6.2.)

(4) If x is indecomposable, and $y \leqq x$, then $y = x\bar{y}$.

For trivially $y \in x\bar{y}$. Write $z = x\bar{y} - y$; if $z \neq o$, 5.7 shows that $zy' \neq o$ for some $y' \sim y'' \leqq y$, and then II gives the existence of $z_1 \leqq z$ and $y_1 \leqq y$ such that $z_1 \sim zy' \sim y_1$. The definition of indecomposability gives $z_1 = o$, whence $zy' = o$, a contradiction.

(5) If x is indecomposable and $y_1 \vee y_2 \leqq x$, and if $y_1 \sim y_2$, then $y_1 = y_2$.

For, from (4) and 5.8, $y_1 = x\bar{y}_1 = x\bar{y}_2 = y_2$.

(6) If x is indecomposable, $y \vee z \leqq x$, and $yz = o$, then $\bar{y}\bar{z} = o$. (From 5.9.)

9.3. Let $\{x_n\}$ be a sequence of indecomposable elements such that $\bar{x}_m \bar{x}_n = o$ whenever $m \neq n$. Then $\bigvee x_n$ is indecomposable.

Let $y \vee z \leqq \bigvee x_n$, and suppose $y \sim z$ and $yz = o$. (We must prove $y = o$.) Let $y_n = yx_n$; thus the elements y_n are disjoint, and $\bigvee y_n = y$. Hence (6.2) we can write $z = \bigvee z_n'$, where the elements z_n' are disjoint and $z_n' \sim y_n$. Let $z_{nm} = z_n' x_m$; from 6.2 again we can write $y_n = \bigvee_m y_{nm}$, where $y_{nm} \sim z_{nm}$. Now, *if* $m \neq n$, we have $y_{nm} \leqq \bar{x}_m$ (from 5.7, since $y_{mn} \sim z_{mn} \leqq x_m$), and so $y_{nm} \leqq x_n \bar{x}_m = o$. And *if* $m = n$, $y_{nm} \vee z_{nm} \leqq x_m$, which is indecomposable; hence again $y_{nm} = o$. Hence $y = \bigvee_{m,n} y_{nm} = o$.

9.4. *If* $b_1 \leqq b_2 \leqq \cdots$ *is an increasing sequence of indecomposable elements, then* $\bigvee b_n$ *is indecomposable.*

Write $x_1 = b_1$, $x_n = b_n - b_{n-1}$; thus x_n is indecomposable (9.2(1)), and $x_m x_n = o$ if $m \neq n$. Since $x_m \vee x_n \leqq b_{m+n}$, which is indecomposable, 9.2(6) shows that $\bar{x}_m \bar{x}_n = o$. So $\bigvee x_n$ is indecomposable (9.3); and $\bigvee x_n = \bigvee b_n$.

9.5. *Given any collection C of indecomposable elements, there exists an indecomposable element b such that* (i) $b \leqq \bigvee \{c \mid c \in C\}$, (ii) *for each* $c \in C$, $[c] \leqq [b]$.

By transfinite induction, using 9.4, we see that there is a *maximal* indecomposable b satisfying (i). We prove that b satisfies (ii) also. In fact, given $c \in C$, there exist (7.5) elements $b_1 \leqq b$, $c_1 \leqq c$, such that $b_1 \sim c_1$ and $\mathrm{Cl}(c - c_1) \cdot \mathrm{Cl}(b - b_1) = o$. Also (9.2(6)) $\mathrm{Cl}(c_1)\mathrm{Cl}(c - c_1) = o$, and $\bar{b}_1 = \bar{c}_1$ (5.8), so that $\mathrm{Cl}(b_1)\mathrm{Cl}(c - c_1) = o$. Again, from 9.2(6), we have $\mathrm{Cl}(b_1)\mathrm{Cl}(b - b_1) = o$. Hence 9.3 shows that the element $p = \mathrm{Cl}(b_1) \vee \mathrm{Cl}(b - b_1) \vee \mathrm{Cl}(c - c_1)$ is indecomposable. But $b \leqq p \leqq \bigvee \{c \mid c \in C\}$. Hence $b = p$, proving that $c - c_1 = o$, that is, that $c \sim b_1 \leqq b$, Q.E.D.

9.6. *Notation.* Let e^o denote the supremum of *all* the indecomposable elements. (Naturally, e^o may be o, and also need not itself be indecomposable.)

Theorem. *There exists a disjoint sequence* $\{b_n\}$ *of indecomposable elements, such that* (i) $\bigvee b_n = e^o$, *and* (ii) *for each* m, n, *either* $[b_m] \leqq [b_n]$ or $[b_n] \leqq [b_m]$. *Further,* (iii) e^o *is invariant.*

We first define a transfinite sequence $\{b_\alpha\}$ of disjoint indecomposable elements, as follows. From 9.5 applied to the collection C_1 of *all* indecomposable elements, there exists an indecomposable element $b_1 \leqq e^o$ such that $[c] \leqq [b_1]$ for every indecomposable c. When b_β has been defined for all $\beta < \alpha$, we define C_α = set of all indecomposable elements c satisfying $c \leqq e^o - \bigvee_{\beta < \alpha} b_\beta$. By 9.5, there exists an indecomposable $b_\alpha \leqq e^o - \bigvee_{\beta<\alpha} b_\beta$ such that $[c] \leqq [b_\alpha]$ for every $c \in C_\alpha$. From postulate 0(2.1) we have $b_{\alpha_0} = o$ for some $\alpha_0 < \omega_1$, and the process is terminated. C_{α_0} must then consist of o only, so that the element $d = e^o - \bigvee_{\alpha<\alpha_0} b_\alpha$ must be o (for otherwise, since $d \leqq e^o$, there would exist an indecomposable c such that $cd \neq o$; and, from 9.2(1), $cd \in C_{\alpha_0}$). On renumbering the elements b_α $(\alpha < \alpha_0)$ into a simple sequence, we see that (i) and (ii) are satisfied.

To prove (iii), suppose $y \sim x \leqq e^o$; it will suffice to prove $y \leqq e^o$. From (i), $x = \bigvee x b_n$; hence (6.2) we can write $y = \bigvee y_n$ where $y_n \sim x b_n$. But $x b_n$ is indecomposable, since b_n is (9.2(1)); hence so is y_n (9.2(3)), so that $y_n \leqq e^o$ for each n. It follows that $y \leqq e^o$.

9.7. Let K (as in 3.1(a)) be the numerical measure algebra generated by $\aleph_0$ atoms p_n $(n = 1, 2, \cdots)$ each of measure 1; that is, the elements of K are the subsets of $\bigcup(p_n)$, and the measure of any such set is the number of elements p_n in it.

THEOREM. *The relative algebra* $(E(e^o), \sim)$ *is isomorphic to a principal ideal in the direct product* $K \otimes U(e^o)$.

(Here $U(e^o)$ denotes, as usual, the principal ideal in U of e^o—which belongs to U, as just shown.)

Let R, S denote the representation spaces of K, $U(e^o)$, respectively (cf. §4). The open-closed subset p_n^* of R which corresponds to $p_n \in K$ is now easily seen to consist of a single point (say) q_n. Write $Q = R - \bigcup(q_n)$; thus (from 4.2(2)) Q is of the first category in R.

Now let b_n be the sequence of indecomposable elements given by 9.6. For each $x \leqq e^o$, define $\psi(x)$ to be the subset $\bigcup_{n=1}^\infty (q_n, (\mathrm{Cl}(x b_n))^*)$ of $T = R \times S$; clearly $\psi(x) \in \mathcal{B}_r(T)$. And define $\theta(x)$ to be the element $\{\psi(x)\}$ of $K \otimes U(e^o)$, that is, the class of all sets $H \in \mathcal{B}_r(T)$ such that $M\{H +_2 \psi(x)\}$ is identically 0, M being the continuous-function-valued "measure" defined in §4. We shall show that θ is the desired isomorphism.

(1) For any sequence $\{x_n\}$ with $x_n \leqq e^o$, $\theta(\bigvee x_n) = \bigvee \theta(x_n)$.

For we have $\mathrm{Cl}(\bigvee_m x_m b_n) = \bigvee_m \mathrm{Cl}(x_m b_n)$ (5.6), and so (4.2(2)) $(\mathrm{Cl}(\bigvee_m x_m b_n))^* = \bigcup_m (\mathrm{Cl}(x_m b_n))^* +_2 Y_n$, where Y_n is of the first category in S. Hence $\psi(\bigvee x_m) = \bigcup \psi(x_m) +_2 \bigcup (q_n, Y_n)$, so that (the sets (q_n, Y_n) being null) $\theta(\bigvee x_m) = \bigvee \theta(x_m)$.

(2) If $x \leqq e^o$, $\theta(e^o - x) = \theta(e^o) - \theta(x)$.

For we have $\mathrm{Cl}((e^o - x) b_n)$ $\mathrm{Cl}(x b_n) = o$ from 9.2(6); and trivially $\mathrm{Cl}((e^o - x) b_n) \vee \mathrm{Cl}(x b_n) = \mathrm{Cl}(b_n)$. Hence

$$\begin{aligned} \psi(e^o) - \psi(x) &= \bigcup(q_n, (\mathrm{Cl}(b_n))^*) - \bigcup(q_n, (\mathrm{Cl}(x b_n))^*) \\ &= \bigcup(q_n, (\mathrm{Cl}((e^o - x) b_n))^*) = \psi(e^o - x), \end{aligned}$$

and (2) follows.

(3) If $x, y \leqq e^o$ and $\theta(x) = \theta(y)$, then $x = y$.

From (1) and (2), it will suffice to prove (on writing $z=x+_2y$) that if $\theta(z)=o$ then $z=o$. We are given, then, that $M\{\bigcup(q_n, (\text{Cl}(zb_n))^*)\}$ is identically 0; hence, for each n, $M\{(q_n, (\text{Cl}(zb_n))^*)\}=0$, and since this function is 1 for each $s\in(\text{Cl}(zb_n))^*$, it follows that $(\text{Cl}(zb_n))^*=0$. That is, $\text{Cl}(zb_n)=o$ for each n. Thus $z=\bigvee zb_n=o$.

(4) If $\{H\}\in K\otimes U(e^o)$ and $\{H\}\leqq\theta(e^o)$, there exists $x\leqq e^o$ such that $\theta(x)=\{H\}$.

We first note that, given any $H\in\mathfrak{B}_r(T)$, there exists $H'\in\{H\}$ and expressible as $H'=\bigcup(q_n, u_n^*)$ (u_n^* open-closed in S). (This is substantially the result of 4.6 in the present case, and follows by the same reasoning applied here.) Now if $\{H\}\leqq\theta(e^o)=\{\bigcup(q_n, \bar{b}_n^*)\}$, the corresponding H' must satisfy $u_n\leqq\bar{b}_n$ for each n. Define $x=\bigvee b_nu_n$. Then $x\leqq e^o$, and (since the elements b_n are disjoint) we have $\text{Cl}(xb_n)=\text{Cl}(b_nu_n)=(\text{Cl}(b_n))u_n$ (5.10) $=u_n$. Thus $\psi(x)=H'$, so that $\theta(x)=\{H\}$.

Properties (1)–(4) show that θ is an algebraic isomorphism between $E(e^o)$ and the principal ideal of $\theta(e^o)$ in $K\otimes U(e^o)$. All that remains is to show that θ is equivalence-preserving (both ways).

(5) If $x, y\leqq e^o$ and $x\sim y$, $M\{\theta(x)\}=M\{\theta(y)\}$.

First suppose $x\leqq b_n$, $y\leqq b_m$. We have $\bar{x}=\bar{y}=u$, say (5.8); and clearly $\psi(x)=\bigcup(q_k, u_k^*)$ where $u_k=o$ if $k\neq n$, and $u_n=u$. Hence, from the definition of M, $M\{\theta(x)\}=M\{\psi(x)\}=0$ for $s\notin u$, and 1 for $s\in u$. The same applies to $M\{\theta(y)\}$, which thus equals $M\{\theta(x)\}$.

In the general case, 6.2 shows that we can write $y=\bigvee y_n$, where the sequence y_n is disjoint and $y_n\sim xb_n$. Now write $y_{nm}=y_nb_m$; by 6.2 again we can write $xb_n=\bigvee_m x_{nm}$ where the elements x_{nm} are disjoint and $x_{nm}\sim y_{nm}$. Since $y_{nm}\leqq b_m$ and $x_{nm}\leqq b_n$, the case already established shows that $M\{\theta(x_{nm})\}=M\{\theta(y_{nm})\}$. But $\bigvee_{m,n} x_{mn}=x$, and $\bigvee_{m,n} y_{mn}=y$; hence, from the countable additivity of M, $M\{\theta(x)\}=M\{\theta(y)\}$.

(6) If $x, y\leqq e^o$ and $M\{\theta(x)\}=M\{\theta(y)\}$, then $x\sim y$.

First suppose $\theta(x)\leqq\{(q_n, \bar{b}_n^*)\}$ and $\theta(y)\leqq\{(q_m, \bar{b}_m^*)\}$. Then $xb_k=o$ unless $k=n$, so that $x\leqq b_n$; and similarly $y\leqq b_m$. The equality of the measure functions now gives $\bar{x}=\bar{y}$. Now (9.6) we can assume without loss of generality that $[b_n]\leqq[b_m]$; and, on applying postulate II, we obtain $x\sim x'\leqq b_m$. Hence $x'=\bar{x}'b_m$ (from 9.2(4)) $=\bar{x}b_m$ (from 5.8) $=\bar{y}b_m=y$ (from 9.2(4) again). That is, $y=x'\sim x$.

The extension to the general case now follows by an argument similar to that used in proving (5). (We apply 6.2 to $K\otimes U(e^o)$.) The proof of the theorem is thus complete.

Remark. In the present case, the direct product (as constructed in §4) can be simplified. It is not hard to see that the abstract measure-algebra $K\otimes U(e^o)$ is isomorphic with the algebra of sequences $\{u_n\}$ ($u_n\in U(e^o)$), in which the algebraic operations are defined "coordinatewise" (cf. 3.1(d)), but in which the equivalence relation is induced by the continuous numerical function $M\{u_n\}$ on S, defined by: the value of $M\{u_n\}$ at $s \in S$ is (except for a first category set of s's) the number of values of n for which $s \in u_n^*$.

10. Reduction to the bounded case.

10.1. The ideal $E(e^o)$ having been disposed of by the theorem just established, we next study the ideal (that is, relative algebra) $E(-e^o)$, in which (from the definition of e^o) no nonzero element can be indecomposable. In effect this means that the indecomposable elements have been eliminated. In the present section we shall make a further reduction which will in effect eliminate the unbounded elements.

10.2. Definitions. We write e' to denote the supremum of all *bounded invariant* elements in $E(-e^o)$, and write $e'' = (-e^o) - e'$. Thus e^o, e' and e'' are disjoint, and $e^o \vee e' \vee e'' = e$.

10.3. Theorem. *e' is bounded and invariant.*

That e' is invariant follows from 5.4. Further, the countable chain condition (postulate 0) gives $e' = \bigvee u_n$ for some sequence $\{u_n\}$ of bounded invariant elements. Define $v_1 = u_1$, $v_n = u_n - \bigvee_1^{n-1} u_i$; thus $e' = \bigvee v_n$ where each v_n is bounded (2.3(2)) and invariant (5.4), and where the sequence $\{v_n\}$ is disjoint. Hence e' is bounded, from 6.5.

10.4. Lemma. *There exists a sequence $\{z_n\}$ ($n = 1, 2, \cdots$ to ∞) of disjoint equivalent bounded elements such that $e'' = \bar{z}_n$ for each n.*

We can obviously assume $e'' \neq o$. Then since e'' is invariant (from 9.6(iii) and 10.3) and disjoint from e', e'' is unbounded, and consequently (6.3) there exists an infinite sequence of disjoint equivalent nonzero elements (say) $t_n \leqq e''$. From postulate III (2.1) there exists $p_{11} \leqq t_1$ such that p_{11} is bounded and nonzero; and then from postulate II there exists $p_{1n} \leqq t_n$ such that $p_{1n} \sim p_{11}$. Thus the elements p_{1n} are disjoint, equivalent, bounded and nonzero, and $\bigvee p_{1n} \leqq e''$. Write $q_2 = \mathrm{Cl}(\bigvee p_{1n}) \leqq e''$; thus $e'' - q_2$ is invariant and therefore unbounded (unless it is o). Thus we obtain (by repeating the argument) a disjoint sequence $\{p_{2n}\}$ of bounded equivalent nonzero elements less than or equal to $e'' - q_2$. The argument is repeated transfinitely; when $p_{\beta n}$ has been defined for all n and all $\beta < \alpha$, we set $q_\alpha = \mathrm{Cl}(\bigvee p_{\beta n})$ ($\beta < \alpha$, $n = 1, 2, \cdots$); if $e'' - q_\alpha$ is not o, it is invariant and unbounded, and so there exists a disjoint infinite sequence $\{p_{\alpha n}\}$ of bounded equivalent nonzero elements less than or equal to $e'' - q_\alpha$. The process must terminate countably; and, on defining $z_n = \bigvee_\alpha p_{\alpha n}$, the elements z_n are disjoint, equivalent (postulate I), bounded (6.5), and satisfy $\mathrm{Cl}(\bigvee z_n) = e''$. But $\mathrm{Cl}(\bigvee z_n) = \bigvee \mathrm{Cl}(z_n)$ (5.6) $= \mathrm{Cl}(z_n)$ for each n (from 5.8).

10.5. Lemma. *Given $u \in U$ such that $o < u \leqq e''$, there exists $v \in U$ such that $o < v \leqq u$ and $v = \bigvee s_n$, where the elements s_n are disjoint, equivalent, and bounded ($n = 1, 2, \cdots$, to ∞).*

Let $\{z_n\}$ be the sequence given by 10.4. Then the elements uz_n are disjoint, equivalent (6.6) and bounded; also they are nonzero, since $\text{Cl}(uz_n) = u\text{Cl}(z_n)(5.10) = u \neq o$. The collection $\{uz_n\}$ can thus be extended to a *maximal* collection of disjoint equivalent (and thus also bounded and nonzero) elements less than or equal to u; and this collection (from postulate 0) will consist of *exactly* $\aleph_0$ elements (say) t_n ($n=1, 2, \cdots$, to ∞). Let y_1, y' be maximal equivalent elements such that $y_1 \leqq t_1$ and $y' \leqq u - \bigvee t_n$, so that (7.5) $\text{Cl}(t_1 - y_1)\ \text{Cl}(u - \bigvee t_n - y') = o$. Define $v = \text{Cl}(t_1 - y_1)$; we shall show that v has the required properties.

Trivially, v is invariant and $v \leqq \bar{t}_1 \leqq u$. Also, $v \neq o$, since otherwise $t_1 = y_1 \sim y'$, contradicting the maximality of the collection $\{t_n\}$ (for y' could then be adjoined to it).

Now, from postulate II, there exists $y_n \leqq t_n$ such that $y_n \sim y_1$, and (since then $y_n \sim y'$) there exists $x_n \leqq y_n$ such that $x_n \sim y'v$. Define $s_1 = y'v \vee (t_1 - y_1)$ and (if $n > 1$) $s_n = x_{n-1} \vee (t_n - y_n)$; thus the elements s_n ($n = 1, 2, \cdots$ to ∞) are disjoint and equivalent (from 6.1 and postulate III). Further, $s_n \sim x_n \vee (t_n - y_n) \leqq t_n$, so that s_n is bounded (6.4). We have $t_n - y_n \leqq v$, from 5.7, and $x_n \leqq v$, from 5.2; thus $\bigvee s_n \leqq v$. On the other hand, $v(u - \bigvee t_n - y') = o$, so that $v \leqq \bigvee t_n \vee y'$, and therefore $v \leqq y'v \vee \bigvee\{(t_n - y_n) \vee (y_n - x_n) \vee x_n\}v$; but $(y_n - x_n)v \sim (y' - y'v)v$ (from 6.6) $= o$, and it follows that $v \leqq \bigvee s_n$. Thus $v = \bigvee s_n$, as required.

10.6. Theorem. *We can write* $e'' = \bigvee f^n$ ($n = 1, 2, \cdots$ *to* ∞), *where the elements* f^n *are disjoint, equivalent, bounded, and satisfy* $\text{Cl}(f^n) = e''$ *for each* n.

Let $\{v_m\}$ be a maximal disjoint (and thus countable) collection of nonzero invariant elements less than or equal to e'', each of which can be written as $v_m = \bigvee s_{mn}$ ($n = 1, 2, \cdots$ to ∞), where the elements s_{mn} are, for each m, disjoint, equivalent and bounded. Let $u = e'' - \bigvee v_m$; then u is invariant (5.4), so that if $u \neq o$ an application of 10.5 would at once contradict the maximality of the collection v_m. Hence $\bigvee_{m,n} s_{mn} = \bigvee v_m = e''$. Now define $f^n = \bigvee_m s_{mn}$; thus $\bigvee f^n = e''$, and the elements f^n are disjoint, equivalent (postulate I) and bounded (from 6.5). Lastly, $e'' = \bigvee \text{Cl}(f^n) = \text{Cl}(f^n)$ for each n, from 5.8.

Corollary. *For each* n, *the algebra* U_{f^n} *of invariant elements relative to* f^n *is isomorphic to the ideal* $U(e'')$.

(From 8.3.)

10.7. *Convention.* The considerations in what follows (till §19) are going to be applied to the relative abstract measure algebras $E(e')$ and $E(f^n)$. Thus we shall assume, throughout §§11–18, that *all elements (except o) are decomposable)*, and that *all elements are bounded.*

11. Decomposition into homogeneous parts.

11.1 Before the desired direct product representation can be obtained, two further reductions are necessary. In this section we carry out the first of these, which will have the effect of enabling us to assume that E is, in a sense to be defined, "homogeneous" with respect to its invariant elements.

DEFINITIONS. Given $x \in E$, and given any nonempty subset $P \subset E$, consider all sets $Q \subset E$ such that $E(x) \subset x\mathcal{B}(P \cup Q)$—so that (8.5) $E(x) = x\mathcal{B}(P \cup Q) = \mathcal{B}_x(xP \cup xQ)$. The smallest cardinal $\mathfrak{m}$ of such a Q is called the *order* of x over P. (Thus, for example, the order of o over any P is 0.) If $x \neq o$, and every y such that $o < y \leqq x$ has order $\mathfrak{m}$ over P, x is said to be *homogeneous* of order $\mathfrak{m}$ over P. Clearly, if $o < y \leqq x$ and x is homogeneous of order $\mathfrak{m}$ over P, then so is y.

11.2. If x has order $\mathfrak{m}$ over P, and $y \leqq x$, then the order of y over P is not greater than $\mathfrak{m}$.

For $E(x) = x\mathcal{B}(P \cup Q)$, where $|Q| = \mathfrak{m}$. Given $z \leqq y$, we have $z \in E(x)$, and so $z = xt$ where $t \in \mathcal{B}(P \cup Q)$. Hence $z = yt$; and thus $E(y) \subset y\mathcal{B}(P \cup Q)$.

11.3. If x is homogeneous of order $\mathfrak{m}$ over P, where $\mathfrak{m}$ is infinite, and if $a \geqq x$, then x is homogeneous of order $\mathfrak{m}$ over aP, and further there exists a set $Q \subset E(x)$, with $|Q| = \mathfrak{m}$, such that $E(x) = x\mathcal{B}(Q \cup aP)$.

We have $E(x) = x\mathcal{B}(P \cup Q')$ for some $Q' \subset E$ with $|Q'| = \mathfrak{m}$. Write $Q = xQ'$; then $x\mathcal{B}(P \cup Q') = x\mathcal{B}(Q \cup aP)$, by 8.5, since both equal $\mathcal{B}_x(Q \cup xP)$. This proves the last part of the statement, and shows that x has order not greater than $\mathfrak{m}$ over aP. To complete the proof, it will suffice to show that if $o < y \leqq x$ than the order of y over aP is not less than $\mathfrak{m}$. Suppose not; then there exists $Q'' \subset E$, with $|Q''| < \mathfrak{m}$, such that $E(y) = y\mathcal{B}(Q'' \cup aP) \subset y\mathcal{B}(Q'' \cup P \cup (a))$. The homogeneity of x over P now gives $|Q'' \cup (a)| \geqq \mathfrak{m}$; and since $\mathfrak{m}$ is infinite this is a contradiction.

11.4. *If* $\mathfrak{m}$ *is an infinite cardinal, and if* x_n *is homogeneous of order* $\mathfrak{m}$ *over* P $(n = 1, 2, \cdots)$, *then so is* $\bigvee x_n$.

Let $\bigvee x_n = x$; we must prove (i) x is of order not greater than $\mathfrak{m}$ over P, and (ii) if $o < y \leqq x$, y is of order not less than $\mathfrak{m}$ over P.

We have $E(x_n) = x_n\mathcal{B}(P \cup Q_n)$ where $|Q_n| = \mathfrak{m}$. Let $\bigcup Q_n \cup \bigcup (x_n) = Q$; thus $|Q| = \mathfrak{m}$ also. Given $y \leqq x$, write $y_n = yx_n \in E(x_n)$; thus $y_n = x_n z_n$ where $z_n \in \mathcal{B}(P \cup Q_n) \subset \mathcal{B}(P \cup Q)$. Since $x_n \in Q \subset \mathcal{B}(P \cup Q)$, this gives that each y_n, and thus also $y = \bigvee y_n$, belongs to $\mathcal{B}(P \cup Q)$, which proves (i).

For (ii), we again write $y_n = yx_n$; then for some n we have $y_n \neq o$. If y has order less than $\mathfrak{m}$ over P, then so does y_n (11.2); but this contradicts the homogeneity of x_n.

11.5. *If* x *is a nonzero element of finite order over* P, *there exists a nonzero element* $y \leqq x$ *of order* 0 *over* P.

We have (say) $E(x) = x\mathcal{B}\{P \cup (a_1) \cup (a_2) \cup \cdots \cup (a_n)\}$. Consider the 2^n elements of the form $b_1 b_2 \cdots b_n$, where each b_i is either a_i or $-a_i$, and all combinations are taken; enumerate them as $t_1, t_2, \cdots, t_N$ $(N = 2^n)$. Let $\mathcal{B}'$ denote the set of all elements expressible as $\bigvee_1^N x_j t_j$ $(x_j \in \mathcal{B}(P))$. It is easy to verify that $\mathcal{B}'$ contains, and thus coincides with, $\mathcal{B}\{P \cup (a_1) \cup \cdots \cup (a_n)\}$. Thus $E(x) = x\mathcal{B}'$. so for at least one j we have $xt_j = y$, say, $\neq o$. Now if $z \in E(y)$, we have $z \in E(x) = x\mathfrak{B}'$, so $z = \bigvee_1^N x_i t_i x$ ($x_i \in \mathfrak{B}(P)$); but $z \leqq y \leqq t_j$, and $t_i t_j = o$ unless $i = j$. Hence $z = x_j t_j x = x_j y$; and this proves that $E(y) \subset y\mathfrak{B}(P)$—that is, y is of order 0 over P.

11.6. *If y is of order* 0 *over U, then y is indecomposable and therefore zero.*

If y were decomposable, there would exist (9.1) $z_1, z_2 \leqq y$ such that $z_1 \sim z_2$, $z_1 z_2 = o$, and $z_i \neq o$ ($i = 1, 2$). Since $E(y) = yU$, by hypothesis, we have $z_i = yu_i$ ($u_i \in U$). Thus $y\bar{z}_i = y(\bar{y}u_i)$ (from 5.10) $= yu_i = z_i$, so that $z_1 = y\bar{z}_1 = y\bar{z}_2$ (5.8) $= z_2$, whence a contradiction. Since all nonzero elements are decomposable (in accordance with the convention introduced in 10.7), $y = o$.

COROLLARY. *Every nonzero element has infinite order over U.*

(From 11.5 and 11.6.)

11.7. THEOREM. *There exists a disjoint sequence $\{a_n\}$ (finite or infinite) such that $\bigvee a_n = e$ and each a_n is homogeneous of infinite order over U*[20].

Let $\mathfrak{m}_1$ be the smallest of the orders of nonzero elements x over U; thus $\mathfrak{m}_1$ is infinite, as just shown, and from 11.2 each x of order $\mathfrak{m}_1$ over U is homogeneous. Let $a_1 = \bigvee \{x \mid x$ is of order $\mathfrak{m}_1$ over $U\}$. The countable chain condition shows that a_1 is the supremum of a countable set of such elements x; and 11.4 then shows that a_1 is homogeneous of order $\mathfrak{m}_1$ over U. The construction is iterated transfinitely, a_α being defined as the supremum of all nonzero elements less than or equal to $-\bigvee_{\beta<\alpha} a_\beta$ (if any) having the smallest possible order $\mathfrak{m}_\alpha$ over U; as before, a_α is homogeneous of order $\mathfrak{m}_\alpha$ over U. The construction terminates for some $\alpha_0 < \omega_1$, for the elements a_α are disjoint and nonzero, and we have $e = \bigvee_{\alpha<\alpha_0} a_\alpha$. All that remains is to reorder the elements a_α ($\alpha < \alpha_0$) into a simple sequence.

12. Scalar multiplication of equivalence classes.

12.1 We could now consider the relative abstract measure algebras $E(a_n)$, in which all nonzero elements are now decomposable, bounded, and (from 11.3 and 8.3) homogeneous over the algebra of relatively invariant elements. However, though it would be possible to prove a representation theorem for each $E(a_n)$, it would not be easy to extend the representation to all of E; this is because (roughly speaking) the elements a_n are not related in any convenient way to their invariant closures in E. In the next section we shall replace them by elements which are "comparable," in the desired way, with their invariant closures in E. Here we shall define and develop the necessary properties of this comparability. It would be possible to elaborate these notions further into a systematic arithmetic of equivalence classes (cf. footnote 17), but we shall confine ourselves to the properties actually needed in the sequel.

12.2. Given $x \neq o$, and a positive integer n, there exist n disjoint equiva-

(20) This theorem generalizes the decomposition of a numerical measure algebra into homogeneous parts; cf. [11].

lent nonzero elements $y_i \leqq x$ $(i=1, \cdots, n)$.

For x is decomposable (cf. 10.7), so there exist $z_1, z_2 \leqq x$ such that $z_i \neq o$, $z_1z_2 = o$, and $z_1 \sim z_2$. Since z_1 is decomposable, there exist $z_{11}, z_{12} \leqq z_1$ such that $z_{1i} \neq o$, $z_{11}z_{12}=o$, and $z_{11} \sim z_{12}$; and from 6.2 there exist disjoint elements z_{21}, $z_{22} \leqq z_2$ such that $z_{21} \sim z_{22} \sim z_{1i}$. Proceeding in this way we obtain, for each m, 2^m disjoint equivalent nonzero elements less than or equal to x; and the result follows.

12.3. In 12.2 we can further suppose $\bigvee y_i = x$.

This follows from 12.2 by the usual "exhaustion" argument.

12.4. If $y_1, y_2, \cdots, y_n$ are disjoint equivalent elements, and if $z_1, z_2, \cdots, z_n$ are disjoint equivalent elements such that $\bigvee y_i \sim \bigvee z_i$, then $y_i \sim z_i$.

Let p_i, q_i be maximal equivalent sub-elements of y_i, z_i (7.5). Since all elements are now bounded (10.7), it readily follows from 7.5(ii) that the $2n$ elements p_i, q_i, are all equivalent. Write $h_i = y_i - p_i$, $k_i = z_i - q_i$; thus $\bar{h}_i\bar{k}_i = o$, and (6.1) we have $h_1 \sim h_2 \sim \cdots \sim h_n$, $k_1 \sim k_2 \sim \cdots \sim k_n$, and $\bigvee h_i \sim \bigvee k_i$. Thus (from 5.6 and 5.8) $\mathrm{Cl}(h_i) = \mathrm{Cl}(\bigvee h_i) = \mathrm{Cl}(\bigvee k_i) = \mathrm{Cl}(k_i)$; and the relation $\bar{h}_i\bar{k}_i = o$ now gives $\bar{h}_i = o = \bar{k}_i$. Thus $y_i = p_i \sim q_i = z_i$.

12.5. Definitions. Given any x and any positive integer n, 12.3 shows that, for each $x' \in [x]$ (that is, $x' \sim x$) we can write $x' = \bigvee_1^n y_i$ where the n elements y_i are disjoint and equivalent; and 12.4 then shows that the equivalence class $[y_1]$ $(=[y_i])$ is uniquely determined. We write $[y_1] = (1/n)[x]$, and more generally $[y_1 \vee y_2 \vee \cdots \vee y_m] = (m/n)[x]$, for each $m \leqq n$. Thus $(m/n)\, x$ is uniquely determined by the equivalence class x and the integers m, n $(0 \leqq m \leqq n \geqq 1)$. Clearly $(0/n)[x] = o$, $(n/n)[x] = [x]$, $(m/n)[o] = [o]$.

The following properties follow easily from this definition, together with the results in §7.

(1) If $p \geqq 1$, $n \geqq 1$, and $0 \leqq m \leqq n$, then $(mp/np)[x] = (m/n)[x]$.

Thus $(m/n)[x]$ depends only on $[x]$ and the *value* of the proper fraction m/n. If $m/n = \rho$, we write $(m/n)[x]$ as $\rho[x]$. In the following statements, ρ, ρ_1, and so on *denote non-negative rational numbers not exceeding* 1.

(2) If $\rho_1 < \rho_2$, then $\rho_1[x] \leqq \rho_2[x]$, with equality only if $[x] = o$.

(Immediate on writing ρ_1 and ρ_2 with a common denominator.)

(3) If $[x] < [y]$, then $\rho[x] \leqq \rho[y]$, with equality only if $\rho = 0$.

(4) If $[y] = \rho_1[x]$ and $[z] = \rho_2[x]$, where $y \vee z \leqq x$, $x \neq o$, and $yz = o$, then $\rho_1 + \rho_2 \leqq 1$ and $[y \vee z] = (\rho_1 + \rho_2)[x]$.

(5) If $[y_n] = \rho[x_n]$, where the elements y_n are disjoint, and the elements x_n are disjoint, then $[\bigvee y_n] = \rho[\bigvee x_n]$.

(6) If $[y] = \rho_1[x]$, then $\rho_2[y] = (\rho_1\rho_2)[x]$.

The next step will consist in extending this definition of "multiplication" of equivalence classes by (rational) scalars, to allow the multipliers to be irrational. Several lemmas are needed.

12.6. If $[x] \leqq \rho[e]$ for arbitrarily small values of ρ, then $x = o$.

For, from (2) above, we have $[x] \leqq (1/2^n)[e]$ for every n. Now (12.3) we can write $e=f_1\vee g_1$ where f_1, g_1 are disjoint and equivalent. Similarly, $g_1=f_2\vee g_2$ where f_2, g_2 are disjoint and equivalent; and so on. The disjoint sequence $\{f_n\}$ is such that $[f_n]=(1/2^n)[e]$; hence there exists $x_n\leqq f_n$ such that $x_n\sim x$. Then $\bigvee_1^\infty x_n\sim\bigvee_2^\infty x_n$; but all elements are now bounded (10.7), and so $x_1=o$, whence $x=o$.

12.7. If $[y_n]\leqq(1/2^n)[x]$ $(n=1, 2, \cdots)$ then $[\bigvee y_n]\leqq[x]$.

This is proved by essentially the same argument as in 12.6, since there is no loss in assuming the elements y_n to be disjoint.

12.8. Suppose x and y are such that, for each positive integer n, there exist p_n, $q_n\in E$ such that $[p_n]\leqq(1/2^{n+1})[e]$, $q_n\leqq(1/2^{n+1})[e]$, and $[x-p_n]\leqq[y\vee q_n]$. Then $[x]\leqq[y]$.

There is no loss in assuming $p_n\leqq x$. Then there exist $x_n\sim x$ and $r_n\sim p_n$ such that $r_n\leqq x_n$ and $x_n-r_n\leqq y\vee q_n$. Thus $x_n\leqq y\vee r_n\vee q_n\leqq y\vee f_n$, where $f_n=\bigvee\{r_m\vee q_m|m\geqq n\}$. Now $[r_n\vee q_n]\leqq(1/2^n)[e]$, from 12.5(4); hence $[f_n]\leqq(1/2^{n-1})[e]$, as readily follows from 12.7. Thus (12.6) $\bigwedge f_n=o$. But (7.2) we have $[x]\leqq[\bigwedge(y\vee f_n)]=[y\vee\bigwedge f_n]=[y]$.

12.9. Given $x\in E$ and a real number σ such that $0\leqq\sigma\leqq 1$, there exists $y\in E$ having the following property: Given any positive rational number $\epsilon<1$, there exist (i) a rational number ρ $(0\leqq\rho\leqq 1)$ such that $|\rho-\sigma|\leqq\epsilon$, (ii) an element $d\in E$ such that $[d]\leqq\epsilon[e]$ and $[y+_2d]=\rho[x]$.

As in the proof of 12.6, we construct a disjoint sequence of elements $f_n\leqq x$ such that, on writing $g_n=x-\bigvee\{f_m|m\leqq n\}$, we have $[f_n]=[g_n]=(1/2^n)[x]$. Expand σ as a binary decimal, say $\sigma=\sum_1^\infty m_n/2^n$, where each m_n is 0 or 1, and define $t_n=f_n$ if $m_n=1$, and $t_n=o$ otherwise. Write $y=\bigvee t_n$. To satisfy the requirements, we have only to choose n so that $2^{-n}<\epsilon$, and take $d=\bigvee\{t_m|m>n\}$ and $\rho=\sum_1^n m_k/2^k$. Clearly $|\rho-\sigma|<2^{-n}$ and (from 12.5(4)) $[y+_2d]=\rho[x]$. Finally, the fact that $[d]\leqq(1/2^n)[e]$ follows from 12.7.

REMARK. The proof shows that a little more is true than was stated; we can further make $d\leqq y$, $\rho\leqq\sigma$, and $[d]\leqq\epsilon[x]$.

12.10. Given $x\in E$ and σ $(0\leqq\sigma\leqq 1)$, suppose that y' also has the same property as y in 12.9; then $y'\sim y$.

For each n we can write

$$[y+_2 d_n]=\rho_n[x],\qquad |\rho_n-\sigma|<1/2^{n+2},\qquad [d_n]\leqq(1/2^{n+2})[e],$$
$$[y'+_2 d_n']=\rho_n'[x],\qquad |\rho_n'-\sigma|<1/2^{n+2},\qquad [d_n']\leqq(1/2^{n+2})[e],$$

and can suppose without loss that $\rho_n'\geqq\rho_n$ for infinitely many values of n. Thus we find $d_n''\leqq y'+_2d_n'$ such that $[d_n'']=(\rho_n'-\rho_n)[x]\leqq(1/2^{n+1})[e]$, and then have (6.1) $[(y'+_2d_n')-d_n'']=\rho_n[x]$. Hence $(y'+_2d_n')-d_n''\sim y+_2d_n$ for arbitrarily large n; and from 12.8 it readily follows that $[y]\leqq[y']\leqq[y]$. Since y is bounded, we obtain $[y']=[y]$.

12.11. DEFINITION. If x, σ and y are related as in 12.9, we write $[y]=\sigma[x]$; from 12.9 and 12.10, this relation determines one and only one equivalence class $[y]$ when $[x]$ and σ $(0 \leqq \sigma \leqq 1)$ are given. Further, if σ is rational this definition reduces to that of 12.5, since in 12.9 we can then take $d=o$ and $\rho=\sigma$.

Throughout the rest of this section, σ, σ_1, and so on will denote real numbers between 0 and 1 (inclusive), and ϵ denotes a real number such that $0<\epsilon<1$. Most of the proofs will be omitted, since they involve no new ideas.

12.12. *If* $y \leqq x$ *and* $[y]=\sigma[x]$, *then* $[x-y]=(1-\sigma)[x]$.

This is immediate from the way in which y was constructed in 12.9, if we note that $\bigvee f_n = x$ there (since $\bigwedge g_n = o$, from 12.6).

12.13. *If* $[y]=\sigma[x]$, *and u is invariant, then* $[uy]=\sigma[ux]$.

Immediate from 6.6 and the definitions—12.5 and 12.11.

12.14. *If* $\sigma_1 \leqq \sigma_2$, $\sigma_1[x] \leqq \sigma_2[x]$.

This follows from 12.8 by an argument similar to that in 12.10.

COROLLARY. *If* $\sigma_1<\sigma_2$, $\sigma_1[x]<\sigma_2[x]$ *unless* $x=o$.

For we can choose rational ρ_1, ρ_2 such that $\sigma_1<\rho_1<\rho_2<\sigma_2$; the corollary now follows from 12.14 and 12.5(2).

12.15. If $[p]=\sigma_1[x]$, $[q]=\sigma_2[x]$, $pq=o$, $p \vee q \leqq x$, and $x \neq o$, then $\sigma_1+\sigma_2 \leqq 1$ and $[p \vee q]=(\sigma_1+\sigma_2)[x]$.

That $\sigma_2 \leqq 1-\sigma_1$ follows easily from 12.12. The remainder of the assertion follows from the definition of $(\sigma_1+\sigma_2)[x]$ and from the rational case (12.5(4)).

12.16. If $\{y_n\}$ is a disjoint sequence such that $y_n \leqq x \neq o$ and $[y_n]=\sigma_n[x]$, then $\sigma \equiv \sum \sigma_n \leqq 1$, and $[\bigvee y_n]=\sigma[x]$.

An easy induction based on 12.15 gives $\sum_1^m \sigma_n \leqq 1$, for each m, so that $\sigma \leqq 1$. It also gives $[\bigvee_1^m y_n]=(\sum_1^m \sigma_n)[x] \leqq \sigma[x]$ (12.14), for each m; hence (7.3) $[\bigvee y_i] \leqq \sigma[x]$. On the other hand, we can assume $\sigma>0$; let ϵ be any positive number less than σ. If N is large enough, we have (on using 12.15 and 12.14) $[\bigvee y_n] \geqq [\bigvee_1^N y_n]=(\sum_1^N \sigma_n)[x] \geqq (\sigma-\epsilon)[x]$; and it readily follows from 12.8 that $[\bigvee y_n] \geqq \sigma[x]$. Thus, since all elements are now bounded, $[\bigvee y_n]=\sigma[x]$.

12.17. If $\{y_n\}$ and $\{z_n\}$ are two sequences, each disjoint, such that $[y_n]=\sigma[z_n]$ $(n=1, 2, \cdots)$, then $[\bigvee y_n]=\sigma[\bigvee z_n]$.

We may suppose $y_n \leqq z_n$. Given any rational $\epsilon>0$ (and <1) we choose a rational number ρ so that $\sigma-\epsilon<\rho<\sigma$, and then (12.9, Remark) choose $d_n \leqq y_n$ so that $[d_n] \leqq (\epsilon/2^n)[e]$ and $[y_n-d_n]=\rho_n[z_n]$, where $\rho<\rho_n<\sigma$ and ρ_n is rational. The rest is routine.

12.18. If $x \geqq y \geqq z$, and $x \neq o$, and if $[y]=\sigma_1[x]$ and $z=\sigma_2[x]$, then $\sigma_1 \geqq \sigma_2$ and $[y-z]=(\sigma_1-\sigma_2)[x]$.

12.19. If $y_1 \leqq y_2 \leqq \cdots \leqq x$, and if $[y_n]=\sigma_n[x]$ and $x \neq o$, then $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq 1$, and $[\bigvee y_n]=(\lim_{n\to\infty} \sigma_n)[x]$.

(From 12.16 and 12.18 applied to the relation $\bigvee y_n = \bigvee (y_n - y_{n-1})$.)

A similar result holds for decreasing sequences.

12.20. If $[y]=\sigma_1[x]$, then $\sigma_2[y]=(\sigma_1\sigma_2)[x]$.

This will be needed only in the case in which σ_1 is rational, when it follows fairly easily from 12.5(6) and the definition (12.11). The extension to the general case can then be derived, though with more trouble.

13. Decomposition into parts comparable with their closures.

13.1. We begin with a fundamental result which, roughly speaking, asserts that the abstract measures can be approximated by "step functions."

THEOREM. *Given $x \in E$ and a positive integer n, there exist n disjoint invariant elements $u_1, u_2, \cdots, u_n$ such that* (i) $\bigvee u_i = e$, *and* (ii) $((i-1)/n)[u_i] \leqq [xu_i] \leqq (i/n)[u_i]$ $(1 \leqq i \leqq n)$.

By 12.3, we can write $e = t_1 \vee t_2 \vee \cdots \vee t_n$, where the elements t_i are disjoint and equivalent. For each i, consider the elements x and $t_1 \vee t_2 \vee \cdots \vee t_i$; from 7.6 there exists $w_i \in U$ such that

(a) $[xw_i] \leqq [w_i(t_1 \vee \cdots \vee t_i)]$, (b) if $v \in U(-w_i)$ then $[xv] \geqq [v(t_1 \vee \cdots \vee t_i)]$, with equality only if $v = o$. Now $[t_1 \vee \cdots \vee t_i] = (i/n)[e]$, by definition; hence, from 12.13, properties (a) and (b) may be restated as:

(a′) $[xw_i] \leqq (i/n)[w_i]$, (b′) if $v \in U(-w_i)$ then $[xv] > (i/n)[v]$ unless $v = o$.

We assert: $o \leqq w_1 \leqq w_2 \leqq \cdots \leqq w_n = e$. For suppose $i < j$ and let $v = w_i - w_j$; if $v \neq o$, we have at once from 12.13 that $[xv] > (j/n)[v] > (i/n)[v]$(12.5(2)) $\geqq [xv]$, a contradiction. Thus $v = o$ and $w_i \leqq w_j$. The fact that $w_n = e$ is immediate from (b′).

Write $w_0 = o$, and define $u_i = w_i - w_{i-1}$ $(1 \leqq i \leqq n)$. Thus the elements $u_1, \cdots, u_n$ are invariant and disjoint, and $\bigvee u_i = e$. From (a′), (b′) and 12.13 we have $((i-1)/n)[u_i] \leqq [xu_i] \leqq (i/n)[u_i]$ as desired; and have in fact more —that $((i-1)/n)[u_i] < [xu_i]$ if $i < n$, unless $u_i = o$.

13.2 LEMMA. *Given $x \neq o$, there exists $y \in E$ such that $o < y \leqq x$ and $[y] = \rho[\bar{y}]$ for some rational ρ* $(0 < \rho \leqq 1)$.

We apply 13.1, and note that if n is large enough $u_1 \neq e$, since otherwise $[x] \leqq (1/n)[e]$ for arbitrarily large n, which is impossible (12.6). Thus for some $i > 1$ we have $u_i \neq o$; and since $((i-1)/n)[u_i] \leqq [xu_i]$ there exists $y \leqq xu_i$ such that $[y] = ((i-1)/n)[u_i]$. Since $i > 1$ and $u_i \neq o$ we have $y \neq o$; and, from 12.13, $[\bar{y}y] = \rho[\bar{y}u_i]$ (where $\rho = (i-1)/n$), that is, $[y] = \rho[\bar{y}]$.

13.3. THEOREM. *Given $x \neq o$, there exists a sequence $\{z_n\}$ (finite or infinite) of disjoint nonzero elements such that* (i) $\bigvee z_n = x$, (ii) $[z_n] = \rho_n[\bar{z}_n]$, *where* $0 < \rho_n \leqq 1$ *and ρ_n is rational.*

Let $\{z_n\}$ be a maximal disjoint collection of nonzero elements $\leqq x$ satisfying (ii). From 13.2 it readily follows that $\bigvee z_n = x$, proving the theorem.

REMARK. The theorem also holds if $x = o$ provided we adopt suitable conventions about "empty" sequences.

13.4. THEOREM. *There exists a disjoint sequence $\{e_n\}$ (finite or infinite) such that* (i) $\bigvee e_n = e$, (ii) *each e_n is homogeneous of infinite order $\mathfrak{m}_n$ over U,* (iii) *for each n, $[e_n] = \rho_n[\bar{e}_n]$, where ρ_n is rational and* $0 < \rho_n \leqq 1$.

From 11.7 there exist disjoint (nonzero) elements a_n such that $\bigvee a_n = e$ and a_n is homogeneous of infinite order over U. Applying 13.3, we write $a_n = \bigvee_m e_{nm}$ where (for each n) the elements e_{nm} are disjoint, nonzero, and $[e_{nm}] = \rho_{nm}[\bar{e}_{nm}]$. From the definition of homogeneity (11.1), each e_{nm} is homogeneous of infinite order over U. We have only to renumber the elements e_{nm} into a single sequence $\{e_n\}$.

This theorem provides the desired decomposition of e ($=e'$ or f^n; cf. 10.7). In the next sections (§§14–17) we shall consider the relative algebras $\{E(e_n), \sim\}$, showing that each is isomorphic with the direct product of a numerical measure algebra with a trivial abstract measure algebra. It will then be relatively easy (§§18, 19) to derive a product representation for all of E.

14. Structure of $E(e_n)$; separable case.

14.1. Let e_n be one of the elements in 13.4, and suppose that its order over U is exactly $\aleph_0$. In this section we shall analyze the structure of the relative algebra $E(e_n)$; and, as all considerations here will be relative to e_n, we shall write $E(e_n)$ simply as E, e_n as e, and the relative sub-algebra of relatively invariant elements, U_{e_n}, as U. From 11.3 and 8.3, we can thus assume that *every nonzero* $x \in E$ *has order* $\aleph_0$ *over* U, *and is also decomposable and bounded* (10.7). We shall deduce the following theorem:

THEOREM. *There exists a sub-algebra P of E such that*

(i) $E = \mathfrak{B}(P \cup U)$,

(ii) $(P, \sim)$ *is naturally isomorphic to a numerical measure algebra* (P, μ), *in such a way that, for each* $p \in P$, $[p] = \mu(p)[e]$,

(iii) (P, μ) *is isometric to* I^1 (cf. 3.1(a)).

14.2. Since the order of e over U is now $\aleph_0$, there exists a countably infinite set $B = \{b_n\}$ ($n = 1, 2, \cdots$) such that $E = \mathfrak{B}(B \cup U)$. The first step in the proof consists in replacing the elements b_n by others with improved properties; and this is done in two stages.

14.3. *The elements* $c(\tau)$. Let τ denote any fraction of the form $i/4^n$ ($n \geqq 0$, $0 \leqq i \leqq 4^n$). We define $c(\tau)$ by induction, as follows: When $n = 0$, we set $c(0) = o$ and $c(1) = e$. When $c(i/4^n)$ has been defined (for a particular $n \geqq 0$ and for all i such that $0 \leqq i \leqq 4^n$), we choose elements $f(i/4^n)$, $g(i/4^n)$, such that $f(i/4^n) \leqq b_{n+1}\{c(i/4^n) - c((i-1)/4^n)\} = h(i/4^n)$, say, $g(i/4^n) \leqq (-b_{n+1})\{c(i/4^n) - c((i-1)/4^n)\} = k(i/4^n)$, and $[f(i/4^n)] = (1/2)[h(i/4^n)]$, $[g(i/4^n)] = (1/2)[k(i/4^n)]$, using 12.3, and then define (for $1 \leqq i \leqq 4^n$)

$$c((4i-4)/4^{n+1}) = c((i-1)/4^n),$$

$$c((4i-3)/4^{n+1}) = c((i-1)/4^n) \vee f(i/4^n),$$

$$c((4i-2)/4^{n+1}) = c((i-1)/4^n) \vee h(i/4^n),$$

and

$$c((4i-1)/4^{n+1}) = c((4i-2)/4^{n+1}) \vee g(i/4^n).$$

It readily follows by induction that

(1) If $\tau_1 \leqq \tau_2$ then $c(\tau_1) \leqq c(\tau_2)$, and that

(2) $[c(i/4^n) - c((i-1)/4^n] \leqq (1/2^n)[e]$.

Again, we have

$$\bigvee_{i=1}^{4^n} \{c((4i-2)/4^{n+1}) - c((4i-4)/4^{n+1})\} = \bigvee h(i/4^n)$$
$$= b_{n+1} \bigvee \{c(i/4^n) - c((i-1)/4^n)\}$$
$$= b_{n+1};$$

hence, on writing C = set of all elements $c(\tau)$, we have $B \subset \mathcal{B}(C)$, and therefore

$$(3) \qquad E = \mathcal{B}(C \cup U).$$

14.4. *The elements d_ρ.* Now let ρ be any rational number such that $0 \leqq \rho \leqq 1$. Choose any y_ρ such that $[y_\rho] = \rho[e]$; an application of 7.6 to the elements $c(\tau)$, y_ρ, then gives an invariant element $u_{\tau\rho}$ such that (on using 12.13)

(4) $[c(\tau)u] \leqq \rho[u]$ whenever $u \in U(u_{\tau\rho})$, and $[c(\tau)v] > \rho[v]$ whenever $v \in U(-u_{\tau\rho})$, unless $v = o$.

It readily follows (by the same argument as in 13.1) that

(5) $u_{\tau_1\rho} \geqq u_{\tau_2\rho}$ if $\tau_1 \leqq \tau_2$; $u_{0\rho} = e$; $u_{1\rho} = o$ unless $\rho = 1$; and $u_{11} = e$.

Similarly

(6) $u_{\tau\rho_1} \leqq u_{\tau\rho_2}$ if $\rho_1 \leqq \rho_2$; $c(\tau)u_{\tau 0} = o$; and $u_{\tau 1} = e$.

Now define, for each n,

$$d_{\rho n} = \bigvee\{c(\tau)u_{\tau\rho} \mid \tau = i/4^n,\ 0 \leqq i \leqq 4^n\},$$

and define

$$d_\rho = \bigvee\{c(\tau)u_{\tau\rho},\ \text{all } \tau\}.$$

It readily follows that $d_{\rho 1} \leqq d_{\rho 2} \leqq \cdots$, and $\bigvee_n d_{\rho n} = d_\rho$. Further, from (6) we have

$$(7) \qquad d_{\rho_1} \leqq d_{\rho_2} \quad \text{if} \quad \rho_1 \leqq \rho_2; \quad \text{and} \quad d_0 = o,\ d_1 = e.$$

We next show that

$$(8) \qquad [d_\rho] = \rho[e].$$

Since $c((i+1)/4^n) \geqq c(i/4^n)$, we have (on writing $u_{\tau\rho} = o$ if $\tau > 1$, so that (5) still holds)

$$d_{\rho n} = \bigvee_{i=0}^{4^n} c(i/4^n)v_{\rho n i}, \quad \text{where} \quad v_{\rho n i} = u_{(i/4^n)\rho} - u_{((i+1)/4^n)\rho}.$$

From (4) and 12.5(5) it follows that $[d_{\rho n}] \leqq \rho[\bigvee_i v_{\rho n i}] = \rho[e]$; and therefore (7.3) we have $[d_\rho] \leqq \rho[e]$.

To obtain the opposite inequality, we note that (from (7)) we may assume $\rho<1$, so that now $u_{1\rho}=o$; thus $v_{\rho n 4^n}=o$, so that

$$d_{\rho n} \leqq \bigvee_{i=0}^{4^n-1} c((i+1)/4^n)v_{\rho n i} = q_{\rho n}, \quad \text{say.}$$

But $q_{\rho n}-d_{\rho n}=\bigvee_{i=0}^{4^n-1}[c((i+1)/4^n)-c(i/4^n)\}v_{\rho n i}$; and from (2) and 12.13 it now follows that $[q_{\rho n}-d_{\rho n}]\leqq(1/2^n)[\bigvee_i v_{\rho n i}]=(1/2^n)[e]$. A fortiori, therefore, $[q_{\rho n}-d_\rho]\leqq(1/2^n)[e]$. But from (4) we have $[c((i+1)/4^n)v_{\rho n i}]\geqq\rho[v_{\rho n i}]$, whence $[q_{\rho n}]\geqq\rho[e]$. It readily follows from 12.8 that $[d_\rho]\geqq\rho[e]$; and thus $[d_\rho]=\rho[e]$, Q.E.D.

Let D denote the set of all elements d_ρ. We shall now prove that $C\subset\mathcal{B}(D\cup U)$. For convenience, we define $d_\rho=o$ for $\rho<0$, and $d_\rho=e$ for $\rho>1$; thus (7) is still maintained.

By 13.1 there exist, for each $\tau(=j/4^m)$ and each $n>0$, n disjoint invariant elements $w_{\tau n i}$ $(1\leqq i\leqq n)$ such that

$$\bigvee_i w_{\tau n i} = e, \tag{9}$$

and

$$((i-1)/n)[w_{\tau n i}] \leqq [c(\tau)w_{\tau n i}] \leqq (i/n)[w_{\tau n i}] \qquad (1\leqq i\leqq n). \tag{10}$$

From 12.13 and (4) we see that, on writing $\rho=(i-2)/n$ (where it is assumed for the moment that $i\geqq 2$), we have $w_{\tau n i}u_{\tau\rho}=o$, and therefore (from (5)) $w_{\tau n i}u_{\tau'\rho}=o$ if $\tau'\geqq\tau$. Hence

$$d_\rho w_{\tau n i} = \bigvee_{\tau'} c(\tau')w_{\tau n i}u_{\tau'\rho} \leqq \bigvee_{\tau'\leqq\tau} c(\tau') \leqq c(\tau),$$

so that

$$d_\rho w_{\tau n i} \leqq c(\tau)w_{\tau n i}, \quad \text{where} \quad \rho=(i-2)/n, \tag{11}$$

a result which holds for *all* $i\geqq 0$ since $d_\rho=o$ if $\rho<0$. It follows that

$$c(\tau) \geqq \bigvee_{i,n} d_\rho w_{\tau n i} \qquad (\rho=(i-2)/n,\ 1\leqq i\leqq n). \tag{12}$$

To obtain the reverse inequality, let $c'(\tau)$ denote the element on the right of (12). Since $[c(\tau)w_{\tau n i}]\leqq(i/n)[w_{\tau n i}]$ (from (10)), while if $i\geqq 2$ $[d_\rho w_{t n i}]=((i-2)/n)[w_{\tau n i}]$ (from (8) and 12.13), we readily obtain $[c(\tau)w_{\tau n i}-d_\rho w_{\tau n i}]\leqq(2/n)[w_{\tau n i}]$, a result which also holds (trivially) for $i=0, 1$. Hence, on summing over i, we obtain (using (9) and 12.17)

$$(2/n)[e] \geqq [c(\tau)-\bigvee_i d_\rho w_{\tau n i}] \geqq [c(\tau)-c'(\tau)];$$

and from 12.6 it follows that $c(\tau)-c'(\tau)=o$. Thus $c(\tau)=c'(\tau)\in\mathcal{B}(D\cup U)$; and (3) now gives

(13) $$E = \mathcal{B}(D \cup U) = \mathcal{B}(P \cup U), \quad \text{where} \quad P = \mathcal{B}(D).$$

14.5. *The finitely additive algebra generated by D.* Consider the set F of all elements x expressible as

$$x = (d_{\rho_1} - d_{\rho_2}) \vee (d_{\rho_3} - d_{\rho_4}) \vee \cdots \vee (d_{\rho_{2n-1}} - d_{\rho_{2n}}),$$

where $1 \geqq \rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_{2n} \geqq 0$, and the numbers ρ_i are rational. It is easy to see that, if x and $y \in F$, then so do $x \vee y$ and $-x$; thus F is a finitely additive sub-algebra of E. Clearly $\mathcal{B}(D) \supset F \supset D$ (for $d_0 = o$—(7)).

From (8), 12.18 and 12.16, we obtain

(14) If $x \in F$, $[x] = \rho[e]$, where $\rho = \sum_1^n (\rho_{2i-1} - \rho_{2i})$ (the numbers ρ_i arising from any expression of x as above, so that ρ is rational and $0 \leqq \rho \leqq 1$). Note that (from 12.14) the number ρ in (14) is determined *uniquely* by x—unless $e = o$, which we shall assume not to be the case.

14.6. *The σ-algebra P′.* Now consider the set P' of elements y which have the following property: For each ϵ (>0 and <1) there exists $z \in E$ such that $y +_2 z \in F$ and $[z] < \epsilon[e]$.

It is easy to verify that, if y and y_i $(1 \leqq i \leqq n)$ belong to P', then so do $-y$ and $\vee y_i$; thus P' is at least a finitely additive sub-algebra of E. Now, we have

(15) If $y \in P'$, there exists σ $(0 \leqq \sigma \leqq 1)$ such that $[y] = \sigma[e]$.

For there exist elements z_n such that $y +_2 z_n \in F$ and $[z_n] \leqq \epsilon_n[e]$, where $\epsilon_n = 1/(n+1)$. Thus, writing $x_n = y +_2 z_n$, we have (14) $[x_n] = \rho_n[e]$, where ρ_n is rational. Further, since $x_m x_n \in F$, we have $[x_m x_n] = \rho_{mn}[e]$, say. Thus $[x_n - x_m x_n] = (\rho_n - \rho_{mn})[e]$, $[x_m - x_m x_n] = (\rho_m - \rho_{mn})[e]$, and so $[x_m +_2 x_n] = (\rho_m + \rho_n - 2\rho_{mn})[e]$. But $[x_m +_2 x_n] = [z_m +_2 z_n] \leqq [z_m \vee z_n] \leqq (\epsilon_m + \epsilon_n)[e]$, and so $(\rho_n - \rho_{mn}) + (\rho_m - \rho_{mn}) \leqq \epsilon_m + \epsilon_n$ (12.5(2)). Thus, since each term on the left is non-negative, we have $\rho_{mn} \leqq \rho_n \leqq \rho_{mn} + \epsilon_m + \epsilon_n$ and $\rho_{mn} \leqq \rho_m \leqq \rho_{mn} + \epsilon_m + \epsilon_n$, whence $|\rho_m - \rho_n| \leqq \epsilon_m + \epsilon_n$. The sequence $\{\rho_n\}$ thus converges to a limit σ $(0 \leqq \sigma \leqq 1)$ as $n \to \infty$; and the definition of $\sigma[e]$ (12.11) shows at once that $[y] = \sigma[e]$.

To prove that P' is a σ-subalgebra of E, we have only to show that if $y_n \in P'$ and $y_1 \leqq y_2 \leqq \cdots$, then $\vee y_n = y$, say, $\in P'$. Now from (15) we have $[y_n] = \sigma_n[e]$ and so (from 12.19) $y = \sigma[e]$ where $\sigma = \lim \sigma_n$. Thus, given ϵ, there exists n such that $[y - y_n] = (\sigma - \sigma_n)[e] \leqq (\epsilon/2)[e]$. Since $y_n \in P'$, there exists z such that $[z] \leqq (\epsilon/2)[e]$ and $y_n +_2 z \in F$. Let $z' = y +_2 y_n +_2 z$; then, since $y +_2 z' \in F$ and $[z'] \leqq \epsilon[e]$, we have $y \in P'$.

14.7. Since $P' \supset F$, we have $P' \supset \mathcal{B}(D) = P$ (in fact $P' = P$, but this is not needed); and (15) then shows that to each $y \in P$ corresponds a unique real number $\mu(y)$, such that $0 \leqq \mu(y) \leqq 1$ and $[y] = \mu(y)[e]$. Properties 12.16, 12.14, show that μ is a countably additive numerical measure on P, vanishing only for o, and such that $\mu(y_1) = \mu(y_2)$ (where y_1, $y_2 \in P$) if and only if $y_1 \sim y_2$. That is, the sub-algebra $(P, \sim)$ of E is naturally isomorphic to the numerical measure algebra (P, μ). We have already shown (13) that $E = \mathcal{B}(P \cup U)$.

Finally, it is not hard to see that (P, μ) is isometric with I^1, the algebra of Lebesgue-measurable sets modulo null sets in the unit interval; one makes the interval $(0, \rho)$ correspond to the element d_ρ, and extends this correspondence in succession to F and P. (We omit the details.) Thus the theorem stated in 14.1 is now proved.

15. **Structure of $E(e_n)$; inseparable case.**

15.1. Now let e_n be one of the elements in 13.4 which has uncountable order over U; we analyze the structure of the relative algebra $E(e_n)$. As in the previous section, all considerations will be relative to e_n; thus we write $E(e_n)$ as E, e_nU as U, and so on, and can now assume that *every nonzero element of E has order $\mathfrak{m} > \aleph_0$ over U, and is also decomposable and bounded.* The fundamental theorem is the same as in the separable case (14.1), except that the measure-algebra is different.

THEOREM. *There exists a sub-algebra P of E such that*

(i) $E = \mathfrak{B}(P \cup U)$,

(ii) $(P, \sim)$ *is naturally isomorphic to a numerical measure algebra* (P, μ), *in such a way that, for each* $p \in P$, $[p] = \mu(p)[e]$,

(iii) (P, μ) *is isometric to* $I^{\mathfrak{m}}$ (cf. 3.1(a)).

15.2. We shall require the following lemma.

PRINCIPAL LEMMA. *Let S be a subalgebra of E such that* (a) $S \supset U$, (b) *every nonzero element of E is of infinite order over S. Then, given any $x \in E - S$, there exists a sub-algebra Q of E such that* (i) $x \in \mathfrak{B}(Q \cup S)$, (ii) $(Q, \sim)$ *is naturally isomorphic to a numerical measure algebra* (Q, μ), *in such a way that, for each* $q \in Q$ *and* $s \in S$, $[qs] = \mu(q)[s]$, *and* (iii) (Q, μ) *is isometric to* I^1.

This lemma will be proved in §16. Taking it for granted, we now deduce the theorem.

15.3. There exists a set $A \subset E$, with $|A| = \mathfrak{m}$, such that $E = \mathfrak{B}(A \cup U)$. Well-order the elements of A as a_α, $1 \leqq \alpha < \Omega$, where Ω is the first ordinal of power $\mathfrak{m}$. From the lemma (applied with $S = U$ and $x = a_1$), there exists a sub-algebra Q_1 such that $a_1 \in \mathfrak{B}(Q_1 \cup U)$ and for each $q \in Q_1$ and $u \in U$ we have $[qu] = \mu_1(q)[u]$; further (Q_1, μ_1) is isometric to I^1. Now suppose that sub-algebras Q_β have been defined for all $\beta < \alpha$ (where $\alpha < \Omega$), in such a way that:

(i) $a_\beta \in \mathfrak{B}(\bigcup_{\gamma \leqq \beta} Q_\gamma \cup U)$;

(ii) $(Q_\beta, \sim)$ is naturally isomorphic to a numerical measure algebra (Q_β, μ_β) in such a way that, on defining $S_\beta = \mathfrak{B}(\bigcup_{\gamma < \beta} Q_\gamma \cup U)$, we have $[bc] = \mu_\beta(b)[c]$ for all $b \in Q_\beta$ and $c \in S_\beta$; and

(iii) Q_β is isometric either to I^1 or to the trivial measure algebra L consisting of o and e only (these elements having measures 0 and 1 respectively).

Define Q_α as follows: *If* $a_\alpha \in S_\alpha$, take Q_α to consist of the elements o and e only, with $\mu_\alpha(o) = 0$ and $\mu_\alpha(e) = 1$. The properties of the transfinite sequence are clearly maintained.

If $a_\alpha \notin S_\alpha$, we first note that every nonzero element $x \in E$ is of infinite order (and in fact of order $\mathfrak{m}$) over S_α; in fact, if $E(x) = x\mathfrak{B}(X \cup S_\alpha)$, there exists an at most countable subset A_β of Q_β such that $Q_\beta = \mathfrak{B}(A_\beta)$ (from the fact that Q_β is isomorphic to I^1 or L), and we have $E(x) = x\mathfrak{B}(\bigcup_{\beta<\alpha} A_\beta \cup U \cup X)$ so that $|\bigcup_{\beta<\alpha} A_\beta \cup X| \geqq \mathfrak{m}$ and therefore $|X| \geqq \mathfrak{m}$. Hence the lemma (15.2) can be applied, taking $S = S_\alpha$ and $x = a_\alpha$; the sub-algebra given by the lemma is taken to be Q_α.

Thus Q_α is defined for all $\alpha < \Omega$; and since $a_\alpha \in S_{\alpha+1}$ we readily obtain

(1) $E = \mathfrak{B}(S_\Omega \cup U)$, where $S_\Omega = \mathfrak{B}(\bigcup Q_\alpha)$.

It follows (by the same order argument as before) that Q_α is isometric to I^1 for $\mathfrak{m}$ values of α.

Suppose $x \in Q_\alpha \cap Q_\beta$ $(\alpha \neq \beta)$. Since $[x] = \mu_\alpha(x)[e]$ and also $[x] = \mu_\beta(x)[e]$, we have (12.14) $\mu_\alpha(x) = \mu_\beta(x)$; hence we may omit the suffix α, and write $\mu_\alpha(x) = \mu(x)$. Again, we may suppose $\beta < \alpha$; then $x \in S_\alpha$, so that (15.3(ii)) $[x] = \mu(x)[x]$. By 12.14, either $x = o$ or $\mu(x) = 1$; in the latter case $[x] = [e]$, and so (e being bounded) $x = e$. Thus:

(2) If $\alpha \neq \beta$, $Q_\alpha \cap Q_\beta = \{o, e\}$.

15.4. The desired sub-algebra P of E will actually be S_Ω; but it is more convenient to define it in another way. Let

D = set of all elements d expressible as $d = b_1 b_2 \cdots b_k$, where $b_i \in Q_{\alpha_i}$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$;

F = set of all elements f expressible as $f = d_1 \vee d_2 \vee \cdots \vee d_n$, where $d_i \in D$ and the elements d_i are disjoint;

$G = \{g \mid g = \bigvee d_i,\ d_i \in D\}$ $(i = 1, 2, \cdots)$;

$P = \{p \mid p = \bigwedge g_n,\ g_n \in G,\ g_1 \geqq g_2 \geqq \cdots\}$.

Clearly $D \subset F \subset G \subset P$. The first step consists in proving:

(3) If $x \in P$, $x = \mu(x)[e]$ for some real number $\mu(x)$ $(0 \leqq \mu(x) \leqq 1)$. ($\mu(x)$ is then unique—from 12.14, Corollary.)

For suppose first $x = d \in D$, say $x = b_1 b_2 \cdots b_k$ as above. Then, since $b_2 \cdots b_k \in S_{\alpha_1}$, $[d] = \mu(b_1)[b_2 \cdots b_k]$ (from 15.3(ii)) $= \mu(b_1)\mu(b_2)[b_3 \cdots b_k]$ $= \mu(b_1)\mu(b_2) \cdots \mu(b_k)[e]$, similarly; thus (3) holds in this case, with $\mu(d) = \prod_1^k \mu(b_i)$.

Next, suppose $x = f \in F$. Then $f = \bigvee_1^n d_i$, where the elements d_i are disjoint and belong to D; by the preceding, $[d_i] = \mu(d_i)[e]$; and then (12.16) $[f] = \mu(f)[e]$ where $\mu(f) = \sum_1^n \mu(d_i)$.

Before considering G, we note that *F is a finitely additive algebra*. In fact, it is clear that if $d_1, d_2 \in D$, then $d_1 d_2 \in D$; and it readily follows that if f, $f' \in F$, then $ff' \in F$. Again, given $d \in D$, we see that $-d \in F$ as follows: We have $d = b_1 b_2 \cdots b_k$, say, where $b_i \in Q_{\alpha_i}$; consider the 2^k elements d_i $(1 \leqq i \leqq 2^k)$ of the form $c_1 c_2 \cdots c_k$ where each c_i is either b_i or $-b_i$. They are disjoint, and we may suppose $d = d_1$, say; then $-d = \bigvee d_i$ $(2 \leqq i \leqq 2^k)$, showing that $-d \in F$ if $d \in D$. From the fact that F is closed under (finite) infima, it now follows that $-f \in F$ if $f \in F$; for we have $f = \vee d_i$ $(d_i \in D)$, and consequently $-f = \wedge(-d_i) \in F$.

It follows that each $g \in G$ is expressible as $g = \vee f_n$, where $f_n \in F$ and the elements f_n are disjoint. (For if $g = \vee d_i$, $d_i \in D$, we take $f_n = \vee_1^n d_i - \vee_1^{n-1} d_i$.) From the preceding, we have $[f_n] = \mu(f_n)[e]$, and therefore (12.16) $[g] = \mu(g)[e]$ where $\mu(g) = \sum_1^\infty \mu(f_n)$.

Finally, given $p \in P$, we have $p = \wedge g_n$, where $g_1 \geqq g_2 \geqq \cdots$, and, as just shown, $[g_n] = \mu(g_n)[e]$. By 12.19, $[p] = \mu(p)[e]$ where $\mu(p) = \lim_{n\to\infty} \mu(g_n)$.

Next,

(4) If $p_n \in P$ $(n = 1, 2, \cdots)$, then $\vee p_n \in P$.

For, from the definition of P, we have $p_n = \wedge_m g_{nm}$, where $g_{n1} \geqq g_{n2} \geqq \cdots$ and $g_{nm} \in G$; and it has just been shown that $\mu(p_n) = \lim_{m\to\infty} \mu(g_{nm})$. Hence for each k we can choose $m = m(k, n)$ so that (i) $\mu(g_{nm}) \leqq \mu(p_n) + 1/2^n k$, (ii) for each k and n, m is the least such number. Write $h_k = \vee_n g_{n,m(k,n)}$; then $h_k \geqq \vee p_n$, $h_k \in G$ (from the definition of G), and (from (ii)) $h_1 \geqq h_2 \geqq \cdots$. Further, 12.7 gives that $[h_k - \vee p_n] \leqq (1/k)[e]$, and so (12.6) $\vee p_n = \wedge h_k$. Thus $\vee p_n \in P$.

Again,

(5) If $p \in P$, then $-p \in P$.

For we have $p = \wedge g_n$, where $g_1 \geqq g_2 \geqq \cdots$ and $g_n \in G$. Thus $-p = \vee(-g_n)$, and by (4) it will suffice to prove that $-g_n \in P$, for each n. But $g_n = \vee_i d_{ni}$, say, where $d_{ni} \in D$. Hence $-g_n = \wedge_i(-d_{ni}) = \wedge_m x_{nm}$, where $x_{nm} = \wedge(-d_i)$ $(1 \leqq i \leqq m)$. Thus $x_{n1} \geqq x_{n2} \geqq \cdots$, and $x_{ni} \in F$ (for F is finitely additive) $\subset G$, so that $-g_n \in P$ by definition.

(4) and (5) show that P is a (σ-) sub-algebra of E; and (3), together with the properties in §12, shows that μ is a countably additive measure on P, (P, μ) being naturally isomorphic to $(P, \sim)$. Since $P \supset D \supset \cup Q_\alpha$ (1) gives $E = \mathfrak{B}(P \cup U)$. Finally, the fact that (P, μ) is isometric to I^m is an immediate consequence of the way in which the values of μ were derived successively on $\cup Q_\alpha$, D, F, G and P; this shows (virtually by definition) that (P, μ) is (to within isometry) the direct product of *all* the measure algebras Q_α, and those Q_α's for which $Q_\alpha = L$ can obviously be disregarded.

16. **Inseparable case continued; proof of the principal lemma.**

16.1. Let S be a sub-algebra of E $(=E(e_n)$, as in §15) such that (a) $S \supset U$, (b) every nonzero element of E is of infinite order over S. The first part of the proof of 15.2 (16.2–16.13) consists essentially in showing that S has properties similar to those of U. Once this is done, the desired numerical measure algebra is constructed in much the same way as in the "separable case" (§14).

16.2. *Notation.* For any $X \subset E$, write $X' = X - (o)$. Thus, if $z \in S'$, $S'(z)$ denotes $\{y \mid y \in S \text{ and } o < y \leqq z\}$.

Throughout 16.2–16.10 we suppose a fixed element $b \in E$ to be given, and we define, for each $z \in S'$,

$$\sigma(z) = \inf \{\eta \mid 0 \leqq \eta \leqq 1 \text{ and } \eta[z] \geqq [bz]\},$$
$$\tau(z) = \sup \{\eta \mid 0 \leqq \eta \leqq 1 \text{ and } \tau[z] \leqq [bz]\}.$$

Thus (from 12.14 and 12.19) we have

$$\sigma(z)[z] \geqq [bz] \geqq \tau(z)[z]$$

and

$$0 \leqq \tau(z) \leqq \sigma(z) \leqq 1.$$

In the next few paragraphs, we derive some properties of the function σ; similar results and arguments will hold throughout for τ. (The immediate goal is 16.7.)

16.3. If $z_n \in S'$ $(n=1, 2, \cdots)$, and the elements z_n are disjoint, then $\sigma(\vee z_n) \leqq \sup \sigma(z_n)$.

For otherwise $\sigma(\vee z_n) > \eta \geqq \sigma(z_n)$ for all n; and then $\eta[z_n] \geqq [bz_n]$, whence $\eta[\vee z_n] \geqq [b(\vee z_n)]$ and therefore $\eta \geqq \sigma(\vee z_n)$—a contradiction.

16.4. If $z \in S'$ and $\sigma(z) > \eta \geqq 0$, there exists $z_0 \in S'(z)$ such that, for all $y \in S'(z_0)$, $\sigma(y) \geqq \eta$.

For if this is false, transfinite induction gives a countable set of disjoint elements z_α such that $\vee z_\alpha = z$ and $\sigma(z_\alpha) < \eta$. From 16.3, $\sigma(z) \leqq \eta$—a contradiction.

16.5. Given $z_0 \in S'$ and $\epsilon > 0$, there exists $y \in S'(z_0)$ such that, whenever $y_1, y_2 \in S'(y)$, we have

$$|\sigma(y_1) - \sigma(y_2)| < \epsilon \quad \text{and} \quad |\tau(y_1) - \tau(y_2)| < \epsilon.$$

Write $\eta = \sup \{\sigma(z) | z \in S'(z_0)\}$, and choose $z_1 \in S'(z_0)$ such that $\sigma(z_1) > \eta - \epsilon$. We can suppose $\eta \neq 0$ (else we merely take $y = z_0$), and thus that $\eta - \epsilon > 0$. On applying 16.4 to z_1, we obtain (say) $y^1 \in S'(z_1) \subset S'(z_0)$ having the desired property as far as the function σ is concerned. An "exhaustion" argument then gives $z_0 = \vee y^n$, where $y^n \in S'(z_0)$ and $|\sigma(y_1) - \sigma(y_2)| < \epsilon$ whenever $y_1, y_2 \in S'(y^n)$. An entirely similar argument applied to the function τ will give $z_0 = \vee t^n$, where $t^n \in S'(z_0)$ and $|\tau(y_1) - \tau(y_2)| < \epsilon$ whenever $y_1, y_2 \in S'(t^n)$. Since $z_0 = \vee_{m,n} y^m t^n$, at least one of the elements $y^m t^n$ is nonzero, and gives the desired element y.

16.6. *Given $z_0 \in S'$ and $\epsilon > 0$, there exists $z \in S'(z_0)$ such that $|\sigma(y_1) - \tau(y_2)| < \epsilon$ whenever $y_1, y_2 \in S'(z)$.*

By 16.5, there exists $y \in S'(z_0)$ such that, whenever $y_1, y_2 \in S'(y)$, $|\sigma(y_1) - \sigma(y_2)| < \epsilon/3$ and $|\tau(y_1) - \tau(y_2)| < \epsilon/3$. We can suppose $yb \neq o$ (for otherwise we merely take $z = y$). Now (13.1) there exist, for each positive integer n, n disjoint invariant elements u_{ni} $(1 \leqq i \leqq n) \in U$ such that $\vee_i u_{ni} = e$ and

$$(1) \qquad ((i-1)/n)[u_{ni}] \leqq [ybu_{ni}] \leqq (i/n)[u_{ni}];$$

and similarly there exist n disjoint elements $v_{nj} \in U$ $(1 \leqq j \leqq n)$ such that

$\mathsf{V}_j v_{nj} = e$ and

(2) $$((j - 1)/n)[v_{nj}] \leqq [yv_{nj}] \leqq (j/n)[v_{nj}].$$

There will exist integers n_0 and i_0, with $n_0 \geqq i_0 \geqq 2$, such that $u_{n_0 i_0} \neq o$; for otherwise $[yb] \leqq (1/n)[e]$ for all n, and thus (12.6) $yb = o$, contrary to our supposition. Choose a positive integer p so large that $p(i_0 - 1) > 2 + 6/\epsilon$.

Write $N = n_0 p$, and consider the elements u_{Ni} $(1 \leqq i \leqq N)$. At least one of them, say u_{Nk}, meets $u_{n_0 i_0}$ (for $\mathsf{V}_i u_{Ni} = e$); and from (1), 12.13 and 12.14 we have $k/N \geqq (i_0 - 1)/n_0$, and so $k > 2 + 6/\epsilon$. Similarly u_{Nk} meets some v_{Nj}; write $u_{Nk} v_{Nj} = u$, and $uy = z$. Thus $z \in S$ and $z \leqq y \leqq z_0$. From (1), (2) and 12.13 we have

$$((k - 1)/N)[u] \leqq [zb] \leqq (k/N)[u]$$

and

$$((j - 1)/N)[u] \leqq [z] \leqq (j/N)[u].$$

Since $u \neq o$, it follows (12.14, Corollary) that $j \geqq k - 1 > 1 + 6/\epsilon$. Thus $z \neq o$. Further, on using 12.5(6) we obtain $((k-1)/j)[z] \leqq ((k-1)/N)[u] \leqq [zb]$, so that $(k-1)/j \leqq \tau(z)$. Again, $\sigma(z) \leqq k/(j-1)$; for this is trivial if $k > j - 1$, and otherwise we have $[zb] \leqq (k/(j-1))((j-1)/N)[u] \leqq (k/(j-1))[z]$. Thus $0 \leqq \sigma(z) - \tau(z) \leqq k/(j-1) - (k-1)/j < \epsilon/3$ (from the choice of p). Finally, if $y_1, y_2 \in S'(z)$, we have $y_1, z \in S'(y)$ and so $|\sigma(y_1) - \sigma(z)| < \epsilon/3$; similarly $|\tau(y_2) - \tau(z)| < \epsilon/3$, and the result follows.

16.7. Given $\epsilon > 0$, there exists a disjoint sequence $\{z_n\}$ such that $z_n \in S'$, $\mathsf{V} z_n = e$, and $|\sigma(y_1) - \tau(y_2)| < \epsilon$ whenever $y_1, y_2 \in S'(z_n)$.

This follows from the preceding by "exhaustion."

16.8. Given a real number η such that $0 < \eta < 1$, there exists $z \in S$ such that $\eta[y] \leqq [by]$ for all $y \in S(z)$, and $\eta[t] \geqq [bt]$ for all $t \in S(-z)$.

On applying the preceding result, with $\epsilon = 1/n$, we obtain elements $z_{ni} \in S'$ such that $z_{ni} z_{nj} = o$ if $i \neq j$, $\mathsf{V}_i z_{ni} = e$, and $|\sigma(y_1) - \tau(y_2)| \leqq 1/n$ whenever $y_1, y_2 \in S'(z_{ni})$. We can further suppose that each $z_{(n+1)i}$ is less than or equal to some z_{nj}, since otherwise we replace (for each n) the elements z_{ni} by all the nonzero elements of the form $z_{1i_1} z_{2i_2} \cdots z_{ni_n}$. Now let I_n denote the set of values of i (possibly empty) for which it is true that $\sigma(y) \leqq \eta$ whenever $y \in S(z_{ni})$, and define

$$s_o = o, \qquad s_n = \mathsf{V}\{z_{ni} \mid i \in I_n\}.$$

Let $z = -\mathsf{V} s_n$. Thus $s_n \in S$, $z \in S$, and $s_1 \leqq s_2 \leqq \cdots$. Hence if $t \in S(-z)$, we have $t = \mathsf{V} t_n$ where $t_n = t(s_n - s_{n-1})$ $(n = 1, 2, \cdots)$; thus $\sigma(t_n) \leqq \eta$, and therefore (16.3) $\sigma(t) \leqq \eta$, so that $\eta[t] \geqq [bt]$.

On the other hand, if $y \in S(z)$ then $ys_n = o$ for each n, and therefore $y = \mathsf{V}\{yz_{ni} \mid i \notin I_n\}$. For each z_{ni} occurring here, there exists (say) $r_{ni} \in S(z_{ni})$ such that $\sigma(r_{ni}) > \eta$. Hence $\tau(yz_{ni}) > \eta - 1/n$, and therefore (from the property of τ analogous to 16.3) $\tau(y) \geqq \eta - 1/n$, so that $\tau(y) \geqq \eta$ and $\eta[y] \leqq [by]$.

16.9. There exists a greatest z having the properties stated in 16.8.

Let z_0 be the supremum of all z's for which 16.8 holds. Then, from the countable chain condition, we have $z_0 = \bigvee z_n$ ($n = 1, 2, \cdots$), where 16.8 holds for each z_n. If $y \in S(z_0)$, we have $y = \bigvee y_n$ where $y_1 = yz_1$ and $y_n = y(z_n - (z_1 \vee \cdots \vee z_{n-1}))$. Thus $\eta[y_n] \leqq [by_n]$, and it readily follows (cf. 12.17) that $\eta[y] \leqq [by]$. If $t \in S(-z_0)$, then $t \in S(-z_1)$, and so $\eta[t] \geqq [bt]$. Thus 16.8 holds for z_0.

16.10. *For each positive integer n, there exist n disjoint elements $s_1, s_2, \cdots, s_n \in S$ such that* (i) $\bigvee s_i = e$, (ii) *whenever* $z \in S(s_i)$, $((i-1)/n)[z] \leqq [zb] \leqq (i/n)[z]$ $(1 \leqq i \leqq n)$.

(Compare 13.1.)

Using 16.8 and 16.9, define z_i to be the greatest element of S such that $(i/n)[y] \leqq [by]$ whenever $y \in S(z_i)$, and $(i/n)[t] \geqq [bt]$ whenever $t \in S(-z_i)$. Thus if $i < j$ we have $(i/n)[z_j - z_i] \geqq [b(z_j - z_i)] \geqq (j/n)[z_j - z_i]$, so that $z_i - z_j = o$; that is, $e \geqq z_1 \geqq z_2 \geqq \cdots \geqq z_n$. We have only to define $s_1 = -z_1$, $s_i = z_{i-1} - z_i$ $(1 < i < n)$, and $s_n = z_{n-1}$.

(The maximality of z_i has not been used, but will be convenient in the next paragraph.)

16.11. *Given $c, d \in E$, there exists $z_0 \in S$ such that $[cy] \leqq [dy]$ whenever $y \in S(z_0)$, and $[ct] \geqq [dt]$ whenever $t \in S(-z_0)$.*

(Compare 7.6.)

Write $N = 2^n$; then, for each n ($= 1, 2, \cdots$) the preceding (applied with $b = c$) gives N disjoint elements s_{ni} $(1 \leqq i \leqq N) \in S$ such that $e = \bigvee_i s_{ni}$ and $((i-1)/N)[z] \leqq [zc] \leqq (i/N)[z]$ for all $z \in S(s_{ni})$. Similarly there exist 2^n disjoint elements t_{nj} $(1 \leqq j \leqq N) \in S$ such that $e = \bigvee_j t_{nj}$ and $((j-1)/N)[z] \leqq [zd] \leqq (j/N)[z]$ for all $z \in S(t_{nj})$. The method of construction used in 16.10 further shows that $s_{(n+1),(2i-1)} \vee s_{(n+1),2i} = s_{ni}$, with a similar result for the t's. Thus, on writing $f_n = \bigvee \{s_{ni}t_{nj} \mid j \geqq i\}$, we have $f_1 \geqq f_2 \geqq \cdots$. Now define $z_0 = \bigwedge f_n$. It is not hard to verify that z_0 has the desired properties.

16.12. *Given $b \in E'$, there exists $y \in E'(b)$ such that, for each $z \in S$, $[yz] \leqq [bz]/2$.*

(Here we use for the first time the assumption that every element of E' has infinite order over S.)

Since b is not of order 0 over S, there exists $h \leqq b$ such that $h \notin bS$.

Applying 16.11 to $c = h$, $d = b - h$, we obtain $z_0 \in S$ such that $[sh] \leqq [s(b-h)]$ if $s \in S(z_0)$, and $[th] \geqq [t(b-h)]$ if $t \in S(-z_0)$. Thus $[sh] \leqq (1/2)[sb]$ and $[t(b-h)] \leqq (1/2)[tb]$. Define $y = z_0 h \vee (-z_0)(b-h)$. Thus clearly $y \leqq b$; and also $y \neq o$, since otherwise we obtain $b - z_0 = h - z_0$ and $hz_0 = o$, whence $h = b - z_0 \in bS$, a contradiction. Finally, given $z \in S$, we can write $yz = sh \vee t(b-h)$ where $s = zz_0$ and $t = z - z_0$. Hence $[yz] = (1/2)[sb \vee tb] \leqq (1/2)[bz]$.

By iteration we now obtain:

16.12′. Corollary. *Given $b \in E'$ and $n > 0$, there exists y_n such that $o < y_n$*

$\leqq b$ and, for each $z \in S$, $[y_n z] \leqq (1/2^n)[bz]$.

16.13. Let $b \in E$, $z \in S$, and η $(0<\eta<1)$ be such that $[by] \geqq \eta[y]$ whenever $y \in S(z)$. Then there exists $b' \leqq zb$ such that $[b'y] = \eta[y]$ whenever $y \in S(z)$.

We can clearly suppose $z \neq o$; thus $bz \neq o$, and so (16.12′) there exists $c \in E'(bz)$ such that, for each $y \in S$, $[cy] \leqq \eta[by]$. Thus c certainly satisfies: (i) $o < c \leqq bz$, (ii) for each $y \in S(z)$, $[cy] \leqq \eta[y]$. An easy "exhaustion" argument now shows that there exists a *maximal* c, say b', satisfying (i) and (ii). Then b' is the desired element. For suppose not; then there exists $y \in S(z)$ such that $[b'y] < \eta[y]$. From 16.10 we obtain, for each n, n disjoint elements $s_{ni} \in S$ $(1 \leqq i \leqq n)$ such that $\bigvee_i s_{ni} = e$ and, for all $s \in S(s_{ni})$,

$$((i-1)/n)[s] \leqq [sb'] \leqq (i/n)[s].$$

Suppose first that, for every n, we have $ys_{ni} = o$ whenever $(i+1)/n < \eta$. It readily follows that $[b'y] \geqq (\eta - 2/n)[y]$ for every n, and so $[b'y] \geqq \eta[y]$, contrary to hypothesis. Thus we may choose n and i so that $(i+1)/n < \eta$ and $ys_{ni} = t$, say, $\neq o$. Then, for every $s \in S(t)$, we have $[sb'] \leqq (i/n)[s] \leqq (\eta - 1/n)[s]$, whence $[s(b-b')] \geqq (1/n)[s]$.

It follows that $t(b-b') \neq o$, so that (16.12′) there exists $d \in E'(t(b-b'))$ such that $[ds] \leqq (1/n)[t(b-b')s] \leqq (1/n)[ts]$ whenever $s \in S$. Define $c' = b' \vee d$; thus $o < c' \leqq b$. Further, given any $y' \in S(z)$, we have $c'y' = b'y't \vee b'y'(-t) \vee dy't$, a supremum of three disjoint elements in which

$$[b'y't] \leqq (\eta - 1/n)[y't],$$

$$[b'y'(-t)] \leqq \eta[y'(-t)] \qquad \text{(from property (ii) of } b'),$$

and

$$[dy't] \leqq (1/n)[y't],$$

from which we obtain $[c'y'] \leqq \eta[y']$, for all $y' \in S(z)$. But this contradicts the maximality of b'.

16.14. *Given $x \in S$, there exists elements $b(\rho) \in E$ $(\rho = i/2^n$, $0 \leqq i \leqq 2^n$, $n = 0, 1, 2, \cdots)$ such that*

(i) *$[b(\rho)z] = \rho[z]$ whenever $z \in S$.*

(ii) *$b(\rho_1) \leqq b(\rho_2)$ if $\rho_1 \leqq \rho_2$; $b(0) = o$, and $b(1) = e$.*

(iii) *$x \in \mathfrak{B}(S \cup B)$, where $B = \bigcup(b(\rho))$.*

For each $\rho = i/2^n$ $(0 \leqq i \leqq 2^n$, $n \geqq 0)$, we define $s(\rho)$ (using 16.9) to be the greatest element of S such that

$$\text{(1)} \quad \begin{aligned} [xz] &\geqq \rho[z] \quad \text{whenever} \quad z \in S(s(\rho)), \quad \text{and} \\ [xz] &\leqq \rho[z] \quad \text{whenever} \quad z \in S(-s(\rho)). \end{aligned}$$

Thus (cf. 16.10) we have

(2) If $\rho_1 \leqq \rho_2$, $e = s(0) \geqq s(\rho_1) \geqq s(\rho_2)$, and $s(1) \leqq x$.

The elements $b(i/2^n)$ are defined by induction over n, starting with

$b(0) \equiv b(0/2^0) = o$, and $b(1) \equiv b(1/2^0) = e$. Suppose that $n \geqq 1$, and that $b(i/2^m)$ has been defined for all $m < n$ and $i \leqq 2^m$ in such a way that properties (i) and (ii) (of the statement of the present proposition) hold, and so that, in addition, we have

$$(3) \qquad b(\rho)s(\rho) \leqq x, \quad \text{and} \quad x - s(\rho) \leqq b(\rho)$$

for all $\rho = i/2^m$, $m < n$. (This will trivially be the case when $n = 1$.) We define $b(2k/2^n) = b(k/2^{n-1})$, so that the definition is consistent. The definition of $b((2k-1)/2^n)$ is more troublesome, and in giving it we suppose (to save suffixes) that k and n are fixed $(1 \leqq k \leqq 2^{n-1})$, and write $\rho^- = (k-1)/2^{n-1}$, $\rho^0 = (2k-1)/2^n$, and $\rho^+ = k/2^{n-1}$. We use s_1, s_2 and s_3 to denote (arbitrary) elements of $S(-s(\rho^0))$, $S(s(\rho^0) - s(\rho^+))$ and $S(s(\rho^+))$, respectively.

First we show

$$(4) \qquad [s_1 x - b(\rho^-)] \leqq (1/2^n)[s_1].$$

For $s_1 = z' \vee z''$ where $z' = s_1 - s(\rho^-)$ and $z'' = s_1\{s(\rho^-) - s(\rho^0)\}$. Now $[z''x] \leqq \rho^0[z'']$ from (1); $[z''xb(\rho^-)] = [z''b(\rho^-)] = \rho^-[z'']$ from (3) and (i); and $z'x - b(\rho^-) = o$ from (3). (4) now follows easily.

A fortiori, $[s_1 x\{b(\rho^+) - b(\rho^-)\}] \leqq (1/2^n)[s_1]$; and since $[s_1\{b(\rho^+) - b(\rho^-)\}] = (1/2^{n-1})[s_1]$ (from (i) and (ii)), we have $[s_1(-x)\{b(\rho^+) - b(\rho^-)\}] \geqq (1/2^n)[s_1]$. Here s_1 is any element of $S(-s(\rho^0))$. Thus, from 16.13, there exists $y_0 \leqq (-x)\{b(\rho^+) - b(\rho^-)\} - s(\rho^0)$ such that $[y_0 s_1] = (1/2^n)[s_1]$ for all s_1 $(\in S(-s(\rho^0)))$. Write $y_1 = \{b(\rho^+) - b(\rho^-)\}(-s(\rho^0)) - y_0$. Thus for all s_1 we have (on using (i) and (ii))

$$(5) \qquad [y_1 s_1] = (1/2^n)[s_1].$$

Similar arguments prove the existence of $y_2 \leqq x\{b(\rho^+) - b(\rho^-)\}\{s(\rho^0) - s(\rho^+)\}$ such that

$$(6) \qquad [y_2 s_2] = (1/2^n)[s_2] \qquad (s_2 \in S\{s(\rho^0) - s(\rho^+)\}),$$

and of $y_3 \leqq \{b(\rho^+) - b(\rho^-)\}s(\rho^+) \leqq x$ such that

$$(7) \qquad [y_3 s_3] = (1/2^n)[s_3] \qquad (s_3 \in S(s(\rho^+))).$$

Define $y = y_1 \vee y_2 \vee y_3$, and $b(\rho^0) = b(\rho^-) \vee y$. Clearly property (ii) is maintained. To establish (i), it will suffice to verify that $[ys] = (1/2^n)[s]$ for all $s \in S$. But we can write $s = s_1 \vee s_2 \vee s_3$; and it is easily verified that $ys_i = y_i s_i$ $(i = 1, 2, 3)$. Since the elements s_i are necessarily disjoint, the desired relation follows from (5), (6) and (7). Property (3) can likewise be verified, and the inductive definition of the elements $b(\rho)$ is complete.

Finally, to prove (iii), we show

$$(8) \qquad x = \bigvee x_n, \quad \text{where} \quad x_n = \bigvee_i b(i/2^n)s(i/2^n).$$

From (3), $x \geqq x_n$ for every n. Now

$$x - x_n \leqq x - x\left\{\bigvee_{i=0}^{2^n-1} b(i/2^n)\{s(i/2^n) - s((i+1)/2^n)\}\right\}$$

$$= \bigvee_0^{2^n-1} \{x - b(i/2^n)\}\{s(i/2^n) - s((i+1)/2^n)\}$$

$$\text{(since } x = \bigvee_i x\{s(i/2^n) - s(i+1)/2^n)\}\text{)}$$

$$= \bigvee t_i, \quad \text{say.}$$

But from (1), $[x\{s(i/2^n)-s((i+1)/2^n)\}] \leqq ((i+1)/2^n)[s(i/2^n)-s((i+1)/2^n)]$; and, from (3) and (i), $[xb(i/2^n)\{s(i/2^n)-s((i+1)/2^n)\}] = (i/2^n)[s(i/2^n) -s((i+1)/2^n)]$. Hence $[t_i] \leqq (1/2^n)[s(i/2^n)-s(i+1)/2^n)]$, and so (12.5(5)) $[x-x_n] \leqq (1/2^n)[e]$. That is, $[x-\bigvee x_n] \leqq (1/2^n)[e]$ for every n; and therefore (12.6) $x=\bigvee x_n$, completing the proof.

16.15. *The "principal lemma"* (15.2). *In* 16.14, *let* $Q=\mathfrak{B}(B)$; *then* $(Q, \sim)$ *is naturally isomorphic to a numerical measure algebra* (Q, μ), *in such a way that, for each* $q \in Q$ *and* $s \in S$, $[qs]=\mu(q)[s]$. *Further*, (Q, μ) *is isometric to* I^1.

This following from 16.14 by essentially the same arguments as in the "separable case" (14.5 and 14.6), the elements $b(i/2^n)$ here playing the role of the elements d_ρ in §14.

17. $E(e_n)$ **as a direct product.**

17.1. As in §§14, 15, we consider the relative algebra $E(e_n)$, e_n being any one of the elements of 13.4. The theorems of 14.1 and 15.1 show that, in every case, $E(=E(e_n))=\mathfrak{B}(P \cup U)$, where P is a (relative) sub-algebra naturally isomorphic with I^1 or I^m, in such a way that (from 12.13) we have the fundamental relation:

$[pu]=\mu(p)[u]$ whenever $p \in P$ and $u \in U$.

We shall now deduce that E *is isomorphic to the direct product* $P \otimes U$ (cf. §4).

17.2. Definitions. Let R, S denote the respective representation spaces of P and U; as in 4.2, we use p^*, u^* to denote the open-closed subsets of R, S respectively which correspond to $p \in P$, $u \in U$. Let Q^* denote the class of all subsets Y of $R \times S$ which are expressible in the form $Y=\bigcup(p_i^* \times u_i^*)$, where the (open-closed) sets u_i^* are disjoint and their union is residual in S (that is, $S-\bigcup u_i^*$ is of first category). Similarly let Q denote the class of all elements $y \in E$ expressible as $y=\bigvee p_i u_i$, where $p_i \in P$, $u_i \in U$, the elements u_i are disjoint, and $\bigvee u_i=e$.

We observe that Q is a finitely additive sub-algebra of E. In fact, if $y_1, y_2 \in Q$ and $y_j=\bigvee_i p_{ij}u_{ij}$ as above, then $-y_1=\bigvee(-p_{i1})u_{i1} \in Q$ and $y_1 \vee y_2 = \bigvee_{i,k} (p_{i1} \vee p_{k2})(u_{i1}u_{k2}) \in Q$. Similarly Q^* is a finitely additive field of sets.

If $y \in Q$ is expressed as above in two ways, say $y=\bigvee p_i u_i=\bigvee q_j v_j$ (where p_i, $q_j \in P$; u_i, $u_j \in U$; $\bigvee u_i=e=\bigvee v_j$, and each of the sequences u_i, v_j, is disjoint), then the "corresponding" sets $\bigcup(p_i^* \times u_i^*)$ and $\bigcup(q_j^* \times v_j^*)$ in Q^* are equal. For we have $(p_i +_2 q_j)u_i v_j=(y+_2 y)u_i v_j=o$, so (from the fundamental property of P) $\mu(p_i +_2 q_j)[u_i v_j] = o$, and therefore $p_i = q_j$ whenever $u_i v_j \neq o$. Thus $p_i^* = q_j^*$ whenever $u_i^* \cap v_j^* \neq 0$, so that

$$\bigcup (p_i^* \times u_i^*) = \bigcup_{i,j} p_i^* \times (u_i^* \cap v_j^*) = \bigcup_{i,j} q_j^* \times (u_i^* \cap v_j^*) = \bigcup_j q_j^* \times v_j^*.$$

Hence we may define a mapping ϕ_1 of Q in Q^* by setting $\phi(\bigvee p_i u_i) = \bigcup (p_i^* \times u_i^*)$. An argument similar to the preceding shows that ϕ_1 is 1-1; and ϕ_1 is evidently a finitely additive algebraic isomorphism between Q and Q^*.

We have (M denoting the continuous-function "measure" in $P \otimes U$—cf. 4.3)

(1) If $Y = \phi_1(y)$ ($y \in Q$, $Y \in Q^*$), then $[y] \leqq \epsilon[e] \rightleftarrows M\{Y\} \leqq \epsilon$ ($0 < \epsilon < 1$).

For suppose $y = \bigvee p_i u_i$ in "normal form," as above. Then if $[y] \leqq \epsilon[e]$ we have (12.13) $[u_i y] \leqq \epsilon[u_i]$; but $[u_i y] = [u_i p_i] = \mu(p_i)[u_i]$, and thus (12.14) $\mu(p_i) \leqq \epsilon$ whenever $u_i \neq o$. Thus $M\{Y\} \leqq \epsilon$ (for each $s \in S$). The converse implication is proved similarly.

17.3. The desired isomorphism between E and $P \otimes U$ is now obtained by "extending" ϕ_1, as follows. Suppose two elements $x \in E$ and $X \in P \otimes U$ are so related, that to every $\epsilon > 0$ (and < 1) there correspond $y \in Q$ and $Y \in Q^*$ such that $[x +_2 y] \leqq \epsilon[e]$, $M\{X +_2 \{Y\}\} \leqq \epsilon$, and $Y = \phi_1(y)$. We then write $X = \phi(x)$. It is easily seen from (1) that this relation is 1-1 (where it exists); and clearly if $x \in Q$ then $\phi(x)$ exists and equals $\{\phi_1(x)\}$. Before proving that ϕ is defined for all $x \in E$, we need some lemmas.

17.4. Let $y = \bigvee y_n$, where $y_n \in Q$ ($n = 1, 2, \cdots$). Then given $\epsilon > 0$ (and < 1), there exists $z \in Q$ such that $[y +_2 z] \leqq \epsilon[e]$.

We may suppose that $y_1 \leqq y_2 \leqq \cdots$ (on replacing y_n by $\bigvee_1^n y_i$). By 7.6 there exists for each n an invariant element w_n such that (i) $[(y - y_n)w_n] \leqq \epsilon[w_n]$, (ii) if v is a nonzero invariant element less than or equal to $-w_n$, $[(y - y_n)v] > \epsilon[v]$. It readily follows that $w_1 \leqq w_2 \leqq \cdots$; further, $\bigvee w_n = e$, since if $e - \bigvee w_n = v_0$, we have $[y - y_n] \geqq [(y - y_n)v_0] \geqq \epsilon[v_0]$ for every n, whence (7.2) $[o] \geqq \epsilon[v_0]$, and so (12.14, Corollary) $v_0 = o$. Now write $v_1 = w_1$, $v_n = w_n - w_{n-1}$ ($n \geqq 2$), so that the elements v_n are disjoint and $\bigvee v_n = e$; and define $z = \bigvee y_n v_n$. Then $y +_2 z = \bigvee (y +_2 y_n)v_n = \bigvee (y - y_n)v_n$ where, since $v_n \leqq w_n$, we have $[(y - y_n)v_n] \leqq \epsilon[v_n]$. Thus (12.17) $[y +_2 z] \leqq \epsilon[e]$. Finally, since $y_n \in Q$, we may write $y_n = \bigvee_i p_{ni} u_{ni}$ in "standard form"; and then $z = \bigvee_{n,i} p_{ni}(u_{ni} v_n) \in Q$.

17.5. *Given* $x \in E$ *and* $\epsilon > 0$ (*and* < 1), *there exists* $y \in Q$ *such that* $[x +_2 y] \leqq \epsilon[e]$.

Let $\mathcal{B}'$ be the set of those elements $x \in E$ for which the above statement is true (for all ϵ). Evidently $\mathcal{B}' \supset Q \supset P \cup U$; also if $x \in \mathcal{B}'$ so does $-x$, and if $x_n \in \mathcal{B}'$ then $\bigvee x_n \in \mathcal{B}'$, as readily follows from 17.4. Thus $\mathcal{B}' \supset \mathcal{B}(P \cup U) = E$.

17.6. *If* $y_n \in E$ *and* $[y_m +_2 y_n] \leqq (1/2^n)[e]$ *whenever* $m \geqq n$, *there exists* $y \in E$ *such that* $[y +_2 y_n] \leqq (1/2^{n-2})[e]$ *for all* $n \geqq 2$; *further, this* y *is unique, and is given by* $y = \bigwedge_n \bigvee_{i \geqq n} y_i$ ($= \lim \sup y_i$).

REMARK. On complementation we obtain that y is also given by $y=\bigwedge_n \bigvee_{i\geqq n} y_i = \lim\inf y_i$; thus the assertion is, roughly speaking, that "Cauchy sequences" converge.

Write $t_n = y_n +_2 \bigvee_{i\geqq n} y_i = \bigvee_{i\geqq n}(y_i - y_n) = \bigvee_{i\geqq n}(y_{i+1} - y_i y_{i+1}) \leqq \bigvee_{i\geqq n}(y_{i+1} +_2 y_i)$. Thus (12.7) $[t_n] \leqq (1/2^{n-1})[e]$. Now define $y = \bigwedge_n \bigvee_{i\geqq n} y_i$. Then $y = \bigwedge_{n\geqq N} \bigvee_{i\geqq n} y_i = \bigwedge_{n\geqq N}(y_n +_2 t_n)$ so that, for each N, $y_N +_2 y \leqq \{y_N \vee \bigwedge_{n\leqq N}(y_n \vee t_n)\} - \{y_N \wedge \bigwedge_{n\geqq N}(y_n - t_n)\} \leqq [y_N \vee \bigvee_{n\geqq N} t_n\} - \{y_N - \bigvee_{n\geqq N} t_n\} = \bigvee_{n\geqq N} t_n$. Hence (12.7 again) $[y_N +_2 y] \leqq (1/2^{N-2})[e]$ for all $N>1$. Finally, y is unique, since if z also has the same property we have $[y+_2 z] \leqq (1/2^{n-3})[e]$ for all $n\geqq 3$, and so (12.6) $y=z$.

17.7. ϕ is a finitely additive algebraic isomorphism between E and $P\otimes U$.

Given $x\in E$, we first show that $\phi(x)$ exists. From 17.5, there exists a sequence $y_n \in Q$ with $[x+_2 y_n] \leqq (1/2^{n+2})[e]$. By 17.2(1), the corresponding sets $Y_n \in Q^*$ (that is, $Y_n = \phi_1(y_n)$) satisfy $M\{Y_m +_2 Y_n\} \leqq 1/2^n$ whenever $m\geqq n$. Hence, by 17.6 applied to the algebra $P\otimes U$, there exists $X\in P\otimes U$ such that $M\{X+_2\{Y_n\}\} \leqq 1/2^{n-2}$. Clearly $X=\phi(x)$, by definition (17.3).

A similar argument (starting from 4.6) shows that $\phi^{-1}(X)$ exists for every $X\in P\otimes U$. Finally, it has already been pointed out that ϕ is 1-1, and the relations $\phi(-x) = -\phi(x)$, $\phi(x_1 \vee x_2) = \phi(x_1) \vee \phi(x_2)$ are obvious.

17.8. If $y_n \in Q$, $\phi(\bigvee y_n) = \bigvee \phi(y_n)$.

It will suffice to prove this assuming $y_1 \leqq y_2 \leqq \cdots$. Given $\epsilon>0$, consider the element $z\in Q$ constructed in 17.4. Thus $[y+_2 z] \leqq \epsilon[e]$, where $y=\bigvee y_n$; further, the construction shows that $z = \bigvee y_n v_n$, where $v_n \in U$, $\bigvee v_n = e$, and the elements v_n are disjoint. Thus $z v_n \leqq y_n$. Now write $Y_n = \phi_1(y_n)$, $Z=\phi_1(z)$; we have $Z\cap(R\times v_n^*) \subset Y_n$, so that $Z\subset \bigcup Y_n \cup (R\times H)$ where H is of the first category in S.

Now, we have from 17.4 that $[yv_n - zv_n] \leqq \epsilon[v_n]$; a fortiori, therefore, $[y_m v_n - z v_n] \leqq \epsilon[v_n]$ when $m\geqq n$, so that (17.2(1)) $M\{(Y_m \cap (R\times v_n^*)) - (Z\cap(R\times v_n^*))\} \leqq \epsilon M\{R\times v_n^*\}$. The countable additivity of M then gives (on summing first over m and then over n)

$$M\{\bigcup Y_n - Z\} \leqq \epsilon.$$

Thus $M\{\bigcup Y_n +_2 Z\} \leqq \epsilon$; and so $\phi(y) = \{\bigcup Y_n\}$ from the definition.

17.9. *If* $x_n \in E$, $\phi(\bigvee x_n) = \bigvee \phi(x_n)$.

As before, we may suppose that $x_1 \leqq x_2 \leqq \cdots$. Let ϵ be given, and choose $y_n \in Q$ so that $[x_n +_2 y_n] \leqq (\epsilon/2^n)[e]$ and $M\{\phi(x_n) +_2 \phi(y_n)\} \leqq \epsilon/2^n$ (using the definition of ϕ). Write $x=\bigvee x_n$, $y=\bigvee y_n$. It is easily verified that $x+_2 y \leqq \bigvee(x_n +_2 y_n)$, so that $[x+_2 y] \leqq \epsilon[e]$. Similarly $M\{X+_2 Y\} \leqq \epsilon$, where $X=\bigvee\phi(x_n)$, $Y=\bigvee\phi(y_n)$. But 17.8 shows that $Y=\phi(y)$; hence there exists $z\in Q$ such that $[y+_2 z] \leqq \epsilon[e]$ and $M\{Y+_2\{Z\}\} \leqq \epsilon$, where $Z=\phi_1(z)$. Thus $[x+_2 z] \leqq (2\epsilon)[e]$ and $M\{X+_2\{Z\}\} \leqq 2\epsilon$, showing that $X=\phi(x)$.

This completes the proof that ϕ is an algebraic (σ-) isomorphism between E and $J\otimes U$. To prove that ϕ is an isomorphism of the abstract measure algebras, we must show that ϕ is "measure-preserving" both ways.

17.10. If $x_1 \sim x_2$, $M\{\phi(x_1)\} = M\{\phi(x_2)\}$.

Given $\epsilon > 0$ (and <1), we choose $y_i \in Q$ $(i=1, 2)$ so that $[x_i +_2 y_i] \leqq \epsilon[e]$ and $M\{X_i +_2 \{Y_i\}\} \leqq \epsilon$, where $X_i = \phi(x_i)$ and $Y_i = \phi_1(y_i)$. Let $y_1 = \bigvee p_n u_n$, $y_2 = \bigvee q_n v_n$, where (as usual) p_n, $q_n \in P$, u_n, $v_n \in U$, $\bigvee u_n = e = \bigvee v_n$, and the sequences $\{u_n\}$, $\{v_n\}$ are each disjoint. We can further suppose $u_n = v_n \neq o$ (on replacing both sequences u_n, v_n by an enumeration of the nonzero elements $u_n v_m$).

First we show

$$\mu(p_n) \leqq \mu(q_n) + 2\epsilon. \tag{1}$$

For we can suppose $\mu(q_n) + 2\epsilon \leqq 1$, else there is nothing to prove. Now, since $x_1 \sim x_2$, there exists $t \leqq x_2$ such that $t \sim x_1 y_1$. For each n we have $[u_n x_1 y_1] = [u_n t] \leqq [u_n y_2 \vee u_n(x_2 - y_2)]$, where $[u_n y_2] = [u_n q_n] = \mu[(q_n)[u_n]$, and $[u_n(x_2 - y_2)] \leqq \epsilon[u_n]$. Thus $[u_n x_1 y_1] \leqq (\mu(q_n) + \epsilon)[u_n]$. Since further $[u_n(y_1 - x_1)] \leqq \epsilon[u_n]$, we have $[u_n y_1] \leqq (\mu(q_n) + 2\epsilon)[u_n]$. But $[u_n y_1] = [u_n p_n] = \mu(p_n)[u_n]$; and (1) follows.

A similar argument applies with p_n and q_n interchanged. Thus we evidently have

$$|M\{Y_1\} - M\{Y_2\}| \leqq 2\epsilon. \tag{2}$$

But $M\{X_i\} = M\{X_i\{Y_i\}\} + M\{X_i - \{Y_i\}\} = M\{Y_i\} - M\{\{Y_i\} - X_i\} + M\{X_i - \{Y_i\}\}$, so that $|M\{X_i\} - M\{Y_i\}| \leqq \epsilon$. Hence $|M\{X_1\} - M\{X_2\}| \leqq 4\epsilon$, and therefore

$$M\{X_1\} = M\{X_2\}.$$

17.11. *If* $M\{\phi(x_1)\} = M\{\phi(x_2)\}$, *then* $x_1 \sim x_2$.

Given ϵ $(0 < \epsilon < 1/9)$, we take y_1, $y_2 \in Q$ so that $[x_i +_2 y_i] \leqq \epsilon[e]$ and $M\{X_i +_2 \{Y_i\}\} \leqq \epsilon$, where $X_i = \phi(x_i)$ and $Y_i = \phi_1(y_i)$, $i = 1, 2$. As in the previous argument, we can suppose $y_1 = \bigvee p_n u_n$, $y_2 = \bigvee q_n u_n$, where p_n, $q_n \in P$, $u_n \in U$, $u_n \neq o$, and the sequence u_n is disjoint. As at the end of 17.10, we have $|M\{X_i\} - M\{Y_i\}| \leqq \epsilon$ and therefore $|M\{Y_1\} - M\{Y_2\}| \leqq 2\epsilon$; and it readily follows that $|\mu(p_n) - \mu(q_n)| \leqq 2\epsilon$, for each n. We may therefore choose $p_n' \in P(p_n)$, $q_n' \in P(q_n)$ so that $\mu(p_n') = \mu(q_n')$, $\mu(p_n) - \mu(p_n') \leqq 2\epsilon$, and $\mu(q_n) - \mu(q_n') \leqq 2\epsilon$. Define $y_1' = \bigvee p_n' u_n$ and $y_2' = \bigvee q_n' u_n$; thus $y_1' \sim y_2'$, $y_i' \leqq y_i$, and $[y_i - y_i'] \leqq (2\epsilon)[e]$. Hence $[x_i +_2 y_i'] \leqq (3\epsilon)[e]$.

Since $y_1' \sim y_2'$, we can write $y_2' = t \vee z_1$, where $t \sim x_1 y_1'$ and $z_1 \sim y_1' - x_1$, so that $[z_1] \leqq (3\epsilon)[e]$; and we then have $y_2' = t x_2 y_2' \vee z_2$ where $z_2 = z_1 \vee t(y_2' - x_2)$, so that $[z_2] \leqq (6\epsilon)[e]$. Now $t x_2 y_2'$ is less than or equal to x_2, and is equivalent to a sub-element of x_1 (for t is); hence if h_1, h_2 denote maximal equivalent sub-elements of x_1, x_2, respectively, 7.5 shows that there exists $k_2 \leqq h_2$ such that $t x_2 y_2' \sim k_2$. Then we have $x_2 - h_2 \leqq x_2 - k_2 \sim x_2 - t x_2 y_2' \leqq (x_2 - y_2') \vee z_2$, so that $[x_2 - h_2] \leqq (9\epsilon)[e]$. This holds for every ϵ sufficiently small, so that $x_2 = h_2$.

Similarly $x_1 = h_1$, and therefore $x_1 \sim x_2$.

The isomorphism between $(E, \sim)$ and $P \otimes U$ is now established[21].

18. Imbedding of $E(e')$ in a direct product.

18.1. Having fully considered the relative algebras $E(e_n)$, we turn now to $E(e')$, where e' is the greatest bounded invariant element disjoint from all indecomposable elements (10.2, 10.3). It was shown in 13.4 that $e' = \bigvee e_n$, that the elements e_n are disjoint, and that $[e_n] = \rho_n[\bar{e}_n]$ for each n (where ρ_n is rational and $0 < \rho_n \leqq 1$). In the preceding section it was shown that each principal ideal $(E(e_n), \sim)$ is isomorphic to the direct product (say) $P_n \otimes U_n$, where P_n is a (relative) sub-algebra of $E(e_n)$ which is (to within isomorphism) a numerical measure algebra (I^1 or I^m), and U_n is the (relative) algebra of relatively invariant elements in $E(e_n)$. Since $U_n = e_n U$ is isomorphic, in a natural way, to the principal ideal $U(\bar{e}_n)$ (8.3), we have an isomorphism ϕ_n mapping $(E(e_n), \sim)$ on $P_n \otimes U(\bar{e}_n)$.

To combine these ϕ_n's into an isomorphism of $E(e')$, we first define a numerical measure algebra J' as follows: J' is the direct sum of the measure algebras $\rho_n P_n$ obtained by multiplying all measures in P_n by ρ_n. That is, J' is the set of all sequences $x = (x_n)$, $x_n \in P_n$, in which algebraic operations are defined in the obvious way (for example, $(x_n) \vee (y_n) = (x_n \vee y_n)$), and in which a (numerical) measure is defined by: $\mu'(x) = \sum \rho_n \mu_n(x_n)$, μ_n being the measure in P_n. It is easy to see that J' $(= (J', \mu'))$ is a non-atomic numerical measure algebra in which the measure is at most countably infinite, so that it satisfies our postulates. We use j' to denote the unit-element in J' (that is, the sequence (e_n)), and j_n to denote the element of J' whose "nth coordinate" is e_n and whose other coordinates are all o. The ideal $J'(j_n)$ is thus isomorphic to (P_n, μ_n); and to save notation we shall usually not distinguish between them. Note that they are not isometric, in general; we have, for $x \in J'(j_n)$, $\mu'(x) = \rho_n \mu_n(x)$. We shall prove that $(E(e'), \sim)$ *is isomorphic to a principal ideal in* $J' \otimes U(e')$.

All considerations in the present section will be relative to e', so we write e' as e and $E(e')$ as E in what follows—noting for future use that, since e' is invariant, the relatively invariant elements form the ideal $U(e')$.

18.2. Let R, S be the representation spaces of J' and U respectively. To the element $j_n \in J'$ there corresponds an open-closed subset $j_n^* \subset R$; and it is easy to see that j_n^* is homeomorphic to the representation space R_n of $J'(j_n)$—that is, of P_n. Thus we can take $R_n = j_n^*$, so that $\bigcup R_n$ is residual in R. Similarly, we can take the representation space S_n of $U(\bar{e}_n)$ to be the open-closed subset $\bar{e}_n^*$ of S which corresponds to $\bar{e}_n \in U$, and have that $\bigcup S_n$ is residual in S. It is easy to see that the direct product $P_n \otimes U(\bar{e}_n)$ is isomorphic

(21) Substantially the same arguments prove that the "direct product" $J \otimes U$, as characterized in 3.1 (e), is isomorphic to the product constructed in §4. More precisely, the arguments apply provided that J is non-atomic and of finite total measure; and the general case can be reduced to this one.

to the principal ideal of $\{R_n \times S_n\}$ in $J' \otimes U$, and we identify them accordingly. We write $M'\{H\}$ for the measure-function on $J' \otimes U$, and $M_n\{H\}$ for the measure-function on $P_n \otimes U(\bar{e}_n)$, with the natural convention that M_n is always 0 for each $s \in S - S_n$. Thus, if $H \subset j_n^* \times e_n^*$, $M'\{H\} = \rho_n M_n\{H\}$.

Note that, while the sets R_n are necessarily disjoint (for the elements j_n are), the sets S_n are not disjoint in general (for the same is true of the elements $\bar{e}_n$).

For each n we have an isomorphism ϕ_n between $E(e_n)$ and the ideal of $\{j_n^* \times e_n^*\}$ $(= \{R_n \times S_n\})$ in $J' \otimes U$. Define, for each $x \in E$, $\phi(x) = \bigvee \phi_n(xe_n)$. It is immediate that ϕ is an algebraic (σ-) isomorphism between E and the principal ideal of $\bigvee\{j_n^* \times e_n^*\}$ in $J' \otimes U$. We have only to verify that ϕ is "measure-preserving" both ways.

18.3. If $x \leqq e_n$, $x' \leqq e_m$, and $x \sim x'$, then $M'\{\phi(x)\} = M'\{\phi(x')\}$.

If $m = n$ this follows from the corresponding property of ϕ_n; thus we can assume $m \neq n$. Now, as was shown in §17, there exists $y \leqq e_n$ such that (i) $[x +_2 y] \leqq \epsilon[e_n]$, (ii) $M_n\{\phi_n(x) +_2 \phi_n(y)\} \leqq \epsilon$, and (iii) $y = \bigvee p_i v_i$ where $p_i \in P_n$, $v_i \in U_{e_n} = e_n U$, and the elements v_i are disjoint and nonzero. Thus $v_i = e_n u_i$, say, where $u_i \in U(\bar{e}_n)$ (cf. 8.2) and the invariant elements u_i are disjoint; and, since $p_i \leqq e_n$, we have $y = \bigvee p_i u_i$. Similarly there exists $y' \leqq e_m$ such that (i′) $[x' +_2 y'] \leqq \epsilon[e_m]$, (ii′) $M_m\{\phi_m(x') +_2 \phi_m(y')\} \leqq \epsilon$, and (iii′) $y = \bigvee q_j u_j'$, where $q_j \in P_m$, $u_j' \in U(\bar{e}_m)$, and the invariant elements u_j' are disjoint. Now enumerate the nonzero elements $u_i u_j'$, $u_i - \bar{e}_m$, $u_j' - \bar{e}_n$, into a sequence w_k; thus the elements w_k are disjoint and invariant, and we may write $y = \bigvee r_k w_k$, $y' = \bigvee s_k w_k$, where $r_k \in P_n$ and $s_k \in P_m$.

We have $[w_k y] = [r_k e_n w_k] = \mu_n(r_k)[e_n w_k]$, from the fundamental property of P_n; but $[e_n] = \rho_n[\bar{e}_n]$, and therefore (12.13) $[e_n w_k] = \rho_n[w_k]$ if $w_k \leqq \bar{e}_n$. If w_k is not $\leqq \bar{e}_n$, then $w_k \leqq -\bar{e}_n$, by construction, so that there is no loss in supposing $r_k = o$. Thus (using 12.20) we see that, in all cases, $[w_k y] = \rho_n \mu_n(r_k)[w_k]$. Similarly $[w_k y'] = \rho_m \mu_m(s_k)[w_k]$. But since $x \sim x'$, we have $[w_k x] = [w_k x']$; and by arguments similar to those in 17.10 it follows that $|\rho_n \mu_n(r_k) - \rho_m \mu_m(s_k)| \leqq 2\epsilon$. Thus $|\rho_n M_n\{\phi_n(y)\} - \rho_m M_m\{\phi_m(y')\}| \leqq 2\epsilon$, whence $|\rho_n M_n\{\phi_n(x)\} - \rho_m M_m\{\phi_m(x')\}| \leqq 4\epsilon$. That is, $|M'\{\phi(x)\} - M'\{\phi(x')\}| \leqq 4\epsilon$ for every ϵ, so that $M'\{\phi(x)\} = M'\{\phi(x')\}$.

18.4. *If* $x \sim x'$, *then* $M'\{\phi(x)\} = M'\{\phi(x')\}$.

From 6.2, we can write $x' = \bigvee x_m'$ where the elements x_m' are disjoint, and $x_m' \sim x e_m$. Similarly we can write $x e_m = \bigvee_n x_{mn}$ where the elements x_{mn} are disjoint and $x_{mn} \sim x_m' e_n$. From 18.3 we have $M'\{\phi(x_{mn})\} = M'\{\phi(x_m' e_n)\}$; and the countable additivity of M' then gives $M'\{\phi(x)\} = M'\{\phi(x')\}$.

18.5. *If* $M'\{\phi(x)\} = M'\{\phi(x')\}$ *then* $x \sim x'$.

By an argument similar to that in 18.4, we see that it will be enough to prove this assuming $x \leqq e_m$ and $x' \leqq e_n$. Let y, y' be maximal equivalent subelements of x, x', respectively (7.5), and write $z = x - y$, $z' = x' - y'$. By 18.4, $M'\{\phi(y)\} = M'\{\phi(y')\}$; also here $M'\{\phi(x)\} < \infty$; hence the additivity of M'

gives $M'\{\phi(z)\}=M'\{\phi(z')\}$. Now it is clear from the construction in §17 that $M'\{\phi(z)\}$ has the value 0 for all $s\in S-\bar{z}^*$; similarly $M'\{\phi(z')\}=0$ outside $\bar{z}'^*$. Since $\bar{z}\bar{z}'=o$, we have $\bar{z}^*\cap\bar{z}'^*=0$, and therefore $M'\{\phi(z)\}=M'\{\phi(z')\}=0$ for *all* $s\in S$. Since $M_m\{\phi_m(z)\}=0$, we have $z=o$; similarly $z'=o$, so that $x=y\sim y'=x'$.

The proof is now complete; but for later use we note the property:

18.6. *If* $u\in U$, $M'\{\phi(u)\}$ *is the characteristic function of* u^*.

We have $[e_n u]=\rho_n[\bar{e}_n u]$, and so, since ϕ is an isomorphism, $M'\{\phi(e_n u)\}=\rho_n M'\{\phi(\bar{e}_n u)\}$. But $M'\{\phi(e_n u)\}=\rho_n M_n\{\phi_n(e_n u)\}$, by definition; hence $M'\{\phi(\bar{e}_n u)\}=M_n\{\phi_n(e_n u)\}$. Since $e_n u$ is invariant relative to e_n, the construction in §17 shows that $M_n\{\phi_n(e_n u)\}$ is the characteristic function of the open-closed subset of S_n which "corresponds" to $e_n u$; and since the (absolute) invariant element corresponding to $e_n u$ in the isomorphism of 8.3 is $\bar{e}_n u$, we here have $M'\{\phi(\bar{e}_n u)\}=$characteristic function of $(\bar{e}_n u)^*$.

It readily follows that $M'\{\phi(u(\bar{e}_n-\bar{e}_1-\bar{e}_2-\cdots-\bar{e}_{n-1}))\}=$characteristic function of $u^*\cap(\bar{e}_n^*-\bar{e}_1^*-\bar{e}_2^*-\cdots-\bar{e}_{n-1}^*)$; and on summing (and neglecting a subset of S of the first category) we obtain $M'\{\phi(u)\}=$characteristic function of u^*.

19. **The representation of** $(E, \sim)$.

19.1 We begin by imbedding $E(e'')$ in a direct product, where e'' is the element defined in 10.2. It has been shown (10.6) that $e''=\bigvee f^n$, where the elements f^n are disjoint, bounded and equivalent, and where $\bar{f}^n=e''$, for each n. Now $E(f^n)$ has, relative to f^n, all the properties which were used in studying $E(e')$. Hence the imbedding obtained in §18 for $E(e')$ applies also to each $E(f^n)$. Remembering that the algebra U_{f^n} of invariant elements relative to f^n is isomorphic to $U(\bar{f}^n)$ (8.3)—that is, to $U(e'')$—we thus have, for each n, an isomorphism ψ_n between $(E(f^n), \sim)$ and a principal ideal in a direct product (say) $J^n\otimes U(e'')$, J^n being a non-atomic numerical measure algebra. Let J'' denote the "direct sum" of the measure algebras J^n—that is, J'' is the set of sequences $x=(x^n)$, $x^n\in J^n$, with algebraic operations defined "co-ordinatewise," and with a (numerical) measure μ'' defined by: $\mu''(x)=\sum\mu^n(x^n)$, μ^n being the measure in J^n. We now show that $E(e'')$ *is isomorphic to a principal ideal in* $J''\otimes U(e'')$.

As in 18.1 and 18.2, we can express the unit-element j'' of J'' as $\bigvee j^n$, where the elements j^n are disjoint and the principal ideal $J''(j^n)$ can be regarded as being J^n. Also as before, the open-closed subset j^{n*} of the representation space R of J'' can be regarded as the representation space R^n of J^n; and the principal ideal of $\{R^n\times S\}$ in $J''\otimes U(e'')$, where S is the representation space of $U(e'')$, can be regarded as being $J^n\otimes U(e'')$. We use M^n for the measure-function in $J^n\otimes U(e'')$, and M'' for the measure-function in $J''\otimes U(e'')$; thus $M''\{H\}=\sum M^n\{H\cap(j^{n*}\times S)\}$.

We define, for each $x\leqq e''$, $\psi(x)=\bigvee\psi_n(xf^n)$; trivially ψ is an algebraic (σ-) isomorphism between $E(e'')$ and the principal ideal of $\bigvee\psi_n(f^n)$ in

$J''\otimes U(e'')$. All that remains is to verify that ψ is "measure-preserving"; and as in doing this all considerations will be relative to e'', we may suppose $e''=e$ in 19.2 and 19.3.

19.2. *If* $x\sim y$, $M''\{\psi(x)\}=M''\{\psi(y)\}$.

By the same argument as in 18.4, it is enough to prove this when $x\leqq f^m$ and $y\leqq f^n$. Now from 13.1 applied relative to f^m (so that the "invariant elements" are of the form uf^m, $u\in U$), there exist, for any given integer N, elements $u_i\in U$ $(1\leqq i\leqq N)$ such that $f^m u_i u_j=o$ when $i\neq j$, $\bigvee u_i\geqq f^m$ (so that $\bigvee u_i=\bar{f}^m=e$), and $((i-1)/N)[u_if^m]\leqq[u_ix]\leqq(i/N)[u_if^m]$. From 5.10 it follows that the elements u_i are actually disjoint. Choose $t_i\leqq u_ix$ with $[t_i]=((i-1)/N)[u_if^m]$, and write $z_i=u_ix-t_i$ and $z=\bigvee z_i$. Thus $x=\bigvee t_i\vee z$, where the elements t_i and z are disjoint, and (from 12.5(5)) $[z]\leqq(1/N)[f^m]$.

Since $y\sim x$, we can (6.2) write $y=\bigvee t_i'\vee z'$, where the elements t_i' and z' are disjoint, $t_i'\sim t_i$, and $z'\sim z$.

Now u_if^m is invariant relative to f^m; hence (18.6) $M^m\{\psi_m(u_if^m)\}$ is the characteristic function of the open-closed subset of S which "corresponds" to u_if^m. Here u_if^m was replaced by $u_i\bar{f}^m=u_i$ in the isomorphism between U_{f^m} and U $(=U(e''))$; thus we have $M^m\{\psi_m(u_if^m)\}=1$ for $s\in u_i^*$, 0 for $s\in S-u_i^*$. Since ψ_m is an isomorphism, the relation $[t_i]=((i-1)/N)[u_if^m]$ gives $M''\{\psi(t_i)\}=(i-1)/N$ for $s\in u_i^*$, 0 for $s\in S-u_i^*$. Similarly we have $M''\{\psi(z)\}\leqq 1/N$ for all s.

Again, we have $f^m\sim f^n$ and therefore (6.6) $u_if^m\sim u_if^n$. Thus $[t_i']=((i-1)/N)[u_if^n]$, so by the same reasoning as before (applied to the isomorphism ψ_n) we have $M''\{\psi(t_i')\}=(i-1)/N$ for $s\in u_i^*$, 0 for $s\in S-u_i^*$, and $M''\{\psi(z')\}\leqq 1/N$.

Thus from the additivity of M'', we have $|M''\{\psi(x)\}-M''\{\psi(y)\}|\leqq 1/N$; and since N here was arbitrary, it follows that $M''\{\psi(x)\}=M''\{\psi(y)\}$.

19.3. *If* $M''\{\psi(x)\}=M''\{\psi(y)\}$, *then* $x\sim y$.

This follows by the same argument as in 18.5. The desired isomorphic imbedding of $(E(e''),\sim)$ is thus established.

19.4. It is now easy to extend the imbedding to one of all of E. For convenience of exposition, we assume that none of the elements e^o, e', e'' (cf. 9.6, 10.2) is zero; the modifications needed otherwise are obvious. We have from 9.7 an isomorphism θ of $E(e^o)$ on a principal ideal in $K\otimes U(e^o)$, and from §18 an isomorphism ϕ of $E(e')$ on a principal ideal in $J'\otimes U(e')$. We have just constructed an isomorphism ψ of $E(e'')$ on a principal ideal in $J''\otimes U(e'')$. Further, the elements e^o, e', e'' are disjoint and invariant, and $e^o\vee e'\vee e''=e$.

Now form the usual direct sum $J=J'\oplus J''$ of the numerical measure algebras J', J''; thus J consists of the pairs (x', x'') $(x'\in J', x''\in J'')$, with measure μ given by $\mu(x',x'')=\mu'(x')+\mu''(x'')$, μ' and μ'' being the measures in J' and J''. As a non-atomic algebra of at most countably infinite measure, J satisfies our postulates. Let $\mathcal{E}$ denote the free direct sum (cf. 3.1(d)) of the direct products $K\otimes U(e^o)$ and $J\otimes U(-e^o)$. To $x\in E$ we make correspond the pair $(\theta(xe^o), \phi(xe')\vee\psi(xe''))$ in $\mathcal{E}$. (The products $J'\otimes U(e')$ and $J''\otimes U(e'')$ are regarded as imbedded in $J\otimes U(e'\vee e'')$ in the obvious way.) This evidently produces an algebraic isomorphism between E and the principal ideal of the element $(\theta(e^o), \phi(e')\vee\psi(e''))$ in $\mathcal{E}$. Further, if $x\sim y$, then (since e^o, e' and e'' are invariant) 6.6 gives $xe^o\sim ye^o$, $xe'\sim ye'$ and $xe''\sim ye''$, so that the corresponding elements of $\mathcal{E}$ are equivalent in $\mathcal{E}$. Conversely, if $x, y\in E$ correspond to equivalent elements in $\mathcal{E}$, we have $xe^o\sim ye^o$ and also $M\{\phi(xe')\}+M\{\psi(xe'')\}=M\{\phi(ye')\}+M\{\psi(ye'')\}$, M denoting the measure function in $J\otimes U(-e^o)$. But $M\{\phi(xe')\}$ is a function on the representation space S of $U(-e^o)$ which vanishes outside the open-closed subset e'^* corresponding to e'. Similarly $M\{\psi(xe'')\}$ vanishes outside e''^*. Since $e'^*\cap e''^*=0$, and $e'^*\cup e''^*=S$, we obtain $M'\{\phi(xe')\}=M'\{\phi(ye')\}$ and $M''\{\psi(xe'')\}=M''\{\psi(ye'')\}$, so that $xe'\sim ye'$ and $xe''\sim ye''$. Hence, finally, $x\sim y$, and the main theorem of this paper is proved: $(E, \sim)$ *is isomorphic to a principal ideal in the free direct sum of* $K\otimes U(e^o)$ *and* $J\otimes U(-e^o)$, *each of the two summands being the direct product of a numerical measure algebra with a trivial abstract measure algebra.*

19.5. The above argument would evidently also prove that $(E, \sim)$ is isomorphic to a principal ideal in the *single* direct product $(K\oplus J)\otimes U$. This way of stating the theorem has the drawback that the numerical measure algebra $K\oplus J$ will in general not satisfy postulate II (for K has atoms of measure 1, and J has in general non-atomic elements of the same measure), so that $(K\oplus J)\otimes U$ will in general not satisfy our postulates. However, it is immediate from this alternative formulation of the theorem that $(E, \sim)$ *is naturally isomorphic to an abstract measure alegbra* (E, M) *in which the values of the abstract measure* M *are equivalence classes of numerical functions* (*more precisely, non-negative real-valued continuous functions on the representation space of* U, *modulo functions which vanish outside a set of first category*), M *being countably additive in the obvious sense.* Incidentally this proves that any equivalence relation satisfying our postulates (2.1) is induced by a "reasonable" countably additive abstract measure λ.

20. **Corollaries and applications.**

20.1. In this section we mention briefly some deductions from the preceding general theory. There is, of course, an application to the theory of abstract-valued integration, which will be treated in a subsequent paper. Of the deductions to be considered here, the decomposition theorem given in 20.4(2) below is probably the most significant.

20.2. *Some special cases.*

(1) Consider the elements a_n given by 11.7; thus they are disjoint, $\bigvee a_n=e'$ (10.2), and a_n is homogeneous of infinite order $\mathfrak{m}_n$ over U. We observe that the arguments of §§14–17 apply to a_n as well as to e_n. Thus *each principal ideal* $(E(a_n), \sim)$ *is isomorphic to the direct product* $I^{\mathfrak{m}_n}\otimes U(a_n)$.

(2) In particular, if all nonzero elements of E are decomposable and bounded (that is, if $e=e'$), and if E is *separable* over U (that is, if e is of order $\leqq\aleph_0$, and thus of order exactly $\aleph_0$, over U), $(E, \sim)$ is isomorphic to $I^1\otimes U$—giving a characterization of the direct product of I^1 with a "trivial" algebra.

20.3. *The numerical case.*

(1) *Characterizations.* It is easy to derive characterizations of *numerical* measure algebras (satisfying our postulates—notably II; cf. 2.2). From the remark in footnote 6, it is enough to restrict attention to the case in which E is non-atomic. We then have the theorem:

If E is a non-atomic algebra, a necessary and sufficient condition that $(E, \sim)$ be isomorphic to a numerical measure algebra is that $(E, \sim)$ satisfies the postulates of 2.1 *and has o and e for its only invariant elements.*

An immediate consequence of this is (cf. [10, p. 423]):

If E is non-atomic, $(E, \sim)$ (satisfying the postulates of 2.1) *is isomorphic to a numerical measure algebra if and only if, for every $x, y\in E$, either $[x]\leqq[y]$ or $[y]\leqq[x]$.*

(2) *Decomposition into homogeneous parts.* From 20.2(1) we have at once the theorem (cf. [11]):

A non-atomic numerical measure algebra (E, μ) (with σ-finite measure, of course) *is the direct sum of a countable number (at most) of homogeneous measure algebras $(E(a_n), \mu)$; and $(E(a_n), \mu)$ is isomorphic to $I^{\mathfrak{m}_n}$* (and thus isometric with $I^{\mathfrak{m}_n}$, to within a multiplicative nonzero constant).

20.4. *Groups of measure-preserving automorphisms.*

(1) *Equivalence with respect to* Φ. Let E be a given algebra (satisfying the countable chain condition), and let a group Φ of automorphisms ϕ of E be given. Write $x\sim_\Phi y$ to mean that there exist sequences $\{x_n\}$, $\{y_n\}$, each disjoint, in E, and automorphisms $\phi_n\in\Phi$, such that $x=\bigvee x_n$, $y=\bigvee y_n$, and $y_n=\phi_n(x_n)$.

One readily sees that $\sim_\Phi$ is an equivalence relation, and that $(E, \sim_\Phi)$ satisfies all the postulates (2.1) except, in general, III. Postulate III will also be satisfied in at least one important case (cf. 20.4(2)).

It is easy to see that $u\in E$ is *invariant* in $(E, \sim_\Phi)$ if and only if $\phi(u)\leqq u$ for every $\phi\in\Phi$—that is, u is "invariant under Φ" in the usual sense.

(2) *Decomposition with respect to* Φ.

THEOREM. *Let (E, m) be a numerical measure algebra in which m is an arbitrary reduced σ-finite numerical measure (cf.* 2.2; *postulate* II *need not be satisfied); and let Φ be a given group of measure-preserving automorphisms of E. Then there exist disjoint nonzero elements $c_n\in E$, each invariant under Φ, and sub-algebras A_n, B_n of $E(c_n)$ and reduced σ-finite numerical measures μ_n on A_n, ν_n on B_n, such that:*

(i) $\bigvee c_n=e$.

(ii) *B_n consists of exactly those elements of $E(c_n)$ which are invariant under Φ.*

(iii) *If $x, y\in A_n$, $\mu_n(x)=\mu_n(y)$ if and only if $x\sim_\Phi y$ (in E).*

(iv) (A_n, μ_n) *is isometric to one of the following*:

(a) $I^{\mathfrak{m}_n}$ (*for some cardinal* $\mathfrak{m}_n$; *cf.* 3.1(a)),

(b) *the direct sum of* $\aleph_0$ *copies of* $I^{\mathfrak{m}_n}$,

(c) *the algebra generated by a finite number of atoms each of measure* 1,

(d) K (*cf.* 3.1(a)),

and no two of the measure algebras (A_n, μ_n) *are isomorphic.*

(v) $(E(c_n), m)$ *is the "direct product" of* (A_n, μ_n) *and* (B_n, ν_n); *that is,* $E(c_n) = c_n\mathcal{B}(A_n \cup B_n)$ *and, whenever* $x \in A_n$ *and* $z \in B_n$, $m(xz) = \mu_n(x)\nu_n(z)$ (*with the convention* $0 \cdot \infty = 0 = \infty \cdot 0$).

REMARKS. Before sketching the proof, we note the following consequences:

(a) For any $x, y \in A_n$ we have $[\mu_n(x) = \mu_n(y)] \leftrightarrow [x \sim_\Phi y] \rightarrow [m(x) = m(y)]$. Hence, if for some n the measure m is σ-finite *on* A_n (that is, $c_n = \bigvee x_k$ where $m(x_k) < \infty$ and $x_k \in A_n$), it follows that μ_n is then a nonzero finite constant multiple of m. If further $m(c_n) < \infty$, the product formula shows that μ_n and ν_n are *both* nonzero finite constant multiples of m. In particular, this is the case if $m(e) < \infty$.

(b) On applying this theorem to the algebra of measurable sets modulo null sets of a separable measure *space*, one can derive sharpened forms of theorems of von Neumann [13 p. 617] and Halmos [7, Theorem 2][22], the direct sum decompositions there given being replaced (roughly speaking) by representations as direct products of measure spaces. It is hoped that details may appear elsewhere.

Proof of theorem. In the present case, $(E, \sim_\Phi)$ can easily be seen to satisfy postulate III, so the preceding theory applies to it. We suppose for simplicity that $e = e'$ (cf. 10.2). Then (11.7, with a slight change in notation) $e = \bigvee c_n$, where the elements c_n are disjoint, nonzero, and homogeneous over U. It is easily seen that each c_n is now invariant. Further, in view of 20.2(1), $(E(c_n), \sim_\Phi)$ is isomorphic to the direct product $A_n \otimes B_n$, where $B_n = U(c_n)$ and $A_n = (A_n, \mu_n)$ is a sub-algebra of $(E(c_n), \sim_\Phi)$ which is also a numerical measure algebra (isometric to $I^{\mathfrak{m}_n}$). We define the measure ν_n on B_n, in the present case, to coincide with m, which can easily be seen to be σ-finite *on* B_n. Thus we need only verify that $m(xz) = \mu_n(x)m(z)$ for $x \in A_n$, $z \in B_n$, and it will suffice to establish this in the special case $\mu_n(x) = 1/k$, k a positive integer. But then there exist k disjoint elements $x_1, x_2, \cdots, x_k$ in A_n such that $x_1 = x$, $\bigvee x_i = c_n$, and $\mu_n(x_i) = 1/k$. Hence $x_i \sim_\Phi x$, so that (z being invariant) $x_i z \sim_\Phi xz$, and therefore $m(x_i z) = m(xz)$. Thus $m(xz) = m(x)/k$, completing the proof.

The ideal $E(e'')$ (10.2) is disposed of similarly, using 10.6; here the measure algebras (A_n, μ_n) will be of the type described in (iv b) of the theorem. Finally $E(e^0)$ (9.6) provides the measure algebras of types (iv c) and (iv d).

(3) *The Banach-Tarski theorem.* Apply the preceding theorem taking

[22] In view of the fact that in [7] and [13] we have the separable case, in which the algebras A_n, B_n can always be realised as sub-fields of the given field of measurable sets.

(E, m) to be the algebra of (Lebesgue) measurable sets modulo null sets in Euclidean space, and Φ to be the group of all rigid motions (or even of all translations). The only invariant elements are o and e, and the algebra E is separable. Hence we here have $c_1 = e$ and $E = A_1$. Property (iii) of the theorem (see remark (a)) then gives: $m(x) = m(y)$ *if and only if* $x \sim_\Phi y$ (m denoting Lebesgue measure)—a well known theorem of Banach and Tarski [1, p. 277].

One can go on to obtain such generalizations as the following: For any measurable subset X of the plane, and any real number a, let $m_X(a)$ denote the (linear) measure of the intersection of X with the line $x = a$. Then if $m_X(x) = m_Y(x)$ for almost all x, we can write $X = Z \cup \bigcup X_n$, $Y = T \cup \bigcup Y_n$, where Z and T are of (plane) measure zero, each of the sequences $\{X_n\}$, $\{Y_n\}$ is disjoint, the sets X_n and Y_n are all measurable, and where for each n the sets X_n, Y_n correspond under some translation parallel to the y-axis. (For let E be the algebra of plane measurable sets modulo null sets, and let Φ be the group of translations parallel to the y-axis; it is not hard to see that the abstract measure function $M\{X\}$, in the representation of $(E, \sim_\Phi)$ given by the theorem of 19.5, is in effect simply the function $m_X(x)$.)

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