# ORTHOGONAL PROPERTIES OF INDEPENDENT FUNCTIONS

BY

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This paper investigates the orthogonal structure of independent systems of functions. The tendency in dealing with independent functions has been largely to establish results relating to the law of large numbers and central limit theorems (see [1])(1). The treatment here is concerned with an analytical study of the orthogonal properties of independent functions and does not consider probability interpretations.

Two functions $f_1(t)$ and $f_2(t)$ are said to be independent over the interval $(0, 1)$, if for any $\alpha_i, \beta_i$ $(i=1, 2)$ we have

$$
\begin{aligned}
& \left| \underset{t}{E} (\alpha_1 < f_1(t) \leqq \beta_1, \alpha_2 < f_2(t) \leqq \beta_2) \right| \\
& \qquad = \left| \underset{t}{E} (\alpha_1 < f_1(t) \leqq \beta_1) \right| \left| \underset{t}{E} (\alpha_2 < f_2(t) \leqq \beta_2) \right|
\end{aligned} \tag{1}
$$

where $|\ \ |$ denotes the measure of the set. The extension to several functions is done in a natural manner [4, pp. 61–63].

The classical example of such a system of functions is the well known system of Rademacher. This system can be defined as follows:

$$
\begin{aligned}
x_n(t) &= \text{sign} \sin (2^{n+1}\pi t) \qquad \text{for } 0 \leqq t \leqq 1, \\
x_0(t) &= 1.
\end{aligned}
$$

Other important systems have been introduced by Steinhaus [2, pp. 23–27], and Kac [3, p. 64].

Some immediate consequences of the definition are

(a) If $x_1(t), \cdots, x_n(t)$, are independent, then

$$
\int_0^1 \prod_1^n x_i(t) dt = \prod_1^n \int_0^1 x_i(t) dt. \tag{2}
$$

If we assume the mean value of each $x_n(t)$ equal to zero, then the orthogonality of $x_n(t)$ results from (2). Another result is that if $x_1 + \cdots + x_n$ belong to $L^p$, then each $x_n(t)$ belongs to $L^p$ for $p \geqq 1$. It is to be remarked now that $x_n(t)$ considered as an orthogonal system is incomplete, that is, there exist nonzero functions (for example, $g(t) = x_1(t)x_2(t)$) which are orthogonal to all $x_n(t)$. Consequently, no uniqueness of expansions can be expected.

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(1) Numbers in brackets refer to the references cited at the end of the paper.

Independent functions are important for their connection with independent random variables, and on the other hand, for any sequence of measurable functions $f_n(t)$, there exists a sequence of independent functions $x_n(t)$ which are equimeasurable to $f_n(t)$ respectively [4, pp. 62–63].

This paper is divided into three main sections dealing with convergence, summability, and lacunarity respectively. In §1 convergence in $L^p$ and ordinary pointwise convergence are discussed. §2 treats relationships of summability to convergence. Finally, in §3 the connection of lacunarity and independence systems with lacunary orthogonal systems is considered.

Whenever an integral is written we presuppose sufficient assumptions to insure the existence of the integral. The assumption of existence of the integral will always be valid if it is assumed that $|x_m(t)| \leqq M_n$, that is, that $x_m(t)$ need not be uniformly bounded. Less restrictive conditions can usually be imposed.

The domain of definition of $x_m(t)$ is (0, 1). The conjugate exponent $p'$ to $p$ will be the number such that $1/p+1/p'=1$.

1. **Convergence.** In this section we deal with convergence properties of independent series. The first part of this section will be devoted to studying modes of convergence of partial sums and their interrelationships.

Let $x_n(t)$ be an independent system of functions each possessing a mean value zero, $\int_0^1 x_n(t)dt=0$. This assumption implies that the $x_n(t)$ can be considered as an orthogonal system of functions. We do not at first assume that the system is normalized.

An essential tool which will be used frequently throughout the paper is the following lemma established by Marcinkiewicz and Zygmund [5, pp. 109–115].

LEMMA 1. *If $\int_0^1 x_n(t)dt=0$, then for all $m$*

$$(1)\qquad \int_0^1 \max_{1\leqq n\leqq m} |s_n(t)|^p dt \leqq K\int_0^1 |s_m(t)|^p dt \qquad \text{for } 1\leqq p\leqq\infty,$$

$$(2)\qquad A_p'\int_0^1\left(\sum_1^m x_n^2\right)^{p/2} \leqq \int_0^1 |s_m(t)|^p dt \leqq A_p\int_0^1 \left(\sum x_n^2\right)^{p/2}$$

*for $1\leqq p<\infty$, where $s_m(t)=\sum_1^m x_n(t)$ and $A_p$, $A_p'$ are constants depending only on $p$.*

We first establish a lemma which is fundamental in all that follows.

LEMMA 2. *A necessary and sufficient condition that $\sum_1^m a_l x_l(t)=s_m(t)$ converge to a function belonging to $L^p$ $(1\leqq p<\infty)$ is that*

$$\int_0^1 |s_m(t)|^p dt \leqq \gamma \qquad \text{for } m = 1, 2, \cdots.$$

**Proof.** *Necessity.* In virtue of Lemma 1 and the fact that $s(t)=\sum_1^\infty a_n x_n(t)$ is in $L^p$, we have

$$\int_0^1 |s_n(t)|^p dt \leqq \int_0^1 \max_n |s_n(t)|^p dt \leqq K\int_0^1 |s(t)|^p dt \leqq \gamma.$$

*Sufficiency.* Since $\int_0^1 |s_n(t)|^p dt \leqq \gamma$ we have by Lemma 1 that

$$\int_0^1 \left| \sum_1^n a_m^2 x_m^2(t) \right|^{p/2} dt \leqq \frac{\gamma}{A_p'}\cdot$$

Hence, $\sum_1^m a_n^2 x_n^2(t)$ converges a.e. to a $p/2$ integrable function. Since

$$G_m(t) = \sum_m^\infty a_l^2 x_l^2(t) \geqq G_{m+1}(t),$$

we obtain in employing Lemma 1 and Lebesgue's convergence theorem that

$$\text{(2a)} \qquad \int_0^1 |s_m(t) - s_n(t)|^p dt \leqq A_p \int_0^1 \left( \sum_m^\infty a_l^2 x_l^2(t) \right)^{p/2} dt \leqq \epsilon$$

provided $n > m \geqq N_0(\epsilon)$. Again, for any $n$ and $m$

$$\text{(3)} \qquad \int_0^1 \max_{m \leqq t \leqq n} |s_t(x) - s_m(x)|^p dx \leqq A_p \int_0^1 |s_n(x) - s_m(x)|^p dx \leqq K.$$

Therefore, if $F_{mn} = \max_{m \leqq t \leqq n} |s_t(x) - s_m(x)|^p$ and since $F_{m,n} \leqq F_{m,n+1} \leqq \cdots$, it follows from (3) that $\lim_{n\to\infty} F_{m,n} = F_m$ exists almost everywhere. Moreover, $F_n \geqq F_{n+1} \geqq \cdots \geqq 0$ which gives on account of (2a) that $\lim_n F_n = 0$ a.e. The conclusion of the proof of the theorem is now immediate.

The theorem is also valid for $p=\infty$. It can be shown that the lemma does not remain true for $p<1$.

We present now several applications of this lemma. For completeness, several definitions will now be given.

A sequence of functions $f_n(t)$ converges weakly in $L^p$ $(1 \leqq p < \infty)$ if for every function $g(t)$ in $L^{p'}(M)$, we have that

$$\lim_{n\to\infty} \int_0^1 f_n(t) g(t) dt$$

converges.

A sequence of functions $f_n(t)$ converges strongly in $L^p$ $(1 \leqq p < \infty)$ if

$$\lim_{m,n\to\infty} \int_0^1 |f_n(t) - f_m(t)|^p dt = 0.$$

A sequence of functions $f_n(t)$ converges in the ordinary sense if $f_n(t)$ converges pointwise for almost every $t$.

Finally, a sequence of functions $f_n(t)$ converges asymptotically to a function $f_0(t)$ if for any $\epsilon > 0$

$$\lim_{n\to\infty} \left| \underset{t}{E}\left[\, | f_n(t) - f_0(t) | \geqq \epsilon_0 \right] \right| = 0,$$

where $|\quad|$ denotes the measure of the set.

A further lemma which is needed will now be given.

LEMMA 2a. *If $s_{n_k}(t)$ converges a.e. to a function $f(t)$ integrable $L^p$, then $s_n(t)$ converges a.e. to $f(t)$.*

REMARK. The subsequence $n_k$ is arbitrary.

**Proof.** The set of functions

$$\begin{aligned} X_{n_1}(t) &= a_1 x_1(t) + \cdots + a_{n_1} x_{n_1}(t), \\ X_{n_2}(t) &= a_{n_1+1} x_{n_1+1}(t) + \cdots + a_{n_2} x_{n_2}(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3a}$$

form an independent system, each having a mean value zero. Moreover, since

$$| s_{n_k} | \leqq \max_k | s_{n_k} |,$$

Lemma 1 implies in virtue of the hypothesis and (3a) that

$$\int_0^1 | s_{n_k}(t) |^p dt \leqq \int_0^1 \max_k | s_{n_k}(t) |^p dt \leqq K \int_0^1 | f(t) |^p dt \leqq \gamma.$$

Also, for $n_k < n < n_{k+1}$

$$\int_0^1 | s_n(t) |^p dt \leqq A_p \int_0^1 \left| \sum_1^n a_k^2 x_k^2(t) \right|^{p/2} dt \leqq A_p A_p' \int_0^1 \left( \sum_1^{n_k+1} a_n^2 x_n^2(t) \right)^{p/2} dt$$

$$\leqq C_p \int_0^1 | s_{n_{k+1}}(t) |^{p/2} dt \leqq \gamma'.$$

This implies in virtue of Lemma 2 that $s_n$ converges to $f(t)$ almost everywhere.

THEOREM 1. *For any $1 \leqq p < \infty$ the following equivalent statements are valid for $\{ s_m(t) \}$:*

(a) *Weak convergence in $L^p$;*

(b) *Ordinary convergence to a function belonging to $L^p$;*

(c) *Strong convergence in $L^p$;*

(d) *Asymptotic convergence to a function in $L^p$.*

**Proof.** (a)→(b). Since weak convergence of $s_m(t)$ in $L^p$ implies

$$\int_0^1 | s_m(t) |^p dt \leqq \gamma$$

for an absolute constant $\gamma$ independent of $m$, an application of Lemma 2 gives the result.

(b)$\rightarrow$(c). This is a consequence of the proof of Lemma 2.

(c)$\rightarrow$(a), (c)$\rightarrow$(d). These are well known.

(d)$\rightarrow$(b). The hypothesis of (iv) yields the existence of a subsequence $s_{n_k}\rightarrow f(t)$, where $f(t)\in L^p$. An application of Lemma 2a completes the proof. Q.E.D.

It is to be remarked that in a certain sense the hypothesis of Theorem 1 are best possible. For it is well known that there exists an independent series $\sum_1^\infty x_n(t)$ which converges almost everywhere but which does not converge in norm of $L^2$. We construct such an example. Consider the set of functions $g_n(t)$ defined on the interval of (0, 1) as follows:

$$g_n(t) = \begin{cases} a_n, & 0 \leqq t \leqq p_n/2, \\ -a_n, & p_n/2 \leqq t \leqq p_n, \\ 0, & p_n \leqq t \leqq 1, \end{cases}$$

where $a_n$ shall denote a sequence of real numbers increasing sufficiently rapidly. There exists a sequence of independent functions $x_n(t)$ equimeasurable with $g_n(t)$ respectively [4, p. 62]. Thus each $x_n(t)$ has a mean value zero. If we consider the set of $t$ where

$$\underset{n}{E} = \underset{t}{E}\,(x_n(t) = 0,\ x_{n+1}(t) = 0, \cdots),$$

then clearly

$$\underset{n}{E} \leqq \underset{n+1}{E} \leqq \underset{n+2}{E} \leqq \cdots .$$

It follows from the hypothesis on $a_n$ that a point of convergence of the series $\sum x_n(t)$ necessarily requires that the value $t$ lie in the set $E_n$ from some $m$ on.

On account of the independence of $x_n(t)$ and the definition of $g_n(t)$, we obtain

$$\text{measure}\left[\prod_m^\infty E_n\right] = \prod_m^\infty \text{measure } m(E_n) = \prod_m^\infty (1 - p_n) \rightarrow 1$$

provided that $\sum p_n < \infty$. If we evaluate

$$\int_0^1 \left| \sum_1^m x_n(t) \right|^2 dt = \sum_1^m a_n^2 p_n,$$

then on choosing $\sum a_n^2 p_n = \infty$, $\sum p_n < \infty$ we find that $\sum x_n(t)$ converges almost everywhere, but does not converge in the norm of $L^2$. In view of Theorem 1, it is clear that $f(t) = \sum_1^\infty x_n(t)$ cannot belong to any $L^p$ class for $p \geqq 1$.

As an immediate consequence of Theorem 1 and Lemma 2a, we have the following corollary.

COROLLARY 1. *Weak, strong or ordinary convergence to a function in* $L^p$ $(1 \leqq p < \infty)$ *of any subsequence* $s_{n_k}$ *implies the same conclusion for the entire sequence.*

This statement asserts that if $s_n(t)$ does not converge almost everywhere to a function in $L^p$, then no subsequence can converge to a function in $L^p$.

We apply Lemma 2 in another direction. It is well known [6] that for $p > 1$, a necessary and sufficient condition for weak convergence in $L^p$ $(p > 1)$ of a sequence of functions $\sum_1^m a_l x_l(t) = s_m(t)$ is that for every $u$, we have

$$\int_0^u s_n(t)dt \rightarrow \int_0^u s(t) \qquad \text{where } s(t) \in L^p, \tag{4}$$

$$\int_0^1 | s_n(t) |^p dt \leqq \gamma. \tag{5}$$

If we consider $x_n(\theta) = r_n(\theta)$ where $r_n(\theta)$ are the Rademacher functions, then if

$$\int_0^u \sum_1^m a_n r_n(\theta) d\theta \rightarrow \int_0^u f(t)dt$$

for every $u$ where $f(t)$ is in $L^p$ $(p \geqq 1)$, then $\sum_1^m a_n r_n(\theta)$ converges almost everywhere to $f(t)$. In virtue of Lemma 2, this implies

$$\int_0^1 \left| \sum_1^m a_n r_n(\theta) \right|^p d\theta \leqq \gamma.$$

Hence, condition (4) above is necessary and sufficient in the case of the Rademacher functions for weak convergence in $L^p$. In the case of a general independent series, we can assert statements of the following form:

If

$$\int_0^1 | s_n(t |) \, dt \leqq \gamma, \tag{6}$$

$$\int_0^u s_n(t)dt \rightarrow \int_0^u s(t)dt \tag{7}$$

where $s(t)$ is in $L^p$ $(p \geqq 1)$, then $s_n(t)$ converges weakly to $s(t)$ in $L^p$. For, indeed, (6) implies in virtue of Theorem 1 that for every $u$

$$\int_0^u s_n(t)dt \rightarrow \int_0^u f(t)dt \quad \text{and} \quad s_n(t) \rightarrow f(t) \tag{8}$$

a.e., where $f(t)$ is in $L$. This together with (7) yields that $f(t)=s(t)$ almost everywhere. Finally, Lemma 2 gives the result.

We proceed now to discuss other questions related to convergence of

$$s_n(t) = \sum_1^n a_l x_l(t).$$

We establish first an important lemma.

LEMMA 3. *If* $|\lambda_n| \leqq M$, *then for* $m=1, 2, \cdots,$

$$\int_0^1 \left| \sum_1^m \lambda_n a_n x_n(t) \right|^p dt \leqq k(p, M) \int_0^1 \left| \sum_1^m a_n x_n(t) \right|^p dt \qquad \text{for } 1 \leqq p < \infty.$$

**Proof.** A double application of Lemma 1 gives

$$\begin{aligned} \int_0^1 \left| \sum_1^m \lambda_n a_n x_n(t) \right|^p dt &\leqq A_p \int_0^1 \left| \sum_1^m a_n^2 \lambda_n^2 x_n^2(t) \right|^{p/2} dt \\ &\leqq M^p A_p^2 \int_0^1 \left| \sum_1^m a_n^2 x_n^2(t) \right|^{p/2} dt \qquad (9) \\ &\leqq M A_p A_p' \int_0^1 \left| \sum_1^m a_n x_n(t) \right|^p dt. \end{aligned}$$

THEOREM 2. *If* $s_n(t)$ *converges to* $f(t)$ *a.e., where* $f(t) \in L^p$ $(1 \geqq p < \infty)$, *then*

$$\int f(t) g(t) dt = \sum_1^\infty a_n b_n,$$

*the sum converges absolutely with* $a_n$ *and* $b_n$ *the Fourier coefficients of* $f(t)$ *and* $g(t)$ *respectively with respect to* $x_n(t)$ [$g(t)$ *is any function in* $L^{p'}$].

**Proof.** The hypothesis implies in virtue of Lemma 1 that $\int_0^1 |s_m(t)|^p dt \leqq \gamma$. Applying Lemma 3, we have

$$\int_0^1 \left| \sum_1^m \lambda_n a_n x_n(t) \right|^p dt \leqq \gamma'.$$

This implies in virtue of Theorem 1 that for any $g(t)$ in $L^{p'}$

$$\sum \lambda_n a_n b_n < \infty$$

for every bounded sequence $|\lambda_n| \leqq 1$. In particular, putting $\lambda_n = \text{sign } a_n b_n$ shows that $\sum |a_n b_n| < \infty$. Moreover, clearly

$$\int fg = \sum a_n b_n.$$

THEOREM 3. *If* $s_n(t) \to f \in L^p$ *almost everywhere for* $p \geqq 1$, *then* $s_n(t)$

$=\sum_1^M a_l x_l(t)$ *converges unconditionally* (*by every rearrangement of its terms* $a_l x_l(t)$) *to* $f(t)$.

**Proof.** In virtue of Lemma 2 and Lemma 3, we obtain for any $|\lambda_n| \leqq 1$

$$\int_0^1 \left| \sum_1^m \lambda_n a_n x_n(t) \right|^p dt \leqq \gamma.$$

Choosing $\lambda_n$ to be 0 or 1, we obtain

$$\int_0^1 \left| \sum_1^m a_{n_i} x_{n_i}(t) \right|^p dt \leqq \gamma$$

for every subseries. This in virtue of a well known result [6, chap. 1] gives the unconditional strong convergence in $L^p$. Applying Theorem 1 yields the ordinary unconditional convergence. To show that every rearrangement converges to the same function, we have by Lemma 1 for $M$ sufficiently large that for any $N > M$

$$\int_0^1 \left| \sum_1^N a_{n(q)} x_{n(q)}(t) - \sum_1^N a_n x_n(t) \right|^p dt \leqq A_p' \int_0^1 \left| \sum_{k=L(M)}^{\infty} a_n^2 x_n^2 \right|^{p/2} dt \leqq \epsilon.$$

Moreover, since

$$\int_0^1 \left| f(t) - \sum_1^N a_n x_n(t) \right|^p dt \to 0, \qquad \int \left| g(t) - \sum_1^N a_{n(q)} x_{n(q)} \right|^p dt \to 0,$$

we infer that $f(t) = g(t)$ almost everywhere.

THEOREM 4. *If* $\left| \sum_1^m a_n x_n(t) \right| \leqq K$ *a.e., then* $\sum_1^\infty |a_n x_n(t)| < \infty$ *a.e.*

**Proof.** We first deal with the case of Rademacher functions $r_n(\theta)$. Let us suppose that $\left| \sum_1^m a_n r_n(\theta) \right| \leqq K$ a.e. We introduce the positive kernel $\prod_1^m (1 + r_k(\theta))$. For any $|\lambda_k| \leqq 1$, on account of the independence of $r_n(\theta)$,

$$\sum_1^m a_n \lambda_n = \int_0^1 \sum_1^m a_n r_n(\theta) \prod_1^m (1 + \lambda_k r_k(\theta)) d\theta$$

$$\leqq \max_m |s_m| \int_0^1 \prod_1^m (1 + \lambda_k r_k(\theta)) d\theta \leqq K.$$

Putting $\lambda_n = \operatorname{sign} a_n$, we get $\sum |a_n| < \infty$. We proceed now to the general case where

$$\left| \sum_1^m a_n x_n(t) \right| \leqq K. \tag{10}$$

In the familiar way, we insert the Rademacher functions and we obtain for a fixed $\theta$

$$\left| \sum_1^m a_n x_n(t) r_n(\theta) \right| \leqq \left| \sum_1^{m_i} a_{n_i} x_{n_i}(t) \right| + \sum_1^{q_i} a_{n_j} x_{n_j}(t). \tag{11}$$

In virtue of Theorem 3, (10), and Lemma 2 with $p=\infty$, we have that for each $\theta$ almost everywhere in $t$

$$\left| \sum_1^m a_n x_n(t) r_n(\theta) \right| \leqq 2K.$$

On account of Fubini's theorem this implies that for almost every $t$

$$\left| \sum_1^m a_n x_n(t) r_n(\theta) \right| \leqq C$$

almost everywhere in $\theta$. In view of the first part of the proof, we obtain almost everywhere in $t$ that $\sum |a_n x_n(t)| < \infty$, which completes the proof.

COROLLARY 1. *If* $\sum_1^m a_n x_n(t)$ *converges, a.e. to a bounded function, then* $\sum |a_n x_n(t)| < \infty$ *a.e.*

This follows immediately from Lemma 2 and the theorem. A further remark in this connection is that if $s_n(t) \geqq -A$ almost everywhere, then we assert that $s_n(t)$ converges almost everywhere to an integrable function. Indeed

$$\int_0^1 |s_n(t)| dt \leqq \int_0^1 |s_n(t) + A| dt + \int_0^1 |A| dt = \int_0^1 s_n(t) dt + 2A = 2A.$$

Applying Lemma 2 we obtain our result.

It is of interest to study the relationship of convergence of $\sum_1^m a_k x_k(t)$ and $\sum a_k^2 < \infty$.

We say that $\{x_n(t)\}$ possesses the property $C_p$ if whenever $\sum_1^m a_k x_k(t)$ converges to a function in $L^p$, then $\sum a_k^2 < \infty$. It is sufficient to study the property $C_p$ only for $1 \leqq p < 2$, the case of $p \geqq 2$ being trivial. We now supplement a result of Marcinkiewicz and Zygmund [4, pp. 65–67].

THEOREM 5. *A necessary and sufficient condition that* $\{x_n(t)\}$ *have property* $C_p$ $(1 \leqq p < 2)$ *is that* $\liminf \int_0^1 |x_n(t)|^p dt \geqq \Delta > 0$.

**Proof.** *Necessity.* Let us suppose the contrary that $\liminf \int_0^1 |x_n(t)|^p dt = 0$, then let $n_k$ be determined so that

$$\int_0^1 |x_{n_k}(t)|^p dt \leqq \frac{1}{k^2} \cdot$$

Choose $a_{n_k} = 1$. Otherwise $a_n = 0$.

Using the inequality that for $x, y, z, \cdots \geqq 0$ and $0 < r < 1$

$$(x + y + z + \cdots)^r < x^r + y^r + z^r + \cdots,$$

since $p/2<1$, we have

$$\int_0^1 \left|\sum_1^m a_n x_n(t)\right|^p dt \leqq A_p \int_0^1 \left|\sum a_n^2 x_n^2(t)\right|^{p/2} dt \leqq A_p \int \sum |a_n x_n(t)|^p dt$$

$$\leqq \int \sum |x_{n_k}(t)|^p dt \leqq \sum \frac{1}{k^2} \leqq \gamma.$$

In virtue of Lemma 2, we obtain that $\sum_1^m a_k x_k(t)$ converges to $f(t) \in L^p$. However, since $a_n = 1$ infinitely often, $\sum a_n^2 < \infty$ which contradicts the hypothesis.

*Sufficiency*. Since

$$\left(\int \left(\sum f_n\right)^k\right)^{1/k} \geqq \sum \left(\int f_n^k\right)^{1/k} \qquad \text{for } 0 < k < 1$$

where $f_n \geqq 0$, we have, since $p/2<1$, that

$$\sum_1^m \left(\int |a_n x_n(t)|^p dt\right)^{2/p} = \sum_1^m \left(\int |a_n^2 x_n^2(t)|^{p/2} dt\right)^{2/p}$$

$$\leqq \left(\int \left|\sum a_n^2 x_n^2\right|^{p/2} dt\right)^{2/p} \tag{12}$$

$$\leqq A_p' \left(\int_0^1 \left|\sum a_n x_n(t)\right|^p\right)^{2/p}.$$

If $\sum_1^m a_n x_n(t) \to f(t) \in L^p$, then by Lemma 2 and (12), we get

$$\sum |a_n|^2 \left(\int |x_n(t)|^p dt\right)^{2/p} \leqq \gamma_p.$$

In virtue of the hypothesis, we obtain $\sum a_n^2 < \infty$, which completes the proof.

As a consequence of the proof, we have the following corollary.

COROLLARY 1. *If* $\liminf \int_0^1 |x_n(t)|^p dt \geqq \Delta > 0$ *for* $1 \leqq p < 2$, *then*

$$\left(\int \left|\sum_1^m a_n x_n(t)\right|^2 dt\right)^{1/2} \leqq C_p \left(\int \left|\sum_1^m a_n x_n(t)\right|^p dt\right)^{1/p}$$

*for any sequence of constants* $a_n$.

It is well known from the general moment problem that given a system of functions $\{\phi_n(t)\}$, a necessary and sufficient condition that for a sequence of constants $a_k$ there exist a function $g(t)$ in $L^p$ with $a_k = \int g\phi_k$ is that

$$\left|\sum_1^m a_k h_k\right| \leqq \gamma \left[\int_0^1 \left|\sum_1^m h_k \phi_k(t)\right|^{p'} dt\right]^{1/p'} \tag{13}$$

where $h_1, \cdots, h_m$ are any numbers and $\gamma$ is an absolute constant. Let us suppose now that $\sum a_n^2 < \infty$; then there exists a function $f(t)$ in $L^2$ with $a_n$ as Fourier coefficients with respect to $x_n(t)$. In virtue of (13) and Corollary 1, we have that there exists a function in $L^{p'}$ having $a_n$ as coefficients provided that

$$\liminf \int_0^1 |x_n(t)|^p dt \geqq \Delta > 0.$$

Thus, it follows that:

COROLLARY 2. *If* $\liminf \int_0^1 |x_n(t)|^p dt \geqq \Delta > 0$, *then if* $\sum a_n^2 < \infty$ *there exists a function* $f(t)$ *in* $L^{p'}(M)$ *with* $a_n$ *as its Fourier coefficients.*

In the next theorem we shall place an additional restriction on the system, namely:

$$\liminf \int_0^1 |x_n(t)| dt \geqq \Delta > 0. \tag{14}$$

We have already encountered the same hypothesis in the previous theorem. In essence, the class of systems satisfying (14) includes most important independent systems. Any uniformly bounded orthogonal system has this property. Moreover, as is shown in §3, any independent system $x_n(t)$ for which there exists a $p > 2$ such that

$$\int_0^1 |x_n(t)|^p dt \leqq \gamma \qquad (\gamma \text{ is an absolute constant})$$

satisfies (14). Any independent system of functions which are equi-integrable [1] possess this property. In general (14) is assumed in proving most theorems concerning the laws of large numbers and central limit theorems. We have now the following lemma.

LEMMA 4. *If* $\liminf \int_0^1 |x_n(t)| dt \geqq \Delta > 0$, *then for any set* $E$ *of positive measure*

$$\liminf \int_E |x_n(t)| dt \geqq \Delta > 0.$$

**Proof.** If $\sum_1^m b_n x_n(t)$ converges in a set of positive measure $E$, it follows from the law of zero and one that it converges almost everywhere. The hypothesis now easily implies that $\sum b_n^2 < \infty$. We have thus shown that $\sum b_n x_n(t)$ possesses the property that whenever $\sum b_n x_n(t)$ converges in $E$, then $\sum b_n^2 < \infty$. This gives as in the proof of Theorem 6 that $\liminf \int_E |x_n(t)| dt \geqq \Delta > 0$.

THEOREM 6. *If* $\liminf \int_0^1 |x_n(t)| dt \geqq \Delta > 0$, $\int_0^1 x_n^2(t) dt = 1$, *and if*

$$\sum | a_n x_n(t) |^p < \infty$$

*in a set E of positive measure* $(1 \leqq p < 2)$, $\sum | a_n x_n^2(t) |^p < \infty$ *almost everywhere.*

**Proof.** Since $\sum | a_n x_n(t) |^p < \infty$ in $E$ where $m(E) > 0$ there exists a set $E' \subset E$ with $m(E') > 0$ such that in $E'$

$$\sum | a_n x_n(t) |^p < K. \tag{15}$$

In virtue of Lemma 5, we have

$$\int_{E'} | x_n(t) |^p dt \geqq \frac{1}{C} \int_E | x_n(t) | \, dt \geqq \Delta' > 0. \tag{16}$$

Integrating (15) over $E'$ with use of (16) yields $\sum | a_n |^p < \infty$. Integrating, we obtain

$$\int_0^1 | \sum a_n x_n(t) |^p dt \leqq \sum | a_n |^p \int x_n^2(t) dt < \infty.$$

This easily yields that $\sum | a_n x_n(t) |^p$ converges almost everywhere.

COROLLARY 1. *If* $\int x_n^2(t) dt = 1$, $\liminf \int_0^1 | x_n(t) | \, dt \geqq \Delta > 0$, *then the absolute convergence of the series* $\sum a_n x_n(t)$ *in a set of positive measure implies the absolute convergence almost everywhere.*

This is precisely the theorem for the case $p = 1$.

2. **Summability.** One of the fundamental concepts essential in studying orthogonal systems is the Lebesgue kernel $L_n(t)$

$$L_n(t) = \int_0^1 \left| \sum_1^n \phi_k(\theta) \phi_k(t) \right| d\theta.$$

It is customary to assume in working with an orthogonal set of functions that the Lebesgue kernel is uniformly summable by some finite row Toeplitz matrix [6, p. 350], that is,

$$\int_0^1 \left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \phi_l(\theta) \phi_l(t) \right| d\theta \leqq M \qquad \text{a.e.}$$

In this connection we turn to investigate the character of the Lebesgue kernel of an independent system of functions. We assume first that $\int_0^1 x_n(t) dt = 0$ for each $n$. We develop many results prior to this investigation.

LEMMA a. *If* $\sigma_i = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l$ *converges for every* $\epsilon_l = \pm 1$, *then* $\sum | a_l | < \infty$.

**Proof.** We first establish that if $\sum_{k=1}^{n_i} A_{ik} \epsilon_k = A_i$ converges for every variation of sign $\epsilon_l = \pm 1$, $\sum_{k=1}^{\infty} | A_{ik} | \leqq C$ for every $i$. If we observe that $\{ (A_{ik})_k = x_i \}_i$ is a sequence of elements in $(l)$ (space of absolutely convergent series), then the hypothesis implies that $x_i$ converges weakly for all functionals $f$ of the form $f=\{\epsilon_k\}$. A result of Banach [11, pp. 138–139] implies that $x_i$ are strongly convergent and hence $\|x_i\| = \sum_{k=1}^{n_i}|A_{ik}| \leqq C$. We now complete the proof. Since

$$\sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l \doteq \sum_{l=1}^{n_i} \epsilon_l \left[ a_l \sum_{k=l}^{n_i} b_{ik} \right] = \sum_{l=1}^{n_i} \epsilon_l A_{il}$$

converges for every $\epsilon_l = \pm 1$, we have in view of the preceding, that

$$\text{(a)} \qquad \left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l \right| \leqq \sum_{l=1}^{n_i} \left| a_l \sum_{k=1}^{n_i} b_{ik} \right| = \sum_{l=1}^{n_i} | A_{il} | \leqq C.$$

If we replace $\epsilon_l$ by $r_l(\theta)$, then the hypothesis states that $\sigma_i(\theta)$ converges for every $\theta$. In virtue of a known result [10, pp. 122–125], this implies that $s_m(\theta) = \sum_{l=1}^{m} a_l r_l(\theta)$ converges a.e. This, combined with (a), yields on account of the proof of Theorem 4 that $\sum|a_n| < \infty$. This completes the proof.

Lemma 5. *If* $\sigma_i(t) = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} a_l x_l(t)$ *and* $\lim_{i,j\to\infty} \int_0^1 |\sigma_i(t) - \sigma_j(t)|^p dt = 0$ *for* $p \geqq 1$, *then*

$$\int_0^1 | s_n(t) |^p dt \leqq \gamma.$$

**Proof.** If $\sigma_i'(t) = \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l x_l(t)$ with $\epsilon_l = \pm 1$ then a double application of Lemma 1 and the hypothesis of the lemma yields that

$$\lim_{i,j\to\infty} \int_0^1 | \sigma_i'(t) - \sigma_j'(t) |^p dt = 0$$

for any $\epsilon_l = \pm 1$. This implies that for any $b_k$ the Fourier coefficients of any functions in $L^{p'}(M)$, we have that

$$\lim_{i} \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} \epsilon_l a_l b_l$$

exists. In virtue of Lemma a, this gives that $\int_0^1 |s_n(t)|^p dt \leqq \gamma$, for otherwise, there exists a function in $L^{p'}(M)$ such that $|\sum a_k b_k| = \infty$, which is impossible.

It is clear in virtue of Lemma 5 and Lemma 2 that:

Lemma 6. *If* $\int |\sigma_i(t) - \sigma_j(t)|^p dt \to 0$ for $p \geqq 0$, *then*

$$\int | s_i(t) - s_j(t) |^p \to 0.$$

Furthermore, we establish the following lemma.

Lemma 7. *If* $\int_0^1 |\sigma_i(t)|^p dt \leqq \gamma$ *for* $p > 1$, *then*

$$\int_0^1 |s_n(t)|^p dt \leqq \gamma' \qquad \text{for } p > 1.$$

**Proof.** The hypothesis implies the existence of a subsequence $\sigma_{i_n}(t)$ which converges weakly in $L^p$. The selection of a subsequence can be considered as a Toeplitz matrix $B$ operating on $\sigma_i(t)$. If $A$ denotes the original Toeplitz matrix then $BA\{s_n(t)\} = \sigma_{i_n}(t)$, where $BA = C$ is a new Toeplitz matrix. Thus, we have shown the existence of a Toeplitz matrix whose elements $\sigma_i'(t) = C\{s_n(t)\}$ converge weakly in $L^p$ $(p>1)$. Due to a result of Saks [9, p. 51], we can select a subsequence from $\sigma_i'(t)$ which is strongly $(C, 1)$ summable in $L^p$, that is,

$$\int_0^1 \left| \frac{\sum_1^m \sigma'_{i_k}(t)}{m} - \frac{\sum_{k=1}^n \sigma'_{i_k}(t)}{n} \right|^p \to 0 \qquad \text{as } m, n \to \infty.$$

Again, as above, this reduces to a new Toeplitz matrix $E$ with the properties

$$E\{s_n(t)\} = \sigma_i''(t) \qquad \text{such that} \qquad \int |\sigma_i'' - \sigma_j''|^p \to 0.$$

In view of Lemma 5, we obtain the result.

The proof of this lemma is not applicable to $p=1$.

We now use a familiar method introduced by Marcinkiewicz and Zygmund [5, p. 111].

LEMMA 8. *If for any Toeplitz matrix, $\sigma_n$ converges almost everywhere to $f(t)$ which is a function in $L^p$ $(p \geqq 1)$, then $s_n$ converges.*

**Proof.** We introduce an independent system $x_1(t)$, $x_1'(t), \cdots, x_n(t)$, $x_n'(t), \cdots$ with $x_n'(t)$ equimeasurable to $-x_n(t)$ respectively [4, p. 63]. We shall consider the symmetric independent system $\overline{x_n(t)} = x_n + x_n'$. In view of the hypothesis, $\sum_1^k a_n \bar{x}_n(t) r_n(\theta)$ is summable for each $\theta$ to a $p$th integrable function $f(t, \theta)$. Moreover, it is clear from the properties of $\bar{x}_n(t)$ that for any $\theta_1$ and $\theta_2$

$$\int_0^1 |f(t, \theta_1)|^p dt = \int_0^1 |f(t, \theta_2)|^p dt.$$

Hence, if we average over $\theta$, we obtain

$$\int_0^1 d\theta \int_0^1 |f(t, \theta)|^p dt \leqq \gamma. \tag{17}$$

In virtue of the Fubini theorem, we have that for almost every $t$, $\sum_1^k a_n \bar{x}_n(t) r_n(\theta)$ is almost everywhere summable to $f(t, \theta)$ in $\theta$. In virtue of a well known result [10, p. 122], we obtain that $\sum_1^k a_n \bar{x}_n(t) r_n(\theta)$ converges almost everywhere in $\theta$ for almost every $t$ to $f(t, \theta)$ in $L^p$. Moreover, in view of

Lemma 2, we obtain

$$\int_0^1 \left| \sum_1^k a_n \bar{x}_n(t) r_n(\theta) \right|^p d\theta \leqq A \int_0^1 |f(t, \theta)|^p d\theta.$$

On account of (17), if we integrate we get

$$\int_0^1 dt \int_0^1 \left| \sum_1^k a_n \bar{x}_n(t) r_n(\theta) \right|^p d\theta \leqq \gamma'.$$

Using Lemma 1 twice, we have that

$$\int_0^1 \left| \sum_1^k a_n \bar{x}_n(t) \right|^p dt \leqq C$$

which in virtue of a known result [5, p. 110] implies

$$\int_0^1 \left| \sum_1^k a_n x_n(t) \right|^p dt \leqq C.$$

Finally, applying Lemma 2 gives the result.

LEMMA 9. *If $\sigma_i(t)$ is asymptotically convergent to a function in $L^p$, then $s_n(t)$ converges.*

**Proof.** This follows easily from Lemma 8 since the hypothesis insures the existence of a subsequence $\sigma_{i_k}$ convergent to a function in $L^p$. This can be represented as a new Toeplitz matrix operating on $s_n(t)$ whose elements converge to a function in $L^p$. It remains only to apply the preceding lemma.

Lemmas 6, 7, 8 and 9 yield the following theorem.

THEOREM 7. *If for any Toeplitz matrix either*

(a) *$\sigma_i(t)$ converges strongly in $L^p$, $p \geqq 1$,*

(b) *$\sigma_i(t)$ converges weakly in $L^p$, $p > 1$,*

(c) *$\sigma_i(t)$ converges a.e. to a function in $L^p$, $p \geqq 1$, or*

(d) *$\sigma_i(t)$ converges asymptotically to a function in $L^p$ ($p \geqq 1$),*

*then $s_n(t)$ converges to a function in $L^p$.*

We now present two corollaries to Theorem 7.

COROLLARY 1. *If $\left| \sum_{k=1}^{m_i} b_{ik} \sum_{l=1}^{k} a_l x_l(t) \right| \leqq M$ almost everywhere, then a.e.*

$$\sum |a_l x_l(t)| < \infty.$$

**Proof.** The hypothesis implies in virtue of Theorem 7 that

$$\left| \sum_1^m a_l x_l(t) \right| \leqq M$$

almost everywhere. An application of Theorem 4 gives the result.

COROLLARY 2. *If for* $p \geqq 1$

$$\int_0^1 \left| \sum_{k=1}^{n_i} b_{ik} \sum_{n=1}^{k} x_n(t) x_n(\theta) \right|^p dt \leqq M$$

*almost everywhere, then*

$$\int_0^1 \left| \sum_1^k x_n(t) x_n(\theta) \right|^p dt \leqq M$$

*a.e., where* $b_{ik}$ *is a finite row matrix.*

**Proof.** It follows easily on account of the hypothesis that a.e.

$$\text{(a)} \qquad \left| \sum_{k=1}^{n_i} b_{ik} \sum_{n=1}^{k} b_n x_n(t) \right| \leqq M$$

for every $b_n$ which are the Fourier coefficients of a function in $L^{p'}$. Moreover, (a) yields that

$$\int_0^1 \left| \sum_{k=1}^{n_i} b_{ik} \sum_{n=1}^{k} b_n x_n(t) \right|^p dt \leqq \gamma.$$

This implies in virtue of Theorem 10 that $s_n(t)$ converges almost everywhere. Consequently, in view of (a), we get a.e.

$$\left| \sum_{n=1}^{k} b_n x_n(t) \right| \leqq K.$$

The conclusion of the corollary follows easily from this fact.

We remark now that in view of Lemma 8 all the preceding theorems of this chapter are valid for infinite Toeplitz matrices.

We indicate now how (b) in Theorem 10 can be extended to the case $p=1$. We do this for a finite row positive Toeplitz matrix. First, we specialize even further to the case where $\sum_{k=1}^{n_i} b_{ik} = 1$ for all $i$. The properties of a Toeplitz matrix insure the existence of a subsequence of rows for which

$$\text{(18)} \qquad \sum_{k=l}^{n_{m_i}} b_{m_i k} \leqq \sum_{k=l}^{n_{m_{i+1}}} b_{m_{i+1}, k} \qquad \text{for every } l.$$

For convenience of notation, we shall assume that all rows of the Toeplitz matrix have this property. In virtue of the hypothesis (b) and Lemma 1, we obtain

$$\int_0^1 \left| \sum_1^{n_i} (a_l x_l(t))^2 \left( \sum_{k=l}^{n_i} b_{ik} \right)^2 \right|^{1/2} dt \leqq \gamma.$$

If $F_i = \sum_{l=1}^{n_i} (a_l x_l(t))^2 (\sum_{k=l}^{n_i} b_{ik})^2$ then from (18) it follows that

$$F_i \leqq F_{i+1} \leqq F_{i+2} \leqq \cdots \qquad \text{with } F_i \in L^{1/2}.$$

Interchanging limit with integral, we obtain

$$\int_0^1 \left| \sum_1^m (a_l x_l(t))^2 \right|^{1/2} dt \leqq \gamma.$$

Again by Lemma 1, we have

$$\int \left| \sum_1^n a_l x_l(t) \right| dt \leqq \gamma'$$

whence on using Lemma 2, the result follows.

The case where $\sum_{k=1}^{n_i} b_{ik} \neq 1$ but $b_{ik} \geqq 0$ can be easily reduced to a similar circumstance so as to yield (22).

THEOREM 8. *If* lim inf $\int_0^1 |x_n(t)| dt \geqq \Delta > 0$ *and $\sigma_i(t)$ converges in a set of positive measure, then $s_n(t)$ converges almost everywhere.*

**Proof.** Using the procedure of Lemma 8, we obtain that $\sum_1^k a_k \bar{x}_k(t) r_k(\theta)$ is summable for each $t$ of a set of positive measure in a set of positive measure of $\theta$. This gives by a known result [10, p. 122] that $\sum a_n^2 \bar{x}_n^2(t) < \infty$ for $t$ in a set of positive measure $E$. Consequently, there exists a set of positive measure $E'$ for which

$$\sum a_n^2 \bar{x}_n^2(t) \leqq k.$$

Since

$$\int_0^1 | \bar{x}_n(t) | \, dt \geqq A_p \int_0^1 | x_n(t) | \, dt \geqq \Delta > 0$$

[5, p. 110], we have, Lemma 5 applied to $\bar{x}_n(t)$, after integrating that $\sum a_n^2 < \infty$. It follows immediately from this that $\sum_1^m a_n x_n(t)$ converges a.e.

We close this chapter with the discussion of the Lebesgue kernel of an independent system.

THEOREM 9. *If* lim inf $\int_0^1 |x_n(t)| dt \geqq \Delta > 0$, *then there exists no finite row Toeplitz matrix which sums the kernel of an independent system. That is to say it is impossible that a.e. we have*

$$\int_0^1 \left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} x_l(t) x_l(\theta) \right| dt \leqq M.$$

**Proof by contradiction.** In virtue of Corollary 2 to Theorem 7, we infer that a.e.

$$\int_0^1 \left| \sum_1^k x_l(t) x_l(\theta) \right| dt \leqq M.$$

In view of Lemma 1, we obtain that a.e.

$$\int_0^1 \left| \sum x_l^2(t) x_l^2(\theta) \right|^{1/2} dt \leqq C$$

or for almost every $\theta$ in $E$ with $m(E)=1$

$$\sum x_l^2(\theta) x_l^2(t)$$

converges almost everywhere in $t$. Applying, for $\theta \in E$, the Egoroff Theorem and Lemma 5, we secure that $\sum x_l^2(\theta)$ converges a.e. Another application of the Egoroff Theorem and Lemma 5 gives that $\sum_1^\infty 1 < \infty$, which is impossible.

In the case when $x_n(t)$ are uniformly bounded, it can be shown that there can exist no Toeplitz matrix such that

$$\int \left| \sum_{k=1}^{n_i} b_{ik} \sum_{l=1}^{k} x_l(\theta) x_l(t) \right| dt \leqq M$$

in a set $E$ of $\theta$ of positive measure.

This result draws a sharp contrast between the kernel of an independent system and the kernel of the Walsh system (completion of Rademacher series) whose kernel is summable by the Cesàro matrix.

3. **Independence and lacunarity.** We now exhibit several connections of lacunary orthogonal systems with independent systems. We first proceed to define what constitutes a lacunary orthogonal system.

We assume throughout this chapter that

$$\int_0^1 x_n(t) dt = 0, \qquad \int_0^1 x_n^2(t) dt = 1.$$

An orthogonal system $\phi_n(t)$ is said to be lacunary of order $p > 2$ if whenever $\sum a_n^2 < \infty$, then $\sum a_n \phi_n(t)$ converges strongly in $L^p$. It has been shown by Steinhaus that this is equivalent to

$$\left( \int \left| \sum_1^m a_n x_n(t) \right|^p dt \right)^{1/p} \leqq \mu_p \left( \sum a_n^2 \right)^{1/2} \tag{19}$$

[6, chap. 7]. We show first:

THEOREM 10. *A necessary and sufficient condition that an independent system $x_n(t)$ be lacunary of order $p$ ($p > 2$) is that $\int_0^1 |x_n(t)|^p dt \leqq \gamma$ for all $n$.*

**Proof.** *Sufficiency.* Using the Minkowski inequality

$$\left( \int \left( \sum f_n \right)^p \right)^{1/p} \leqq \sum \left( \int f_n^p \right)^{1/p} \qquad \text{for } p > 1 \text{ and } f_n \geqq 0,$$

we have in virtue of Lemma 1, since $p/2 > 1$,

$$\left(\int_0^1 \left|\sum a_n x_n(t)\right|^p dt\right)^{2/p} \leqq A_p\left(\int \left|\sum a_n^2 x_n^2(t)\right|^{p/2} dt\right)^{2/p}$$

$$\leqq A_p \sum\left(\int (a_n^2 x_n^2(t))^{p/2}\right)^{2/p}$$

$$\leqq A_p \sum a_n^2\left(\int_0^1 |x_n(t)|^p dt\right)^{2/p} \leqq \gamma_p \sum a_n^2.$$

Thus (19) has been established, which completes the proof of sufficiency.

*Necessity.* If $f \in L^{p'}$ $(p>2)$ and $a_n = \int_0^1 f x_n(t) dt$, then

$$\sum_1^m a_n^2 = \int_0^1 f \sum_1^m a_n x_n(t) dt \leqq \left(\int_0^1 |f|^{p'}\right)^{1/p'} \left(\int \left|\sum_1^m a_n x_n(t)\right|^p dt\right)^{1/p}.$$

This implies with the aid of (19) that

$$\sum_1^m a_n^2 \leqq \left(\int |f|^{p'}\right)^{1/p'} \mu_p \left(\sum_1^m a_n^2\right)^{1/2}$$

or

$$\left(\sum_1^m a_n^2\right)^{1/2} \leqq \mu_p \left(\int |f|^{p'}\right)^{1/p'}.$$

Consequently, $\sum a_n^2 < \infty$, hence $|a_n| \leqq M$. Suppose lim sup $\int_1^0 |x_n(t)|^p dt = \infty$, then there exists a function $f \in L^{p'}$ such that lim sup $\left|\int_1^0 f x_n(t) dt\right| = \infty$, which is a contradiction. Q.E.D.

In particular if $x_n(t)$ is uniformly bounded the system is lacunary of every order greater than 2. This also follows from Lemma 1. Moreover, the proof of the sufficiency establishes the following corollary.

COROLLARY 1. *If* $\int_0^1 |x_n(t)|^p dt \leqq \gamma$ *for* $p>2$, *then*

$$\left(\int_0^1 \left|\sum_1^m a_n x_n(t)\right|^p dt\right)^{1/p} \leqq \gamma_p \left(\int \left|\sum_1^m a_n x_n(t)\right|^2\right)^{1/2}.$$

The hypothesis $\int_0^1 |x_n(t)|^p dt \leqq \gamma$ if applied to any orthogonal system guarantees the existence of a subsequence $x_{n_k}$ which forms a lacunary system [6, chap. 7].

COROLLARY 2. *If for every function* $f \in L^{p'}$ $(1<p'<2)$ *with* $a_n = \int f x_n$ *we have* $|a_n| \leqq M_f$, *then* $\sum a_n^2 < \infty$ *for every function in* $L^{p'}$.

**Proof.** The hypothesis clearly implies $(\int |x_n(t)|^p dt)^{1/p} \leqq \gamma$, whence as in the proof we obtain $\sum a_n^2 < \infty$.

COROLLARY 3. *If* $x_n(t)$ *is lacunary of order* $p>2$, *and if* $\sum |a_n x_n(t)|^q < \infty$

*for a set E of positive measure, then* $\sum |a_n x_n(t)|^q < \infty$ *almost everywhere.*

**Proof.** Since $x_n(t)$ is lacunary, it follows that $\int_0^1 |x_n(t)| \geqq \Delta > 0$ [6, chap. 7]. The remainder of the proof follows as in Theorem 6.

We present other conditions for lacunarity.

THEOREM 11. *If* $\int_0^1 |x_n(t)|^{p'} dt \geqq \Delta > 0$ $(p>2)$, *then a necessary and sufficient condition that* $x_n(t)$ *is lacunary of order* $p$ *is that the expansion of every* $f \in L^p$ *converge to a function in* $L^p$.

REMARK. Since $\sum a_n^2 < \infty$, the expansion of every $f \in L^p$ necessarily converges.

*Necessity.* Since $x_n(t)$ is lacunary, we have for any $a_n$ the Fourier coefficient of $f \in L^p$

$$\int_0^1 \left| \sum_1^m a_n x_n(t) \right|^p dt \leqq k_p \left(\sum a_n^2\right)^{p/2} \leqq \gamma.$$

An application of Lemma 2 gives the result.

*Sufficiency.* In virtue of Lemma 2, we have for every $f \in L^p$

$$\int_0^1 |s_n(t)|^p dt \leqq \gamma.$$

In view of Theorem 1, we have for every $f$ in $L^p$

$$\int_0^1 |s_n(f) - s_m(f)|^p dt \to 0.$$

This gives for every $g(t)$ in $L^{p'}$

$$\int_0^1 |s_n(g) - s_m(g)|^{p'} dt \to 0.$$

Whence, we have

$$|b_n| \left(\int_0^1 |x_n(t)|^{p'}\right)^{1/p'} \leqq k$$

or

$$|b_n| \leqq k_f.$$

As a consequence of Corollary 2 to Theorem 12 the lacunarity of order $p$ follows easily from this fact.

It is well known that lacunary trigonometric series behave like independent series and we expect most of the results of the three chapters to remain valid for such series. It is first necessary to remark that for general Fourier series, we have the inequality

$$\int_0^1 \max_k |s_{n_k}(t)|^p dt \leqq k_p \int |f|^p \qquad \text{for } p > 1 \tag{20}$$

where

$$s_{n_k} = \frac{a_0}{2} + \sum_1^{n_k} [a_n \cos nx + b_n \sin nx],$$

$$n_{k+1} \geqq \lambda n_k \quad \text{for} \quad \lambda > 1, \qquad \begin{matrix} a_n \\ b_n \end{matrix} = \int f \begin{matrix} \cos nx \\ \sin nx \end{matrix} dx.$$

In the case of lacunary trigonometric series the ordinary partial sums of such series coincide with $s_{n_k}$. Moreover, for lacunary trigonometric series, we have

$$A_p(\sum a_n^2 + b_n^2)^{p/2} \leqq \int |s_n|^p \leqq A_p'(\sum a_n^2 + b_n^2)^{p/2}. \tag{21}$$

Clearly, (20) and (21) occupy the analogous role for the inequalities of Lemma 1. In the case of Lemma 5, it suffices to remark that if a lacunary trigonometric series is summable by a Toeplitz matrix in a set of positive measure it converges almost everywhere.

Thus the methods of this paper yield many new proofs of known theorems concerning lacunary trigonometric series.

For a general lacunary orthogonal system it is now known whether

$$\int_0^1 \max_{1 \leqq n \leqq m} |s_n(t)|^p dt \leqq k_p \int_0^1 |s_m(t)|^p dt$$

holds and hence we cannot assert the validity of the results of this paper for such systems of functions.

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