

HILBERT'S CHARACTERISTIC FUNCTION AND THE ARITHMETIC GENUS OF AN ALGEBRAIC VARIETY

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1. Introduction. The properties of the characteristic function of Hilbert associated with a doubly homogeneous ideal $\mathfrak{a}$ in a ring $R = k[X, Y]$ of polynomials in two sets of indeterminates (X) and (Y) were studied by van der Waerden in [1]⁽²⁾. Since such an ideal can be regarded as defining an algebraic correspondence between the varieties $U = \mathcal{V}(\mathfrak{a} \cap k[X])$ and $V = \mathcal{V}(\mathfrak{a} \cap k[Y])$, van der Waerden's results can be looked upon as belonging to the general theory of algebraic correspondences. In case U and V are birationally equivalent normal varieties⁽³⁾, further results can be given. It is the object of this note to study the properties of the Hilbert characteristic function which is associated with such a pair of normal varieties. A new proof of the invariance of the arithmetic genus of two- and three-dimensional varieties as well as a proof of the Riemann-Roch theorem for a large class of linear systems on an algebraic surface are among the results obtained.

2. Enumerative functions. Let U and V be models of a field Σ of degree of transcendency r over a ground field k . The field k is assumed to be algebraically closed but otherwise arbitrary. Let $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_g)$ and $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_h)$ be nonhomogeneous coordinates of the generic points of U and V respectively, and let x_0 and y_0 be quantities which are algebraically independent over Σ . Let $x_i = x_0 \bar{x}_i$, $i = 1, 2, \dots, g$; $y_j = y_0 \bar{y}_j$, $j = 1, 2, \dots, h$, and form the ring $\mathfrak{o} = k[x_0, x_1, \dots, x_g; y_0, y_1, \dots, y_h]$. This ring is an integral domain with quotient field $\Sigma(x_0, y_0)$. If $R = k[X_0, X_1, \dots, X_g; Y_0, Y_1, \dots, Y_h]$ (where (X) and (Y) are indeterminates) and if τ is the homomorphic mapping of R onto $\mathfrak{o}$ which sends X_i into x_i , Y_j into y_j , then the kernel $\mathfrak{a}$ of τ is a doubly homogeneous prime ideal in R . The characteristic function $\chi(m, n; \mathfrak{a})$ defined in [1] as the number of doubly homogeneous forms of degree m in (X) and of degree n in (Y) which are linearly independent (over k) modulo $\mathfrak{a}$ is an enumerative function determined uniquely by the pair of models (U, V) of the field Σ . Van der Waerden's main result as-

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⁽²⁾ Numbers in brackets refer to the bibliography.

⁽³⁾ Throughout this paper the term "normal variety" will mean "locally normal variety" in the sense of [5, Definition 3].

serts that there exist integers a_{ij} , M , N such that

$$(2.1) \quad \chi(m, n; \mathfrak{a}) = \sum_r a_{ij} C_{m,i} C_{n,j}, \quad \text{when } m \geq M, n \geq N,$$

where $\sum_r$ denotes the sum over all integers i, j such that $i+j \leq r$. We shall denote the polynomial $\sum_r a_{ij} C_{m,i} C_{n,j}$ associated with the pair (U, V) by $\rho(m, n)$.

Let $\{A_m\}^{(*)}$ ($\{B_n\}$) denote the system of $(r-1)$ -dimensional subvarieties cut out on U (V) by the hypersurfaces of order m (n) in its ambient space S_θ (S_h). We regard each of these systems as lying on the join W of U and V , and there we consider the minimal sum $\{A_m + B_n\}$ of $\{A_m\}$ and $\{B_n\}$ together with the complete sum $|A_m + B_n|$. Two enumerative functions $r(m, n)$ and $s(m, n)$ associated with the pair (U, V) are defined as follows:

$$\begin{aligned} r(m, n) &= 1 + \dim |A_m + B_n|, \\ s(m, n) &= 1 + \dim \{A_m + B_n\}. \end{aligned}$$

Since it is clear from the above definition of the ideal $\mathfrak{a}$ that $\chi(m, n; \mathfrak{a}) = s(m, n)$, equation (2.1) can be written in the form

$$(2.2) \quad s(m, n) = \rho(m, n), \quad \text{when } m \geq M, n \geq N.$$

Our main interest is in the function $r(m, n)$. We shall show that under suitable restrictions on the models U and V and on the birational correspondence between them, the minimal sum $\{A_m + B_n\}$ is complete for all m when n is large or for all n when m is large. Moreover, for such values of m and n , the equations $r(m, n) = s(m, n) = \rho(m, n)$ hold.

3. Varieties of dimension one. As a starting point for induction proofs to be undertaken later we consider the special case in which Σ is of degree of transcendency one over k . We recall a well known lemma of Castelnuovo which gives a sufficient condition for the minimal sum of two linear series on an algebraic curve to be complete.

CASTELNUOVO'S LEMMA. *If g_μ is a linear series without fixed points on a curve Γ , and if g_ν is a complete non-special series on Γ which partially contains g_μ and is such that the residual series $g_\nu - g_\mu$ is non-special, then the minimal sum of g_ν and g_μ is complete.*

This lemma leads immediately to the following one.

LEMMA 1. *If the variety V (of the pair (U, V)) is a normal curve, there exists an integer n_0 such that the minimal sum $\{A_m + B_n\}$ is complete if $n \geq n_0$ and m is arbitrary.*

Proof. Let π be the genus of Σ and let μ and ν be the orders of U and V respectively. The system $\{A_m\}$ is then a $g_{m\mu}$ while $\{B_n\}$ is a $g_{n\nu}$. Since V is

(*) The notation $\{C\}$ is used to denote a specified linear system of generic member C , and the notation $|C|$ is reserved for the complete system determined by C .

normal, $\{B_n\}$ is complete if n is sufficiently large. We fix n_0 so that (a) $n\nu - \mu > 2\pi - 2$, (b) $g_{n\nu}$ partially contains g_μ , (c) $g_{n\nu}$ is complete, when $n \geq n_0$. We proceed by induction on m since for $m=0$ the lemma is trivially true in view of (c). We therefore assume that the minimal sum of $g_{n\nu}$ and $g_{m\mu}$ is a complete series g_N ($N = m\mu + n\nu$). We observe that g_N is non-special and that it partially contains g_μ . By condition (a), $N - \mu > 2\pi - 2$ so that $g_N - g_\mu$ is also non-special. By Castelnuovo's lemma the minimal sum of g_N and g_μ is complete, and since this minimal sum coincides with the minimal sum of $g_{(m+1)\mu}$ and $g_{n\nu}$, the lemma is proved.

LEMMA 2. *Under the hypotheses of Lemma 1, the ρ -function associated with the pair (U, V) is given by the formula*

$$(3.1) \quad \rho(m, n) = m\mu + n\nu - \pi + 1.$$

Moreover, $r(m, n) = \rho(m, n)$ whenever $m\mu + n\nu > 2\pi - 2$.

Proof. If $n \geq n_0$, then $\{A_m + B_n\}$ is complete so that $r(m, n) = s(m, n)$ and $r(m, n) = \rho(m, n)$ for large values of m and n . Since the complete system $|A_m + B_n|$ is non-special when m and n are large, the Riemann-Roch theorem for curves yields $\rho(m, n) = r(m, n) = m\mu + n\nu - \pi + 1$. The expression $m\mu + n\nu - \pi + 1$ must then give the value of $\rho(m, n)$ for all m and n since ρ is a polynomial. It gives the value of $r(m, n)$ whenever $m\mu + n\nu > 2\pi - 2$, q.e.d.

4. The virtual arithmetic genus. A class of varieties which we shall call *sectionally normal* is defined inductively as follows.

DEFINITION 1. *An irreducible 1-dimensional variety is said to be sectionally normal if it is nonsingular. An irreducible r -dimensional variety is sectionally normal if almost all⁽⁵⁾ of its sections by the hypersurfaces of its ambient space are sectionally normal varieties of dimension $r-1$.*

We next define a class of birational transformations.

DEFINITION 2. *A birational transformation T defined on an irreducible variety $W \subset S_n$ will be called a proper transformation if $T(W \cap S_t)$ is normal for almost all linear subspaces S_t of S_n ($t = n - r + 1, \dots, n$). In particular, $T(W)$ is required to be normal.*

We point out that any normal curve or surface is sectionally normal as is any nonsingular variety of arbitrary dimension⁽⁶⁾; moreover, if T is a regular birational transformation defined on a sectionally normal variety W , then T is proper. We shall have occasion to point out in the next section that there are proper transformations which are not regular.

⁽⁵⁾ The term "almost all" here means all with the possible exception of a proper algebraic subsystem.

⁽⁶⁾ It has recently been proved by A. Seidenberg (but not yet published) that almost all hyperplane sections of a normal variety V are normal varieties. This implies that every normal variety is sectionally normal.

THEOREM 1. *If the transformation $T: U \rightarrow V$ is proper then there exists an integer n_0 such that the minimal sum of $\{A_m\}$ and $\{B_n\}$ is complete when $n \geq n_0$, $m \geq 0$.*

Proof. The proof is by induction with respect to the dimension r of the field Σ of rational functions on U and V . By Lemma 1 the theorem is true if Σ is of dimension one over k and we assume that it is true if Σ is of dimension $r-1$ over k . The passage from $r-1$ to r is effected by induction with respect to m . Since T is proper, the variety V is normal so that the minimal sum $\{A_m + B_n\}$ is complete if $m=0$ and n is sufficiently large. We assume that the minimal sum $\{A_{m-1} + B_n\}$ is complete if n is large.

Let A be an irreducible hyperplane section of U such that $A' = T(A)$ is normal, and the transformation $T': A \rightarrow A'$ induced by T is proper. (Such hyperplane sections exist in view of the theorem of Bertini [4] and the fact that T is proper⁽⁷⁾.) If $\{\bar{A}_m\}$ and $\{\bar{B}_n\}$ are the linear systems cut out on A and A' by the hypersurfaces of their respective ambient spaces, then these systems are cut out on A and A' by the systems $\{A_m\}$ and $\{B_n\}$. By the induction assumption the minimal sum $\{\bar{A}_m + \bar{B}_n\}$ is complete if $n \geq \bar{n}_0$. We denote its dimension (increased by one) by $\bar{r}(m, n)$, and we fix n_0 ($\geq \bar{n}_0$) so that $\{B_n\}$ is complete when $n \geq n_0$. If G is the minimal sum of $\{A_m\}$ and $\{B_n\}$, then G cuts out the complete system $|\bar{A}_m + \bar{B}_n|$ on A , the dimension of which is $\bar{r}(m, n) - 1$. The residual system of G with respect to A is the minimal sum $\{A_{m-1} + B_n\}$, which by the induction assumption is complete. Since the dimension of $|A_{m-1} + B_n|$ is given by $r(m-1, n) - 1$ we conclude that

$$1 + \dim G = r(m-1, n) + \bar{r}(m, n).$$

⁽⁷⁾ The theorem of Bertini is proved in [4] under the hypothesis that k is of characteristic zero. In the special case of the hyperplane sections of an irreducible variety the following considerations remove this restriction on the characteristic. Let V/k be an irreducible variety in S_n (k is algebraically closed) and let $\mathfrak{p}$ be the prime homogeneous ideal of V in $k[Y]$ ($=k[Y_0, Y_1, \dots, Y_n]$). Let $(u) = (u_0, u_1, \dots, u_n)$ be a set of $n+1$ indeterminates (algebraically independent quantities over k) and let K denote the field generated over k by the quotients of the u 's. If L denotes the linear form $\sum u_i Y_i$, then it is easily seen that the ideal $(\mathfrak{p}, L)$ in $k(u)[Y]$ is prime. This implies that the intersection W of V with the hyperplane $L=0$ is irreducible over K . The variety W is of dimension $r-1$. If m is the order of V , any linear S_{n-r} which is general over k meets V in m distinct points. Hence any linear S_{n-r+1} which is general over K meets W in m distinct points. It follows that also W is of order m .

Let P be a general point of W/K . Then it is clear that P is also a general point of V/k and that K is a pure transcendental extension of $k(P)$ (of transcendence degree $n-1$). Hence K is maximally algebraic in $K(P)$ and therefore W is an absolutely irreducible variety. Also the associated form F of W (in the sense of Chow-van der Waerden) is therefore absolutely irreducible (the coefficients of F may be assumed to be forms in $k[u]$). Hence F remains irreducible for almost all specializations $(u) \rightarrow (\bar{u})$, $\bar{u}_i \in k$. Since in any such specialization, F is specialized to the associated form $\bar{F}$ of the corresponding hyperplane section $\bar{W}$ of V (here $\bar{W}$ is to be thought of as a cycle, not a variety), it follows at once that almost all hyperplane sections of V are irreducible varieties of order m .

However, if H is the complete system $|A_m + B_n|$ determined by G , the same reasoning applied to H would lead to the conclusion that

$$1 + \dim H = r(m-1, n) + \bar{r}(m, n).$$

It follows that $G=H$, q.e.d.

COROLLARY 1. *Under the hypothesis of Theorem 1, the function $r(m, n)$ satisfies the addition formula*

$$(4.1) \quad r(m, n) = r(m-1, n) + \bar{r}(m, n)$$

when $n \geq n_0$.

COROLLARY 2. *The r - and ρ -functions associated with the pair (U, V) are equal when $n \geq n_0$ and m is arbitrary.*

Proof. Let $\bar{\rho}(m, n)$ be the ρ -function associated with the pair (A, A') introduced above. Since the statement is true for varieties of dimension one by Lemma 2, we assume for purposes of induction that $\bar{r}(m, n) = \bar{\rho}(m, n)$ if $n \geq \bar{n}_0$ and $m \geq 0$. Since the minimal sum $\{A_m + B_n\}$ is complete, $r(m, n) = s(m, n)$, so that if both m and n are large, $r(m, n) = \rho(m, n)$. It follows from (4.1) that

$$(4.2) \quad \rho(m, n) = \rho(m-1, n) + \bar{\rho}(m, n)$$

if both m and n are large, and since the ρ -functions are polynomials, (4.2) is valid for all values of m and n . Equations (4.1) and (4.2) together imply that if $r(m, n) = \rho(m, n)$ then also $r(m-1, n) = \rho(m-1, n)$. This completes the proof.

Let W be a normal variety defined by a homogeneous prime ideal $\mathfrak{p}$ in the ring $S = k[X_0, X_1, \dots, X_h]$, and let $\{C_n\}$ be the linear system cut out on W by the hypersurfaces of order n in the ambient space S_h of W . If $r(n) - 1$ is the dimension of the complete system $|C_n|$ and if $\chi(n; \mathfrak{p})$ is the number of forms of degree n in S which are linearly independent modulo $\mathfrak{p}$, then $r(n) = \chi(n; \mathfrak{p})$ if n is sufficiently large, since W is normal. The function $\chi(n; \mathfrak{p})$ is a polynomial in n if n is large;

$$\chi(n; \mathfrak{p}) = \rho(n) = \sum_{i=0}^r a_i C_{n,i},$$

where the coefficients, $a_0, a_1, \dots, a_r$ are integers uniquely determined by W , and r is the dimension of W . We call $\rho(n) - 1$ the *virtual dimension* of $|C_n|$, and note that for large values of n the *effective dimension* $r(n) - 1$ equals the virtual dimension $\rho(n) - 1$.

DEFINITION 3. *The virtual arithmetic genus $p_a(W)$ of the normal r -dimensional variety W is the integer $(-1)^r(a_0 - 1)$.*

THEOREM 2. *If U and V are normal models of Σ and if $T: U \rightarrow V$ and*

$T^{-1}: V \rightarrow U$ are both proper transformations, then $p_a(U) = p_a(V)$.

Proof. Let $r_1(m), \rho_1(m); r_2(n), \rho_2(n)$ be the effective and virtual dimensions (increased by one) of $|A_m|$ and $|B_n|$ respectively, and let $r(m, n), \rho(m, n)$ be the r - and ρ -functions associated with the pair (U, V) . Let $\rho_1(m) = \sum c_{1i} C_{m,i}$, $\rho_2(n) = \sum c_{2j} C_{n,j}$. By definition, $r_1(m) = r(m, 0)$, $r_2(n) = r(0, n)$. Since both T and T^{-1} are proper, it follows by Corollary 2 of Theorem 1 that there exist integers m_0 and n_0 such that

$$\begin{aligned} r(m, 0) &= \rho(m, 0), & \text{if } m \geq m_0, \\ r(0, n) &= \rho(0, n), & \text{if } n \geq n_0. \end{aligned}$$

It follows that

$$\rho_1(m) = \rho(m, 0), \quad \rho_2(n) = \rho(0, n),$$

for all values of m and n . If a_{00} is the constant term in $\rho(m, n)$, then $c_{10} = a_{00} = c_{20}$. It follows that $p_a(U) = p_a(V)$, q.e.d.

COROLLARY 1. *If U and V are sectionally normal models in regular birational correspondence, then $p_a(U) = p_a(V)$. In other words, the virtual arithmetic genus of a sectionally normal model is a relative birational invariant.*

Proof. We have pointed out above that a regular birational transformation defined on a sectionally normal model is proper. Since the inverse of a regular transformation is also regular, the corollary follows.

5. Varieties of dimension two and three. In this article we confine our attention to the cases in which Σ is of dimension two or three over k and we restrict our remarks to *nonsingular* models of such fields. We assume throughout the remainder of the text that k is of characteristic zero.

If U and V are nonsingular models of Σ and if V dominates U , $U < V^{(*)}$, then it is immediately clear that $T: U \rightarrow V$ is proper. In fact, the system $T(\{A_1\})$ on V has no base points (since $U < V$) and since V has no singularities, the general member of $T(\{A_1\})$ has no singularities in view of Bertini's theorem [6]. If Σ is of dimension two this remark suffices to prove that T is proper. If Σ is of dimension three we observe in addition that a generic A_1 and its transform $T(A_1)$ are nonsingular and that $A_1 < T(A_1)$. Hence T induces a proper transformation on a generic A_1 so that it is itself proper.

We consider *quadratic* and *monoidal* transformations in the next two lemmas. These terms are used in the sense of [5, article 11].

LEMMA 3. *If U and V are nonsingular, and if V is obtained from U by a*

(*) The statement " V dominates U " implies that the birational transformation $T^{-1}: V \rightarrow U$ has no fundamental points on V ; or equivalently, the local ring $Q(P')$ contains the local ring $Q(P)$, when $P(\subset U)$ and $P'(\subset V)$ are a pair of corresponding points in the birational correspondence T . This relationship is denoted by the symbols $U < V$.

quadratic transformation with center at a point P of U , then both $T:U \rightarrow V$ and $T^{-1}:V \rightarrow U$ are proper transformations.

Proof. In this case $U < V$ so that T is proper. Let $(\eta_0, \eta_1, \dots, \eta_s)$ be homogeneous coordinates of the general point of U and let $\mathfrak{p}$ be the prime ideal of P in the ring $\mathfrak{o} = k[\eta]$. If $\phi_0, \phi_1, \dots, \phi_h$ form a linear basis for the forms of degree ν in $\mathfrak{p}$, and if ν is sufficiently large, then the ideal $\mathfrak{a} = \sum \mathfrak{o}\phi_i$ differs from $\mathfrak{p}$ by at most an irrelevant component and $(\phi_0, \phi_1, \dots, \phi_h)$ can be regarded as the general point of V . Hence the linear system defined by $\phi(\lambda) = 0$, where $\phi(\lambda) = \lambda_0\phi_0 + \lambda_1\phi_1 + \dots + \lambda_h\phi_h$, $\lambda_i \in k$, is the inverse transform of the system $\{B_1\}$ of hyperplane sections of V . The point P is the only base point of this system, and since U is nonsingular, the generic member of this system has no singular points except possibly at P itself. If we assume that P is not on the hyperplane section $\eta_0 = 0$, and if we put $\psi_i = \phi_i/\eta_0^\nu$, then the quantities $\psi_0, \psi_1, \dots, \psi_h$ will form a basis for the prime ideal of P in the quotient ring $Q(P/U)$. It follows that among the ψ_i there is at least one whose leading form at P is linear so that almost all members of $\{\phi(\lambda)\}$ have a simple point at P (see [6, p. 137]). The generic member of $\{\phi(\lambda)\}$ is therefore nonsingular and hence normal. This proves that T^{-1} is proper if U and V are surfaces.

If Σ is of dimension three, it is necessary to prove in addition that the general characteristic curve of the system $\{\phi(\lambda)\}$ is nonsingular, since such a curve is the transform by T^{-1} of a section $V \cap S_{h-2}$, where S_h is the ambient space of V . However, it is proved in [5] that a quadratic transformation with center at a point P of U induces a quadratic transformation on any subvariety of U through P . If we apply this remark to a generic member $\bar{\phi}$ of the system $\{\phi(\lambda)\}$ we conclude as above that the generic characteristic curve $\bar{\phi} \cap \phi(\lambda)$ has no singularities, q.e.d.

If U and V are three-dimensional varieties, and if T is a monoidal transformation with center along an irreducible curve $\Delta \subset U$, then since U is nonsingular, every point of Δ is a simple point of U . If in addition every point of Δ is a simple point of Δ , then we shall call T a *nonsingular* monoidal transformation.

LEMMA 4. *If U and V are nonsingular and if $T:U \rightarrow V$ is a nonsingular monoidal transformation, then T and T^{-1} are proper.*

Proof. Since $U < V$, T is proper. The proof that T^{-1} is proper is similar to the proof given in Lemma 3 for quadratic transformations. We use the same notations as in Lemma 3, except that now $\mathfrak{p}$ is the prime ideal of Δ in $\mathfrak{o}$. The general member of the system $\{\phi(\lambda)\}$ has no singularities except possibly at points of Δ . If P is a point of Δ not on the surface $\eta_0 = 0$, then $\psi_0, \psi_1, \dots, \psi_h$ will form a basis for the prime ideal of Δ in $\mathfrak{T} = Q(P/U)$. Then, since P is simple both for Δ and for U , it follows that among the ψ_i there will be at least

one whose leading form at P is linear (see [5, footnote 34]). Hence P cannot be a singular base point of $\{\phi(\lambda)\}$. It follows that the generic member of $\{\phi(\lambda)\}$ is nonsingular.

To complete the proof we must show that the generic characteristic curve of $\{\phi(\lambda)\}$ is nonsingular, and for this it suffices to show, in view of Bertini's theorem, that the system cut by $\{\phi(\lambda)\}$ on a generic member ϕ (outside the fixed curve Δ) has no base points on Δ . The total transform $T(\Delta)$ of Δ is an irreducible surface F on V which is "ruled" by a pencil $\{f\}$ of rational curves. The curves f are in 1:1 correspondence with the points of Δ and each f is the total transform of the point of Δ to which it corresponds [7, article 5]. We choose a hyperplane section B_1 of V such that the section $F \cap B_1$ is an irreducible curve Γ which is not a component of any member of the pencil $\{f\}$. We further assume that $T^{-1}(B_1)$ is a nonsingular irreducible member $\bar{\phi}$ of $\{\phi(\lambda)\}$. If $\phi(\lambda) \in \{\phi(\lambda)\}$, we write $\phi(\lambda) \cap \bar{\phi} = \Delta + R(\lambda)$. Let us assume that $P \in \Delta$ is a base point of the residual system $\{R(\lambda)\}$. It then follows that every hyperplane section B'_1 of V meets the section B_1 in a point of the variety $T(P) \cap \Gamma$. Since Γ is not a component of any member of $\{f\}$, it follows that $T(P) \cap \Gamma$ consists of a finite set of points, so that the conclusion that every hyperplane B'_1 meets Γ in a point of $T(P) \cap \Gamma$ is absurd, q.e.d.

These lemmas yield the following further corollary to Theorem 2.

COROLLARY 2. *The virtual arithmetic genus of a nonsingular model of a field Σ of dimension two or three over k is invariant under quadratic and nonsingular monoidal transformations.*

The considerations of part IV of [7] show in particular that if U and V are nonsingular models of a three-dimensional field Σ , then there exist models U_1 and V_1 in regular birational correspondence such that U_1 (V_1) is obtained from U (V) by a sequence of quadratic and nonsingular monoidal transformations. It follows that $p_a(U) = p_a(U_1) = p_a(V_1) = p_a(V)$. Since a similar statement is true for nonsingular surfaces (see [3]) we can assert the following theorem.

THEOREM 3. *If Σ is a field of dimension two or three over k , then any two nonsingular models of Σ have the same virtual arithmetic genus.*

The virtual arithmetic genus of a nonsingular model can therefore be regarded as a character of the field rather than of a particular model. This character is called the *arithmetic genus* of Σ and is denoted by $p_a(\Sigma)$.

6. Normal surfaces. Our objective in this section is to show that if U is any normal model of a field Σ of dimension two over k , then the virtual arithmetic genus of U is not less than the arithmetic genus of Σ , $p_a(U) \geq p_a(\Sigma)$. Since it is known that there exists a nonsingular model V of Σ such that $V > U$ [2], the following theorem will yield the desired result.

THEOREM 4. *If U and V are normal models of Σ (Σ of dimension two), and if $U < V$, then $p_a(U) \geq p_a(V)$.*

The major portion of the proof of this theorem is contained in the following two lemmas.

LEMMA 5. *Let Γ be an irreducible algebraic curve of order ν in S_n , and let $g_{m\nu}$ be the series cut out on Γ by the hypersurfaces of order m in S_n . If g_μ is a linear series on Γ without fixed points, then there exists an integer M such that if $m \geq M$, the deficiency of the minimal sum of $g_{m\nu}$ and g_μ is not greater than the deficiency of $g_{m\nu}$, in symbols, $\delta(g_{m\nu} + g_\mu) \leq \delta(g_{m\nu})$.*

Proof. Let G_μ be a set of the series g_μ and choose nonhomogeneous coordinates on Γ so that no point of G_μ is at infinity. If $\mathfrak{a}$ is the ideal in the ring $\mathfrak{o} = k[x_1, x_2, \dots, x_n]$ of nonhomogeneous coordinates on Γ which is determined by G_μ , then the length of $\mathfrak{a}$ is μ , and there exist μ elements $\phi_1, \phi_2, \dots, \phi_\mu$ in $\mathfrak{o}$ which form an independent k -basis for $\mathfrak{o}$ modulo $\mathfrak{a}$. If m exceeds the degree of each of the polynomials ϕ_i , then the set G_μ will impose μ independent conditions on the series $g_{m\nu}$. That is, if $\bar{g}_{m\nu}$ is the subsystem of $g_{m\nu}$ consisting of those sets of $g_{m\nu}$ which contain G_μ , then $\dim \bar{g}_{m\nu} = \dim g_{m\nu} - \mu$.

Fix M so large that the above condition holds when $m \geq M$. Increase M if necessary so that $g_{m\nu} \supset g_\mu$, and the residual series $g_{m\nu} - g_\mu$ is non-special when $m \geq M$. Let G'_μ be another fixed set of g_μ which has no point in common with G_μ . Consider the two series $g_{m\nu} + G_\mu$ and $g_{m\nu} + G'_\mu$ obtained by adding the fixed sets G_μ and G'_μ to the series $g_{m\nu}$. The union of these two series is contained in the minimal sum of $g_{m\nu}$ and g_μ . The common part of these series is the series $\bar{g}_{m\nu}$. Hence

$$m\nu + \mu - \pi - \delta(g_{m\nu} + g_\mu) \geq 2(m\nu - \pi - \delta(g_{m\nu})) - (m\nu - \pi - \delta(g_{m\nu}) - \mu),$$

and $\delta(g_{m\nu} + g_\mu) \leq \delta(g_{m\nu})$, q.e.d.

LEMMA 6. *If U and V are normal models of Σ such that $U < V$, then there exist integers m_0, d such that*

$$(6.1) \quad \rho(m, n) + d \geq r(m, n) \geq \rho(m, n)$$

when $m \geq m_0$ and $n \geq 0$.

Proof. Since $U < V$, and since U and V are normal, it follows that $T: U \rightarrow V$ is proper, so that by Corollary 2 of Theorem 1 there exists an integer n_0 such that $r(m, n) = \rho(m, n)$ when $n \geq n_0, m \geq 0$. Let B_1 be an irreducible non-singular hyperplane section of V , and let $\bar{B}_1 = T^{-1}(B_1)$. If $r^*(m, n)$ and $\rho^*(m, n)$ are the r - and ρ -functions associated with the pair $(\bar{B}_1, B_1)$, then, by Lemma 2, $r^*(m, n) = \rho^*(m, n)$ for all m if n is large or for all n if m is large. The image of B_1 on the join of U and V is a normal curve, so that it follows as in the proof of Theorem 1 that $|A_m + B_n|$ will cut a complete series on B_1 if n is large. Since the residual system of $|A_m + B_n|$ with respect to B_1 is

the complete system $|A_m + B_{n-1}|$, we conclude that $r(m, n) = r(m, n-1) + r^*(m, n)$ for large values of n . It follows that $\rho(m, n) - \rho(m, n-1) = \rho^*(m, n)$ for all values of m, n since the ρ -functions are polynomials.

Let $g_{m\nu}$ be the series cut out on $\overline{B}_1$ by the system $\{A_m\}$. If we apply Lemma 5 to the case in which the g_μ of that lemma is the series g_ν , we find $\delta(g_{(m+1)\nu}) \leq \delta(g_{m\nu})$ for large values of m . It follows that $\delta(g_{m\nu})$ remains constant when m is large, say $\delta(g_{m\nu}) = \delta$ if $m \geq M_0$. Let $g_{s\mu}$ be the series cut on B_1 by $\{B_s\}$, $s = 1, 2, \dots, n_0$, consider it as a series on $\overline{B}_1$, and apply Lemma 5 to each of the series $g_{s\mu}$ in turn. This yields a set of integers $M_1, M_2, \dots, M_{n_0}$. Let $m_0 = \max(M_0, M_1, \dots, M_{n_0})$. We can then assert that if $m \geq m_0$, then $\delta(g_{m\nu} + g_{s\mu}) \leq \delta$, $s = 1, 2, \dots, n_0$.

The system cut by $|A_m + B_s|$ on B_1 ($m \geq m_0$, $s = 1, 2, \dots, n_0$) contains the minimal sum $g_{m\nu} + g_{s\mu}$ and is totally contained in the complete series determined by $g_{m\nu} + g_{s\mu}$. Since the residual system of $|A_m + B_s|$ with respect to B_1 is the complete system $|A_m + B_{s-1}|$, the following inequalities are valid:

$$r(m, s-1) + r^*(m, s) \geq r(m, s) \geq r(m, s-1) + r^*(m, s) - \delta, \\ s = 1, 2, \dots, n_0.$$

On combining these inequalities with the equalities $r(m, n) = \rho(m, n)$, $n \geq n_0$, $m \geq 0$; $r^*(m, n) = \rho^*(m, n)$, $m \geq m_0$, $n \geq 0$, and using the addition formula $\rho(m, n) - \rho(m, n-1) = \rho^*(m, n)$, we find

$$\rho(m, s) + (n_0 - s)\delta \geq r(m, s) \geq \rho(m, s), \quad s = 0, 1, \dots, n_0; m \geq m_0.$$

If we put $d = \delta n_0$, the lemma follows, q.e.d.

Theorem 4 now follows quite readily. As in the proof of Theorem 2, let $r_1(m)$, $\rho_1(m)$; $r_2(n)$, $\rho_2(n)$ be the effective and virtual dimensions (increased by one) of $|A_m|$ and $|B_n|$, and let

$$\rho_1(m) = \sum_{i=0}^2 c_{1i} C_{m,i}, \quad \rho_2(n) = \sum_{j=0}^2 c_{2j} C_{n,j}, \quad \rho(m, n) = \sum_2 a_{ij} C_{m,i} C_{n,j}.$$

Since $r_2(n) = r(0, n) = \rho(0, n)$ if $n \geq n_0$, it follows that $c_{20} = a_{00}$. On the other hand, $\rho_1(m) = r_1(m) = r(m, 0)$ for large values of m so that, by Lemma 6, $\rho(m, 0) + d \geq \rho_1(m) \geq \rho(m, 0)$. It follows that the polynomials $\rho(m, 0)$ and $\rho_1(m)$ differ only by a constant and that $c_{10} \geq a_{00}$. Hence $p_a(U) \geq p_a(V)$, q.e.d.

If W is an arbitrary normal model of a two-dimensional field Σ , if the system $\{C\}$ of hyperplane sections of W is of degree ν and genus π , and if $p_a(W) = p_a(\Sigma) + s$, where $s \geq 0$, then by a straightforward computation it is found that the function $\rho(n)$ associated with W is given by the expression

$$\rho(n) = \nu C_{n,2} + (\nu - \pi + 1)n + p_a(\Sigma) + s + 1.$$

It follows that

$$\dim |nC| = \nu C_{n,2} + (\nu - \pi + 1)n + p_a(\Sigma) + s$$

if n is sufficiently large. This is the Riemann-Roch theorem for the complete system $|nC|$.

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