THE WEDDERBURN PRINCIPAL THEOREM FOR JORDAN ALGEBRAS

BY

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1. Introduction. A nonassociative algebra $\mathfrak{A}$, of finite dimension over a field $\mathfrak{F}$, is called a *Jordan algebra* if $\mathfrak{A}$ is commutative, and if the identity

$$a(a^2b) = a^2(ab) \tag{1.1}$$

holds for all elements a, b in $\mathfrak{A}$. The identity (1.1) is equivalent, in a commutative algebra, to the identity

$$[(ab)u]v + [(vb)u]a + [(av)u]b = (ab)(uv) + (vb)(ua) + (av)(ub) \tag{1.2}$$

for arbitrary elements a, b, u, v in $\mathfrak{A}$, provided that $\mathfrak{F}$ is not of characteristic 2 or 3 [4, pp. 546, 549](1).

Let $\mathfrak{A}^*$ be an associative algebra over a field $\mathfrak{F}$ not of characteristic 2. Replace the associative multiplication $a \cdot b$ in $\mathfrak{A}^*$ by the quasi-multiplication

$$ab = (a\cdot b + b\cdot a)/2. \tag{1.3}$$

If a subspace $\mathfrak{A}$ of $\mathfrak{A}^*$ is closed with respect to this new multiplication, then $\mathfrak{A}$ is a Jordan algebra relative to the new multiplication. A *special Jordan algebra*(2) is a Jordan algebra isomorphic to one obtained from an associative algebra in the above manner.

The title of this paper is the same as that of a paper by A. A. Albert [3]. However, in that paper Albert considers only special Jordan algebras, while we prove the principal theorem here for the "general" Jordan algebras defined by commutativity and the identity (1.1). In each case the base field is assumed to be of characteristic 0.

Considering the Jordan algebra $\mathfrak{B}$, we define(3)

$$\mathfrak{B}^{(1)} = \mathfrak{B}, \qquad \mathfrak{B}^{(k+1)} = \mathfrak{B}^{(k)}\mathfrak{B}^{(k)} = (\mathfrak{B}^{(k)})^2 \qquad \text{for } k = 1, 2, 3, \cdots .$$

Then $\mathfrak{B}$ is said to be solvable in case there exists an integer m such that $\mathfrak{B}^{(m)} = 0$. If, furthermore, $\mathfrak{B}$ is an ideal of the Jordan algebra $\mathfrak{A}$, then $\mathfrak{B}$ is called a solvable ideal of $\mathfrak{A}$.

A Jordan algebra $\mathfrak{A}$ is called *simple* if $\mathfrak{A}$ contains no ideals except $\mathfrak{A}$ it-

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(1) Numbers in brackets refer to the references cited at the end of the paper.

(2) The special Jordan algebras have been called "Jordan algebras of linear transformations" [2].

(3) The product $\mathfrak{B}\mathfrak{C}$ of two subspaces $\mathfrak{B}$ and $\mathfrak{C}$ of the algebra $\mathfrak{A}$ is the set of all sums $\sum bc$, b in $\mathfrak{B}$, c in $\mathfrak{C}$. In particular, $\mathfrak{B}^2 = \mathfrak{B}\mathfrak{B}$.

self and the ideal (0) and if $\mathfrak{A}$ is not the trivial one-dimensional algebra over $\mathfrak{F}$ such that $\mathfrak{A}^2=(0)$. $\mathfrak{A}$ is called *semisimple* if the only solvable ideal of $\mathfrak{A}$ is the ideal (0).

A. A. Albert has recently developed a structure theory for Jordan algebras over a field of characteristic 0. In [4] he shows that every Jordan algebra $\mathfrak{A}$ contains a unique maximal solvable ideal $\mathfrak{N}$, called the *radical* of $\mathfrak{A}$, that $\mathfrak{A}-\mathfrak{N}$ is semisimple, and that every semisimple Jordan algebra over $\mathfrak{F}$ is uniquely expressible as the direct sum of simple Jordan algebras.

In continuation of the general structure theory developed in [4], we shall prove here the following theorem.

PRINCIPAL THEOREM. *Let $\mathfrak{A}$ be a Jordan algebra over a field $\mathfrak{F}$ of characteristic* 0, *and let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. Then there exists a subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ such that $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$, $\mathfrak{S}\cap\mathfrak{N}=0$, and $\mathfrak{S}\cong\mathfrak{A}-\mathfrak{N}$.*

This is the analogue for Jordan algebras of the well known Wedderburn principal theorem for associative algebras. The proof, as with associative algebras, is essentially effected in two stages. We first consider the case $\mathfrak{N}^2\neq 0$. By means of an inductive procedure similar to that used for associative algebras we reduce the theorem to the case $\mathfrak{N}^2=0$. The latter case is then resolved by considerations on the *split Jordan algebras*, to be discussed in detail below.

Albert's paper [3] contains many hints for the solution of the more general problem considered here. However, the proofs in the present paper differ considerably in some important respects from those given in [3].

I should like at this point to thank Professor R. D. Schafer, of the University of Pennsylvania, for his guidance during the preparation of this paper. Professor Schafer proposed this problem as a doctoral research project.

2. **Reduction to the case** $\mathfrak{N}^2=0$. We shall prove the principal theorem for Jordan algebras over a field $\mathfrak{F}$ of characteristic 0. In this section, however, we need assume only that $\mathfrak{F}$ is not of characteristic 2. For then the identity (1.2) holds [4, p. 549].

Let $\mathfrak{B}$ be any subspace of a Jordan algebra $\mathfrak{A}$. Since $\mathfrak{A}$ is commutative, we have $\mathfrak{B}^2\mathfrak{B}=\mathfrak{B}\mathfrak{B}^2$, and there is no ambiguity in denoting this subspace by $\mathfrak{B}^3$. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, we have $\mathfrak{B}^3\leqq\mathfrak{B}^2\leqq\mathfrak{B}$.

LEMMA 2.1. *Let $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ be ideals of a Jordan algebra $\mathfrak{A}$ over a field $\mathfrak{F}$ of characteristic not* 2. *Then $\mathfrak{T}=(\mathfrak{B}\mathfrak{C})\mathfrak{D}+(\mathfrak{D}\mathfrak{B})\mathfrak{C}+(\mathfrak{C}\mathfrak{D})\mathfrak{B}$ is an ideal of $\mathfrak{A}$.*

Let a be any element of $\mathfrak{A}$. To show $[(\mathfrak{B}\mathfrak{C})\mathfrak{D}]\mathfrak{A}\leqq\mathfrak{T}$, it is sufficient to show $[(bc)d]a\in\mathfrak{T}$ for any elements b in $\mathfrak{B}$, c in $\mathfrak{C}$, d in $\mathfrak{D}$. By (1.2), we have

$$[(bc)d]a = -[(ac)d]b - [(ba)d]c + (bc)(da) + (ac)(db) + (ba)(dc).$$

Because $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ are ideals, all terms on the right are in $\mathfrak{T}$. Similarly, $[(\mathfrak{D}\mathfrak{B})\mathfrak{C}]\mathfrak{A}\leqq\mathfrak{T}$ and $[(\mathfrak{C}\mathfrak{D})\mathfrak{B}]\mathfrak{A}\leqq\mathfrak{T}$, and the lemma is proved.

LEMMA 2.2. *If $\mathfrak{A}$ is a Jordan algebra over a field $\mathfrak{F}$ of characteristic not 2, and if $\mathfrak{B}$ and $\mathfrak{C}$ are ideals in $\mathfrak{A}$, then $\mathfrak{B}_1=\mathfrak{A}(\mathfrak{B}\mathfrak{C})+\mathfrak{B}\mathfrak{C}$ is an ideal in $\mathfrak{A}$.*

Clearly, $\mathfrak{A}(\mathfrak{B}\mathfrak{C})\leqq\mathfrak{B}_1$. Hence, to prove that $\mathfrak{A}\mathfrak{B}_1=\mathfrak{B}_1\mathfrak{A}\leqq\mathfrak{B}_1$, it is sufficient to show that $[(bc)a_1]a_2$ is in $\mathfrak{B}_1$ for b in $\mathfrak{B}$, c in $\mathfrak{C}$, and a_1, a_2 in $\mathfrak{A}$. Using (1.2) once more, we obtain

$$[(bc)a_1]a_2 = -[(a_2c)a_1]b-[(ba_2)a_1]c+(bc)(a_1a_2)+(a_2c)(a_1b)+(ba_2)(a_1c).$$

The first two and last two terms on the right are in $\mathfrak{B}\mathfrak{C}$, since $\mathfrak{B}$ and $\mathfrak{C}$ are ideals; hence all terms on the right are in $\mathfrak{B}_1$, and $[(bc)a_1]a_2$ is in $\mathfrak{B}_1$. This proves the lemma.

In particular, we have shown that, if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{B}^3$ and $\mathfrak{A}\mathfrak{B}^2+\mathfrak{B}^2$ are ideals of $\mathfrak{A}$ contained in $\mathfrak{B}$. Moreover, we shall prove that, if $\mathfrak{N}$ is the radical of $\mathfrak{A}$, and $\mathfrak{N}\neq 0$, then $\mathfrak{A}\mathfrak{N}^2+\mathfrak{N}^2$ is properly contained in $\mathfrak{N}$. In order to accomplish this, we must establish some preliminary lemmas.

Given a basis for $\mathfrak{B}$, an ideal of $\mathfrak{A}$, we can find a set $\mathfrak{W}$ of linearly independent elements $t_1, t_2, \cdots, t_\lambda$ in $\mathfrak{A}$ and not in $\mathfrak{B}$ such that the set $\mathfrak{W}$ together with the basis for $\mathfrak{B}$ constitute a basis for $\mathfrak{A}$. That is,

$$\mathfrak{A} = \mathfrak{F}t_1+\mathfrak{F}t_2+\cdots+\mathfrak{F}t_\lambda+\mathfrak{B}, \tag{2.1}$$

where $\lambda=\dim\mathfrak{A}-\dim\mathfrak{B}$. If $\dim\mathfrak{B}^2=\nu$, let the set

$$\mathfrak{B}^{(0)} = [v_1^{(0)}, v_2^{(0)}, \cdots, v_\nu^{(0)}]$$

be a basis for $\mathfrak{B}^2$; that is

$$\mathfrak{B}^2 = \mathfrak{F}v_1^{(0)}+\mathfrak{F}v_2^{(0)}+\cdots+\mathfrak{F}v_\nu^{(0)}. \tag{2.2}$$

Define, for $q=0, 1, 2, \cdots$, the set $\mathfrak{B}^{(q+1)}$ to be the collection of all products

$$v^{(q+1)} = (t_iv_j^{(q)})(t_kv_l^{(q)}),$$

where $i, k=1, 2, \cdots, \lambda$, and $v_j^{(q)}$, $v_l^{(q)}$ run through all elements of $\mathfrak{B}^{(q)}$. The products $v^{(q)}$ in explicit form are products of elements from $\mathfrak{W}$ and from $\mathfrak{B}^{(0)}$ associated in a particular fashion. In the explicit form there will appear, in any product $v^{(q)}$, $2(2^q-1)$ elements (not necessarily distinct) from $\mathfrak{W}$ and 2^q elements (not necessarily distinct) from $\mathfrak{B}^{(0)}$. Furthermore, $\mathfrak{B}^{(q)}$ is contained in $\mathfrak{B}^2$ for every value of q.

LEMMA 2.3. *All products $t_i[(t_iv_j^{(q)})(t_kv_l^{(q)})]$ are in $\mathfrak{B}^2$, for t_i, t_k in $\mathfrak{W}$ and $v_j^{(q)}$, $v_l^{(q)}$ in $\mathfrak{B}^{(q)}$, where $q=0, 1, 2, \cdots$.*

Let $t_kv_l^{(q)}=x\in\mathfrak{B}$. Using (1.2) with $a=v=t_i$, $b=v_j^{(q)}$, $u=x$, we have

$$2[(t_iv_j^{(q)})x]t_i = -[(t_i^2)x]v_j^{(q)}+2(t_iv_j^{(q)})(xt_i)+t_i^2(xv_j^{(q)}).$$

Now x and t_i^2x are in $\mathfrak{B}$, whence $[(t_i^2)x]v_j^{(q)}$ and $xv^{(q)}$ are in $\mathfrak{B}^3$. Since $\mathfrak{B}^3$ is an ideal, $(t_i^2)(xv_j^{(q)})$ is in $\mathfrak{B}^3$. Also $t_iv_j^{(q)}$ and xt_i are in $\mathfrak{B}$, whence the middle term on the right is in $\mathfrak{B}^2$. It follows that $t_i[(t_iv_j^{(q)})(t_kv_l^{(q)})]$ is in $\mathfrak{B}^2$, and the lemma is proved.

LEMMA 2.4. *If the element t_i appears as one of the $2(2^s-1)$ elements from $\mathfrak{W}$ in the explicit form of the product $v^{(s)}$, then $t_iv^{(s)}$ is in $\mathfrak{B}^2$, for $s=1, 2, 3, \cdots$.*

The proof is by induction on s, the case $s=1$ following from Lemma 2.3 by setting $q=0$. We suppose the lemma true for s. We have

$$v^{(s+1)} = (t_jv_m^{(s)})(t_kv_n^{(s)}), \tag{2.3}$$

for some t_j, t_k in $\mathfrak{W}$ and $v_m^{(s)}$, $v_n^{(s)}$ in $\mathfrak{B}^{(s)}$. If $t_i=t_j$, or $t_i=t_k$, then the required conclusion follows from Lemma 2.3. We shall assume, therefore, that t_i appears as one of the $2(2^s-1)$ elements from $\mathfrak{W}$ in the explicit form of one of products $v^{(s)}$ in (2.3), say $v_m^{(s)}$. Let $t_kv_n^{(s)}=x\in\mathfrak{B}$. Then, using (1.2), we have

$$\begin{aligned} t_iv^{(s+1)} &= [(t_jv_m^{(s)})x]t_i = -[(t_iv_m^{(s)})x]t_j - [(t_jt_i)x]v_m^{(s)} \\ &\quad + (t_jv_m^{(s)})(xt_i) + (t_iv_m^{(s)})(xt_j) + (t_jt_i)(xv_m^{(s)}). \end{aligned}$$

By the induction hypothesis $t_iv_m^{(s)}$ is in $\mathfrak{B}^2$. Then $(t_iv_m^{(s)})x$ is in $\mathfrak{B}^3$. So is $xv_m^{(s)}$. Since $\mathfrak{B}^3$ is an ideal, the first and last terms on the right are in $\mathfrak{B}^3$. The other terms on the right are clearly in $\mathfrak{B}^2$, and it follows that $t_iv^{(s+1)}$ is in $\mathfrak{B}^2$. This proves the lemma.

Now, for any ideal $\mathfrak{B}$ in the Jordan algebra $\mathfrak{A}$, we define

$$\mathfrak{B}_0 = \mathfrak{B}, \qquad \mathfrak{B}_{k+1} = \mathfrak{A}\mathfrak{B}_k^2 + \mathfrak{B}_k^2, \qquad k = 0, 1, 2, \cdots.$$

We are now in a position to prove the following theorem.

THEOREM 2.5. *If $\mathfrak{A}$ is a Jordan algebra over a field $\mathfrak{F}$ of characteristic not 2, and if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then there exists an integer k such that $\mathfrak{B}_k\leqq\mathfrak{B}^2$.*

From (2.1) and (2.2), we see that $\mathfrak{A}\mathfrak{B}^2$ is spanned by $\mathfrak{B}^3$ and elements of the form $t_iv^{(0)}$. Since $\mathfrak{B}^3\leqq\mathfrak{B}^2$, a basis for $\mathfrak{B}_1=\mathfrak{A}\mathfrak{B}^2+\mathfrak{B}^2$ may be chosen consisting of elements of $\mathfrak{B}^2$ together with elements of the form $t_iv^{(0)}$ not in $\mathfrak{B}^2$. It follows by an evident induction that for any positive integer s a basis may be chosen for $\mathfrak{B}_{s+2}$, consisting of elements of $\mathfrak{B}^2$ together with elements of the form $t_iv^{(s+1)}$ not in $\mathfrak{B}^2$, where the factors $t_jv_m^{(s)}$ and $t_kv_n^{(s)}$ of $v^{(s+1)}$ in (2.3) appear among the elements in the basis already chosen for $\mathfrak{B}_{s+1}$, but are not in $\mathfrak{B}^2$. Since $t_iv^{(s+1)}$ would be in $\mathfrak{B}^2$, by Lemma 2.4, if t_i were to appear in the explicit factorization of $v^{(s+1)}$, we can use, for the basis elements of $\mathfrak{B}_{s+2}$ not in $\mathfrak{B}^2$, only those elements $t_iv^{(s+1)}$ for which t_i does not appear in $v^{(s+1)}$. Hence, at the formation of each new $\mathfrak{B}_{s+2}$ the basis elements $t_iv^{(s+1)}$ not in $\mathfrak{B}^2$ have acquired a *new* element t_i from $\mathfrak{W}$. Since there is only a finite number λ of elements t_i in $\mathfrak{W}$, it follows that no such element $t_iv^{(\lambda)}$ exists outside of $\mathfrak{B}^2$, whence $\mathfrak{B}_{\lambda+1}\leqq\mathfrak{B}^2$. If we take $k=\lambda+1$, the theorem is proved.

COROLLARY 2.6. *If $\mathfrak{A}$ is a Jordan algebra over a field $\mathfrak{F}$ of characteristic not 2, and if $\mathfrak{N}\neq 0$ is the radical of $\mathfrak{A}$, then $\mathfrak{A}\mathfrak{N}^2+\mathfrak{N}^2$ is an ideal of $\mathfrak{A}$ properly contained in $\mathfrak{N}$.*

Define ideals $\mathfrak{N}_k$ in the same way as the $\mathfrak{B}_k$ were defined for the preceding theorem. Then there exists an integer k such that $\mathfrak{N}_k \leqq \mathfrak{N}^2$. But, if $\mathfrak{N}=\mathfrak{N}_1 =\mathfrak{A}\mathfrak{N}^2+\mathfrak{N}^2$, we shall have

$$\mathfrak{N} = \mathfrak{N}_1 = \mathfrak{N}_2 = \cdots = \mathfrak{N}_k \leqq \mathfrak{N}^2,$$

whence

$$\mathfrak{N} = \mathfrak{N}^{(2)} = \mathfrak{N}^{(3)} = \cdots = \mathfrak{N}^{(t)} = 0,$$

for some integer t. This contradicts the assumption $\mathfrak{N}\neq 0$. Hence, it must be true that $\mathfrak{N}_1 < \mathfrak{N}$.

We now reduce the proof of the principal theorem to the case $\mathfrak{N}^2=0$. In a proof by induction on the dimension of $\mathfrak{A}$, we assume that the principal theorem has already been established for algebras having dimensionality less than dim $\mathfrak{A}$. The theorem is trivial for $\mathfrak{A}$ of dimension 1. Suppose $\mathfrak{N}^2\neq 0$. Then $\mathfrak{N}_1=\mathfrak{A}\mathfrak{N}^2+\mathfrak{N}^2\neq 0$, and we have $\mathfrak{N}>\mathfrak{N}_1>0$. Since $(\mathfrak{A}-\mathfrak{N}_1)-(\mathfrak{N}-\mathfrak{N}_1) \cong\mathfrak{A}-\mathfrak{N}$, we see that $\mathfrak{N}-\mathfrak{N}_1$ is the radical of the algebra $\mathfrak{A}-\mathfrak{N}_1$. Since dim $(\mathfrak{A}-\mathfrak{N}_1) <$ dim $\mathfrak{A}$, the induction hypothesis implies the existence of a subalgebra $\mathfrak{S}_0$ of $\mathfrak{A}-\mathfrak{N}_1$ such that $\mathfrak{S}_0\cong\mathfrak{A}-\mathfrak{N}$. Then there exists a subalgebra $\mathfrak{S}_1$ of $\mathfrak{A}$ such that $\mathfrak{N}_1<\mathfrak{S}_1$, and $\mathfrak{S}_1-\mathfrak{N}_1\cong(\mathfrak{A}-\mathfrak{N}_1)-(\mathfrak{N}-\mathfrak{N}_1)\cong\mathfrak{A}-\mathfrak{N}$. This implies that $\mathfrak{N}_1$ is the radical of $\mathfrak{S}_1$, and dim $\mathfrak{S}_1<$ dim $\mathfrak{A}$. Then there is a subalgebra $\mathfrak{S}$ of $\mathfrak{S}_1$ such that $\mathfrak{S}\cong\mathfrak{S}_1-\mathfrak{N}_1\cong\mathfrak{A}-\mathfrak{N}$.

The existence in $\mathfrak{A}$ of a subalgebra $\mathfrak{S}\cong\mathfrak{A}-\mathfrak{N}$ is sufficient to establish the principal theorem. Since we shall have occasion to use this fact again in later sections, we give a short proof here. Firstly, $\mathfrak{S}\cap\mathfrak{N}$ is a solvable ideal of $\mathfrak{S}$, and must be zero, since $\mathfrak{S}$ is semisimple. Therefore, dim $(\mathfrak{S}+\mathfrak{N})$ $=$ dim $(\mathfrak{A}-\mathfrak{N})+$dim $\mathfrak{N}=$ dim $\mathfrak{A}$, while $\mathfrak{S}+\mathfrak{N}$ is a subspace of $\mathfrak{A}$. This implies $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$.

The induction hypothesis above contains the implicit assumption that the principal theorem is true for $\mathfrak{N}^2=0$. We have, therefore, reduced the proof of the principal theorem to the case $\mathfrak{N}^2=0$.

We note further that if the principal theorem is true for Jordan algebras with a unity element, then it is true for all Jordan algebras. For, if $\mathfrak{A}$ does not have a unity element, adjoin a unity element e to form $\mathfrak{A}_1=\mathfrak{A}+\mathfrak{F}e$. It may be easily verified that $\mathfrak{A}_1$ is a Jordan algebra and that the radical $\mathfrak{N}$ of $\mathfrak{A}$ is also the radical of $\mathfrak{A}_1$(4). If the principal theorem is true for Jordan algebras with a unity element, then there is a subalgebra $\mathfrak{S}_1$ of $\mathfrak{A}$ such that $\mathfrak{S}_1\cong\mathfrak{A}_1$

(4) Here, we use the fact that every element in the radical of $\mathfrak{A}_1$ is nilpotent. In a Jordan algebra $\mathfrak{A}$, the powers a^k of any element a in $\mathfrak{A}$ are unambiguously defined, hence the nilpotent elements in $\mathfrak{A}$ are those elements which have a power equal to 0.

$-\mathfrak{N}$, $\mathfrak{S}_1 \cap \mathfrak{N} = 0$, $\mathfrak{A}_1 = \mathfrak{S}_1 + \mathfrak{N}$. But then $\mathfrak{S} = \mathfrak{S}_1 \cap \mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ such that $\mathfrak{S} \cap \mathfrak{N} = 0$, and $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$. It is well known that these latter statements imply $\mathfrak{S} \cong \mathfrak{A} - \mathfrak{N}$.

We shall assume in the sequel, therefore, that $\mathfrak{A}$ contains a unity element.

3. **Reduction to the case $\mathfrak{A} - \mathfrak{N}$ a split algebra.** We assume henceforth that the base field $\mathfrak{F}$ has characteristic 0. If a Jordan algebra $\mathfrak{A}$ contains an idempotent z, then $\mathfrak{A}$ may be decomposed into the direct sum of subspaces $\mathfrak{A}_z(1)$, $\mathfrak{A}_z(1/2)$, and $\mathfrak{A}_z(0)$, where, for $\lambda = 0, 1/2, 1$, $\mathfrak{A}_z(\lambda)$ is the set of all elements b in $\mathfrak{A}$ such that $bz = \lambda b$. For $\lambda = 0, 1$, $\mathfrak{A}_z(\lambda)$ is a subalgebra of $\mathfrak{A}$ whose radical $\mathfrak{N}_z(\lambda)$ is the intersection of $\mathfrak{A}_z(\lambda)$ with $\mathfrak{N}$, the radical of $\mathfrak{A}$ [4, Theorem 7].

Let $\overline{\mathfrak{S}} = \mathfrak{A} - \mathfrak{N}$, and $\bar{z}$ be the image in $\overline{\mathfrak{S}}$ of the idempotent z. Then $\overline{\mathfrak{S}}$ is a Jordan algebra and has the decomposition

$$\overline{\mathfrak{S}} = \overline{\mathfrak{S}}_{\bar{z}}(1) + \overline{\mathfrak{S}}_{\bar{z}}(1/2) + \overline{\mathfrak{S}}_{\bar{z}}(0).$$

For $\lambda = 0, 1$, we have $\mathfrak{A}_z(\lambda) - \mathfrak{N}_z(\lambda) \cong \overline{\mathfrak{S}}_{\bar{z}}(\lambda)$.

LEMMA 3.1. *If $\bar{n}$ is an idempotent element in $\overline{\mathfrak{S}} = \mathfrak{A} - \mathfrak{N}$, then there is an idempotent z in $\mathfrak{A}$ such that $\bar{z} = \bar{n}$.*

Let a be any element in $\mathfrak{A}$ such that $\bar{a} = \bar{n}$. Then $\bar{a}$ is not nilpotent, whence $\mathfrak{F}[a]$ is a non-nilpotent associative algebra, and must contain an idempotent element $z = f(a)$. We then have $\bar{z} = \beta\bar{n}$, β in $\mathfrak{F}$. Since z is an idempotent, z cannot be in $\mathfrak{N}$, whence $\beta \neq 0$. Now $\bar{z}^2 = (\beta\bar{n})^2 = \beta^2\bar{n} = \bar{z} = \beta\bar{n}$, whence $(\beta^2 - \beta)\bar{n} = 0$, $\beta^2 - \beta = 0$, $\beta = 1$, and $\bar{z} = \bar{n}$.

LEMMA 3.2. *Let $\bar{n}_1, \bar{n}_2, \cdots, \bar{n}_m$ be pairwise orthogonal idempotents in $\overline{\mathfrak{S}}$ and e be an idempotent in $\mathfrak{A}$ such that $\bar{e} = \bar{n} = \bar{n}_1 + \bar{n}_2 + \cdots + \bar{n}_m$. Then there exist pairwise orthogonal idempotents $e_1, e_2, \cdots, e_m$ in $\mathfrak{A}$ such that $\bar{e}_i = \bar{u}_i$ for $i = 1, \cdots, m$ and $e = e_1 + \cdots + e_m$.*

The case $m = 1$ is contained in the hypothesis. Assume the lemma true for $m - 1$. Since $\bar{n}\bar{n}_i = \bar{n}_i$ for $i = 1, \cdots, m$, the elements $\bar{n}_i$ are all in $\overline{\mathfrak{S}}_{\bar{n}}(1)$. But $\mathfrak{A}_e(1) - \mathfrak{N}_e(1) \cong \overline{\mathfrak{S}}_{\bar{n}}(1)$. By the induction hypothesis, it is possible to find p.o.i.(5) $e_1, \cdots, e_{m-1}$ in $\mathfrak{A}_e(1) \leq \mathfrak{A}$ such that $\bar{e}_i = \bar{n}_i$ for $i = 1, \cdots, m-1$. Let $e' = e_1 + \cdots + e_{m-1}$. Let $e_m = e - e'$. Now e' and e_m are in $\mathfrak{A}_e(1)$, so that $e'e = e'$, and $e_m e = e_m$. We have $e_i e_m = e_i e - e_i e' = e_i - e_i = 0$ for $i = 1, \cdots, m-1$, while $e_m^2 = (e - e')^2 = e - e' = e_m$; $\bar{e}_m = \bar{n} - (\bar{n}_1 + \cdots + \bar{n}_{m-1}) = \bar{n}_m$. This completes the induction, and the lemma is proved.

We know that $\overline{\mathfrak{S}}$ can be expressed in a unique manner as the direct sum of ideals which are simple Jordan algebras; that is, $\overline{\mathfrak{S}} = \overline{\mathfrak{S}}_1 \oplus \overline{\mathfrak{S}}_2 \oplus \cdots \oplus \overline{\mathfrak{S}}_m$. For each value of i, $\overline{\mathfrak{S}}_i$ has a unity element $\bar{n}_i$, and the elements $\bar{n}_i$ are p.o.i. The element $\bar{n} = \bar{n}_1 + \cdots + \bar{n}_m$ is the unity element for $\overline{\mathfrak{S}}$. By Lemma 3.2, there are p.o.i. $e_1, e_2, \cdots, e_m$ in $\mathfrak{A}$ such that $\bar{e}_i = \bar{n}_i$ for $i = 1, \cdots, m$. Let

(5) We shall use the abbreviation p.o.i. for "pairwise orthogonal idempotents."

$\mathfrak{N}_{e_i}(1)$ be the radical of $\mathfrak{A}_{e_i}(1)$. Since $\overline{\mathfrak{S}}_i \cong \overline{\mathfrak{S}}_{\bar{e}_i}(1)$, we have $\mathfrak{A}_{e_i}(1) - \mathfrak{N}_{e_i}(1) \cong \overline{\mathfrak{S}}_i$. If the principal theorem is true for $\mathfrak{A} - \mathfrak{N}$ a simple algebra, we may assert that $\mathfrak{A}_{e_i}(1)$ contains a subalgebra $\mathfrak{S}_i$ isomorphic to $\overline{\mathfrak{S}}_i$ for $i = 1, \cdots, m$. Let $\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2 + \cdots + \mathfrak{S}_m$. This sum is direct, since $\mathfrak{S}_i$ is contained in $\mathfrak{A}_{e_i}(1)$ and the sum $\mathfrak{A}_{e_1}(1) + \cdots + \mathfrak{A}_{e_m}(1)$ is direct. It follows that $\mathfrak{S} \cong \mathfrak{A} - \mathfrak{N}$, and we have further reduced the proof of the principal theorem to the case where $\mathfrak{A} - \mathfrak{N}$ is a simple algebra. In §2, we reduced it to the case where $\mathfrak{A}$ is an algebra with a unity element in which $\mathfrak{N}^2 = 0$.

Now, it is known [4, Theorem 5A] that if $\mathfrak{A}$ is a Jordan algebra over a field $\mathfrak{F}$ of characteristic 0, if $\mathfrak{N}$ is the radical of $\mathfrak{A}$, and if $\mathfrak{K}$ is a scalar extension of $\mathfrak{F}$, then $\mathfrak{N}_{\mathfrak{K}}$ is the radical of $\mathfrak{A}_{\mathfrak{K}}$, $(\mathfrak{A} - \mathfrak{N})_{\mathfrak{K}} = \mathfrak{A}_{\mathfrak{K}} - \mathfrak{N}_{\mathfrak{K}}$, and $\mathfrak{A}_{\mathfrak{K}}$ is semisimple if and only if $\mathfrak{A}$ is semisimple. Moreover, if $\mathfrak{A}$ is semisimple over $\mathfrak{F}$, there is a finite scalar extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}$ is the direct sum of simple ideals which are split algebras (to be defined and discussed in §5) over $\mathfrak{K}$. By the result of the preceding paragraph, we may assume that $(\mathfrak{A} - \mathfrak{N})_{\mathfrak{K}}$ is a split algebra. If the principal theorem holds in case $\mathfrak{A} - \mathfrak{N}$ is a split algebra, as we shall show in §§6, 7, 8, then $\mathfrak{A}_{\mathfrak{K}}$ contains a subalgebra $\mathfrak{B}$ such that $\mathfrak{B} \cong (\mathfrak{A} - \mathfrak{N})_{\mathfrak{K}}$, and the remainder of the proof is the same as that of the associative proof [1, p. 47], since associativity is not used there.

Since we may take the scalar extension $\mathfrak{K}$ above to contain $(-1)^{1/2}$, we may assume wherever necessary in what follows that the base field $\mathfrak{F}$ contains $(-1)^{1/2}$.

4. **Decomposition relative to several idempotents.** Let the unity element of the Jordan algebra $\mathfrak{A}$ be the sum of the p.o.i. $e_1, \cdots, e_n$. Then $\mathfrak{A}$ may be decomposed into the direct sum of the subspaces $\mathfrak{A}_{ij} = \mathfrak{A}_{ji}$, $i, j = 1, \cdots, n$, where $\mathfrak{A}_{ii} = \mathfrak{A}_{e_i}(1)$, $i = 1, \cdots, n$, and $\mathfrak{A}_{ij} = \mathfrak{A}_{e_i}(1/2) \cap \mathfrak{A}_{e_j}(1/2)$ for $i \neq j$; $i, j = 1, \cdots, n$. The subspaces $\mathfrak{A}_{ij}$ have the following multiplicative properties [4, Theorem 12], where i, j, k, l are distinct:

$$(4.1)\qquad \begin{aligned} &\text{(i)} && \mathfrak{A}_{ii}^2 \leqq \mathfrak{A}_{ii},\\ &\text{(ii)} && \mathfrak{A}_{ii}\mathfrak{A}_{ij} \leqq \mathfrak{A}_{ij},\\ &\text{(iii)} && \mathfrak{A}_{ij}^2 \leqq \mathfrak{A}_{ii} + \mathfrak{A}_{jj},\\ &\text{(iv)} && \mathfrak{A}_{ij}\mathfrak{A}_{jk} \leqq \mathfrak{A}_{ik},\\ &\text{(v)} && \mathfrak{A}_{ii}\mathfrak{A}_{jk} = \mathfrak{A}_{ij}\mathfrak{A}_{kl} = 0. \end{aligned}$$

The idempotent e_i is the identity for $\mathfrak{A}_{ii}$, while for every element a_{ij} in $\mathfrak{A}_{ij}$, $i \neq j$, we have

$$(4.2)\qquad e_i a_{ij} = e_j a_{ij} = a_{ij}/2.$$

We note that $\mathfrak{A}_{ii} + \mathfrak{A}_{ij} + \mathfrak{A}_{jj} = \mathfrak{A}_z(1)$, where $z = e_i + e_j$. Now let $\mathfrak{N}_{ij}$ be the radical of $\mathfrak{A}_z(1)$, $\mathfrak{N}$ the radical of $\mathfrak{A}$, $\overline{\mathfrak{S}} = \mathfrak{A} - \mathfrak{N}$. We may decompose $\overline{\mathfrak{S}}$ into subspaces $(\overline{\mathfrak{S}})_{ij}$ relative to the p.o.i. $\bar{e}_i$. Then $\mathfrak{A}_z(1) - \mathfrak{N}_{ij} \cong (\overline{\mathfrak{S}})_{\bar{z}}(1)$; that is, (4.3) $$(\mathfrak{A}_{ii}+\mathfrak{A}_{ij}+\mathfrak{A}_{jj})-\mathfrak{N}_{ij}\cong(\overline{\mathfrak{S}})_{ii}+(\overline{\mathfrak{S}})_{ij}+(\overline{\mathfrak{S}})_{jj}.$$

LEMMA 4.1. *If a_{ij} and b_{ij} are in $\mathfrak{A}_{ij}$, and if a_{jk} is in $\mathfrak{A}_{jk}$, where i, j, k are distinct, then*

(4.4) $$(a_{ij}a_{jk})b_{ij}+a_{ij}(a_{jk}b_{ij})=(a_{ij}b_{ij})a_{jk}.$$

For, it follows from (4.1) that $(a_{ij}a_{jk})b_{ij}$ is in $\mathfrak{A}_{jk}$, so that $[(a_{ij}a_{jk})b_{ij}]e_i=0$. Using this fact, we put $a=a_{ij}$, $b=a_{jk}$, $u=b_{ij}$, $v=e_i$ in (1.2) and obtain

$$[(e_ia_{jk})b_{ij}]a_{ij}+[(a_{ij}e_i)b_{ij}]a_{jk}$$
$$=(a_{ij}a_{jk})(b_{ij}e_i)+(e_ia_{jk})(b_{ij}a_{ij})+(a_{ij}e_i)(b_{ij}a_{jk}).$$

From (4.2) we obtain the result (4.4).

Under the hypotheses of Lemma 4.1, we obtain as direct consequences of (4.4) the following propositions, which we shall use in the concluding sections of this paper:

(4.5) $(a_{ij}a_{jk})a_{ij}=(1/2)a_{ij}^2a_{jk}$;

(4.6) *if* $(a_{ij}b_{ij})a_{jk}=0$, *then* $(a_{ij}a_{jk})b_{ij}=-a_{ij}(a_{jk}b_{ij})$;

(4.7) *if* $a_{ij}(a_{jk}b_{ij})=0$, *then* $(a_{ij}a_{jk})b_{ij}=(a_{ij}b_{ij})a_{jk}$;

(4.8) *if* $a_{ij}(a_{jk}b_{ij})=(a_{ij}b_{ij})a_{jk}=0$, *then* $(a_{ij}a_{jk})b_{ij}=0$.

Let $a_{ij}, b_{ij}\in\mathfrak{A}_{ij}$; $a_{kj}, a_{jk}, b_{kj}, b_{jk}, c_{jk}\in\mathfrak{A}_{jk}$, $c_{jl}\in\mathfrak{A}_{jl}$, where i, j, k, l are distinct. We substitute $a=u=a_{ij}$, $b=b_{jk}$, $v=c_{jl}$ in (1.2) and obtain

$$[(a_{ij}b_{jk})a_{ij}]c_{jl}+[(c_{jl}b_{jk})a_{ij}]a_{ij}+[(a_{ij}c_{jl})a_{ij}]b_{jk}$$
$$=2(a_{ij}b_{jk})(a_{ij}c_{jl})+a_{ij}^2(b_{jk}c_{jl}).$$

The second term on each side is 0 by virtue of (4.1). We apply (4.5) to the remaining terms on the left to obtain:

(4.9) $(a_{ij}b_{jk})(a_{ij}c_{jl})=(1/4)[(a_{ij}^2b_{jk})c_{jl}+(a_{ij}^2c_{jl})b_{jk}]$;

(4.10) *if* $a_{ij}^2=\alpha(e_i+e_j)$, α *in* $\mathfrak{F}$, *then* $(a_{ij}b_{jk})(a_{ij}c_{jl})=(1/4)\alpha b_{jk}c_{jl}$;

(4.11) *if* $a_{ij}^2=0$, *then* $(a_{ij}b_{jk})(a_{ij}c_{jl})=0$.

Setting $a=a_{ij}$, $b=b_{jk}$, $u=a_{ij}$, $v=c_{jk}$ in (1.2), we obtain:

(4.12) *if* $a_{ij}^2=\alpha(e_i+e_j)$, *and* $b_{jk}c_{jk}=\beta(e_j+e_k)$, α, β *in* $\mathfrak{F}$, *then* $(a_{ij}b_{jk})(a_{ij}c_{jk})=(1/4)\alpha\beta(e_i+e_j)$.

In particular, if we set $c_{jk}=b_{jk}$ in (4.12), we obtain:

(4.13) *if* $a_{ij}^2=\alpha(e_i+e_j)$, $b_{jk}^2=\beta(e_j+e_k)$, α, β *in* $\mathfrak{F}$, *then* $(a_{ij}b_{jk})^2=(1/4)\alpha\beta\cdot(e_i+e_k)$.

From (1.2) we also obtain the results:

(4.14) *if* $a_{ij}^2=b_{jk}^2=0$, *then* $(a_{ij}b_{jk})^2=0$;

(4.15) *if* $(a_{ij}b_{ji})=\alpha(e_i+e_j)$, $b_{jk}a_{jk}=\beta(e_j+e_k)$, *and* $a_{ij}a_{kj}=b_{ji}b_{jk}=0$, α, β *in* $\mathfrak{F}$, *then* $(a_{ij}b_{jk})(a_{kj}b_{ji})=(1/2)\alpha\beta(e_i+e_k)$;

(4.16) *if* $a_{ij}^2=\alpha(e_i+e_j)$, *and* $b_{jk}c_{jk}=0$, α *in* $\mathfrak{F}$, *then* $(a_{ij}b_{jk})(a_{ij}c_{jk})=0$.

5. **The split algebras**[6]. Any simple Jordan algebra is central simple (that is, simple for all scalar extensions) over its center. Albert proved in [2]

[6] The classification of the split algebras given here is taken from [5, pp. 141, 142].

and [4] that if $\mathfrak{A}$ is a central simple Jordan algebra over $\mathfrak{F}$ of characteristic 0, then there is a finite extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}_{\mathfrak{K}}$ is one of the following split algebras.

A. *The algebra of all $n\times n$ matrices over $\mathfrak{K}$ relative to the multiplication* (1.3). This algebra, of degree(7) n and dimension n^2, has the usual matrix units e_{ij}, $i, j=1, \cdots, n$, as a basis. The elements e_{ii}, $i=1, \cdots, n$, are p.o.i. whose sum is the identity of this algebra. Let $\mathfrak{A}$ be the algebra, and let $\mathfrak{A}_{ij}=\mathfrak{A}_{ji}$, $i, j=1, \cdots, n$, be the subspaces arising in the decomposition of $\mathfrak{A}$ relative to the p.o.i. e_{ii}. Then the element e_{ij} is in $\mathfrak{A}_{ij}$ for $i, j=1, \cdots, n$. The elements e_{ij} have the following additional properties, where i, j, k are distinct:

$$(5.1)\qquad \begin{aligned} &\text{(i)}\quad e_{ij}e_{ji} = (e_{ii}+e_{jj})/2; \quad e_{ij}^2 = 0;\\ &\text{(ii)}\quad e_{ij}e_{jk} = e_{ik}/2; \quad e_{ij}e_{ik} = e_{ij}e_{kj} = 0. \end{aligned}$$

B. *The algebra of all $n\times n$ symmetric matrices over $\mathfrak{K}$ relative to the multiplication* (1.3). This algebra, of degree n and dimension $n(n+1)/2$, is a subalgebra of the split algebra of class A of degree n, and has a basis consisting of elements $u_{ij}=u_{ji}=(e_{ij}+e_{ji})/2$, $i, j=1, \cdots, n$. The elements u_{ii}, $i, j=1, \cdots, n$, are p.o.i. whose sum is the identity for the algebra. Let $\mathfrak{A}$ be this split algebra and let $\mathfrak{A}_{ij}=\mathfrak{A}_{ji}$, $i, j=1, \cdots, n$, be the subspaces arising from the decomposition of $\mathfrak{A}$ relative to the idempotents u_{ii}. Then the element u_{ij} is in the subspace $\mathfrak{A}_{ij}$ for $i, j=1, \cdots, n$. The elements u_{ij} have the following additional properties, where i, j, k are distinct:

$$(5.2)\qquad \begin{aligned} &\text{(i)}\quad u_{ij}^2 = (u_{ii}+u_{jj})/4;\\ &\text{(ii)}\quad u_{ij}u_{jk} = u_{ik}/4. \end{aligned}$$

C. *The algebra, relative to the multiplication* (1.3), *of all $2n\times 2n$ matrices X over $\mathfrak{K}$ such that $X=P^{-1}X'P$, where X' is the transpose of X, and*

$$P = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

I_n being the $n\times n$ identity matrix. This algebra, of degree n and dimension $2n^2-n$, is a subalgebra of the split algebra of class A of degree $2n$, and has a basis consisting of elements h_{ii}, $i=1, \cdots, n$, which are p.o.i., and elements h_{ij}, $f_{ij}=-f_{ji}$, $d_{ij}=-d_{ji}$, $i\neq j$, $i, j=1, \cdots, n$, such that the elements h_{ij}, f_{ij}, d_{ij} are in the subspace $\mathfrak{A}_{ij}$, $i, j=1, \cdots, n$, arising from the decomposition of this split algebra relative to the p.o.i. h_{ii}. The elements h_{ii}, h_{ij} generate the split algebra of degree n of class A and have the same multiplication prop-

(7) The degree of a split algebra $\mathfrak{A}$ is the number of pairwise orthogonal primitive idempotents whose sum is the identity of $\mathfrak{A}$.

erties as the e_{ij} in A above, and the product of any two f's or any two d's is 0. The basal elements have the following additional properties, where i, j, k are distinct:

$$(5.3)\quad \begin{aligned} &\text{(i)}\quad f_{ij}d_{ji} = (h_{ii} + h_{jj})/2; \quad h_{ij}f_{ij} = h_{ij}d_{ij} = 0;\\ &\text{(ii)}\quad h_{ij}f_{jk} = f_{ik}/2; \quad h_{ij}d_{ik} = d_{jk}/2;\\ &\qquad f_{ij}d_{jk} = h_{ik}/2;\\ &\qquad h_{ij}f_{ik} = h_{ij}d_{jk} = 0. \end{aligned}$$

D. *The algebra of degree 2 and dimension $q+1$ over $\mathfrak{K}$ with the basis s_0, $s_1, \cdots, s_q$, where*

$$(5.4)\qquad s_0s_i = s_i; \qquad s_is_j = \delta_{ij}s_0, \qquad i, j = 1, \cdots, q,$$

where δ_{ij} is the Kronecker delta.

E([8]). *The algebra, relative to the multiplication* (1.3), *of all 3×3 hermitian matrices over $\mathfrak{D}$, where $\mathfrak{D}$ is the Cayley-Dickson algebra over $\mathfrak{K}$ with the basis $u_0, u_1, \cdots, u_7$ such that*

$u_0^2=u_0$;

$u_0u_\alpha=u_\alpha$, $u_\alpha^2=-u_0$, $u_\alpha u_\beta=-u_\beta u_\alpha$, *for* $\alpha\neq\beta$, $\alpha, \beta=1, \cdots, 7$;

$u_1u_2=u_3$, $u_2u_3=u_1$, $u_3u_1=u_2$;

$u_4u_\alpha=u_{\alpha+4}$;

$(u_\alpha+u_4u_\beta)(u_\gamma+u_4u_\delta)=(u_\alpha u_\gamma-u_\delta\bar{u}_\beta)+u_4(\bar{u}_\alpha u_\delta+u_\gamma u_\beta)$, *for* $\alpha, \beta, \gamma, \delta=0, 1, 2, 3$.

In $\mathfrak{D}$, the conjugate $\bar{d}$ of any element $d=\alpha_0u_0+\alpha_1u_1+\cdots+\alpha_7u_7$ is given explicitly by

$$\bar{d} = 2\alpha_0u_0 - d = \alpha_0u_0 - \alpha_1u_1 - \cdots - \alpha_7u_7.$$

Let the elements e_{ij} be the matrix units. The split algebra of class E is of degree 3 and dimension 27 and has a basis consisting of the p.o.i. e_{ii}, $i=1, 2, 3$, and the following elements

$$\begin{aligned} w_{ij0} &= w_{ji0} = (e_{ij} + e_{ji}),\\ w_{ij\alpha} &= -w_{ji\alpha} = (e_{ij} - e_{ji})u_\alpha,\\ &\qquad \text{for } i \neq j;\ i, j = 1, 2, 3;\ \alpha = 1, 2, \cdots, 7. \end{aligned}$$

Let this split algebra be decomposed into the subspaces $\mathfrak{A}_{ij}$, $i, j=1, 2, 3$, relative to the p.o.i. e_{ii}. The elements w_{ij0}, $w_{ij\alpha}$ are in $\mathfrak{A}_{ij}$ for $i, j=1, 2, 3$; $\alpha=1, 2, \cdots, 7$. The basis elements have the following additional properties, where i, j, k are distinct and have the values 1, 2, 3:

$$(5.5)\qquad w_{ij\alpha}^2 = (e_{ii} + e_{jj})/2, \quad w_{ij\alpha}w_{ij\beta} = 0, \quad \alpha \neq \beta,\ \alpha, \beta = 0, \cdots, 7;$$

([8]) This discussion of the split algebra of class E over $\mathfrak{F}$ is based on the discussion given in [6].

(5.6) $$w_{ij\alpha}w_{jk\beta} = \epsilon_\gamma(w_{ik\gamma}/2), \qquad \text{where } u_\alpha u_\beta = \epsilon_\gamma u_\gamma,\ \epsilon_\gamma = \pm 1.$$

We note that the elements u_0, u_1, u_2, u_3 in $\mathfrak{D}$ generate the ordinary quaternion algebra $\mathfrak{Q}$ over $\mathfrak{K}$, and the Jordan algebra of all 3×3 hermitian matrices over $\mathfrak{Q}$ is a subalgebra $\mathfrak{B}$ of the split algebra of class E. It is known that if $\mathfrak{K}$ contains $(-1)^{1/2}$, then $\mathfrak{B}$ is the split algebra of class C and degree 3 over $\mathfrak{K}$. In §8, we shall use the following properties, which are consequences of(5.6):

(5.7) $$\begin{aligned} w_{ij4}w_{jk\alpha} &= w_{ik,\alpha+4}, \\ w_{ij\alpha}w_{jk,\beta+4} &= 2w_{ij4}(w_{ij\alpha}w_{ik\beta}), \quad i \neq j;\ i, j = 1, 2, 3;\ \alpha, \beta = 0, 1, 2, 3. \\ w_{ij,\alpha+4}w_{jk,\beta+4} &= w_{ij\beta}w_{jk\alpha}, \end{aligned}$$

6. **The case $\mathfrak{A}-\mathfrak{N}$ a split algebra of class** D. We shall assume henceforth that $\mathfrak{N}^2=0$, and that $\mathfrak{A}$ is a Jordan algebra with a unity element e. Then $\bar{e}$ is the unity element for $\overline{\mathfrak{S}}$.

LEMMA 6.1. *Let b be an element of $\mathfrak{A}$ such that $\bar{b}^2=\bar{e}$, and let $c=(3b-b^3)/2$ in $\mathfrak{A}$. Then $\bar{c}=\bar{b}$, and $c^2=e$.*

We note first that $b^2=e+y$, for y in $\mathfrak{N}$, and $\bar{b}^3=\bar{b}$. Hence, $\bar{c}=(3\bar{b}-\bar{b}^3)/2 =(3\bar{b}-\bar{b})/2=\bar{b}$. Also, $b^4=e+2y$, $b^6=e+3y$, since $y^2=0$, being in $\mathfrak{N}^2=0$. Then $c^2=(9b^2-6b^4+b^6)/4=e$.

The lemma implies that, if $\overline{\mathfrak{S}}$ contains an element $\bar{s}_1$ such that $\bar{s}_1^2=\bar{e}$, then there is an element t_1 in $\mathfrak{A}$ such that $\bar{t}_1=\bar{s}_1$ and $t_1^2=e$. This is the case $q=1$ of the following lemma.

LEMMA 6.2. *Let $\bar{s}_1, \bar{s}_2, \cdots, \bar{s}_q$ be elements in $\overline{\mathfrak{S}}$ such that $\bar{s}_i^2=\bar{e}$, $\bar{s}_i\bar{s}_j=0$ for $i\neq j$; $i, j=1, \cdots, q$. For any k such that $1\leqq k\leqq q$, let $t_1, \cdots, t_k$ be elements in $\mathfrak{A}$ such that $\bar{t}_i=\bar{s}_i$, $t_i^2=e$, $t_it_j=0$ for $i\neq j$; $i, j=1, \cdots, k$. Then there exist elements $t_{k+1}, t_{k+2}, \cdots, t_q$ in $\mathfrak{A}$ such that $\bar{t}_i=\bar{s}_i$, $t_i^2=e$, $t_it_j=0$, for $i\neq j$; $i, j=1, \cdots, q$.*

The proof is by induction on q. Assume the lemma true for $q-1$. Then there are elements $t_{k+1}, t_{k+2}, \cdots, t_{q-1}$ in $\mathfrak{A}$ such that $\bar{t}_i=\bar{s}_i$, $t_i^2=e$, $t_it_j=0$, for $i\neq j$; $i, j=1, \cdots, q-1$. Let a be an element of $\mathfrak{A}$ such that $\bar{a}=\bar{s}_q$. Let

$$b = a - (at_1)t_1.$$

Then $bt_1=at_1-[(at_1)t_1]t_1=0$, since from (1.2) we find that $[(at_1)t_1]t_1=at_1$ by substituting $a=a$, $b=u=v=t_1$. Since $bt_1=0$, it follows from (1.2) by setting $a=u=b$, $v=t_1$, that $b^3t_1=0$. We note that $\bar{a}\bar{t}_1=\bar{s}_q\bar{s}_1=0$, whence at_1 and $(at_1)t_1$ are in $\mathfrak{N}$, so that $\bar{b}=\bar{a}=\bar{s}_q$. This implies that $\bar{b}^2=\bar{e}$. By Lemma 6.1, the element $c=(3b-b^3)/2$ has the properties $\bar{c}=\bar{b}=\bar{s}_q$, and $c^2=e$. Moreover, $ct_1 =(bt_1-b^3t_1)/2=0$. Suppose that there exists an element c in $\mathfrak{A}$ such that $\bar{c}=\bar{s}_q$, $c^2=e$, and $ct_i=0$ for $i=1, \cdots, h-1$, where $h\leqq q$. Let $d=c-(ct_h)t_h$. We can show $ct_h=0$ in the same way that we showed $bt_1=0$. For $i\leqq h-1$, we obtain $[(ct_h)t_h]t_i=0$ by setting $a=c$, $b=u=t_h$, $v=t_i$ in (1.2) and noting that $ct_i=t_ht_i=0$. Then $dt_i=d^3t_i=0$, for $i=1, \cdots, h$. We also have $\bar{d}=\bar{c}=\bar{s}_q$, so that $\bar{d}^2=\bar{e}$. Then the element $w=(3d-d^3)/2$ has the properties $\bar{w}=\bar{s}_q$, $w^2=e$, $wt_i=0$ for $i=1, \cdots, h$. This completes the inductive proof of the fact that there is an element t_q in $\mathfrak{A}$ such that $\bar{t}_q=\bar{s}_q$, $t_q^2=e$, $t_qt_i=0$, for $i=1, \cdots, q-1$, and the lemma is proved.

LEMMA 6.3. *If $\mathfrak{A}$ is a Jordan algebra with a unity element such that $\mathfrak{N}^2=0$ and $\mathfrak{A}-\mathfrak{N}$ is a split algebra of class* D, *then there is a subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ such that $\mathfrak{S}\cong\mathfrak{A}-\mathfrak{N}$; that is, the principal theorem holds in this case.*

The proof follows directly from the definition of the split algebra of class D and the application of Lemmas 6.1 and 6.2 with $k=1$.

Finally, we note that if $\bar{e}$ is the identity of $\bar{\mathfrak{S}}$, a split algebra of class D, then $\bar{e}=\bar{e}_1+\bar{e}_2$, where e_1 and e_2 are p.o.i. We may decompose $\bar{\mathfrak{S}}$ so that $\bar{\mathfrak{S}}=(\bar{\mathfrak{S}})_{11}+(\bar{\mathfrak{S}})_{12}+(\bar{\mathfrak{S}})_{22}$ relative to the p.o.i. $\bar{e}_1$ and $\bar{e}_2$. Also, $\mathfrak{A}=\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{22}$ relative to the p.o.i. e_1 and e_2. Now, whenever $\bar{s}_1=\bar{e}_1-\bar{e}_2$, the basis elements $\bar{s}_2, \bar{s}_3, \cdots, \bar{s}_q$ will all be in $(\bar{\mathfrak{S}})_{12}$. We choose $t_1=e_1-e_2$, and the elements $t_2, t_3, \cdots, t_q$ will all be in $\mathfrak{A}_{12}$.

7. **The case $\mathfrak{A}-\mathfrak{N}$ a split algebra of class** A, B, **or** C. If $\bar{\mathfrak{S}}=\mathfrak{A}-\mathfrak{N}$ is a split algebra over $\mathfrak{F}$ of degree n of class A, B, C, or E (in the last case $n=3$), then the identity of $\bar{\mathfrak{S}}$ is the sum of p.o.i. $\bar{e}_{11}, \bar{e}_{22}, \cdots, \bar{e}_{nn}$. It is possible to find p.o.i. $e_{11}, e_{22}, \cdots, e_{nn}$ in $\mathfrak{A}$ such that $\bar{e}_{ii}$ is the image in $\bar{\mathfrak{S}}$ of e_{ii} for $i=1, \cdots, n$, and the identity element e of $\mathfrak{A}$ is $e=e_{11}+e_{22}+\cdots+e_{nn}$.

Let $\mathfrak{A}$ be decomposed into subspaces $\mathfrak{A}_{ii}$, $\mathfrak{A}_{ij}=\mathfrak{A}_{ji}$, $i\neq j$; $i, j=1, \cdots, n$, relative to the p.o.i. $e_{11}, \cdots, e_{nn}$, and let $\bar{\mathfrak{S}}$ be decomposed into the subspaces $(\bar{\mathfrak{S}})_{ii}$, $(\bar{\mathfrak{S}})_{ij}$ relative to the p.o.i. $\bar{e}_{11}, \cdots, \bar{e}_{nn}$. Now $\bar{\mathfrak{S}}$ contains a subalgebra $\bar{\mathfrak{B}}_1$ which is a split algebra of degree n and class B, where $\bar{\mathfrak{B}}_1$ has a basis $\bar{u}_{ii}=\bar{e}_{ii}$, $\bar{u}_{ij}=\bar{u}_{ji}$, $i\neq j$, $i, j=1, \cdots, n$, whose multiplication table is given in §5. If $\mathfrak{N}_{1i}$ is the radical of $\mathfrak{A}_{11}+\mathfrak{A}_{1i}+\mathfrak{A}_{ii}$, then

$$(\mathfrak{A}_{11}+\mathfrak{A}_{1i}+\mathfrak{A}_{ii})-\mathfrak{N}_{1i}\cong\overline{(\mathfrak{S})}_{11}+\overline{(\mathfrak{S})}_{1i}+\overline{(\mathfrak{S})}_{ii}.$$

Also, $\bar{u}_{1i}$ is in $(\bar{\mathfrak{S}})_{1i}$, and for $i\neq 1$, we have $(2\bar{u}_{1i})^2=\bar{u}_{11}+\bar{u}_{ii}$, the unity element of $\mathfrak{A}_{11}+\mathfrak{A}_{1i}+\mathfrak{A}_{ii}$. Define $u_{ii}=e_{ii}$, $i=1, \cdots, n$. Since $2\bar{u}_{1i}\in\bar{\mathfrak{S}}_{1i}$, there is an element v_{1i} in $\mathfrak{A}_{1i}$ such that $\bar{v}_{1i}=2\bar{u}_{1i}$. Set $2u_{1i}=(3v_{1i}-v_{1i}^3)/2$. Then u_{1i} is in $\mathfrak{A}_{1i}$, since v_{1i}^3 is in $\mathfrak{A}_{1i}$ by (4.1). Also Lemma 6.1 implies that u_{1i} has the image $\bar{u}_{1i}$ in $\bar{\mathfrak{S}}$ and $(2u_{1i})^2=u_{11}+u_{ii}$. We define

$$u_{ij}=u_{ji}=4(u_{1i}u_{1j}),$$

for $i\neq j$; $i, j=2, \cdots, n$. Then u_{ij} is in $\mathfrak{A}_{ij}$, and $\bar{u}_{ij}$ is the image in $\bar{\mathfrak{S}}$ of u_{ij}. From (4.13), with $\alpha=\beta=1/4$, we have $u_{ij}^2=(u_{ii}+u_{jj})/4$. By (4.5), $u_{1i}u_{1j}=4u_{1i}(u_{1i}u_{1j})=u_{1j}/4$. By (4.10), $u_{ij}u_{jk}=16(u_{1i}u_{1j})(u_{1j}u_{1k})=u_{1i}u_{1k}=u_{1k}/4$. The elements u_{ii}, u_{ij} are linearly independent over $\mathfrak{F}$ since the elements $\bar{u}_{ii}$, $\bar{u}_{ij}$ are. It follows that the elements u_{ii}, u_{ij} are the basis for a subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$

such that $\mathfrak{B}_1$ is isomorphic to $\overline{\mathfrak{B}}_1$. This proves the principal theorem in the case where $\overline{\mathfrak{S}}$ is a split algebra of class B; that is, of dimension $n(n+1)/2$.

As we remarked at the end of §3, we may assume that $\mathfrak{F}$ contains $(-1)^{1/2}$. If $\mathfrak{S}$ is of dimension greater than $n(n+1)/2$, then $\overline{\mathfrak{S}}$ contains a split algebra of dimension n^2; that is, a subalgebra $\overline{\mathfrak{B}}_2$ of class A with a basis consisting of the p.o.i. $\bar{e}_{ii}$, $i=1, \cdots, n$, and elements $e_{ij} \in \mathfrak{A}_{ij}$, $i \neq j$; $i, j=1, \cdots, n$, with multiplication table as given in §5. The elements $\bar{u}_{ii}=\bar{e}_{ii}$, $\bar{u}_{ij}=(\bar{e}_{ij}+\bar{e}_{ji})/2$ form a basis for an algebra $\overline{\mathfrak{B}}_1$ of class B contained in $\overline{\mathfrak{B}}_2 \leqq \overline{\mathfrak{S}}$. By the preceding paragraph, there is a subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$ isomorphic to $\overline{\mathfrak{B}}_1$ and having basis elements $u_{ii}=e_{ii}$, $u_{ij}=u_{ji}$ for $i \neq j$, with respective images $\bar{u}_{ii}$, $\bar{u}_{ij}$ in $\overline{\mathfrak{S}}$. The algebra $(\overline{\mathfrak{S}})_{11}+(\overline{\mathfrak{S}})_{12}+(\overline{\mathfrak{S}})_{22}$ contains a subalgebra with the basis $\bar{s}_0=\bar{e}_{11}+\bar{e}_{22}$, $\bar{s}_1=\bar{e}_{11}-\bar{e}_{22}$, $\bar{s}_2=\bar{e}_{12}+\bar{e}_{21}=2\bar{u}_{12}$, $\bar{s}_3=(-1)^{1/2}(\bar{e}_{12}-\bar{e}_{21})$. This is the basis for a split algebra of class D. Let $t_1=e_{11}-e_{22}$, $t_2=2u_{12}$. Since

$$(\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{22})-\mathfrak{N}_{12} \cong (\overline{\mathfrak{S}})_{11}+(\overline{\mathfrak{S}})_{12}+(\overline{\mathfrak{S}})_{22},$$

while $e_{11}+e_{22}$ is the unity element for $\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{22}$, we may assert, on the basis of Lemma 6.2 with $k=2$, that there exists an element t_3 in $\mathfrak{A}_{12}$ such that $t_1t_3 =t_2t_3=0$, $\bar{t}_3=\bar{s}_3$, $t_3^2=e_{11}+e_{22}$. Define

$$e_{12} = (t_2-(-1)^{1/2}t_3)/2; \qquad e_{21} = (t_2+(-1)^{1/2}t_3)/2.$$

Then e_{12} and e_{21} are in $\mathfrak{A}_{12}$, and their respective images in $\overline{\mathfrak{S}}$ are $\bar{e}_{12}$ and $\bar{e}_{21}$; also $e_{12}^2=e_{21}^2=0$, $e_{12}e_{21}=(e_{11}+e_{22})/2$. Let $e_{1i}=4(u_{2i}e_{12})$; $e_{i1}=4(u_{2i}e_{21})$, for $i \geqq 3$. Let

$$e_{ij} = 2(e_{i1}e_{1j}) \qquad \text{for } i \neq j;\ i, j = 2, \cdots, n.$$

Note that (4.1) implies that each element so defined is in the subspace $\mathfrak{A}_{ij}$ corresponding to the subscripts. By (4.16), $e_{1i}^2=e_{i1}^2=0$. By (4.12), $e_{1i}e_{i1} =(e_{11}+e_{ii})/2$. Using (4.5), and noting that $e_{12}^2=e_{21}^2=0$, we have $e_{12}e_{1i}=e_{21}e_{i1}$ $e_{j2}e_{12}=e_{21}e_{2j}=0$. Then we obtain, by (4.5), that $e_{1j}e_{ij}=e_{i1}e_{ij}=0$. By (4.14), $e_{ij}^2=0$. By (4.15), $e_{ij}e_{ji}=(e_{ii}+e_{jj})/2$. From (4.11), we find that $e_{ij}e_{kj}=e_{ij}e_{ik} =0$. From (4.7) it follows that $e_{12}e_{2i}=e_{1i}/2$, $e_{j2}e_{21}=e_{j1}/2$. By (4.7), $e_{1i}e_{ij} =e_{1j}/2$, $e_{ij}e_{j1}=e_{i1}/2$. Substituting $a=e_{i1}$, $b=e_{1j}$, $u=e_{j1}$, $v=e_{1k}$ in (1.2), we obtain $e_{ij}e_{jk}=4(e_{i1}e_{1j})(e_{j1}e_{1k})=e_{ik}/2$. Hence, the elements e_{ij} satisfy all of the multiplication laws required of the split algebra of degree n of class A. If $\overline{\mathfrak{S}}$ is of dimension n^2; that is, class A, we have proved the principal theorem for this case.

If $\overline{\mathfrak{S}}$ has dimension greater than n^2, then $\overline{\mathfrak{S}}$ contains a split algebra $\overline{\mathfrak{B}}_3$ of class C and dimension $2n^2-n$, and $\overline{\mathfrak{B}}_3$ has a basis consisting of elements $\bar{h}_{ii}=\bar{e}_{ii}$, $\bar{h}_{ij}$, $\bar{f}_{ij}=-\bar{f}_{ji}$, $\bar{d}_{ij}=-\bar{d}_{ji}$, $i \neq j$; $i, j=1, \cdots, n$, where $\bar{h}_{ij}$, $\bar{f}_{ij}$, $\bar{d}_{ij}$ are in $(\overline{\mathfrak{S}})_{ij}$. The elements $\bar{h}_{ii}$, $\bar{h}_{ij}$ generate a split algebra $\overline{\mathfrak{B}}_2$ of class A with basis $h_{ii}=e_{ii}$, h_{ij}, $i \neq j$; $i, j=1, \cdots, n$, such that the images of h_{ii}, h_{ij} in $\overline{\mathfrak{S}}$ are $\bar{h}_{ii}$, $\bar{h}_{ij}$, respectively. The algebra $(\mathfrak{A})_{11}+(\mathfrak{A})_{12}+(\mathfrak{A})_{22}$ contains a subalgebra with basis $\bar{s}_0=\bar{h}_{11}+\bar{h}_{22}$, $\bar{s}_1=\bar{h}_{11}-\bar{h}_{22}$, $\bar{s}_2=\bar{h}_{12}+\bar{h}_{21}$, $\bar{s}_3=(-1)^{1/2}(\bar{h}_{12}-\bar{h}_{21})$, $\bar{s}_4=\bar{f}_{12}+\bar{f}_{21}$, $\bar{s}_5=(-1)^{1/2}(\bar{f}_{12}-\bar{d}_{21})$. This is a split algebra of class D. Since $(\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{22})-\mathfrak{N}_{12}\cong(\bar{\mathfrak{S}})_{11}+(\bar{\mathfrak{S}})_{12}+(\bar{\mathfrak{S}})_{22}$, we may conclude, by an application of Lemma 6.2 similar to that above with $k=3$, that $\mathfrak{A}_{12}$ contains an element q_{12} such that $\bar{q}_{12}=\bar{f}_{12}+\bar{d}_{21}$, $h_{12}q_{12}=h_{21}q_{12}=0$, and $q_{12}^2=h_{11}+h_{22}$. We define first, for $i\geqq 3$,

$$f_{1i} = -f_{i1} = 2(h_{i2}q_{12}); \qquad d_{i1} = -d_{1i} = 2(h_{2i}q_{12}).$$

Then we define

$$f_{12} = -f_{21} = 2(h_{23}f_{13}); \qquad d_{21} = -d_{12} = 2(h_{32}d_{31}).$$

Finally, we define

$$f_{ij} = 2(h_{i1}f_{1j}); \qquad d_{ji} = 2(h_{1i}d_{j1})$$

for $i\neq j$; $i, j=2, \cdots, n$. Note that each element defined above is in the subspace $\mathfrak{A}_{ij}$ corresponding to the subscripts.

We first show that $f_{12}=2(h_{2i}f_{1i})$ for $i=4, \cdots, n$, as well as for $i=3$. Setting $a=h_{i3}$, $b=h_{32}$, $u=q_{12}$, $v=h_{2i}$ in (1.2), we obtain

$$\begin{aligned} h_{2i}f_{1i} &= 4[(h_{i3}h_{32})q_{12}]h_{2i} \\ &= 4\{-[(h_{2i}h_{32})q_{12}]h_{i3} - [(h_{i3}h_{2i})q_{12}]h_{32} \\ &\quad + (h_{i3}h_{32})(q_{12}h_{2i}) + (h_{2i}h_{32})(q_{12}h_{i3}) + (h_{i3}h_{2i})(q_{12}h_{32})\}. \end{aligned}$$

It follows from (4.1) that the first and fourth terms on the right are 0. Also, by (4.2), $h_{i3}h_{2i}=h_{23}/2$, $h_{i3}h_{32}=h_{i2}/2$. Applying (4.4) to the second and third terms, we obtain

$$\begin{aligned} h_{2i}f_{1i} &= 4\{h_{23}(q_{12}h_{32})/2 - q_{12}/8 - (h_{i2}q_{12})h_{2i}/2 + q_{12}/8 + h_{23}(q_{12}h_{32})/2\} \\ &= 2h_{23}f_{13} - h_{2i}f_{1i}. \end{aligned}$$

Hence $f_{12}=2(h_{23}f_{13})=2(h_{2i}f_{1i})$, for $i\geqq 4$. Similarly, $d_{21}=2(h_{32}d_{31})=2(h_{i2}d_{i1})$, for $i\geqq 4$. We observe that $f_{1i}^2=d_{i1}^2=0$ by (4.16), for $i\geqq 3$, while $f_{12}^2=d_{21}^2=0$ by (4.14). Also by (4.14), we have $f_{ij}^2=d_{ji}^2=0$. Using (4.5), we may first show that $f_{1i}f_{12}=d_{i1}d_{21}=0$, then $f_{1j}f_{ij}=d_{j1}d_{ji}=0$. From (4.10) we obtain $f_{1i}f_{1j}=d_{i1}d_{j1}=0$, and from (4.11) we obtain $f_{ij}f_{kj}=d_{ji}d_{jk}=0$. Therefore, since $f_{ij}=-f_{ji}$, and $d_{ji}=-d_{ij}$, as we shall show, it follows that the product of any two f's or any two d's is 0.

Now, substituting $a=h_{i2}$, $b=h_{21}$, $u=h_{j2}$, $v=q_{12}$ in (1.2), we obtain, for $i\neq j$; $i, j\geqq 3$,

$$\begin{aligned} f_{ij} + f_{ji} &= 2\{(h_{i1}f_{1j}) + (h_{j1}f_{1i})\} \\ &= 8\{(h_{i2}h_{21})(h_{j2}q_{12}) + (h_{j2}h_{21})(h_{i2}q_{12})\} \\ &= 8\{-(h_{j2}h_{i2})(h_{21}q_{12}) + [(h_{i2}h_{21})h_{j2}]q_{12} \\ &\quad + [(h_{i2}q_{12})h_{j2}]h_{21} + [(q_{12}h_{21})h_{j2}]h_{i2}\}. \end{aligned}$$

By (4.1), and the fact that $q_{12}h_{21}=0$, we conclude that every term on the right is 0, and $f_{ij}=-f_{ji}$. Similarly, $d_{ji}=-d_{ij}$.

We shall next prove that $f_{12}h_{12}=f_{13}h_{13}=\cdots=f_{1n}h_{1n}=0$. Using (1.2) again, we have

$$\begin{aligned} f_{12}h_{12} &= 2(f_{1i}h_{2i})h_{12} \\ &= 4[(h_{i2}q_{12})h_{2i}]h_{12} \\ &= 4\{-[(h_{12}q_{12})h_{2i}]h_{i2} - [(h_{i2}h_{12})h_{2i}]q_{12} \\ &\quad + (h_{i2}q_{12})(h_{2i}h_{12}) + (h_{12}q_{12})(h_{2i}h_{i2}) + (h_{i2}h_{12})(h_{2i}q_{12})\}. \end{aligned}$$

On the right, only the third term survives, whence we have

$$f_{12}h_{12} = 4(h_{i2}q_{12})(h_{2i}h_{12}) = f_{1i}h_{1i}, \qquad \text{for } i \geqq 3.$$

This implies $f_{12}h_{12}=f_{1i}h_{1i}$ is in $\mathfrak{A}_{11}=(\mathfrak{A}_{11}+\mathfrak{A}_{22})\cap(\mathfrak{A}_{11}+\mathfrak{A}_{ii})$. But, again by (1.2),

$$\begin{aligned} (h_{i2}q_{12})(h_{2i}h_{12}) &= -(h_{2i}q_{12})(h_{i2}h_{12}) - (h_{i2}h_{2i})(q_{12}h_{12}) \\ &\quad + [(h_{i2}h_{12})q_{12}]h_{2i} + [(h_{2i}h_{i2})q_{12}]h_{12} + [(h_{2i}h_{12})q_{12}]h_{i2} \\ &= [(h_{2i}h_{12})q_{12}]h_{i2}, \end{aligned}$$

which is in $\mathfrak{A}_{22}+\mathfrak{A}_{ii}$, by (4.1). We have shown that $f_{1i}h_{1i}$ is in $\mathfrak{A}_{11}\cap(\mathfrak{A}_{22}+\mathfrak{A}_{ii})=0$, so that $f_{12}h_{12}=f_{1i}h_{1i}=0$ for $i=3,\cdots,n$.

Then we may prove in succession, using the identities in §4, that $h_{i2}f_{1i}=h_{12}f_{1i}=0$ for $i\geqq 3$; $h_{i2}f_{12}=f_{1i}/2$, $i\geqq 3$; $f_{12}h_{1i}=0$, $h_{1i}f_{1j}=0$ for all $i, j\geqq 2$; $h_{i1}f_{1i}=0$ for $i=3,\cdots,n$; $h_{21}f_{12}=0$. The remaining laws for the multiplication of the h's by the f's follow in a straightforward fashion. Because of the parallel definitions for the d's, we may also establish the rules for the multiplication of the h's by the d's in a similar manner. We then establish successively that

$$\begin{aligned} f_{1i}d_{i1} &= (h_{11}+h_{ii})/2, & i &= 3,\cdots,n, \\ f_{12}d_{21} &= (h_{11}+h_{22})/2, & & \\ f_{ij}d_{ji} &= (h_{ii}+h_{jj})/2, & i &\neq j;\ i,j=2,\cdots,n. \end{aligned}$$

The remaining products of the f's by the d's are also established in a straightforward manner.

It follows that the elements $h_{ii}, h_{ij}, f_{ij}, d_{ji}, i\neq j; i, j=1,\cdots,n$, generate a subalgebra $\mathfrak{B}_3$ of $\mathfrak{A}$ which is the split algebra of degree n of class C, and the principal theorem has been proved for the case where $\mathfrak{A}-\mathfrak{N}$ is a split algebra of class C. There remains the possibility that $\overline{\mathfrak{S}}$ has dimension greater than $2n^2-n$. But then $n=3$, and $\overline{\mathfrak{S}}$ is the exceptional Jordan algebra of class E having dimension 27 over $\mathfrak{F}$. We prove the principal theorem for this case in the concluding section.

8. **The case $\mathfrak{A}-\mathfrak{N}$ a split algebra of class** E. Let $\overline{\mathfrak{S}}=\mathfrak{A}-\mathfrak{N}$ be the split algebra over $\mathfrak{F}$ of class E. Then $\overline{\mathfrak{S}}$ has a basis consisting of p.o.i. $\bar{e}_{11}, \bar{e}_{22}, \bar{e}_{33}$, and elements $\bar{w}_{ij\alpha}$, $i\neq j$; $i, j=1, 2, 3$; $\alpha=0, 1, \cdots, 7$, where $\bar{w}_{ij\alpha}\in(\overline{\mathfrak{S}})_{ij}$,

$\bar{w}_{ij0}=\bar{w}_{ji0}$, $\bar{w}_{ij\alpha}=-\bar{w}_{ji\alpha}$ for $\alpha=1, 2, \cdots, 7$. The elements $\bar{e}_{ii}$, $\bar{w}_{ij\alpha}$, $i\neq j$; $i, j=1, 2, 3$; $\alpha=0, 1, 2, 3$, are the basis for an algebra $\overline{\mathfrak{B}}_3$ which is the Jordan algebra of all 3×3 hermitian matrices over the quaternions. But, since we assume that $\mathfrak{F}$ contains $(-1)^{1/2}$, this algebra $\overline{\mathfrak{B}}_3$ is isomorphic to the split algebra of class C and degree 3 over $\mathfrak{F}$.

By what was proved in §7, with $n=3$, there is in $\mathfrak{A}$ a subalgebra $\mathfrak{B}_3$, with a basis consisting of the p.o.i. e_{ii}, $i=1, 2, 3$, together with elements $w_{ij\alpha}$, $i\neq j$; $i, j=1, 2, 3$; $\alpha=0, 1, 2, 3$, such that $w_{ij0}=w_{ji0}$ and $w_{ij\alpha}=-w_{ji\alpha}$, $\alpha=1, 2, 3$, are in $\mathfrak{A}_{ij}$; multiplication of these basal elements is defined in §5. Their images in $\overline{\mathfrak{S}}$ are, respectively, $\bar{e}_{ii}$, $\bar{w}_{ij0}$, $\bar{w}_{ij\alpha}$.

Now, $(\overline{\mathfrak{S}})_{11}+(\overline{\mathfrak{S}})_{12}+(\overline{\mathfrak{S}})_{22}$, of dimension 10, is a split algebra of class D and has basis elements $\bar{s}_0=\bar{e}_{11}+\bar{e}_{22}$, $\bar{s}_1=\bar{e}_{11}-\bar{e}_{22}$, $\bar{s}_{\alpha+2}=\bar{w}_{12\alpha}$, $\alpha=0, \cdots, 7$. Since $(\mathfrak{A}_{11}+\mathfrak{A}_{12}+\mathfrak{A}_{22})-\mathfrak{N}_{12}\cong(\overline{\mathfrak{S}})_{11}+(\overline{\mathfrak{S}})_{12}+(\overline{\mathfrak{S}})_{22}$, we may assert, by virtue of Lemma 6.2 with $k=5$, that there is in $\mathfrak{A}_{12}$ an element w_{124} mapping into $\bar{w}_{124}$ such that $w_{124}^2=e_{11}+e_{22}$, $w_{12\alpha}w_{124}=0$ for $\alpha=0, 1, 2, 3$. We define

$$\begin{aligned} w_{13,\alpha+4} &= -w_{31,\alpha+4} = 2(w_{124}w_{23\alpha}), \\ w_{23,\alpha+4} &= -w_{32,\alpha+4} = -2(w_{124}w_{13\alpha}), \end{aligned} \qquad \alpha = 0, 1, 2, 3;$$

$$w_{12,\alpha+4} = 2(w_{13,\alpha+4}w_{230}) = -w_{21,\alpha+4}, \qquad \alpha = 1, 2, 3.$$

The rules for multiplication which require verification follow:

$$\begin{aligned} w_{13,\alpha+4}^2 &= 4(w_{124}w_{23\alpha})^2 = e_{11}+e_{33}, \\ w_{23,\alpha+4}^2 &= 4(w_{124}w_{13\alpha})^2 = e_{22}+e_{33}, \\ w_{12,\alpha+4}^2 &= 4(w_{13,\alpha+4}w_{230})^2 = e_{11}+e_{22}, \end{aligned}$$

by(4.13);

$$w_{134}w_{230} = 2(w_{124}w_{230})w_{230} = w_{124}/2,$$

by (4.5). Then

$$w_{12,\alpha+4} = 2(w_{13,\alpha+4}w_{230}) \qquad \text{for } \alpha = 0, 1, 2, 3.$$

Using (4.6), we obtain

$$\begin{aligned} w_{134}w_{12\alpha} &= w_{23,\alpha+4}/2, & w_{234}w_{12\alpha} &= -w_{13,\alpha+4}/2, \\ w_{134}w_{23\alpha} &= -w_{12,\alpha+4}/2, & w_{234}w_{13\alpha} &= w_{12,\alpha+4}/2. \end{aligned}$$

The latter follows from the fact that $w_{234}w_{230}=4(w_{124}w_{130})(w_{120}w_{130})=0$, by (4.16). Thus we have established that

$$w_{ij4}w_{jk\alpha} = w_{ik,\alpha+4}/2, \qquad \alpha = 0, 1, 2, 3.$$

Now

$$w_{ik,\alpha+4}w_{ik,\beta+4} = 4(w_{ij4}w_{jk\alpha})(w_{ij4}w_{jk\beta}) = 0,$$

for $\alpha\neq\beta$, by (4.16). Also,

$$w_{23\alpha}w_{234} = 4(w_{12\alpha}w_{130})(w_{124}w_{130}) = 0,$$

and

$$w_{13\alpha}w_{134} = 4(w_{12\alpha}w_{230})(w_{124}w_{230}) = 0,$$

for $\alpha=0, 1, 2, 3$, by (4.16). Therefore,

$$w_{ij\alpha}w_{ij4} = 0 \qquad \text{for } i, j = 1, 2, 3;\ \alpha = 0, 1, 2, 3.$$

Now, since there exists $w_{ij\gamma}$, $0\leqq\gamma\leqq 3$, such that $w_{ik\alpha}=\pm 2(w_{ij\beta}w_{jk\gamma})$ for α, β given, we have

$$\begin{aligned} w_{ik\alpha}w_{ik,\beta+4} &= 2w_{ik\alpha}(w_{ij4}w_{jk\beta}) \\ &= \pm\, 4(w_{ij\gamma}w_{jk\beta})(w_{ij4}w_{jk\beta}) = 0, \qquad \text{by (4.16).} \end{aligned}$$

Finally,

$$\begin{aligned} w_{ij,\alpha+4}w_{jk,\beta+4} &= 4(w_{ik4}w_{kj\alpha})(w_{ik4}w_{ij\beta}) \\ &= 2\{-w_{ik4}^2(w_{ij\beta}w_{kj\alpha}) + w_{ij\beta}w_{kj\alpha}/2 \\ &\quad + w_{ik4}[w_{ik4}(w_{ij\alpha}w_{kj\beta})]\} \\ &= -\,w_{ij\beta}w_{kj\alpha} = w_{ij\beta}w_{jk\alpha} \qquad \text{for } \alpha, \beta = 0, 1, 2, 3; \\ w_{ij\alpha}w_{jk,\beta+4} &= -\,2w_{ij\alpha}(w_{ij4}w_{ik\beta}) \\ &= 2w_{ij4}(w_{ij\alpha}w_{ik\beta}) \qquad \text{by (4.6).} \end{aligned}$$

This concludes the proof of the principal theorem for the case $\mathfrak{S}=\mathfrak{A}-\mathfrak{N}$, the split algebra of class E, and completes the case $\mathfrak{N}^2=0$.

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