THE REGULAR REPRESENTATION OF A RESTRICTED DIRECT PRODUCT OF FINITE GROUPS

BY

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1. **Introduction.** Let $\mathfrak{G}$ be a countable group and $L_2(\mathfrak{G})$ the Hilbert space of those complex-valued functions $f(G)$ on $\mathfrak{G}$ for which

$$(1.1) \qquad \|f\|^2 = \sum_{G\in\mathfrak{G}} |f(G)|^2 < \infty.$$

We wish to study the *left-regular representation* $\Gamma\to R(\Gamma)$ of the group $\mathfrak{G}$; that is, with every element Γ of $\mathfrak{G}$ we associate the unitary operator

$$(1.2) \qquad R(\Gamma): f(G) \to f(\Gamma^{-1}G)$$

defined for all $f(G)\in L_2(\mathfrak{G})$. Let W be the smallest self-adjoint algebra of bounded linear operators on $L_2(\mathfrak{G})$ which is closed in the weak topology for operators (that is, a ring of operators in the sense of [1](1)) and contains the operator $R(G)$ for all $G\in\mathfrak{G}$. Let us denote by Z the center of W, that is, the set of those elements of W which commute with every element of W. In accordance with Theorem VII of [5] we can perform the "central decomposition" of $L_2(\mathfrak{G})$ under W. We obtain a *generalized direct sum* (=*direct integral*) *decomposition* (as defined in [5])

$$(1.3) \qquad L_2(\mathfrak{G}) = \int_{\oplus} \mathfrak{H}_t$$

of $L_2(\mathfrak{G})$ into Hilbert- (or finite-dimensional) spaces $\mathfrak{H}_t$ over the complex numbers. To every element f of $L_2(\mathfrak{G})$ corresponds a vector-valued function $f(t)$ which is measurable in a certain sense (cf. [5]) and for any given t the value $f(t)$ is an element of the space $\mathfrak{H}_t$.

We shall denote this by

$$(1.4) \qquad f \sim f(t).$$

To define such a direct integral, the set of points t must form a measure space. Since we are dealing only with separable spaces $\mathfrak{H}_t$ and $L_2(\mathfrak{G})$ in the present paper, we can assume (following [5]) that t varies over the real line and that the measure is a Lebesgue-Stieltjes measure given by a nondecreasing semi-continuous real-valued function $s(t)$ satisfying $\int_{-\infty}^{+\infty} ds(t)=1$ (cf. part I of [5] for more details).

The particular direct integral (1.3) which we wish to study is character-

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

ized by the condition that the ring Z "belongs" to it in the sense of Theorem IV of [5]. This means that the elements C of Z are exactly those operators on $L_2(\mathfrak{G})$ for which there exists a complex-valued s-measurable essentially bounded function $c(t)$ such that

(1.5) $Cf=f'$ in $L_2(\mathfrak{G})$ if and only if $c(t)f(t)=f'(t)$ in s-almost every $\mathfrak{H}_t$, where $f\sim f(t)$ and $f'\sim f'(t)$.

Roughly speaking, this means that the elements of Z are exactly those bounded operators on $L_2(\mathfrak{G})$ which decompose into scalar multiples $c(t)I_t$ of the identity operator I_t of $\mathfrak{H}_t$ for almost every t.

Such central decompositions have been studied for arbitrary rings of operators in separable Hilbert space by von Neumann in part IV of [5] and it has been proved there (cf. [5, Theorem VII]) that the ring W then decomposes into essentially unique rings of operators $W(t)$ such that $W(t)$ is for a.e. t a *factor* in the space $\mathfrak{H}_t$ in the sense of [1].

The case where W is the above "group-ring" generated by the operators $R(G)$ of the regular representation has been studied in Chapter 2 of [7] and also in [8] and [9]. And it has been seen there that some of the most important properties of the unitary representations which one obtains from the above $\int\oplus$, such as the nature of the generalized Peter-Weyl-Fourier inversion formula, also the uniqueness of the irreducible constituent representations, depend on the following problem: *What are the kinds of algebras* $W(t)$? More precisely: To what type do the factors $W(t)$ belong in the sense of the classification of [1]?

In the present paper we shall solve this problem in the case where $\mathfrak{G}$ is *a restricted direct product* of countably many finite groups $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n, \cdots$. By this we mean that $\mathfrak{G}$ is the subgroup of those elements

$$G = (g_1, g_2, \cdots, g_n, \cdots), \qquad g_n \in \mathfrak{g}_n, \tag{1.6}$$

of the direct product of the groups $\mathfrak{g}_n$, for which

$$g_n = 1_n \qquad \text{for all but a finite number of } n \tag{1.7}$$

where 1_n is the identity element of $\mathfrak{g}_n$.

It is known (cf. part IV of [5] or Theorem 2.3 of [7]) that almost all the $W(t)$ which one obtains in the above manner from the regular representation of a countable discrete group are factors of type I_r $(r<\infty)$ or II_1 in the sense of [1]. It is easy to see that whenever all but a finite number of the groups $\mathfrak{g}_n$ are commutative, then almost all the factors $W(t)$ obtained from the regular representation of the restricted direct product $\mathfrak{G}$ of the groups $\mathfrak{g}_n$ are of type I_r $(r<\infty)$, that is almost every $W(t)$ is a finite-dimensional simple algebra over the complex numbers. For in this case $\mathfrak{G}$ is clearly the direct product of a commutative and a finite group.

The remaining case is taken care of by the following theorem.

THEOREM 1. *Let* $\mathfrak{G}$ *be the restricted direct product of countably many finite* *groups* $\mathfrak{g}_n$. *If infinitely many of the* $\mathfrak{g}_n$ *are noncommutative, then almost all the algebras* $W(t)$ *which one obtains from the central decomposition of the regular representation of* $\mathfrak{G}$ *in the above manner are factors of type* II_1 *in the sense of* [1].

We shall prove this theorem in §§2–6. In §7 we consider a consequence of the results obtained, which can be described roughly by the statement that for a restricted direct product of finite groups almost every character occurs in the central decomposition of the regular representation (in fact with multiplicity one). Compare Theorem 2 for the precise formulation.

Theorem 1 together with the results of [9] strengthens our belief in the conjecture that the regular representation of "most" discrete groups gives rise to factors of class II_1 (except possibly one a set of measure zero). Thus for discrete groups the central decomposition of the regular representation seems to behave quite differently from that of Lie groups (cf. [8] especially Theorem 6). We intend to discuss this point and various consequences and related questions in a later publication.

Another conjecture in this connection is the following: If a locally compact group has at least one continuous infinite-dimensional irreducible unitary representation, then $L_2(\mathfrak{G})$ decomposes under the operators $R(G)$ into infinite-dimensional irreducible spaces almost everywhere(2). The results of [9] and the methods of the present paper seem to make this problem accessible for discrete groups.

Our present knowledge of unitary representations together with Theorem 1 above and [9] would tempt one to make the following further conjecture: Almost all the factors $W(t)$ which one obtains from the central decomposition of the regular representation of a (separable unimodular) locally compact group are of the same class in the sense of the classification of factors given in [1]. For countable discrete groups these last two conjectures can be shown to be equivalent, but for nondiscrete groups neither seems to imply the other easily.

Something should be said about the relation of Theorem 1 above with von Neumann's theory of almost periodic functions on an arbitrary group [14]. It is easily shown that a (restricted) direct product of finite groups has a separating system of finite-dimensional (irreducible) unitary representations. On the other hand it can be proved that Theorem 1 implies that almost all the irreducible representations which one obtains from any decomposition of $L_2(\mathfrak{G})$ into irreducible representation-spaces (under the operators $R(G)$) must be infinite-dimensional. Thus if we combine this with the observation made at the end of §2 of Chapter 2 of [8], we see that there is a very large

(2) *Added in proof.* I. Kaplansky has informed me that there exists an example of an infinite discrete group due to B. H. Neumann and I. Kaplansky whose weakly closed group algebra is the direct sum of a commutative ideal and a factor of type II_1. Therefore this and the next conjecture are not true in the generality stated here. Nevertheless there seems to be a large class of locally compact groups for which they are true.

class of maximally almost periodic groups whose regular representations decompose into infinite-dimensional irreducible representations (almost everywhere); in fact this shows that the class of such groups is much larger than Chapter 2 of [8] alone would allow one to conclude.

2. **Characters.** Let $g \to u(g)$ be an irreducible (unitary) representation of a finite group $\mathfrak{g}$ in a vector space of dimension d over the complex numbers. We shall call the function

$$x(g) = (1/d) \text{ trace } u(g) \tag{2.1}$$

a *character* of $\mathfrak{g}$.

This differs from the usual definition by the factor $1/d$, but the normalization $x(1) = 1$ will be needed in the following. On the (finite) set of characters of $\mathfrak{g}$ we introduce a measure μ by defining the measure $\mu((x))$ of the set (x) consisting of *one* character x to be

$$\mu((x)) = d^2/ | \mathfrak{g} | \tag{2.2}$$

where $|\mathfrak{g}|$ is the order of $\mathfrak{g}$.

Now consider again the above finite groups $\mathfrak{g}_1, \mathfrak{g}_2, \cdots$ and let $\mathfrak{X}_n$ be the set of characters of $\mathfrak{g}_n$. Denote by $\mathfrak{X}$ the Cartesian product of the sets $\mathfrak{X}_n$ and introduce in $\mathfrak{X}$ the product measure M of the measures μ_n in $\mathfrak{X}_n$. Let S_n be an arbitrary subset of $\mathfrak{X}_n$ and let S be the Cartesian product of the sets S_n. If $S_n = \mathfrak{X}_n$ for all but a finite number of n, then we call S a *rectangle* and put

$$M(S) = \prod_n \mu_n(S_n). \tag{2.3}$$

Note that $\mu_n(\mathfrak{X}_n) = 1$ for all $n = 1, 2, \cdots$, because for any finite group $\mathfrak{g}$ it is well known that $\sum_x d_x^2 = |\mathfrak{g}|$, where $\sum_x$ denotes the sum over all characters x of $\mathfrak{g}$. Therefore

$$M(\mathfrak{X}) = 1.$$

According to [11] M has a unique extension to a countably additive nonnegative set function defined for all sets in the countably additive(3) Boolean algebra $\mathfrak{B}_0$ generated by the "rectangles" S; we denote this extension again by M. We *complete* $\mathfrak{B}_0$ to the Boolean algebra $\mathfrak{B}'(\mathfrak{X})$, by requiring in the usual manner that $Y \subseteq Y' \subseteq \mathfrak{X}$, $Y' \in \mathfrak{B}_0$, and $M(Y') = 0$ shall imply $Y \in \mathfrak{B}'(\mathfrak{X})$. We shall call the elements of $\mathfrak{B}'(\mathfrak{X})$ the *mesurable* subsets of $\mathfrak{X}$.

Given any sequence $x_1, x_2, \cdots$ of characters of $\mathfrak{g}_1, \mathfrak{g}_2, \cdots$ respectively we can define a function $X(G)$ on $\mathfrak{G}$ by

$$X(G) = \prod_{n=1}^{\infty} x_n(g_n) \tag{2.4}$$

where $G = (g_1, g_2, \cdots)$ and $g_n \in \mathfrak{g}_n$. Note that $x_n(1_n) = 1$ by definition and

(3) Usually called a σ-Boolean algebra.

$g_n = 1_n$ for all but a finite number of n, therefore $X(G)$ is well defined and finite for each $G \in \mathfrak{G}$. If $X'(G) = \prod_n x_n'(g_n)$ and $X'(G) = X(G)$ for all $G \in \mathfrak{G}$, then clearly $x_n'(g_n) = x_n(g_n)$ for all n and all $g_n \in \mathfrak{g}_n$. Thus the functions $X(G)$ are in one-one correspondence with the elements X of the product space $\mathfrak{X}$. We shall use this one-one correspondence to identify the elements X of $\mathfrak{X}$ and the functions $X(G)$ defined by (2.4).

3. Certain algebras of class functions. To each element A of the operator algebra W corresponds a function $a(g) \in L_2(\mathfrak{G})$ such that

$$(3.1) \qquad (Af)(G) = \sum_{H \in \mathfrak{G}} a(GH^{-1})f(H)$$

for all $f \in L_2(\mathfrak{G})$ (cf. [4, Lemma 5.3.2]); moreover A is in the center Z of W if and only if $a(G)$ is a *class function*: $a(GG') = a(G'G)$. Let Z^0 be the subset of those elements C of Z for which the corresponding function $c(G)$ vanishes outside a finite subset of $\mathfrak{G}$. Whenever $C \in Z^0$ we can define(4)

$$(3.2) \qquad \hat{c}(X) = \sum_G c(G)\overline{X}(G).$$

There exists an integer k such that $c(G) = 0$ for $G \notin \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$. Therefore

$$\hat{c}(X) = \sum_{g_j \in \mathfrak{g}_j} c(g_1, g_2, \cdots, g_k)\bar{x}_1(g_1) \cdots \bar{x}_k(g_k) \prod_{n=k+1}^{\infty} \bar{x}_n(1_n)$$

where $c(g_1, g_2, \cdots, g_k)$ is the restriction of $c(G)$ to $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$. Therefore $\hat{c}(X)$ considered as a function $\hat{c}(x_1, x_2, \cdots)$ of the variables $x_1, x_2, \cdots$ depends on the first k variables only:

$$(3.3) \qquad \hat{c}(X) = \hat{c}(x_1, \cdots, x_k) = \sum_{g_j \in \mathfrak{g}_j} c(g_1, \cdots, g_k)\bar{x}_1(g_1) \cdots \bar{x}_n(g_n).$$

Hence $\hat{c}(X)$ is a measurable function of X.

Also, since $c(g_1, \cdots, g_k)$ is obviously a class function on the finite (normal) subgroup $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ of $\mathfrak{G}$, the well known theory of characters of finite groups tells us immediately that the mapping $c(G) \to \hat{c}(X)$ is one-one, in fact that

$$(3.4) \qquad \sum_G |c(G)|^2 = \int_{\mathfrak{X}} |\hat{c}(X)|^2 dX$$

where dX refers to the measure M defined in §2. Indeed since $\hat{c}(X)$ depends only on $x_1, \cdots, x_k$ we have

$$\int_{\mathfrak{X}} |\hat{c}(X)|^2 dX = \sum_{x \in \mathfrak{X}_j, j \leq k} |\hat{c}(x_1, \cdots, x_k)|^2 \mu_1(x_1) \cdots \mu_k(x_k),$$

which is well known to be equal to $\sum |c(G)|^2$, where the sum is taken over

(4) $\bar{x}$ denotes the complex conjugate of x throughout.

all $G \in \mathfrak{G}^{(k)}$, as required. Here $\mathfrak{G}^{(k)}$ denotes $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$.

Now let $c_1(G)$ be any other class function on $\mathfrak{G}$ with $c_1(G)=0$ for $G \notin \mathfrak{G}^{(k)}$. Then

$$c_2(G) = (c * c_1)(G) = \sum_{H \in \mathfrak{G}} c(GH^{-1})c_1(H) = \sum_{H \in \mathfrak{G}^{(k)}} c(GH^{-1})c_1(H).$$

Hence according to well known properties of the characters of finite groups we obtain

$$\hat{c}_2(X) = \hat{c}(X)\hat{c}_1(X). \tag{3.5}$$

Let C_1 be that element of $Z^0 \subset Z$ to which the function $c_1(G)$ corresponds in the sense of equation (3.1). It is easily seen that the function corresponding to the operator CC_1 is then the above convolution $(c * c_1)(G)$. Since the mapping $C \rightarrow \hat{c}(X)$ is obviously linear we have proved the following lemma.

LEMMA 3.1. *The mapping $C \rightarrow \hat{c}(X)$ defined by equations* (3.1) *and* (3.2) *is an isomorphism of the operator algebra Z^0 into the algebra of bounded measurable functions of X.*

Suppose $c^{(j)}(G)$ is a class function on $\mathfrak{G}$ vanishing outside the subgroup $\mathfrak{g}_j$. Then $\hat{c}^{(j)}(X)$ depends on x_j only, and as $c^{(j)}(G)$ varies over all such class functions, $\hat{c}^{(j)}(X)$ varies over *all* functions depending on x_j only. From this it follows easily that as C varies over Z^0, the corresponding functions $\hat{c}(X)$ vary over a dense subset of the space $L_2(\mathfrak{X})$ of all measurable functions $\phi(X)$ of X satisfying $\int |\phi(X)|^2 dX < \infty$. Since $c(G)$ obviously varies over a dense subset of the space $L_2(\mathfrak{C})$ of square summable *class functions* on $\mathfrak{G}$, we have the following lemma.

LEMMA 3.2. *The above mapping $c(G) \rightarrow \hat{c}(X)$ is an isometry between dense subsets of the Hilbert spaces $L_2(\mathfrak{C})$ and $L_2(\mathfrak{X})$. It has therefore a unique extension to a unitary mapping from $L_2(\mathfrak{C})$ onto $L_2(\mathfrak{X})$.*

We denote the image of *any* class function $c(G)$ with $\sum_G |c(G)|^2 < \infty$ under this unitary mapping again by $\hat{c}(X)$ and have

$$\hat{c}(X) = \text{l.i.m.} \sum_{G \in \mathfrak{G}} c(G)\overline{X}(G). \tag{3.7}$$

It is easily seen (cf. for instance p. 790 of [4]) that whenever $C \in W$, the corresponding function $c(G)$ satisfies $\sum |c(G)|^2 < \infty$, therefore (3.6) defines in particular a linear mapping of Z into $L_2(\mathfrak{X})$. Moreover it is readily verified that equation (3.5) remains true for arbitrary $c_1(G) \in L_2(\mathfrak{C})$ and almost all $X \in \mathfrak{X}$ whenever $c(G)$ is such that $C \in Z$.

Let us now consider the idempotents $E \in Z: E^2 = E$ whence $E = E^*$ is the adjoint of E. They form a countably additive Boolean algebra $\mathfrak{B}(Z)$. The corresponding $\hat{e}(X)$ is then the characteristic function of some measurable subset $\mathfrak{E}$ of $\mathfrak{X}$:

$$\hat{e}(X) = \begin{cases} 1 & \text{for} \quad X \in \mathfrak{E}, \\ 0 & \text{for} \quad X \notin \mathfrak{E}, \end{cases}$$

where $\mathfrak{E}$ is uniquely determined up to an arbitrary set of M-measure zero. The above implies now immediately that the mapping

$$(3.7) \qquad E \to \mathfrak{E}$$

is an isomorphism of the Boolean algebra $\mathfrak{B}(Z)$ into the Boolean quotient algebra $\mathfrak{B}'$ of all measurable subsets of $\mathfrak{X}$ modulo sets of M-measure zero. Let $\mathfrak{E}$ be an arbitrary measurable subset of $\mathfrak{X}$ and $\eta(x)$ its characteristic function. Since $M(X)=1$ we have $\int|\eta(X)|^2dX<\infty$. Therefore there exists by Lemma 3.2 a class function $c(G)\in L_2(\mathfrak{G})$ with $\eta(X)=\hat{c}(X)$, and $\eta(X)^2=\eta(X)$ implies $(c*c)(G)=c(G)$. This proves that the isomorphism (3.7) is *onto* $\mathfrak{B}'$. We now show that it is countably additive. Let $E_1, E_2, \cdots$ be a sequence of pairwise orthogonal idempotents in Z and $\sum_{r=1}^{\infty} E_r=E$. Then the corresponding functions on $\mathfrak{G}$ satisfy $\|e-e_r\|^2=\sum_G|e(G)-e_r(G)|^2\to 0$ as $r\to\infty$. The corresponding elements $\mathfrak{E}_r$ of $\mathfrak{B}'$ are pairwise disjoint. Denote by $\mathfrak{E}'$ their union: $\mathfrak{E}'=\bigcup_{r=1}^{\infty}\mathfrak{E}_r$. If $\eta'(X)$ is the characteristic function of $\mathfrak{E}'$ we have

$$\int \left| \eta(X) - \sum_{r=1}^{N} e_r(X) \right|^2 dX \to 0 \qquad \text{as } N \to \infty,$$

since the measure M is countably additive. Hence Lemma 3.2 implies $\eta'(X)=\hat{e}(X)$ a.e., that is, $\bigcup_r\mathfrak{E}_r$ is the image of $\sum_r E_r$ under (3.7) as desired.

We have therefore proved the following lemma.

LEMMA 3.3. *The mapping* (3.7), *that is, equation* (3.6), *defines a countably additive*(5) *isomorphism of the Boolean algebra* $\mathfrak{B}(Z)$ *of idempotents in* Z *onto the Boolean quotient algebra* $\mathfrak{B}(\mathfrak{X})$ *of the algebra of all measurable subsets of* $\mathfrak{X}$ *modulo the ideal of sets of* M*-measure zero.*

Moreover if E *is any idempotent element of* Z, $e(G)$ *the corresponding function in* $L_2(\mathfrak{G})$, *and* $\mathfrak{E}$ *the image of* E *in* $\mathfrak{B}'$ *under* (3.7), *then*

$$(3.8) \qquad e(1) = \sum_G |e(G)|^2 = M(\mathfrak{E}).$$

4. The central decomposition. Let us now consider the central decomposition of the regular representation of $\mathfrak{G}$, as described in §1 in more detail. Define an operator E_λ on $L_2(\mathfrak{G})$ by

$$(4.1) \qquad E_\lambda f = f' \qquad \text{with} \quad f'(t) = \begin{cases} f(t) & \text{for } t \leqq \lambda, \\ 0 & \text{for } t > \lambda. \end{cases}$$

The E_λ form a spectral family called in [5] *the resolution of the identity belonging to the given* $\int\oplus$. We can now restate Theorem III of [5]: The countably

(5) That is, a σ-isomorphism in the usual terminology.

additive Boolean algebra generated by the operators E_λ is σ-isomorphic to the Boolean algebra $\mathfrak{B}(s)$. By $\mathfrak{B}(s)$ we denote the Boolean quotient algebra of the algebra of all s-measurable subsets of the real line (in the sense of the Lebesgue-Stieltjes measure determined by the function $s(t)$ introduced in §1) modulo the ideal of sets of s-measure zero. On the other hand the definition of the ring belonging to a direct integral as given in §11 of [5] implies easily that the σ-Boolean algebra generated by the E_λ is the Boolean algebra $\mathfrak{B}(Z)$ of *all* idempotent elements E of Z.

LEMMA 4.1. *The Boolean quotient algebra $\mathfrak{B}(s)$ of all s-measurable subsets of the real line modulo the ideal of sets of s-measure zero is σ-isomorphic to the Boolean algebra $\mathfrak{B}(Z)$ of all idempotent elements of Z.*

Moreover if we denote this isomorphism by J then the function $s(t)$ can be assumed to be such that for every $E \in \mathfrak{B}(Z)$ we have(6)

$$\text{(4.2)} \qquad \textit{Lebesgue-Stieltjes-s-measure of } J(E) = e(1)$$

where $e(G)$ is the function corresponding to E in the sense of equation (3.1).

Proof. The existence of the isomorphism J has been deduced from the results of [5]. That the function $s(t)$ can be chosen such that equation (4.2) becomes true has been proved in [7, p. 21].

If we combine Lemma 4.1 with Lemma 3.3 we obtain at once the following lemma.

LEMMA 4.2. *Let us denote again by $\mathfrak{B}(\mathfrak{X})$ the Boolean algebra of all measurable subsets of the product space $\mathfrak{X}$ modulo the sets of M-measure zero, then the composition of the isomorphisms* (3.7) *and J is a measure-preserving σ-isomorphism between $\mathfrak{B}(\mathfrak{X})$ and $\mathfrak{B}(s)$.*

The direct integral (1.3) is such that every operator $R(G)$ defined by (1.2) decomposes, that is, there exists for each $G \in \mathfrak{G}$ an operator-valued function $U(G, t)$ measurable in t (in the sense of [5]) such that

$$(R(G)f)(t) = U(G, t)f(t) \qquad \text{for a.a. } t.$$

Since $\mathfrak{G}$ is countable, it is obvious that after a possible change on a t-set of s-measure zero the mapping $G \to U(G, t)$ is a unitary representation of $\mathfrak{G}$ for every t. After another possible change on a null set, the ring $W(t)$ is generated by the operators $U(G, t)$ and $W(t)$ is a factor of finite type (cf. Theorem IX of [5] and Theorem 2.4 of [7]). There exists then a relative trace Tr_t on $W(t)$ (cf. [1]) which can be normalized such that

$$\mathrm{Tr}_t(I_t) = 1.$$

We define

(6) We shall assume from now until the last but one section of §6 that our s-measure is so chosen as to satisfy (4.2).

$$(4.3)\qquad X_t(G) = \mathrm{Tr}_t(U(G, t)).$$

If $c(G)$ is any class function on G vanishing outside some finite subset of $\mathfrak{G}$, then the operator

$$\sum_G c(G)U(G, t)$$

is easily seen to commute with all elements of $W(t)$, that is, is in the center of $W(t)$, which consists of the scalar multiples of I_t. Therefore

$$(4.4)\qquad \sum_G c(G)U(G, t) = \mathfrak{c}(t)I_t.$$

It is clear that $\mathfrak{c}(t)$ coincides for a.a. t with the function $c(t)$ defined by equation (1.5). However $\mathfrak{c}(t)$ is uniquely defined for all t and it follows at once that the mapping

$$(4.5)\qquad C \to \mathfrak{c}(t)$$

is for each t a homomorphism of Z^0 to the complex numbers.

Let us denote by $x_{t,n}(g_n)$ the restriction of $X_t(G)$ to the subgroup $\mathfrak{g}_n$. Let Z_n be the algebra of class functions on g_n (with convolution as multiplication), that is, Z_n is the center of the group algebra of $\mathfrak{g}_n$. The restriction of the homomorphism $C \to \mathfrak{c}(t)$ to Z_n is defined by the function $x_{t,n}(g_n)$:

$$\sum_{g_n \in \mathfrak{g}_n} c(g_n)x_{t,n}(g_n) = \mathfrak{c}(t)I_t.$$

It is a well known fact about finite groups that this equation implies that $x_{t,n}(g_n)$ is a character of $\mathfrak{g}_n$ as defined in §2. (Note that our normalization $\mathrm{Tr}_t(I_t) = 1$ implies that the normalization expressed by equation (2.1) is satisfied: $x_{t,n}(1_n) = 1$.) Now put

$$X_t'(G) = \prod_{n=1}^{\infty} x_{t,n}(g_n)$$

for $G = (g_1, g_2, \cdots)$. Then

$$\sum_G c(G)X_t'(G) = c'(t)I_t$$

defines for every t also a homomorphism $C \to c'(t)$ of Z^0 to the complex numbers. Clearly $X_t'(g_n) = X_t(g_n)$ for every $g_n \in \mathfrak{g}_n$ and all n. Hence whenever $c(G)$ vanishes outside some one $\mathfrak{g}_n$ we have $c(t) = c'(t)$ for all t. But since $\mathfrak{G}$ is the restricted direct product of the normal subgroups $\mathfrak{g}_n$, it is clear that Z^0 is the algebra generated by its subalgebras Z_n. Therefore the two homomorphisms $C \to c(t)$ and $C \to c'(t)$ coincide for *all* $C \in Z^0$. Hence $X_t'(G) = X_t(G)$ for all $G \in \mathfrak{G}$ and all t, therefore we have proved the following lemma.

LEMMA 4.3. *Let* $L_2(\mathfrak{G}) = \int \oplus \mathfrak{H}_t$ *be the central decomposition of the regular rep-*

resentation of $\mathfrak{G}$. *Then there exists for almost every real t and every integer n a character* $x_{t,n}(g_n)$ *of the group* $\mathfrak{g}_n$ *such that*

$$X_t(G) = \mathrm{Tr}_t(U(G, t)) = \prod_{n=1}^{\infty} x_{t,n}(g_n) \qquad \textit{for a.a. } t \tag{4.6}$$

where $G = (g_1, g_2, \cdots)$.

We make the usual definition.

Definition. A character $x(g)$ of a finite group $\mathfrak{g}$ is called *linear* if it is a homomorphism of $\mathfrak{g}$. That is, the linear characters are the (traces of the) one-dimensional representations of $\mathfrak{g}$ over the complex numbers.

We shall now prove the following lemma.

Lemma 4.4. *After a possible omission of a t-set of measure zero,* $\mathfrak{H}_t$ *is finite-dimensional if and only if* $x_{t,n}(g_n)$ *is for this t a linear character of* $\mathfrak{g}_n$ *for all but a finite number of n.*

Proof. Put

$$f_1(G) = \begin{cases} 1, & G = 1, \\ 0, & G \neq 1. \end{cases}$$

Consider f_1 as an element of $L_2(\mathfrak{G})$; then under the given $\int \oplus \mathfrak{H}_t$ there corresponds to f_1 a vector-valued function $f_1(t)$ determined for a.a t. It has been shown in the course of the proof of Theorem 2.3 of [7] that there exists a t-set $\mathfrak{N}$ of s-measure zero such that(7)

$$\mathrm{Tr}_t(A) = (Af_1(t), f_1(t)) \tag{4.7}$$

for all $A \in W(t)$ and all $t \notin \mathfrak{N}$; moreover $\mathfrak{N}$ is independent of A.

Now put

$$x_t^{(N)}(G) = \prod_{n=1}^{N} x_{t,n}(g_n)$$

for $G = g_1 g_2 \cdots g_N$ ($g_n \in \mathfrak{g}_n$), that is, $G \in \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_N = \mathfrak{G}^{(N)}$. It is clear that $x_t^{(N)}$ is a character of $\mathfrak{G}^{(N)}$.

Let us now apply $U(G, t)$ to $f_1(t)$ for all $G \in \mathfrak{G}^{(N)}$ and denote the closure in $\mathfrak{H}_t$ of the finite linear combinations of the vectors $U(G, t)f_1(t)$ ($G \in \mathfrak{G}^{(N)}$) by $\mathfrak{H}_t^{(N)}$. Given any positive definite function $\phi(G)$ on an arbitrary group (that is, a function satisfying $\sum_{j,k} \phi(G_j G_k^{-1}) z_j \bar{z}_k \geqq 0$ for any finite number of group elements $G_1, G_2, \cdots$ and arbitrary complex numbers $z_1, z_2, \cdots$), then there exists, according to Gelfand and Raikov [12], a unitary representation $U(G)$ (uniquely determined up to unitary equivalence) of this group, and a generating vector v in the representation space such that

(7) By (,) we denote the inner product in any Hilbert- (or finite-dimensional) space.

$$\phi(G) = (U(G)v, v).$$

Equation (4.7) implies immediately that $x_t^{(N)}(G)$ is a positive definite function on $\mathfrak{G}^{(N)}$. Hence the unitary representation of $\mathfrak{G}^{(N)}$ induced in the space $\mathfrak{H}_t^{(N)}$ is determined by the positive definite function $x_t^{(N)}(G)$ with $f_1(t)$ as generating vector, in the above sense of Gelfand and Raikov. On the other hand $x_t^{(N)}(G)$ is a character of the finite group $\mathfrak{G}^{(N)}$, therefore there exists an irreducible representation of $\mathfrak{G}^{(N)}$ of a certain dimension $d^{(N)}$ such that equation (2.1) is satisfied by $x_t^{(N)}(G)$. Also it is easily seen that

$$d^{(N)} = d_1 d_2 \cdot \cdots \cdot d_N$$

where d_n is the dimension of the irreducible representation of $\mathfrak{g}_n$ which defines the character $x_{t,n}$ of $\mathfrak{g}_n$ by means of equation (2.1).

Since $x_t^{(N)}(G)$ is a character of $\mathfrak{G}^{(N)}$, the construction of Gelfand and Raikov (12) leads to a representation of dimension $(d^{(N)})^2$. Thus

$$\dim \mathfrak{H}_t^{(N)} = (d^{(N)})^2 = d_1^2 d_2^2 \cdots d_N^2. \tag{4.8}$$

On the other hand it is obvious that

$$\mathfrak{H}_t^{(1)} \subseteq \mathfrak{H}_t^{(2)} \subseteq \cdots$$

and $\sum_N \mathfrak{H}_t^{(N)}$ is a dense linear subspace of $\mathfrak{H}_t$. Hence

$$\dim \mathfrak{H}_t = \lim_{N\to\infty} \dim \mathfrak{H}_t^{(N)} = \lim_{N\to\infty} (d^{(N)})^2.$$

Hence if $d_n > 1$ for infinitely many n, then $\lim_{N\to\infty} d^{(N)} = \infty$, since $d_n \geqq 1$ for all n by definition of d_n. Thus we have proved that (after possible omission of a suitable t-set of measure zero) $\mathfrak{H}_t$ is infinite-dimensional if for this t the character $x_{t,n}(g_n)$ of $\mathfrak{g}_n$ is nonlinear for infinitely many n.

To prove the converse we have to assume that, for a given value of t, $d_n = 1$ for all n except $n_1, n_2, \cdots, n_r$ say, $r < \infty$. But under these conditions equation (4.8) implies

$$\dim \mathfrak{H}_t^{(N)} \leqq d_{n_1}^2 d_{n_2}^2 \cdot \cdots \cdot d_{n_r}^2$$

for this value of t ($n_1, \cdots, n_r$ depend of course on t but we need not indicate this dependence). Therefore for this t the spaces $\mathfrak{H}_t^{(N)}$ are all equal for $N > (d_{n_1} \cdot \cdots \cdot d_{n_r})^2$. Hence

$$\dim \mathfrak{H}_t \leqq (d_{n_1} d_{n_2} \cdot \cdots \cdot d_{n_r})^2 \qquad \text{for this } t,$$

provided t is outside of a certain set of measure zero. This completes the proof of Lemma 4.4.

We shall also want the following lemma.

LEMMA 4.5. *The algebra $W(t)$ is a factor of class* II$_1$ *for s-almost those t for* *which* dim $\mathfrak{H}_t = \infty$.

Proof. This has already been proved at the end of [9]. For the sake of completeness, we outline the easy argument: As has been pointed out, it follows easily from the results of part I of [5] that the transforms of $f_1(t)$ under the elements of $W(t)$ are dense in $\mathfrak{H}_t$ for a.e. t. Hence for any $A \in W(t)$ the mapping $A \rightarrow Af_1(t)$ is a linear mapping of $W(t)$ to a dense linear subspace of $\mathfrak{H}_t$ for a.a. t. Therefore if $W(t)$ were of finite order over the field of complex numbers, dim $\mathfrak{H}_t$ would have to be finite too. Hence $W(t)$ is of infinite order over the field of complex numbers. Therefore it must be a factor of II_1 for any such t, because it is known (cf. §22 of [5] or Theorem 2.4 of [7]) that factors of class II_1 are the only factors of infinite order which can occur in the central decomposition of the regular representation of a countable discrete group (except possibly on a set of measure zero).

5. **Relation between the two measures.** We have seen in §4 that after a change of the spaces $\mathfrak{H}_t$ (and the operators acting in $\mathfrak{H}_t$) on a t-set of s-measure zero the function $X_t(G) = \mathrm{Tr}_t(U(G, t))$ is of the form

$$X_t(G) = \prod_{n=1}^{\infty} x_{t,n}(g_n)$$

where $x_{t,n}(g_n)$ is a character of the finite group $\mathfrak{g}_n$. We assume from now on that this change has been made. Then we have a single-valued mapping ξ

$$(5.1) \qquad \xi: \quad t \rightarrow X_t$$

defined for all $t > -\infty, < +\infty$ with values in the product space $\mathfrak{X}$. The mapping ξ induces a mapping ξ^* of the space of all complex-valued functions of X into the space of complex-valued functions of t; the mapping ξ^* is defined by

$$(5.2) \qquad (\xi^*\psi)(t) = \psi(X_t).$$

Now let $\gamma(X)$ be an element of $L_2(\mathfrak{X})$. Then there exists according to §3 a class function $c(G)$ on $\mathfrak{G}$ such that $\gamma(X) = \hat{c}(X)$ for a.a. X, that is,

$$\gamma(X) = \text{l.i.m.} \sum_G c(G)\overline{X}(G) \qquad \text{for a.a. } X.$$

Denote by $L_2^0(\mathfrak{X})$ the subspace of those $\gamma \in L_2(\mathfrak{X})$ for which $c(G)$ vanishes outside some finite subset of $\mathfrak{G}$. After a change of $\gamma(X)$ on an X-set of M-measure zero we have

$$\gamma(X) = \sum_G c(G)\overline{X}(G) \qquad \text{for all } X \in \mathfrak{X}$$

whenever $\gamma \in L_2^0(\mathfrak{X})$ with the $\sum_G$ now finite. Clearly

$$(5.3) \qquad (\xi^*\gamma)(t) = \sum_G c(G)\overline{X}_t(G).$$

On the other hand, we have seen in §4 that

$$c(t) = \sum_G c(G)X_t(G) \qquad \text{for a.a. } t$$

and that

$$\sum_G |c(G)|^2 = \int_{-\infty}^{+\infty} |c(t)|^2 ds(t);$$

therefore

$$\overline{c(t)} = (\xi^*\gamma)(t) \qquad \text{for a.a. } t$$

and equation (3.4) together with Lemma 3.2 imply

$$(5.4) \qquad \int |\gamma(X)|^2 dX = \int |(\xi^*\gamma)(t)|^2 ds(t)$$

not only for $\gamma(X) \in L_2^0(\mathfrak{X})$, but for *any* $\gamma(X) \in L_2(\mathfrak{X})$ and

$$(5.5) \qquad (\xi^*\gamma)(t) = \text{l.i.m.} \sum_G c(G)\overline{X}_t(G) \qquad \text{for a.a. } t.$$

Hence, in particular, $(\xi^*\gamma)(t)$ is an s-measurable function of t.

Now let Y be any measurable subset of $\mathfrak{X}$ and $\eta(X)$ its characteristic function. Since $M(\mathfrak{X}) = 1$,

$$\int \eta(X)dX = \int |\eta(X)|^2 dX < \infty.$$

Therefore $(\xi^*\eta)(t)$ is an s-measurable function of t and

$$\int (\xi^*\eta)(t)ds(t) = \int |(\xi^*\eta)(t)|^2 ds(t) = \int |\eta(X)|^2 dX = \int \eta(X)dX.$$

On the other hand, $(\xi^*\eta)(t)$ is obviously the characteristic function of the set $\xi^{-1}(Y)$. Hence $\xi^{-1}(Y)$ is an s-measurable set and

$$(5.6) \qquad s\text{-measure of } \xi^{-1}(Y) = M(Y).$$

6. **Completion of the proof of Theorem 1.** Let us denote by Ξ the set of those $X(G) = \prod_n x_n(g_n)$ for which $x_n(g_n)$ is a *linear* character of $\mathfrak{g}_n$ for all but a finite number of n.

We shall now prove the following lemma.

LEMMA 6.1. *If infinitely many of the groups* $\mathfrak{g}_n$ *are noncommutative, then*

$$(6.1) \qquad M(\Xi) = 0,$$

where M *denotes the product measure defined in* §2.

Proof. We denote by $\mathfrak{X}_n'$ the set of linear characters of the finite group $\mathfrak{g}_n$. Let $[\mathfrak{g}_n, \mathfrak{g}_n]$ be the commutator subgroup of $\mathfrak{g}_n$ and $\mathfrak{g}_n'$ the factor group

$$\mathfrak{g}_n' = \frac{\mathfrak{g}_n}{[\mathfrak{g}_n, \mathfrak{g}_n]}. \tag{6.2}$$

Then $\mathfrak{g}_n'$ is a finite commutative group, and the characters of $\mathfrak{g}_n'$ are exactly the linear characters of $\mathfrak{g}_n$. Since the degree of a linear character is one (by definition), we have by definition of the measure μ_n (cf. equation (2.2))

$$\mu_n(X_n') = \frac{|\mathfrak{g}_n'|}{|\mathfrak{g}_n|}, \tag{6.3}$$

where $|\mathfrak{g}_n'|$ is the order of $\mathfrak{g}_n'$, and $|\mathfrak{g}_n|$ is the order of $\mathfrak{g}_n$, because a finite commutative group has as many characters (over the complex numbers) as it has elements.

Let now Ξ_k be the set of those $X \in \mathfrak{X}$ whose nth components x_n are elements of $\mathfrak{X}_n'$ for all $n > k$. Clearly

$$\Xi = \bigcup_{k=1}^{\infty} \Xi_k$$

and

$$\Xi_k = \mathfrak{X}_1 \times \cdots \times \mathfrak{X}_k \times \prod_{k=n+1}^{\infty} \mathfrak{X}_n' \tag{6.4}$$

where $\times$ and $\prod$ denote Cartesian products.

If $\mathfrak{g}_n$ is not commutative then we have obviously

$$|\mathfrak{g}_n'| \leqq |\mathfrak{g}_n|/2;$$

hence $\mu_n(\mathfrak{X}_n') \leqq 1/2$. Therefore if infinitely many of the groups $\mathfrak{g}_n$ are non-commutative, then equation (6.4) shows that

$$M(\Xi_k) = 0.$$

Hence $M(\Xi) = 0$ in this case. This proves Lemma 6.1.

Let us now denote by Φ the set of those t for which $\dim \mathfrak{H}_t < \infty$. By Lemma 4.4 we have $t \in \Phi'$ if and only if $X_t \in \Xi$ where Φ' is a suitable set differing from Φ by a set of s-measure zero. Thus

$$\xi^{-1}(\Xi) = \Phi'. \tag{6.5}$$

Hence if we now combine equations (5.6) and (6.5) with Lemma 6.1, we see at once that *the s-measure of the set* Φ *of those t for which* $\dim \mathfrak{H}_t < \infty$ *is zero.* But according to Lemma 4.5, Φ differs from the set of those t for which $W(t)$ is not a factor of type II_1 by a set of s-measure zero. Therefore we have proved that whenever infinitely many of the groups $\mathfrak{g}_n$ are noncommutative, then almost all the rings $W(t)$ are factors of type II_1. This completes the proof of Theorem 1 in the case where the Lebesgue-Stieltjes measure defined by the function $s(t)$ satisfies equation (4.2).

Suppose that we are now given an arbitrary direct integral which belongs to the ring Z (= center of W) in the sense of [5, p. 432]. According to Theorem IV of [5] any such direct integral must, because of our Lemma 4.1, be equivalent (in the sense of [5, Definition 3]) to the direct integral (1.3) for which the s-measure satisfies (4.2). It now follows from [13, pp. 430–431] that there exists a one-one point transformation (defined almost everywhere) of the real line such that the two measures become mutually absolutely continuous with respect to each other. Hence in particular sets of measure zero correspond to sets of measure zero. Therefore Theorem 1 is proved for the most general direct integral (in the sense of [5]) to which the center Z belongs (in the sense of condition (1.5) of §1 above).

7. **Some remarks and consequences.** If $\mathfrak{G}$ is a compact topological group, then it is very well known that every continuous irreducible representation of $\mathfrak{G}$ occurs in the regular representation of $\mathfrak{G}$. If $\mathfrak{G}$ is not compact, then the decomposition of $L_2(\mathfrak{G})$ no longer takes place in a discrete manner, therefore there does not seem to be an obvious way of even *formulating* the above statement for arbitrary locally compact groups. Moreover such examples as the Lorentz group indicate that for any reasonable formulation of the property "to occur in the regular representation," not all irreducible representations do occur.

A second difficulty is the fact that when factors of class II occur in the central decomposition of $L_2(\mathfrak{G})$, then the decomposition into irreducible representations is by no means unique. Therefore, quite apart from the difficulty of the decomposition being non-discrete, there does not seem to be much sense in asking whether a given irreducible representation occurs, for one would have to add, under what particular decomposition.

However the central decomposition is unique (cf. parts III and IV of [5]). Therefore, for it, the second difficulty does not occur. Moreover in our case where $\mathfrak{G}$ is the restricted direct product of finite groups there is a "natural" measure on the space of characters $X(G)$, namely, the product measure M which we introduced in §2 above. In this particular case both of the above difficulties can be overcome.

Let again Z^0 be the algebra of those class functions which vanish outside finite subsets of $\mathfrak{G}$, and X a homomorphism of Z^0 to the complex numbers. Let C be any (necessarily finite) class of conjugate elements of $\mathfrak{G}$ considered as an element of Z^0, and $\mathrm{X}(C)$ its image under the homorphism X. Denote for any $G \in C$

$$\mathrm{X}(C) = X(G). \tag{7.1}$$

Definition. We shall call the functions $X(G)$ so obtained the characters of $\mathfrak{G}$.

Lemma 7.1. *Every character $X(G)$ of a restricted direct product $\mathfrak{G}$ of finite groups is of the form* (2.4); *that is, the characters of $\mathfrak{G}$ are exactly the elements of the product space $\mathfrak{X}$ of* §2.

Proof. Denote by $x_n(g_n)$ the restriction of $X(G)$ to the finite subgroup $\mathfrak{g}_n$ of $\mathfrak{G}$, then $x_n(g_n)$ is a character of $\mathfrak{g}_n$. Now put

$$X'(G) = \prod_n x_n(g_n) \qquad \text{for } G = (g_1, g_2, \cdots).$$

Then $c \rightarrow \sum_G c(G)X'(G)$ defines a homomorphism of Z^0 to the complex numbers which clearly coincides with X whenever the function $c(G)$ vanishes outside one subgroup $\mathfrak{g}_n$; but such class functions obviously generate the algebra Z^0, hence the two homomorphisms coincide for all elements of Z^0 and therefore $X(G) = X'(G)$ for all $G \in \mathfrak{G}$.

We shall now prove the following theorem.

Theorem 2. *If $\mathfrak{G}$ is the restricted direct product of countably many finite groups, then almost every character occurs in the central decomposition of $L_2(\mathfrak{G})$. More precisely: Perform the central decomposition of $L_2(\mathfrak{G})$ with respect to a suitable Lebesgue-Stieltjes measure $ds(t)$. For s-almost every t we obtain a character $X_t(G)$ of $\mathfrak{G}$ of the form* (2.4). *As t varies over the real line minus a suitable set of s-measure zero, X_t varies over $\mathfrak{X} - \mathfrak{X}_0$, where $\mathfrak{X}_0$ is a set of M-measure zero in the product space $\mathfrak{X}$ defined in* §2.

Proof. This theorem is an easy consequence of the above results together with a result of [13] which states that any measure-preserving σ-isomorphism between the Boolean algebras of measurable sets of two complete separable metric spaces $\mathfrak{S}_1$, $\mathfrak{S}_2$ is induced a.e. by a point-transformation, provided the measures m_j $(j = 1, 2)$ satisfy the following two conditions:

(i) the measure of any open set is positive;

(ii) for every measurable set $\mathfrak{E}_j$ of the space $\mathfrak{S}_j$ we have

$$m_j(\mathfrak{E}_j) = \inf m_j(\mathfrak{O}_j)$$

where $\mathfrak{O}_j$ is an arbitrary open set in $\mathfrak{S}_j$ $(j = 1, 2)$.

Let us take for $\mathfrak{S}_1$ the real line and for the measure m_1 the Lebesgue-Stieltjes measure defined by the function $s(t)$, where we assume for the moment that equation (4.2) is satisfied. For $\mathfrak{S}_2$ we take the product-space $\mathfrak{X}$ introduced in §2. Since $\mathfrak{X}$ is the Cartesian product of finite sets, it is easy to introduce a metric in $\mathfrak{X}$ such that the above conditions (i) and (ii) are satisfied if we put m_2 equal to the product measure M defined on $\mathfrak{X}$ in §2.

We have seen in §5 that the set mapping which one obtains from the mapping ξ^* by restricting ξ^* to the characteristic functions of measurable subsets of $\mathfrak{X}$ defines a measure-preserving σ-isomorphism of the Boolean algebra $\mathfrak{B}(\mathfrak{X})$ onto the Boolean algebra $\mathfrak{B}(s)$. Hence we may conclude, using the above mentioned result of [13], that there exists a subset $\mathfrak{X}^0$ of $\mathfrak{X}$ of M-measure zero, a subset Ω of the real line of s-measure zero, and a one-one mapping ξ' of $X-X^0$ onto $(-\infty, +\infty)-\Omega$ which induces the above Boolean algebra isomorphism. But this implies (as is easily seen) that after possible omission of another subset of measure zero of X and one of $(-\infty, +\infty)$ we have

$$\xi' = \xi^{-1}.$$

This proves Theorem 2 when the Lebesgue-Stieltjes measure, which is used to define the direct integral (1.3), satisfies equation (4.2).

However, according to Theorems III and IV of [5], any other Lebesgue-Stieltjes measure, by means of which an equivalent direct integral can be defined in the sense of [5], is after a suitable one-one point-transformation mutually absolutely continuous with the above s-measure. Hence we obtain again two suitable sets Ω_0 and $\mathfrak{X}_0$ (say) of measure zero such that the mapping $t \rightarrow X_t$ is one-one from $(-\infty, +\infty)-\Omega_0$ onto $\mathfrak{X}-\mathfrak{X}_0$.

REMARK. We have proved a little more than Theorem 2 asserts: Almost every character of $\mathfrak{G}$ *occurs* in fact *with multiplicity one.*

The above suggests the following problem: For what class of (locally compact) groups can the assertion of Theorem 2 be formulated at all? And for what subclass of groups is the assertion true or false? It would seem even of interest to have an example of a group where the assertion can be formulated *in a natural manner* and proved to be false. The group of real or complex 2×2 matrices of determinant one seems likely to be such an example. For this group, the second of the two difficulties mentioned at the beginning of this section does not occur because of Theorem 6.1 of [8].

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