THE STRUCTURE OF VALUATIONS OF THE RATIONAL FUNCTION FIELD $K(x)$[1]

BY

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1. **Introduction.** The following problem was suggested to the writer by Professor Saunders MacLane: Given a valuation $(V_0K=\Gamma_0, \mathfrak{K})$ of a field K with value group Γ_0 and residue class field $\mathfrak{K}$; (A) to determine the nature of Γ and $\mathfrak{L}$ for any extension $(VL=\Gamma, \mathfrak{L})$ of V_0 from K to L/K; and conversely (B) to construct valuations of an extension L/K with value groups and residue class fields which conform to the requirements of (A). The present paper considers this problem in the case when L is a simple transcendental extension $K(x)$ of K. The valuations are of arbitrary rank (cf. [2][2]).

It is well known that (1) the sum of the transcendence degree $T[\mathfrak{L}/\mathfrak{K}]$ of $\mathfrak{L}$ over $\mathfrak{K}$ and the rational rank (cf. [6, footnote 3]) $R[\Gamma/\Gamma_0]$ of the factor group Γ/Γ_0 cannot exceed $T[L/K]$, here equal to 1. Also, (2) if

$$T[\mathfrak{L}/\mathfrak{K}] + R[\Gamma/\Gamma_0] = T[L/K],$$

then $\mathfrak{L}$ and Γ are finitely generated over $\mathfrak{K}$ and Γ_0, respectively. To these conditions we add (3) $\mathfrak{L}$ and Γ must be at most denumerably generated over $\mathfrak{K}$ and Γ_0; and (4) if $T[\mathfrak{L}/\mathfrak{K}]=1$, then $\mathfrak{L}$ must be a rational function field in one variable over a finite algebraic extension of $\mathfrak{K}$. The possible forms for Γ and $\mathfrak{L}$ are given explicitly in Theorems 7.1 and 8.1.

The construction of extensions $(VK(x)=\Gamma, \mathfrak{L})\supseteq(V_0K=\Gamma_0, \mathfrak{K})$ with Γ and $\mathfrak{L}$ satisfying conditions (1) to (4) is given in §9, except for the case where Γ/Γ_0 is finite and $\mathfrak{L}$ is a finite algebraic extension of $\mathfrak{K}$. §12 contains a note on the extension of these results to finitely generated purely transcendental extensions of K.

Two approaches have been made to the study of rank 1 valuations of $K(x)$. One, used by Ostrowski [7], represents x as the limit of a pseudo-convergent sequence in the algebraic completion of K. The other, used by MacLane [3] and [4] and based on work of Rella [8], represents each discrete valuation of $K(x)$ by a simple sequence of approximating subvaluations of $K(x)$, in which each approximant is derived from the preceding by a certain "key" polynomial. It is an exploitation of Gauss' Lemma.

Following the latter method, we show (§6) that every valuation V of

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(1) This paper includes some of the results given in the author's doctoral dissertation (Harvard, 1947).

(2) Numbers in brackets refer to the bibliography at the end of the paper.

$K(x)$ (of arbitrary rank) can be approximated by a well-ordered system of "inductive" valuations. This yields MacLane's structure theory [3] for V, and the results given above(3).

The existence of extensions $(VK(x)=\Gamma, \mathcal{L})$ of V_0 for which Γ/Γ_0 is finite and $\mathcal{L}$ is a finite algebraic extension of $\mathcal{K}$ depends on the presence of certain transcendental pseudo-convergent sets in K or an algebraic extension of K. Some incomplete conditions for this case are given in §10. They are analogous to Kaplansky's conditions [1] for the special case of immediate extensions.

2. **Augmented valuations.** Let V_0 be a general valuation of a base field K with value group Γ_0; and let Γ be any ordered abelian group containing Γ_0. If $f(x)=\sum_{i=0}^{n} c_i x^i$, $c_i \in K$, then the function V_1,

$$V_1(f) = \min_i \left[V_0(c_i) + i\gamma\right], \tag{2.1}$$

where γ is any element in Γ, defines a "first stage" valuation [3, §3; 7, p. 363; 8, pp. 35–36] of the polynomial ring $K[x]$; $V_1 = V_0$ on K. V_1 is denoted by $[V_0, V_1(x)=\gamma]$. For a valuation of this type, any linear polynomial in x can be used in the role of x.

Any valuation W of $K[x]$ can be augmented to other valuations of $K[x]$ by means of certain *key* polynomials which are monic, equivalence-irreducible, and equivalence-minimal in the following sense. Two polynomials f and g are equivalent in W, or $f \sim g$, if $w(f-g) > w(f)$; f equivalence-divides g if there exists h in $K[x]$ such that $fh \sim g$; f is equivalence-irreducible if the equivalence-divisibility of a product by f implies that of a factor. The polynomial f is equivalence-minimal if the degree (in x) of every polynomial equivalence-divisible by f is not less than the degree of f.

Let W be a valuation of $K[x]$ with value group $\Gamma' \subseteq \Gamma$, and let $\phi = \phi(x)$ be a key polynomial over W. If we write the polynomial f in the form $\sum_{i=0}^{m} f_i \phi^i$, where $f_i \in K[x]$ and $\deg f_i < \deg \phi$(4), then the function V,

$$V(f) = \min_i \left[W(f_i) + i\gamma\right],$$

where $\gamma \in \Gamma$, $\gamma > W(\phi)$, is an *augmented* valuation [3, §4], and is denoted by $V = [W, V(\phi) = \gamma]$. It has the following property(3):

I. *For* $f \neq 0$, $W(f) \leqq V(f)$; $W(f) < V(f)$ *if and only if* f *is equivalence-divisible by* ϕ *in* W. *In particular* $W(f) = V(f)$ *if* $\deg f < \deg \phi$.

If we build a finite sequence $\{V_\mu\}$ of augmented valuations(5),

$$V_\mu = [V_{\mu-1}, V_\mu(\phi_\mu) = \gamma_\mu], \qquad \mu = 2, 3, \cdots, k, \tag{2.2}$$

(3) Many of MacLane's proofs carry over to the general case with only minor modifications. In such instances his results are quoted without proof.

(4) $\deg f$ will always mean the degree of $f(x)$ in x.

(5) Here V_1 may be any valuation of $K[x]$. The subscript 1 is not reserved for first stage valuations.

with the conditions

$$\deg \phi_\mu \geqq \deg \phi_{\mu-1}, \tag{2.3}$$

$$\phi_\mu \sim \phi_{\mu-1} \text{ in } V_{\mu-1} \text{ is false,} \tag{2.4}$$

then $\{V_\mu\}$ has the following properties:

II. *For each f in $K[x]$, $V_\mu(f) \leqq V_\lambda(f)$ for $\mu < \lambda$. If $V_\nu(f) = V_{\nu+1}(f)$, then $V_\nu(f) = V_\omega(f)$ for all $\omega > \nu$.*

III. *If* $\deg \phi_\mu = \deg \phi_\lambda$ *for $1 \leqq \eta < \mu$ and all $\lambda > \mu$, then*:

(a) $V_\eta(\phi_\lambda - \phi_\mu) = \gamma_\mu < \gamma_\lambda$;

(b) $V_\eta(\phi_\mu) = V_\eta(\phi_\lambda)$; $V_\mu(\phi_\lambda) = \gamma_\mu$;

(c) $V_\lambda = [V_\mu, V(\phi_\lambda) = \gamma_\lambda]$.

3. **Limit valuations.** Another type of valuation of $K[x]$ can be obtained as follows: Suppose that a well-ordered set of valuations $\{V_\mu\}$ has been defined for all μ less than some limit ordinal σ, and that $\{V_\mu\}$ has property II.

If, for each $f \in K[x]$, there is an ordinal ν such that $V_\nu(f) = V_{\nu+1}(f)$, let $\nu(f)$ be the first such. The function W_σ:

$$W_\sigma(f) = V_{\nu(f)}(f) \tag{3.1}$$

defines a valuation of $K[x]$; we denote it by $W_\sigma = [\{V_\mu\}, \mu < \sigma]$.

Otherwise, there must exist a polynomial g such that $V_\mu(g) < V_\lambda(g)$ for all $\mu < \lambda$. A monic polynomial of minimum degree with this property will be called a *pseudo-key* for $\{V_\mu\}$. A pseudo-key is irreducible in $K[x]$. Expanding any f in terms of such a pseudo-key s,

$$f = \sum_{i=0}^{m} f_i s^i, \qquad \deg f_i < \deg s, \tag{3.2}$$

we can define the function V_σ,

$$V_\sigma(f) = \min_i [W_\sigma(f_i) + i\gamma_\sigma], \tag{3.3}$$

where $\gamma_\sigma > V_\mu(s)$ for all μ.

THEOREM 3.1. *The function V_σ defined by* (3.2) *and* (3.3) *is a valuation of $K[x]$; it is denoted by*

$$V_\sigma = [\{V_\mu\}, \mu < \sigma, V_\sigma(s) = \gamma_\sigma].$$

Proof. For the triangle and product laws to hold for V_σ, it is sufficient that (cf. [3, Theorem 4.2] or [8])

(A) the triangle law hold for polynomials of degree $< \deg s$, and

(B) if f and g are polynomials of degree less than $\deg s$ with the expansion (3.2), $fg = qs + r$, then

$$V_\sigma(f) + V_\sigma(g) = V_\sigma(r) < V_\sigma(q) + \gamma_\sigma.$$

It is necessary only to verify (B). For some ordinal ν, $V_\nu(r) = V_{\nu+1}(r) = V_\sigma(r)$ and $V_\nu(fg) = V_{\nu+1}(fg) = V_\sigma(f) + V_\sigma(g)$. Now $V_{\nu+1}(qs) > V_\nu(qs) \geqq \min\ [V_\nu(fg), V_\nu(r)] = V_{\nu+1}(fg) = V_{\nu+1}(r)$. Hence $V_\sigma(q) + \gamma_\sigma > V_{\nu+1}(qs) > V_\sigma(r) = V_\sigma(f) + V_\sigma(g)$. Q.E.D.

Both W_σ and V_σ are called *limit valuations*.

To show that properties I and II hold for V_σ, we need Ostrowski's Lemma [7, p. 371, III; 1, p. 306, Lemma 4].

LEMMA 3.2. *Let $\beta_0, \beta_1, \cdots, \beta_m$ be any elements of an ordered Abelian group Γ, and let $\{\alpha_\mu\}$ be a well-ordered set of elements of Γ (without a last element) such that $\alpha_\sigma < \alpha_\lambda$ for all $\sigma < \lambda$. Then there exist an integer e $(0 \leqq e \leqq m)$ and an ordinal η such that $\beta_i + i\alpha_\mu > \beta_e + e\alpha_\mu$ for all $i \neq e$ and $\mu > \eta$.*

THEOREM 3.3. *Given the limit valuation $V_\sigma = [\{V_\mu\},\ \mu < \sigma,\ V_\sigma(\phi_\sigma) = \gamma_\sigma]$ with the pseudo-key ϕ_σ. For $f \neq 0$, $V_\mu(f) \leqq V_\sigma(f)$ for all μ. The following statements are equivalent*:

(i) *$V_\mu(f) < V_\lambda(f)$ for all $\mu < \lambda < \sigma$;*

(ii) *$V_\mu(f) < V_\sigma(f)$ for all $\mu < \sigma$;*

(iii) *ϕ_σ equivalence-divides f in all V_μ for μ greater than some ordinal η.*

Proof. Let $f = \sum_{i=0}^{m} f_i \phi_\sigma^i$ be the expansion (3.2) for f. By II and Lemma 3.2, there exist an integer e and an ordinal η such that $V_\mu(f_e\phi_\sigma^e) < V_\mu(f_i\phi_\sigma^i)$ for all $\mu > \eta$ and all $i \neq e$. Thus, for $\mu > \eta$, $V_\mu(f) = V_\mu(f_e\phi_\sigma^e) = \min_i\ [V_\mu(f_i\phi_\sigma^i)] \leqq \min_i\ [V_\sigma(f_i\phi_\sigma^i)] = V_\sigma(f)$. Moreover, the inequality sign holds if and only if $e \neq 0$, which in turn is true if and only if $V_\mu(f) < V_\lambda(f)$ for all $\eta < \mu < \lambda$ (or for all $\mu < \lambda$, by II). If $e \neq 0$, then ϕ_σ equivalence-divides f in V_μ, $\mu > \eta$. Conversely if $V_\mu(f - q\phi_\sigma) > V_\mu(f) = V_\mu(q\phi_\sigma)$ for some $q \in K[x]$, then for $\lambda > \mu$, $V_\lambda(f) \geqq \min\ [V_\lambda(f - q\phi_\sigma),\ V_\lambda(q\phi_\sigma)] > \min\ [V_\mu(q\phi_\sigma),\ V_\mu(q\phi_\sigma)] = V_\mu(f)$. Q.E.D.

Note. Theorem 3.3 proves that II holds for the set $\{V_\mu\}$, $\mu \leqq \sigma$. It further shows that I holds for V_σ if we make the convention that W is to be interpreted as representing all V_μ for μ greater than some ordinal η; η depends on f. The pseudo-key ϕ_σ takes the place of a key for V_σ. The next theorem shows that augmenting a limit-valuation with a key of sufficiently high degree preserves II.

THEOREM 3.4. *If* deg $\phi_\sigma \leqq$ deg $\phi_{\sigma+1}$ *in the valuation $V_{\sigma+1} = [\{V_\mu\},\ \mu < \sigma,\ V_\sigma(\phi_\sigma) = \gamma_\sigma,\ V_{\sigma+1}(\phi_{\sigma+1}) = \gamma_{\sigma+1}]$, then $V_\mu(f) = V_\sigma(f)$ for some $\mu < \sigma$ implies $V_\sigma(f) = V_{\sigma+1}(f)$.*

Proof. If $V_\mu(f) = V_\sigma(f)$, then in the expansion (3.2) in terms of ϕ_σ, $V_\sigma(f) = V_\sigma(f_0) < V_\sigma(f - f_0)$ (cf. the preceding proof). But $V_{\sigma+1}(f - f_0) \geqq V_\sigma(f - f_0)$, and $V_{\sigma+1}(f_0) = V_\sigma(f_0)$, by I. Therefore $V_{\sigma+1}(f - f_0) > V_{\sigma+1}(f_0)$, which implies $V_{\sigma+1}(f) = V_{\sigma+1}(f_0) = V_\sigma(f)$.

4. Inductive valuations.

DEFINITION 4.1. *A ρth stage inductive valuation V_ρ of $K[x]$ is any valuation obtained by a well-ordered sequence of valuations $\{V_\sigma\}$, $\sigma \leqq \rho$, where*

(i) $V_1 = [V_0, V_1(\phi_1) = \gamma_1]$, *$\phi_1$ linear;*

(ii) *if σ is not a limit-ordinal, $\sigma > 1$, $V_\sigma = [V_{\sigma-1}, V_\sigma(\phi_\sigma) = \gamma_\sigma]$;*

(iii) *if σ is a limit-ordinal, then V_σ is the limit valuation $[\{V_\mu\}, \mu < \sigma]$, or $[\{V_\mu\}, \mu < \sigma, V_\sigma(\phi_\sigma) = \gamma_\sigma]$, where ϕ_σ is a pseudo-key for $\{V_\mu\}$;*

(iv) deg $\phi_\mu \leqq$ deg ϕ_λ *for all ordinals $\mu < \lambda \leqq \rho$;*

(v) *if* deg $\phi_\mu =$ deg ϕ_λ, *$\phi_\mu \sim \phi_\lambda$ in V_μ is false.*

If ρ is a limit ordinal, V_ρ is called a *constant degree limit valuation* when the set $\{\deg \phi_\sigma\}$, $\sigma < \rho$, is bounded; otherwise, *increasing degree*(6).

An inductive valuation V_ρ has property I, and the set of subvaluations $\{V_\sigma\}$, $\sigma \leqq \rho$, has properties II and III. Any augmented valuation $V_{\rho+1}$ is an inductive valuation, provided that the key $\phi_{\rho+1}$ satisfies conditions (iv) and (v). However, we have the following theorem.

THEOREM 4.2. *The limit valuation $W_\rho = [\{V_\mu\}, \mu < \rho]$ cannot be augmented to an inductive valuation V.*

Proof. Let ϕ be a prospective key for V. We write $\phi = q\phi_{\nu(\phi)+1} + r$ (cf. (3.1)), where deg $r <$ deg $\phi_{\nu(\phi)+1}$. By I and II, we have $W_\rho(r) = V_{\nu(\phi)}(r) < V_{\nu(\phi)+1}(q\phi_{\nu(\phi)+1}) \leqq W_\rho(q\phi_{\nu(\phi)+1}) \leqq V(q\phi_{\nu(\phi)+1})$. By condition (iv), deg $\phi >$ deg r; hence $W_\rho(r) = V(r)$, and $V(\phi) = V(r) = W_\rho(\phi)$. This contradicts the requirement that $V(\phi) > W_\rho(\phi)$.

5. **Conditions for limit valuations.** If ρ is a limit ordinal, and $\{V_\sigma\}$, $\sigma < \rho$, is a well-ordered set of inductive valuations

$$(5.1)\quad V_\sigma = [V_{\sigma-1}, V_\sigma(\phi_\sigma) = \gamma_\sigma] \quad \text{or} \quad V_\sigma = [\{V_\mu\}, \mu < \sigma, V_\sigma(\phi_\sigma) = \gamma_\sigma],$$

then $\{V_\sigma\}$ has property II. If $\{V_\sigma\}$ has a pseudo-key s, there exists an integer d such that deg $\phi_\sigma = d$ for all σ not less than some ordinal ω. Moreover, the value group Γ_σ of V_σ with respect to $K(x)$ equals Γ_ω, for $\sigma > \omega$, by III. Now the inductive valuation $[\{V_\sigma\}, \sigma < \rho, V_\rho(s) = \gamma_\rho]$ can be constructed if γ_ρ can be chosen greater than all $V_\sigma(s)$. This can be done (without increasing the rank of the valuation) if and only if the set $\{V_\sigma(s)\}$ is bounded in Γ_ω.

THEOREM 5.1(7). *In the set $\{V_\sigma\}$, $\sigma < \rho$, of valuations defined by* (5.1), *the set $\{V_\sigma(s)\}$ is bounded by some element of Γ_ω if and only if the same is true for $\{\gamma_\sigma\}$, $\sigma > \omega$*(8).

Proof. We expand s in terms of each ϕ_σ, $\sigma > \omega$,

(6) Henceforth the term limit valuation refers only to inductive valuations.

(7) This theorem has particular relevance to the rank 1 case. In this case the set $\{V_\sigma\}$, $\omega \leqq \sigma < \rho$, can always be replaced by a cofinal denumerable sequence $\{V_\mu\}$ (2.2). A limit value can be defined (as in [3]) on $K(x)$ by the function V: $V(f) = \lim_{\mu \to \infty} V_\mu(f)$. This function may be nonfinite in the sense that it assigns to some nonzero polynomials the value ∞. Our Theorem 5.1 and MacLane's Theorem 7.1 [3] together give a NAS condition for the finiteness of V. On the other hand, the latter theorem gives a NAS condition for the existence of a pseudo-key for $\{V_\mu\}$ and hence for $\{V_\sigma\}$, when $\lim_{\mu \to \infty} \gamma_\mu = \infty$.

(8) Note that by III, $\gamma_\sigma < \gamma_\lambda$ for $\omega < \sigma < \lambda$.

$$s = \sum_{i=0}^{m} b_{i\sigma}\phi_\sigma^i, \qquad \deg b_{i\sigma} < d.$$

By III, $V_\sigma = [V_\omega,\ V_\sigma(\phi_\sigma) = \gamma_\sigma]$; and $V_\omega(\phi_\sigma) = \gamma_\omega$. As noted in the proof of Lemma 3.4 of [4],

$$V_\omega(s) = \min_i [V_\omega(b_{i\sigma}) + i\gamma_\omega].$$

For some index e there exists a well-ordered set $\{\sigma(\alpha)\}$ of ordinals cofinal in the set $\{\sigma\}$, $\omega < \sigma < \rho$, such that

$$V_\omega(s) = V_\omega(b_{e,\sigma(\alpha)}) + e\gamma_\omega \leqq V_\omega(b_{i,\sigma(\alpha)}) + i\gamma_\omega,$$

for all i. Thus, for each α, $V_\omega(b_{e,\sigma(\alpha)}) =$ a constant δ, and, for all i and α, $V_\omega(b_{i,\sigma(\alpha)}) \geqq$ some lower bound ξ. By II, each $V_\sigma(s) \leqq V_{\sigma(\alpha)}(s) \leqq \delta + e\gamma_{\sigma(\alpha)}$ for some α.

On the other hand

$$\begin{aligned} V_{\sigma(\alpha)}(s) &= \min_i [V_\omega(b_{i,\sigma(\alpha+1)}) + i\gamma_{\sigma(\alpha)}] \\ &= V_\omega(b_{e(\alpha),\sigma(\alpha+1)}) + e(\alpha)\gamma_{\sigma(\alpha)} \\ &\geqq \xi + e(\alpha)\gamma_{\sigma(\alpha)}, \end{aligned}$$

for all α and some index $e(\alpha)$, depending on α. Moreover $e(\alpha) \neq 0$; for otherwise $V_{\sigma(\alpha+1)}(s) = V_{\sigma(\alpha)}(s)$. Thus each $\gamma_\sigma \leqq \gamma_{\sigma(\beta)} \leqq (V_{\sigma(\beta)}(s) - \xi)/e(\beta)$, for some ordinal β. This completes the proof.

6. **The sufficiency of inductive valuations.** From the proof of Theorem 8.1 of [3] we borrow the following result:

LEMMA 6.1. *Let W be any valuation of $K[x]$. Let V_σ be an inductive valuation $[V_{\sigma-1},\ V_\sigma(\phi_\sigma) = W(\phi_\sigma) = \gamma_\sigma]$ or $[\{V_\mu\},\ \mu < \sigma,\ V_\sigma(\phi_\sigma) = W(\phi_\sigma) = \gamma_\sigma]$ such that*

IV. (a) $W(f) \geqq V_\sigma(f)$ *for all f in $K[x]$*,
(b) $W(f) = V_\sigma(f)$ *if* $\deg f < \deg \phi_\sigma$.

Then any monic polynomial ϕ of minimum degree such that $W(\phi) > V_\sigma(\phi)$ defines an inductive valuation $V = [V_\sigma,\ V(\phi) = W(\phi) = \gamma]$ which satisfies IV. *Moreover, $V_\sigma(f) = V(f)$ implies $V_\sigma(f) = W(f)$.*

THEOREM 6.2. *Every valuation W of $K[x]$ can be represented as an inductive valuation.*

Proof. First, $V_1 = [V_0,\ V_1(x) = \gamma_1 = W(x)]$ is an inductive valuation satisfying IV.

Now suppose that V_σ is an inductive valuation with property IV and such that $V_\sigma(f) = W(f)$ for all f of degree less than n. We proceed by induction on n. If there exists a polynomial h of degree n such that $V_\sigma(h) < W(h)$, we let $\mathcal{N}_n$ be the set of all monic polynomials of degree n with this property, and let $\mathcal{M}_n$ be the corresponding set of W-values.

Case (1). If $\mathcal{M}_n$ has a maximum element γ, we choose a member of $\mathcal{N}_n$ with value γ, call it $\phi_{\sigma+1}$ and define $V_{\sigma+1}=[V_\sigma,\ V_{\sigma+1}(\phi_{\sigma+1})=\gamma]$. By Lemma 6.1, this is an inductive valuation satisfying IV. Moreover, if $\deg f=n$, $V_{\sigma+1}(f)=W(f)$; for $f=c\phi_{\sigma+1}+f_0$, where $c\in K$, $\deg f_0<n$; and $W(f)>V_{\sigma+1}(f)$ implies $W(f/c)>W(\phi_{\sigma+1})$, contradicting the choice of $\phi_{\sigma+1}$.

Case (2). If $\mathcal{M}_n$ has no maximum element, we choose from $\mathcal{N}_n$ a set of polynomials whose values form a well-ordered cofinal subset $\{\gamma_{\sigma+\mu}\}$ in $\mathcal{M}_n$, $\gamma_{\sigma+\mu}<\gamma_{\sigma+\omega}$ for $\mu<\omega$. These we label $\phi_{\sigma+1},\cdots,\phi_{\sigma+\mu},\cdots;\ \mu<\lambda$. Using transfinite induction and Lemma 6.1, we construct a well-ordered set $\{V_{\sigma+\mu}\}$, $\mu<\lambda$ of inductive valuations, each with property IV. If μ is not a limit-ordinal, $V_{\sigma+\mu}=[V_{\sigma+\mu-1},\ V_{\sigma+\mu}(\phi_{\sigma+\mu})=\gamma_{\sigma+\mu}]$; otherwise $V_{\sigma+\mu}=[\{V_{\sigma+\omega}\},\ \omega<\mu,\ V_{\sigma+\mu}(\phi_{\sigma+\mu})=\gamma_{\sigma+\mu}]$. This yields a limit valuation: either $W_{\sigma+\lambda}=[\{V_{\sigma+\mu}\},\ \mu<\lambda]$, or $V_{\sigma+\lambda}=[\{V_{\sigma+\mu}\},\ \mu<\lambda,\ V_{\sigma+\lambda}(s)=W(s)]$, where s is a pseudo-key for $\{V_{\sigma+\mu}\}$. Now $W_{\sigma+\lambda}=W$, or $V_{\sigma+\lambda}$ possesses IV, by the last statement of Lemma 6.1. If $\deg f=n$, $f=c\phi_{\sigma+\mu}+f_\mu$ for each μ; $c\in K$, $\deg f_\mu<n$. Now $W(f)>V_{\sigma+\lambda}(f)$ implies $W(f/c)>W(\phi_{\sigma+\mu})$ for all $\mu<\lambda$, contradicting the choice of the set $\{\phi_{\sigma+\mu}\}$. Hence $W(f)=V_{\sigma+\lambda}(f)$ for all f of degree not greater than n.

This completes the induction on n. If the process does not stop at some finite degree, it will go on indefinitely to give an increasing degree limit valuation equal to W.

7. **The value group.** If V_σ is an inductive valuation $[V_{\sigma-1},\ V_\sigma(\phi_\sigma)=\gamma_\sigma]$, and if Γ_σ is the value group of V_σ with respect to $K(x)$, then $\Gamma_\sigma=\Gamma_{\sigma-1}(\gamma_\sigma)$, that is, all elements of the form $\gamma+m\gamma_\sigma$, where $\gamma\in\Gamma_{\sigma-1}$ and m is an integer. If V_σ is a limit valuation $[\{V_\mu\},\ \mu<\sigma,\ V_\sigma(\phi_\sigma)=\gamma_\sigma]$, then by III, $\Gamma_\sigma=\Gamma_\omega(\gamma_\sigma)$, where ϕ_ω is the first key in the set $\{\phi_\mu\}$, $\mu<\sigma$, of highest degree. Moreover, if γ_σ is incommensurable with Γ_0, that is, no multiple of γ_σ is in Γ_0, then V_σ can not be augmented to a new inductive valuation [3, Theorem 6.7]. An induction argument gives the following theorem.

THEOREM 7.1. *Let V_0 be a valuation of K with value group Γ_0, and let V_ρ be an inductive valuation of $K(x)$ with keys and pseudo-keys $\{\phi_\sigma\}$; then Γ_ρ has one of the forms*:

(a) $\Gamma_\rho=\Gamma_0(\gamma_1,\gamma_2,\cdots,\gamma_n)$,

(b) $\Gamma_\rho=\Gamma_0(\gamma_1,\gamma_2,\cdots,\gamma_{n-1},\zeta_n)$,

(c) $\Gamma_\rho=\Gamma_0(\gamma_1,\gamma_2,\cdots)$, *not reducible to form* (a),

where γ_i or ζ_i is the V_ρ-value of the last key or pseudo-key of degree i (when such actually is present); γ_i is commensurable with Γ_0; ζ_i is not.

8. **The residue class field.** The structure theorems in §§9–14 of [3] may be verified for the more general inductive valuations considered here. The proofs are not sufficiently different from MacLane's to warrant their repetition here. The pertinent results are as follows:

Let $\mathfrak{K}$ be the residue class field of K with respect to V_0; let $\mathfrak{L}_\sigma$ be the residue field of $L=K(x)$ with respect to V_σ; and let H_σ be the corresponding homomorphism mapping the valuation ring in $K(x)$ onto $\mathfrak{L}_\sigma$. If V_σ is commensurable, then $\mathfrak{L}_\sigma = F_\sigma(y)$, where F_σ is an algebraic extension of the field $\mathfrak{K}$, and y is transcendental over F_σ; if V_σ is incommensurable, $\mathfrak{L}_\sigma = F_\sigma$. If V_σ is augmented to V, the resulting residue field $\mathfrak{L}$ is $F(z)=F_\sigma(\theta, z)$, where θ and z are algebraic and transcendental over F_σ, respectively, and are determined by the augmenting key ϕ. Corresponding to ϕ there exists a polynomial $p(x)$ such that $V_\mu(p)=V_\sigma(p)=-V_\sigma(\phi)$ for some $\mu<\sigma$; $p(x)$ will be called a V_σ-*deflater* of ϕ. Then θ is a root of $H_\sigma(p\phi)$, a polynomial of degree $[\deg \phi/(\tau_\sigma \deg \phi_\sigma)]$ in the ring $F_\sigma[y]$; where τ_σ is the commensurability number of V_σ, that is, the order of $\Gamma_\sigma/\Gamma_{\sigma-1}$ or $\Gamma_\sigma/\Gamma_\omega$ (cf. §7); and $z=H(q\phi^\tau)$, where q is a V-deflater of ϕ^τ.

Analogous to property III we have the condition that if $\deg \phi_\sigma = \deg \phi$, then $F_\sigma = F$. If V_σ is a limit valuation $[\{V_\mu\}, \mu<\sigma]$ without a pseudo-key, $\mathfrak{L}_\sigma$ is the union $\bigcup_{\mu<\sigma} F_\mu$ of the fields F_μ. If V_σ is a limit valuation with pseudo-key s, then $\mathfrak{L}_\sigma$ is $F_\sigma(z)$ or F_σ, as before, where now $F_\sigma = \bigcup_{\mu<\sigma} F_\mu$.

THEOREM 8.1. *Let* $(V_\rho K(x)=\Gamma_\rho, \mathfrak{L}_\rho)$ *be an extension of* $(V_0K=\Gamma_0, \mathfrak{K})$ *with keys and pseudo-keys* $\{\phi_\sigma\}$; *then* $\mathfrak{L}_\rho$ *has one of the forms*:

(i) $\mathfrak{L}_\rho = \mathfrak{K}(\alpha_1, \alpha_2, \cdots, \alpha_m)$;

(ii) $\mathfrak{L}_\rho = \mathfrak{K}(\alpha_1, \alpha_2, \cdots, \alpha_{m-1}, y)$;

(iii) $\mathfrak{L}_\rho = \mathfrak{K}(\alpha_1, \alpha_2, \cdots)$;

where the α_i *are algebraic over* $\mathfrak{K}$; *y is transcendental. If* Γ_ρ *is not commensurable with* Γ_0, $\mathfrak{L}_\rho$ *must have form* (i). *If* $\mathfrak{L}_\rho$ *has form* (ii), Γ_ρ/Γ_0 *must be finite. The number of adjoined elements is not greater than the number of degrees represented in the set* $\{\phi_\sigma\}$.

From Theorem 7.1 and 8.1 the possible combinations of Γ_ρ and $\mathfrak{L}_\rho$ are (a)(i), (a)(ii), (a)(iii), (b)(i), (c)(i), and (c)(iii).

9. **The existence of inductive valuations with a prescribed structure.** Given $(V_0K=\Gamma_0, \mathfrak{K})$, the construction of $(VK(x)=\Gamma_\rho, \mathfrak{L}_\rho)$ with Γ_ρ and $\mathfrak{L}_\rho$ satisfying the conditions of Theorems 7.1 and 8.1 is given in most cases(9) by the following theorem of MacLane [3, Theorem 13.1].

LEMMA 9.1. *In a given inductive valuation* $(V_\sigma K(x)=\Gamma_\sigma, \mathfrak{L}_\sigma)$, *let* $\psi(y)\neq y$ *be a monic polynomial of degree* $m>0$, *irreducible in* $F_\sigma[y]$. *Then there is one and, except for equivalent polynomials in* V_σ, *only one* $\phi(x)$ *which is a key over* V_σ *and which has* $H_\sigma(p\phi)=\psi(y)$ *for a suitable* V_σ-*deflater* p *of* ϕ.

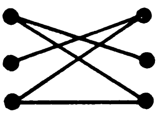
THEOREM 9.2. *Given* $(V_0K=\Gamma_0, \mathfrak{K})$; *let* $\Gamma\neq 0$ *and* $\mathfrak{L}$ *be extensions of* Γ_0 *and* $\mathfrak{K}$, *respectively, such that* (*schematically*) Γ *and* $\mathfrak{L}$ *occur in any of the following combinations* (*cf. Theorems* 7.1 *and* 8.1):

(9) The excluded case is the combination (a)(i).

(a) $\Gamma_0(\gamma_1, \cdots, \gamma_n)$

(b) $\Gamma_0(\gamma_1, \cdots, \gamma_{n-1}, \zeta_n)$

(c) $\Gamma_0(\gamma_1, \gamma_2, \cdots)$



(i) $\mathcal{K}(\alpha_1, \cdots, \alpha_m)$

(ii) $\mathcal{K}(\alpha_1, \cdots, \alpha_{m-1}, y)$

(iii) $\mathcal{K}(\alpha_1, \alpha_2, \cdots)$.

Then there exists an extension $(VK(x)=\Gamma, \mathcal{L})$ *of* V_0.

10. **A special case.** Further conditions are needed for the existence of type (ai), that is, an extension $(VK(x)=\Gamma, \mathcal{L})$ of $(V_0K=\Gamma_0, \mathcal{K})$ with Γ/Γ_0 finite and $\mathcal{L}$ a finite algebraic extension of $\mathcal{K}$.

First let K be algebraically closed. Then $\Gamma=\Gamma_0$ and $\mathcal{L}=\mathcal{K}$; that is, the extension must be immediate. Any inductive representation V_ρ of V must have an infinite number of linear keys (§8) and none of higher degree. Suppose V_ρ is defined by the well-ordered set $\{V_\sigma\}$, $\sigma<\rho$, where $V_\sigma=[V_{\sigma-1}; V_\sigma(x-a_\sigma)=\gamma_\sigma]$ or $V_\sigma=[\{V_\mu\}, \mu<\sigma, V_\sigma(x-a_\sigma)=\gamma_\sigma]$. For $\sigma<\lambda$, $V_0(a_\lambda-a_\sigma)=\gamma_\sigma$, hence $V_0(a_\lambda-a_\sigma)>V_0(a_\sigma-a_\mu)$ for $\mu<\sigma<\lambda$, that is, $\{a_\sigma\}$ is a *pseudo-convergent set* in K(10). For every $a\in K$, the set $\{a-a_\sigma\}$ ultimately attains a constant value; otherwise the valuation V_ρ would equal the first stage valuation $W_1=[V_0, W_1(x-a)=V_\rho(x-a)]$. This is to say that $\{a_\sigma\}$ has no limit in K. Since K is closed, it further implies that $\{a_\sigma\}$ is of transcendental type. Conversely, any transcendental pseudo-convergent set in K without a limit in K (*t.p.c.s.w.l.*) defines a valuation $(V_\rho K(x)=\Gamma_0, \mathcal{K})$. Hence, *if K is algebraically closed, there exists an immediate (the only type* (ai)) *extension to $K(x)$ if and only if there exists a t.p.c.s.w.l. in K*(11).

If K is arbitrary and A is its algebraic closure, then any type (ai) valuation of $K(x)$ can always be extended (cf. [7, p. 300, II]) to an (ai) valuation of $A(x)$. Hence, *for each* (ai) *extension of* $(V_0K=\Gamma_0, \mathcal{K})$ *to* $K(x)$ *there is a t.p.c.s.w.l. in A (with respect to some extension of V_0 to A).*

A partial converse is given by the following theorem.

THEOREM 10.1. *Let* $(V_0K=\Gamma_0, \mathcal{K})$ *be any valuation of K. Let Γ be a finite commensurable extension of Γ_0 and $\mathcal{L}$ a finite algebraic extension of $\mathcal{K}$. Let M be any algebraic extension of K with a valuation* $(V_0'M=\Gamma, \mathcal{L})$ *which is an extension of V_0. If* (1) *M is a simple extension of K, and* (2) *M contains a t.p.c.s.w.l., then there exists an extension* $(VK(x)=\Gamma, \mathcal{L})$.

Note. M must always exist, but it is not always a simple extension of K. The latter is true in the important case when M/K is separable; in particular, when K has characteristic 0.

(10) For these definitions cf. [1, §2]. If $\{a_\sigma\}$ is pseudo-convergent, then for each $f\in K[x]$, eventually either (1) $V_0(f(a_\lambda))=V_0(f(a_\sigma))$ or (2) $V_0(f(a_\lambda))>V_0(f(a_\sigma))$ for all $\lambda>\sigma$. The set $\{a_\sigma\}$ is of transcendental or algebraic type according as (1) does or does not hold for *all* f.

(11) This follows from Theorems 1, 2, and 3 of [1]. In fact Kaplansky proves this result under his hypothesis A, a weaker condition than closure. It is of interest to note that V_ρ is equal to the valuation which assigns to $f(x)$ the ultimately constant value of $f(a_\sigma)$ (cf. [1, Theorem 2] and [7, §65]).

Proof. Since M contains a t.p.c.s.w.l. there exists an extension $(V'M(x_1)=\Gamma, \mathcal{L})$ of $(V_0'M, \Gamma, \mathcal{L})$ to $M(x_1)$; x_1 transcendental over M [1, Theorem 2]. Moreover, there exists in $M(x_1)$ an element x_2, transcendental over M, such that $V'(x_2-1)$ is arbitrarily large.

Suppose $\Gamma=\Gamma_0(\gamma_1, \cdots, \gamma_m)$, $\mathcal{L}=\mathcal{K}(\alpha_{m+1}, \cdots, \alpha_n)$ and $M=K(v)$. From M we select u_i such that $V_0'(u_i)=\gamma_i$, $i=1, \cdots, m$, and $H_0'(u_i)=\alpha_i$, $i=m+1, \cdots, n$. Suppose $u_i=\sum_{j=0}^{r} a_{ij}v^j$, $a_{ij}\in K$. We set $u_i'=\sum_{j=0}^{r} a_{ij}v^j x_2^j$, $i=1, 2, \cdots, n$. Then we have $u_i'-u_i=(x_2-1)[a_{i1}v+a_{i2}v^2(x_2+1)+\cdots+a_{ir}v^r(x_2^{r-1}+\cdots+1)]$. Since $V'(x_2)=0$, $V'(x_2^j+\cdots+1)\geqq 0$ for all positive integers j. Hence $V'(u_i'-u_i)\geqq V'(x_2-1)+\min_{j=1,\cdots,r}[V_0'(a_{ij}v^j)]$ for $i=1, 2, \cdots, n$. If we choose x_2 so that $V'(x_2-1)>\max_i[V'(u_i)-\min_j\{V_0'(a_{ij}v^j)\}]$, then $V'(u_i'-u_i)>V'(u_i)$, for $i=1, \cdots, m, \cdots, n$. It follows that $V'(u_i')=\gamma_i$ for $i=1, \cdots, m$, and $H'(u_i')=H_0'(u_i)=\alpha_i$ for $i=m+1, \cdots, n$. Now $K(x)$, where $x=vx_2$, is a transcendental extension of K with the desired valuation.

11. **The existence of limit valuations with pseudo-keys.** In constructing extensions $(VK(x), \Gamma, \mathcal{L})$ with prescribed Γ and $\mathcal{L}$ (Theorem 9.2), it is not necessary to use limit valuations with pseudo-keys. One might therefore ask if there actually exist such valuations which can not be represented by a finite set of keys. Are pseudo-keys really necessary? The answer is yes, even in the rank 1 case.

Let $(V_0K=\Gamma_0, \mathcal{K})$ be of rank 1; and let $\{a_j\}$ be an algebraic [10] pseudo-convergent sequence in K without a limit in K such that $\lim_{j\to\infty} V_0(a_{j+1}-a_j)<\infty$ [12]. We construct the inductive valuations $V_j=[V_{j-1}, V_j(x-a_j)=\gamma_j]$, $j=1, 2, \cdots$, where $\gamma_j=V_0(a_{j+1}-a_j)$. Let $q(x)$ be any monic polynomial in $K[x]$ for which $V_0(q(a_j))<V_0(q(a_{j+1}))$ for j greater than some integer j_0. For each j we expand

$$q(x) = \sum_{i=0}^{m} q_i(a_j)(x - a_j)^i, \tag{11.1}$$

where $q_0=q$ and $q_n=1$. Now $V_j(q(x))=\min_i[V_0(q_i(a_j))+i\gamma_j]$. For j greater than some j_1, the value of each term of (11.1) increases with j. It follows that $V_j(q(x))<V_{j+1}(q(x))$ for all j. This implies that $\{V_j\}$ has a pseudo-key s. Furthermore, s cannot be of degree 1, namely, $x-a$, $a\in K$; for then a would be a limit of $\{a_j\}$. Finally the V_j -values of s are bounded, by Theorem 5.1. Hence the valuation $[\{V_j\}, V(s)=\lim_{j\to\infty}V_j(s)]$ is the desired valuation.

12. **Extensions to $L_n=K(x_1, x_2, \cdots, x_n)$.** The results of §§7–10 can be extended at once to the case of a purely transcendental extension $L_n=K(x_1, \cdots, x_n)$ of degree n. When $T[\mathcal{L}/\mathcal{K}]+R[\Gamma/\Gamma_0]<n$ (§1), our results are not complete, but for rank 1 valuations they can be improved by the following lemma.

[12] The existence of a field with such a sequence is implied by the first counterexample in §5 of [1].

LEMMA 12.1. *Let $V_1 = [V_0, V_1(x_1) = \epsilon > 0]$ be a first stage (rank 1) valuation of $K(x_1)$; then, for any positive integer q, there exists an immediate extension of V_1 to the field $K(x_1, \cdots, x_q)$, where the x_i are algebraically independent over K.*

Proof. In the power series field $K\{\{x_1\}\}$ it is possible to select $(q-1)$ series which are algebraically independent over $K(x_1)$ and in which the nonzero coefficients have zero V_0-value (cf. [6, §3], especially method II). But these series lie in an immediate extension of $K(x_1)$.

THEOREM 12.2. *Let $(V_0K = \Gamma_0, \mathcal{K})$ be a rank 1 valuation, and let $\mathcal{L}, \Gamma$ be at most denumerably generated extensions of $\mathcal{K}, \Gamma_0$ with $U = T[\mathcal{L}/\mathcal{K}] + R[\Gamma/\Gamma_0] < n$. There exists an extension $(VL_n = \Gamma, \mathcal{L})$ if*

(i) *when $\mathcal{L}$ and Γ can be finitely generated over $\mathcal{K}$ and Γ_0, $U \geqq 2$ and, whenever $R[\Gamma/\Gamma_0] = 0$, $\mathcal{L}$ is a rational function field over an extension of $\mathcal{K}$;*

(ii) *otherwise, $U \geqq 1$.*

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