

ON A CLASS OF MARKOV PROCESSES⁽¹⁾

BY

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1. Introduction. Let $x_k(t)$, $x_k(0)=0$, $0 \leq t < \infty$ ($k=1, \dots, n$) be elements of n independent Wiener spaces. Let $\bar{x}(t)$ be an element of the product space of the n independent Wiener spaces. Throughout the paper we shall assume $V(t, \bar{x})$ to be a Borel measurable function of t and the n -dimensional vector $\bar{x}=(x_1, \dots, x_n)$ that is bounded in every finite Euclidean sphere of the (t, \bar{x}) space, $0 \leq t < \infty$. These assumptions will not be restated in the rest of the paper. We shall often assume that $V(t, \bar{x})$ satisfies additional regularity conditions specified later on in the paper.

Let

$$(1.1) \quad y(t) = \int_0^t V(\tau, \bar{x}(\tau)) d\tau.$$

We wish to relate the study of the Markov process

$$(1.2) \quad (\bar{x}(t), y(t))$$

to the study of certain differential and integral equations. Such a study is of interest for the following reasons. We obtain information about Markov processes of type (1.2). For certain choices of the function $V(t, \bar{x})$, the study yields interesting information about the problem of the absorbing barrier in diffusion theory. Moreover, if we look at the problem from a different point of view, we obtain a general method of getting limit theorems of a certain type [2]⁽²⁾.

Consider the distribution function

$$(1.3) \quad \sigma(\alpha; t) = \Pr \{ y(t) \leq \alpha \}.$$

Let $\bar{G}_m = (G_m^{(1)}, \dots, G_m^{(n)})$, where $G_1^{(k)}, G_2^{(k)}, \dots$ ($k=1, \dots, n$) are independent, normally distributed random variables with mean 0 and variance 1. The distribution function of

$$\frac{1}{m} \sum_{k/m \leq t} V\left(\frac{k}{m}, \frac{\bar{G}_1 + \dots + \bar{G}_k}{m^{1/2}}\right)$$

is the same as that of

Received by the editors January 5, 1950 and May 11, 1950 and, in revised form, October 28, 1950.

⁽¹⁾ Work done under ONR contract.

⁽²⁾ Numbers in brackets refer to the bibliography at the end of the paper.

$$(1.4) \quad \frac{1}{m} \sum_{k/m \leq t} V\left(\frac{k}{m}, \bar{x}\left(\frac{k}{m}\right)\right)$$

by definition of Wiener measure. The distribution function of (1.4) approaches the distribution function $\sigma(\alpha; t)$ at every continuity point of the latter as $m \rightarrow \infty$, given that $V(t, \bar{x})$ is a limit of continuous functions.

In many instances, it is possible to show that $\sigma(\alpha; t)$ is not only the limiting distribution of (1.4) but also that of

$$(1.5) \quad \frac{1}{m} \sum_{k/m \leq t} V\left(\frac{k}{m}, \frac{\bar{X}_1 + \cdots + \bar{X}_k}{m^{1/2}}\right)$$

where $X_1^{(k)}, X_2^{(k)}, \dots (k=1, \dots, n)$ are general, independent, identically distributed random variables with mean 0 and variance 1 and $\bar{X}_m = (X_m^{(1)}, \dots, X_m^{(n)})$. In the last section we shall consider the case of a function $V(t, \bar{x}) = V(\bar{x})$ for which such an invariance is easily proved.

Let $V(t, \bar{x})$ be bounded below by the constant M in (t, \bar{x}) -space. Then

$$(1.6) \quad Q(t, \bar{x}) = E\{\exp(-uy(t)) \mid \bar{x}(t) = \bar{x}\} \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}, \quad u \geq 0,$$

exists since

$$(1.7) \quad \begin{aligned} Q(t, \bar{x}) &\leq E\{\exp(u \mid M \mid t) \mid \bar{x}(t) = \bar{x}\} \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}} \\ &\leq \exp(u \mid M \mid t) \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}. \end{aligned}$$

Set

$$(1.8) \quad C(\bar{\xi}, \tau; \bar{x}, t) = \frac{\exp(-|\bar{x} - \bar{\xi}|^2/2(t - \tau))}{(2\pi(t - \tau))^{n/2}}.$$

If $V(t, \bar{x})$ is bounded in (t, \bar{x}) -space, $Q(t, \bar{x})$ satisfies the following integral equation

$$(1.9) \quad Q(t, \bar{x}) + u \int_0^t \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau = C(\bar{0}, 0; \bar{x}, t)$$

where the integration from $-\infty$ to ∞ denotes integration over all of \bar{x} space and $\bar{0}$ denotes the origin of the \bar{x} -space. M. Kac proved this result in the 1-dimensional case and the proof for the n -dimensional case follows in a completely analogous manner [1].

DEFINITION 1. The function $g(t, \bar{x})$ is said to satisfy a Hölder condition at (t', \bar{x}') if there are numbers $K, \alpha > 0$ such that

$$(1.10) \quad |g(t, \bar{x}) - g(t', \bar{x}')| \leq K\{|t - t'|^\alpha + |\bar{x} - \bar{x}'|^\alpha\}$$

in a neighborhood of (t', \bar{x}') .

DEFINITION 2. The function $g(t, \bar{x})$ is said to satisfy a uniform Hölder condition at (t', \bar{x}') if there are numbers $K, \alpha, \delta > 0$ such that

$$|g(t + \Delta t, \bar{x} + \bar{\Delta}\bar{x}) - g(t, \bar{x})| \leq K\{|\Delta t|^\alpha + |\bar{\Delta}\bar{x}|^\alpha\}$$

when

$$|\Delta t| + |\bar{\Delta}\bar{x}| < \delta$$

for all points (t, \bar{x}) in a neighborhood of (t', \bar{x}') .

The boundedness of $V(t, \bar{x})$ in (t, \bar{x}) -space is assumed throughout §2. The boundedness of $V(t, \bar{x})$ and a set of basic estimates imply that $Q(t, \bar{x})$ satisfies a uniform Hölder condition at all $(t, \bar{x}) \neq (0, \bar{0})$. $Q(t, \bar{x})$ is then shown to satisfy the differential equation

$$(1.11) \quad \frac{1}{2} \Delta Q - \frac{\partial Q}{\partial t} - uV(t, \bar{x})Q = 0$$

at every point $(t, \bar{x}) \neq (0, \bar{0})$ at which $V(t, \bar{x})$ satisfies a Hölder condition. ΔQ denotes the Laplacian of Q in \bar{x} -space. Continuity properties of the derivatives of Q are then examined.

Stronger conditions are assumed in §3 to insure the applicability of Green's theorem in obtaining further results. We assume $V(t, \bar{x})$ satisfies a uniform Hölder condition everywhere except in a regular set S . The definition of a regular set is given in §3. We require that $V(t, \bar{x})$ be bounded below in (t, \bar{x}) space. $Q(t, \bar{x})$ is then a solution of equation (1.11) at all points (t, \bar{x}) not in the set S and satisfies

$$\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}| < \epsilon} Q(t, \bar{x}) d\bar{x} = 1 \quad \text{for all } \epsilon > 0$$

and a few other auxiliary conditions. $Q(t, \bar{x})$ is the unique solution of equation (1.11), satisfying these conditions if $V(t, \bar{x}) \geq 0$. Additional remarks are made indicating that an analogous theorem holds for

$$(1.12) \quad F(t, \bar{x}) = E\{\exp(iu y(t)) \mid \bar{x}(t) = \bar{x}\} \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}.$$

The transform

$$(1.13) \quad q(\bar{x}, s) = \int_0^\infty e^{-st} Q(t, \bar{x}) dt, \quad s \geq 0,$$

is considered in §4 when $V(t, \bar{x}) = V(\bar{x}) \geq 0$. We again assume that $V(\bar{x})$ satisfies a uniform Hölder condition everywhere except in a regular set S . $q(\bar{x}, s)$ then is the unique solution of

$$(1.14) \quad 2^{-1} \Delta q - (s + uV(\bar{x}))q = 0$$

at all points $\bar{x} \neq \bar{0}$ not in S , satisfying

$$\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}|=\epsilon} \frac{\partial q}{\partial n} ds = -2 \quad \text{for all } \epsilon > 0$$

and a few other auxiliary conditions. $\partial q / \partial n$ is the derivative of q normal to the sphere $|\bar{x}| = \epsilon$. M. Kac proved this theorem in the 1-dimensional case [2].

Let

$$(1.15) \quad \sigma(\bar{x}, \alpha, t) = \Pr \{y(t) \leq \alpha \mid \bar{x}(t) = \bar{x}\}.$$

The integral equation satisfied by $\sigma(\bar{x}, \alpha, t)$ when $V(t, \bar{x})$ is bounded is derived in §5.

Note that

$$(1.16) \quad \int_0^\infty e^{-u\alpha} d_\alpha \sigma(\alpha; t) = \int_{-\infty}^\infty Q(t, \bar{x}) d\bar{x}$$

when $V(t, \bar{x}) \geq 0$. Moreover, if $V(t, \bar{x}) = V(\bar{x}) \geq 0$,

$$(1.17) \quad \int_0^\infty \int_0^\infty e^{-u\alpha - s't} d_\alpha \sigma(\alpha; t) dt = \int_{-\infty}^\infty q(\bar{x}, s) d\bar{x}.$$

2. The parabolic differential equation.

LEMMA 1. Let $|V(t, \bar{x})| \leq M$. Then

$$(2.1) \quad \lim_{\epsilon \rightarrow 0} \int_{|\bar{x}| \leq \epsilon} Q(t, \bar{x}) d\bar{x} = 1$$

for all $\epsilon > 0$. Moreover, $Q(t, \bar{x})$ satisfies the uniform Hölder condition

$$(2.2) \quad |Q(t + \Delta t, \bar{x} + \bar{\Delta x}) - Q(t, \bar{x})| \leq M(t, \bar{x}) \{ |\bar{\Delta x}| + |\Delta t| |\lg \Delta t| \}$$

at all $(t, \bar{x}) \neq (0, \bar{0})$.

Equation (1.9) is basic for the required estimates. The function $C(\bar{0}, 0; \bar{x}, t)$ is infinitely differentiable in t and the components of \bar{x} at all points $(t, \bar{x}) \neq (0, \bar{0})$. Moreover $\lim_{t \rightarrow 0} \int_{|\bar{x}| \leq \epsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x} = 1$ for all $\epsilon > 0$. Hence we need only consider

$$(2.3) \quad G(t, \bar{x}) = \int_0^t \int_{-\infty}^\infty C(\bar{\xi}, \tau; \bar{x}, t) V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau.$$

Inequality (1.7) implies that

$$\int_{|\bar{x}| \leq \epsilon} G(t, \bar{x}) d\bar{x} \leq M e^{u|M|t} \int_{|\bar{x}| \leq \epsilon} C(\bar{0}, 0; \bar{x}, t) d\bar{x}$$

so that equation (2.1) is easily verified.

The derivatives

$$(2.4) \quad \frac{\partial G(t, \bar{x})}{\partial x_k} = \int_0^t \int_{-\infty}^{\infty} \frac{\partial C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k} V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau$$

($k=1, \dots, n$) exist and are continuous everywhere. Hence, the derivatives $\partial Q(t, \bar{x})/\partial x_k$ ($k=1, \dots, n$) exist and are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$. This in turn implies that $Q(t, \bar{x})$ satisfies the uniform Hölder condition

$$(2.5) \quad |Q(t, \bar{x} + \bar{\Delta x}) - Q(t, \bar{x})| \leq M(t, \bar{x}) |\bar{\Delta x}|$$

at all $(t, \bar{x}) \neq (0, \bar{0})$. We derive the Hölder condition in t again making use of inequality (1.7).

$$\begin{aligned} G(t + \Delta t, \bar{x}) - G(t, \bar{x}) &= \int_t^{t+\Delta t} \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t + \Delta t) V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \{C(\bar{\xi}, \tau; \bar{x}, t + \Delta t) - C(\bar{\xi}, \tau; \bar{x}, t)\} V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ &= I_1 + I_2, \end{aligned} \quad \Delta t > 0.$$

Now

$$\begin{aligned} |I_1| &\leq C(\bar{0}, 0; \bar{x}, t + \Delta t) M e^{u M t} \Delta t, \\ I_2 &= \int_{t-\Delta t}^t \int_{-\infty}^{\infty} + \int_0^{t-\Delta t} \int_{-\infty}^{\infty} + \int_{\Delta t}^{t-\Delta t} \int_{-\infty}^{\infty} (C(\bar{\xi}, \tau; \bar{x}, t + \Delta t) \\ &\quad - C(\bar{\xi}, \tau; \bar{x}, t)) V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ &= I_3 + I_4 + I_5. \end{aligned}$$

Clearly

$$|I_3 + I_4| \leq 2 \{C(\bar{0}, 0; \bar{x}, t + \Delta t) + C(\bar{0}, 0; \bar{x}, t)\} M e^{u M t} \Delta t$$

while

$$\begin{aligned} |I_5| &\leq \Delta t M e^{u M t} \int_{\Delta t}^{t-\Delta t} \int_{-\infty}^{\infty} \left| \frac{\partial C(\bar{\xi}, \tau; \bar{x}, t + \theta \Delta t)}{\partial t} \right| C(\bar{0}, 0; \bar{\xi}, \tau) d\bar{\xi} d\tau \\ &\leq \Delta t M e^{u M t} \int_{\Delta t}^{t-\Delta t} \int_{-\infty}^{\infty} \frac{n}{2} \frac{(t - \tau + \Delta t)^{n/2}}{(t - \tau)^{(n+2)/2}} C(\bar{\xi}, \tau; \bar{x}, t + \Delta t) C(\bar{0}, 0; \bar{\xi}, \tau) d\bar{\xi} d\tau \\ &\quad + \Delta t M e^{u M t} \int_{\Delta t}^{t-\Delta t} \int_{-\infty}^{\infty} \frac{1}{2} \frac{|\bar{x} - \bar{\xi}|^2 (t - \tau + \Delta t)^{n/2}}{(t - \tau)^{(n+4)/2}} C(\bar{\xi}, \tau; \bar{x}, t + \Delta t) \\ &\quad \cdot C(\bar{0}, 0; \bar{\xi}, \tau) d\bar{\xi} d\tau = I_6 + I_7, \end{aligned} \quad 0 \leq \theta \leq 1.$$

Now

$$\begin{aligned}
 |I_6| &\leq \Delta t M e^{uMt} C(\bar{0}, 0; \bar{x}, t + \Delta t) \int_{\Delta t}^{t-\Delta t} \frac{(t + \Delta t - \tau)^{n/2}}{(t - \tau)^{(n+2)/2}} d\tau \\
 &\leq M(t, \bar{x}) \Delta t |\lg \Delta t|, \\
 |I_7| &\leq \frac{4\Delta t M e^{uMt}}{(\pi t)^{(n+4)/2}} \int_{\Delta t}^{t/2} \int_{-\infty}^{\infty} |\bar{x} - \bar{\xi}|^2 C(\bar{0}, 0; \bar{\xi}, \tau) d\bar{\xi} d\tau \\
 &\quad + \frac{\Delta t M e^{uMt}}{(\pi t)^{n/2}} \int_{t/2}^{t-\Delta t} \frac{(t + \Delta t - \tau)^{(n+2)/2}}{(t - \tau)^{(n+4)/2}} d\tau \int_{-\infty}^{\infty} \bar{u}^2 e^{-\bar{u}^2/2} d\bar{u} \\
 &\leq M(t, \bar{x}) \Delta t |\lg \Delta t|.
 \end{aligned}$$

Therefore

$$(2.6) \quad |G(t + \Delta t, \bar{x}) - G(t, \bar{x})| \leq M(t, \bar{x}) \Delta t |\lg \Delta t|.$$

Inequalities (2.5), (2.6) imply that $Q(t, \bar{x})$ satisfies the uniform Hölder condition (2.2) at all $(t, \bar{x}) \neq (0, \bar{0})$. Lemma 1 is thereby proved.

LEMMA 2. Let $V(t, \bar{x})$ be a bounded function satisfying a Hölder condition at $(t, \bar{x}) \neq (0, \bar{0})$. Then the function

$$H(t, \bar{x}) = \int_0^t \int_{|\bar{x} - \bar{\xi}| \leq a} V(\tau, \bar{\xi}) C(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau, \quad a > 0,$$

satisfies the differential equation

$$\frac{1}{2} \Delta H - \frac{\partial H}{\partial t} + V(t, \bar{x}) = 0$$

at (t, \bar{x}) . The derivatives $\partial^2 H / \partial x_k^2$ ($k=1, \dots, n$) are given by

$$\frac{\partial^2 H}{\partial x_k^2} = \int_0^t \int_{|\bar{x} - \bar{\xi}| \leq a} \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k^2} (V(\tau, \bar{\xi}) - V(t, \bar{x})) d\bar{\xi} d\tau.$$

This lemma is proved on p. 229 of Levi [4] when \bar{x} is 1-dimensional. The proof in the n -dimensional case completely parallels the proof of Levi.

LEMMA 3. Let $V(t, \bar{x})$ be a bounded function satisfying a Hölder condition at a point $(t, \bar{x}) \neq (0, \bar{0})$. $Q(t, \bar{x})$ then satisfies the differential equation (1.11) at (t, \bar{x}) .

$V(t, \bar{x})Q(t, \bar{x})$ satisfies a Hölder condition at (t, \bar{x}) by Lemma 1. The function $C(\bar{0}, 0; \bar{x}, t)$ satisfies the differential equation

$$\frac{1}{2} \Delta C - \frac{\partial C}{\partial t} = 0.$$

Equation (1.9) and Lemma 2 then imply that $Q(t, \bar{x})$ satisfies equation (1.11) at (t, \bar{x}) .

LEMMA 4. *Let $V(t, \bar{x})$ be a bounded function satisfying a uniform Hölder condition at a point $(t, \bar{x}) \neq (0, \bar{0})$. The derivatives $\partial^2 Q / \partial x_k^2$ ($k=1, \dots, n$), $\partial Q / \partial t$ exist in a neighborhood of (t, \bar{x}) and are continuous at (t, \bar{x}) .*

The derivatives $\partial^2 Q / \partial x_k^2$ ($k=1, \dots, n$), $\partial Q / \partial t$ exist in a neighborhood of (t, \bar{x}) by Lemma 3. Now

$$\frac{\partial Q}{\partial t} = \frac{1}{2} \Delta Q - uVQ$$

so that we need only verify the continuity of $\partial^2 Q / \partial x_k^2$ ($k=1, \dots, n$) at (t, \bar{x}) .

$$\begin{aligned} & \frac{\partial^2 G(t, \bar{x})}{\partial x_k^2} - \frac{\partial^2 G(t + \Delta t, \bar{x} + \bar{\Delta} \bar{x})}{\partial x_k^2} \\ &= \left\{ \iint_{|\bar{x} - \bar{\xi}| + |t - \tau| < \delta} + \iint_{|\bar{x} - \bar{\xi}| + |t - \tau| \geq \delta} \right\} \frac{(\partial^2 C(\bar{\xi}, \tau; \bar{x}, t))}{\partial x_k^2} (V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) \\ & \quad - V(t, \bar{x}) Q(t, \bar{x})) - \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x} + \bar{\Delta} \bar{x}, t + \Delta t)}{\partial x_k^2} (V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) \\ & \quad - V(t + \Delta t, \bar{x} + \bar{\Delta} \bar{x}) Q(t + \Delta t, \bar{x} + \bar{\Delta} \bar{x})) d\bar{\xi} d\tau \\ &= I_8 + I_9. \end{aligned}$$

I_9 vanishes as $|\Delta t|, |\bar{\Delta} \bar{x}| \rightarrow 0$. The uniform Hölder condition allows us to obtain the same estimate for each of the two terms of I_8 . We find that

$$\begin{aligned} |I_8| &\leq 2 \iint_{|\bar{x} - \bar{\xi}| + |t - \tau| < 2\delta} \frac{\partial^2 C(\bar{\xi}, \tau; \bar{x}, t)}{\partial x_k^2} (V(\tau, \bar{\xi}) Q(\tau, \bar{\xi}) - V(t, \bar{x}) Q(t, \bar{x})) d\bar{\xi} d\tau \\ &\leq 2K \int_{|t - \tau| < 2\delta} \int_{-\infty}^{\infty} e^{-u^2} (1 + u_k^2) \{ |t - \tau|^{\alpha-1} + |u_k|^\alpha |t - \tau|^{\alpha/2-1} \} d\bar{u} d\tau \\ &\leq \frac{M\delta^{\alpha/2}}{\alpha}. \end{aligned}$$

The continuity is verified on letting $\delta \rightarrow 0$.

3. Unbounded $V(t, \bar{x})$.

DEFINITION 3. *We shall say that the set S is regular if the following conditions are satisfied for every pair (T, \bar{y}) , $T \geq 0$, and all $b > b(T, \bar{y}) > 0$:*

(1) *For every $\epsilon > 0$ there is a set $R(\epsilon)$, the sum of a finite number of nonoverlapping closed parallelepipeds each of diameter less than ϵ with sides parallel to the coordinate axes, in $\{0 \leq t \leq T, |\bar{x} - \bar{y}| \leq b\}$ and covering*

$$\{0 \leq t \leq T, |\bar{x} - \bar{y}| \leq b\} \cap S.$$

- (2) The volume of $R(\epsilon)$ is less than ϵ .
 (3) The distance between the complement of $R(\epsilon)$ in $\{0 \leq t \leq T, |\bar{x} - \bar{y}| \leq b\}$ and S is positive.
 (4) The total surface area of the parallelepipeds of $R(\epsilon)$ is bounded for all ϵ .

THEOREM 1. Let $V(t, \bar{x})$ be a function bounded below and satisfying a uniform Hölder condition everywhere except in a regular set S . Then $Q(t, \bar{x})$ is a solution of equation (1.11) at all points $(t, \bar{x}) \neq (0, \bar{0})$ not in S and satisfies the following conditions:

- (1) $Q(t, \bar{x}) \rightarrow 0$ as $|\bar{x}| \rightarrow \infty$.
 (2) $\lim_{t \rightarrow 0} \int_{|\bar{x}| \leq \epsilon} Q(t, \bar{x}) d\bar{x} = 1$ for all $\epsilon > 0$.
 (3) $\partial Q / \partial x_k$ ($k=1, \dots, n$) are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$. $\partial^2 Q / \partial x_k^2$ ($k=1, \dots, n$), $\partial Q / \partial t$ are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$ not in S .

If $V(t, \bar{x}) \geq 0$, $Q(t, \bar{x})$ is the unique solution of equation (1.11), satisfying conditions (1)–(3).

Let

$$V_N(t, \bar{x}) = \begin{cases} V(t, \bar{x}) & \text{if } |V(t, \bar{x})| \leq N, \\ N & \text{otherwise} \end{cases}$$

and

$$Q_N(t, \bar{x}) = E \left\{ \exp \left(-u \int_0^t V_N(\tau, \bar{x}(\tau)) d\tau \right) \middle| \bar{x}(t) = \bar{x} \right\} \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}.$$

Lemma 3 implies that $Q_N(t, \bar{x})$ satisfies

$$\frac{1}{2} \Delta Q_N - \frac{\partial Q_N}{\partial t} - u V_N(t, \bar{x}) Q_N = 0$$

at all points $(t, \bar{x}) \neq (0, \bar{0})$ not in S . Inequality (1.7) and Lemmas 1 and 4 imply that $Q_N(t, \bar{x})$ satisfies conditions (1) to (3). Let

$$(3.1) \quad f(\bar{\xi}, \tau; \bar{x}, t) = C(\bar{\xi}, \tau; \bar{x}, t) - \int_0^{t-\tau} \frac{e^{-b^2/2(t-\tau-z)}}{(2\pi(t-\tau-z))^{n/2}} v_n(|\bar{x} - \bar{\xi}|, z) dz$$

where

$$(3.2) \quad v_n(r, t) = \sum_{j=1}^{\infty} 2 \exp(-\alpha_j^{(n)2} t / 2b^2) J_{(n-2)/2}(\alpha_j^{(n)} r / b) / J'_{(n-2)/2}(\alpha_j^{(n)})$$

(see §6). The numbers $\alpha_j^{(n)}$ ($j=1, \dots$) are the positive zeros of the Bessel function $J_{(n-2)/2}(x)$. $f(\bar{\xi}, \tau; \bar{x}, t)$ is the fundamental solution of

$$\frac{1}{2} \Delta f(\cdot, \cdot; \bar{x}, t) - \frac{\partial f}{\partial t} = 0, \quad \frac{1}{2} \Delta f(\bar{\xi}, \tau; \cdot, \cdot) + \frac{\partial f}{\partial \tau} = 0$$

with boundary value zero at the spheres of radius b about $\bar{\xi}$, \bar{x} respectively.

Let $b > 0$ be such that we can find sets $R(\epsilon)$ satisfying conditions (1) to (4) of definition 3 covering

$$\{0 \leq \tau \leq t, |\bar{x} - \bar{\xi}| \leq b\} \cap S.$$

Let

$$R(\epsilon, \delta) = R(\epsilon) \cap \{\delta \leq \tau \leq t - \delta, |\bar{x} - \bar{\xi}| \leq b\}, \quad \delta > 0.$$

Call the surfaces consisting of the upper and lower faces respectively of the parallelepipeds of $R(\epsilon, \delta)$ perpendicular to the t axis, $U(\epsilon)$ and $L(\epsilon)$. Call the surface consisting of the faces of the parallelepipeds of $R(\epsilon, \delta)$ parallel to the t axis $P(\epsilon)$. Let

$$B(W) = \frac{1}{2} \Delta W - \frac{\partial W}{\partial \tau} - u V_N W,$$

$$C(W) = \frac{1}{2} \Delta W + \frac{\partial W}{\partial \tau}.$$

We apply Green's theorem making use of the continuity of $\partial^2 Q_N / \partial x_k^2$ ($k=1, \dots, n$), $\partial Q / \partial t$ away from the set S and obtain

$$\begin{aligned} 0 &= \left\{ \int_{\delta}^{t-\delta} \int_{|\bar{x}-\bar{\xi}| \leq b} - \iint_{R(\epsilon, \delta)} \right\} (f(\bar{\xi}, \tau; \bar{x}, t) B(Q_N(\tau, \bar{\xi})) \\ &\quad - Q_N(\tau, \bar{\xi}) C(f(\bar{\xi}, \tau; \cdot, \cdot))) d\bar{\xi} d\tau \\ &= - \frac{1}{2} \int_{\delta}^{t-\delta} \int_{|\bar{x}-\bar{\xi}| \leq b} \frac{\partial f(\bar{\xi}, \cdot; \cdot, \cdot)}{\partial n} Q_N ds d\tau \\ &\quad - \left\{ \int_{\delta}^{t-\delta} \int_{|\bar{x}-\bar{\xi}| \leq b} - \iint_{R(\epsilon, \delta)} \right\} u f V_N Q_N d\bar{\xi} d\tau \\ &\quad + \frac{1}{2} \int_{P(\epsilon)} \left(\frac{\partial f(\bar{\xi}, \tau; \cdot, \cdot)}{\partial n} Q_N - \frac{\partial Q_N(\tau, \bar{\xi})}{\partial n} f \right) ds \\ &\quad - \left\{ \int_{U(\epsilon)} - \int_{L(\epsilon)} \right\} f(\bar{\xi}, \tau; \cdot, \cdot) Q_N ds \\ &\quad + \int_{|\bar{x}-\bar{\xi}| \leq b} (f(\bar{\xi}, \delta; \bar{x}, t) Q_N(\delta, \bar{\xi}) - f(\bar{\xi}, t - \delta; \bar{x}, t) Q_N(t - \delta, \bar{\xi})) d\bar{\xi}. \end{aligned}$$

Let $\epsilon \rightarrow 0$. Condition (2) of Definition 3 and the boundedness of $f V_N Q_N$ over the $R(\epsilon, \delta)$ imply that the volume integral over $R(\epsilon, \delta)$ vanishes in the limit. The bounded surface area of $P(\epsilon)$, $U(\epsilon)$, and $L(\epsilon)$, the symmetry of the parallelepipeds, and the continuity of $\partial f / \partial n$, $\partial Q_N / \partial n$, Q_N , f imply that the surface

integrals over $P(\epsilon)$, $U(\epsilon)$, $L(\epsilon)$ vanish in the limit. We then let $\delta \rightarrow 0$ and obtain

$$(3.3) \quad Q_N(t, \bar{x}) + u \int_0^t \int_{|\bar{x} - \bar{\xi}| \leq a} f(\bar{\xi}, \tau; \bar{x}, t) Q_N(\tau, \bar{\xi}) V_N(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ = f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x} - \bar{\xi}| = a} \frac{\partial f(\bar{\xi}, \tau; \bar{x}, t)}{\partial n} Q_N(\tau, \bar{\xi}) ds d\tau.$$

Now

$$\lim_{N \rightarrow \infty} Q_N(t, \bar{x}) = Q(t, \bar{x})$$

which is finite since $V(t, \bar{x})$ is bounded below. On taking the limit of equation (3.3) as $N \rightarrow \infty$, we have

$$(3.4) \quad Q(t, \bar{x}) + u \int_0^t \int_{|\bar{x} - \bar{\xi}| \leq a} Q(\tau, \bar{\xi}) f(\bar{\xi}, \tau; \bar{x}, t) V(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ = f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x} - \bar{\xi}| = a} \frac{\partial f(\bar{\xi}, \cdot; \cdot, \cdot)}{\partial n} Q(\tau, \bar{\xi}) ds d\tau.$$

This integral equation plays the same role that equation (1.9) did in §2. The surface integral of equation (3.4) is infinitely differentiable in t and the components of \bar{x} . It satisfies

$$\frac{1}{2} \Delta C - \frac{\partial C}{\partial t} = 0.$$

The counterparts of Lemmas 1 to 4 with $f(\bar{\xi}, \tau; \bar{x}, t)$ in place of $C(\bar{\xi}, \tau; \bar{x}, t)$ are proved in exactly the same manner as before. Hence $Q(t, \bar{x})$ satisfies equation (1.11) and conditions (1) to (3) of the theorem.

A uniqueness argument for $Q(t, \bar{x})$ as a solution of equation (1.11) can be carried out if $V(t, \bar{x})$ is non-negative. We shall give an example of such a uniqueness argument in the proof of Theorem 2.

THEOREM 2. *Let $V(t, \bar{x})$ satisfy a uniform Hölder condition at all points (t, \bar{x}) not in a regular set S . Then $F(t, \bar{x})$ is the unique solution of equation*

$$(3.5) \quad \frac{1}{2} \Delta F - \frac{\partial F}{\partial t} + iuV(t, \bar{x})F = 0$$

at all points $(t, \bar{x}) \neq (0, \bar{0})$ not in S with $F(t, \bar{x})$ satisfying the following conditions:

- (1) $F(t, \bar{x}) \rightarrow 0$ as $|\bar{x}| \rightarrow \infty$.
- (2) $\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}| \leq \epsilon} F(t, \bar{x}) d\bar{x} = 1$ for all $\epsilon > 0$.
- (3) $\partial F / \partial x_k$ ($k = 1, \dots, n$) are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$. $\partial^2 F / \partial x_k^2$

($k=1, \dots, n$), $\partial F/\partial t$ are continuous at all $(t, \bar{x}) \neq (0, \bar{0})$ not in S .

The proof that $F(t, \bar{x})$ is a solution of (3.5) and satisfies conditions (1) to (3) parallels the proof of Theorem 1. One need not bound $V(t, \bar{x})$ below since

$$|F(t, \bar{x})| \leq \frac{\exp(-\bar{x}^2/2t)}{(2\pi t)^{n/2}}.$$

The analogue of equation (3.4) is

$$\begin{aligned} (3.6) \quad F(t, \bar{x}) - iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leq a} f(\bar{\xi}, \tau; \bar{x}, t) F(\tau, \bar{\xi}) V(\tau, \bar{\xi}) d\bar{\xi} d\tau \\ = f(\bar{0}, 0; \bar{x}, t) - \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=a} F(\tau, \bar{\xi}) f(\bar{\xi}, \tau; \bar{x}, t) ds d\tau. \end{aligned}$$

We now prove that $F(t, \bar{x})$ is the unique solution of equation (3.5) satisfying conditions (1) to (3). Let $F = F_1 - F_2$ be the difference of two such solutions. Let \bar{F} be the complex conjugate of F . Clearly

$$\frac{1}{2} \Delta \bar{F} - \frac{\partial \bar{F}}{\partial t} - iuV(t, \bar{x}) \bar{F} = 0.$$

Let

$$D(W) = \frac{1}{2} \Delta W - \frac{\partial W}{\partial t} + iuV(t, \bar{x})W.$$

We apply Green's theorem as in the proof of Theorem 1 and obtain

$$\begin{aligned} (3.7) \quad \int_0^t \int_{|\bar{x}-\bar{\xi}| \leq b} \{f(\bar{\xi}, \tau; \bar{x}, t) D(|F(\tau, \bar{\xi})|^2) - |F(\tau, \bar{\xi})|^2 C(f(\bar{\xi}, \tau; \bar{x}, t))\} d\bar{\xi} d\tau \\ = -\frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}|=b} |F(\tau, \bar{\xi})|^2 \frac{\partial f(\bar{\xi}, \cdot; \cdot, \cdot)}{\partial n} ds d\tau - |F(t, \bar{x})|^2 ds d\tau \\ + iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leq b} V(\tau, \bar{\xi}) |F(\tau, \bar{\xi})|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau \\ = \frac{1}{2} \int_0^t \int_{|\bar{x}-\bar{\xi}| \leq b} f(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^n \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau \\ + iu \int_0^t \int_{|\bar{x}-\bar{\xi}| \leq b} V(\tau, \bar{\xi}) |F(\tau, \bar{\xi})|^2 f(\bar{\xi}, \tau; \bar{x}, t) d\bar{\xi} d\tau. \end{aligned}$$

Since $|F(t, \bar{x})| \leq M$, on letting $b \rightarrow \infty$ in equation (3.7), we obtain

$$-|F(t, \bar{x})|^2 = \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) \sum_{k=1}^n \left| \frac{\partial F(\tau, \bar{\xi})}{\partial \xi_k} \right|^2 d\bar{\xi} d\tau.$$

This cannot be so unless $F(t, \bar{x}) = F_1 - F_2 \equiv 0$. The uniqueness proof is complete. The theorem is thereby proved.

Consider

$$Q(\tau, \bar{\xi}; t, \bar{x}) = E\{\exp(-u(y(t) - y(\tau))) \mid \bar{x}(t) = \bar{x}, \bar{x}(\tau) = \bar{\xi}\} \\ \cdot \frac{\exp(-|\bar{x} - \bar{\xi}|^2/2(t - \tau))}{(2\pi(t - \tau))^{n/2}},$$

$0 \leq \tau \leq t$, $0 \leq V(t, \bar{x})$. We again assume $V(t, \bar{x})$ satisfies a uniform Hölder condition at all points (t, \bar{x}) not in a regular set S . $Q(\tau, \bar{\xi}; t, \bar{x})$ exists and satisfies equation (1.11). Moreover $Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the following conditions:

(1) $Q(\tau, \bar{\xi}; t, \bar{x}) \rightarrow 0$ as $|\bar{x}| \rightarrow \infty$.

(2) $\lim_{t \rightarrow \tau+} \int_{|\bar{x} - \bar{\xi}| \leq \epsilon} Q(\tau, \bar{\xi}; t, \bar{x}) d\bar{x} = 1$ for all $\epsilon > 0$.

$Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the adjoint differential equation and the corresponding conditions in the backward variables $\tau, \bar{\xi}$. The proof is essentially the proof of Theorem 1. Note that $Q(\tau, \bar{\xi}; t, \bar{x})$ satisfies the Chapman-Kolmogorov equation. It does not, however, have the norming of a probability density.

4. The elliptic differential equation. Let

$$(4.1) \quad \psi(s, r) = \frac{1}{((2s)^{1/2}r)^{(n-2)/2}} \left\{ K_{(n-2)/2}((2s)^{1/2}r) \right. \\ \left. - \frac{I_{(n-2)/2}((2s)^{1/2}r) K_{(n-2)/2}((2s)^{1/2}b)}{I_{(n-2)/2}((2s)^{1/2}b)} \right\}$$

(see §6). $\psi(s, |\bar{x}|)$ is the fundamental solution of

$$\frac{1}{2} \Delta \psi - s\psi = 0$$

with boundary value zero at $|\bar{x}| = b$. Note that

$$(4.2) \quad \lim_{\epsilon \rightarrow 0} \int_{|\bar{x}| = \epsilon} \frac{\partial \psi}{\partial n} ds = -2.$$

LEMMA 5. *Let $V(\bar{x})$ be a bounded function satisfying a Hölder condition at the point $\bar{x} \neq 0$. Then the function*

$$H(\bar{x}) = \int_{|\bar{x} - \bar{\xi}| \leq a} V(\bar{\xi}) \psi(s, |\bar{x} - \bar{\xi}|) d\bar{\xi}$$

satisfies the differential equation

$$\frac{1}{2} \Delta H - sH + V(\bar{x}) = 0$$

at \bar{x} .

The proof parallels an argument of Kellogg on p. 153 [3] proving that

$$h(\bar{x}) = \int_{|\bar{x}-\bar{\xi}| \leq a} \frac{V(\bar{\xi})}{|\bar{x}-\bar{\xi}|} d\bar{\xi}, \quad \bar{x} = (x_1, x_2, x_3),$$

satisfies

$$\frac{1}{2} \Delta h + V(\bar{x}) = 0$$

at $\bar{x} \neq 0$ if $V(\bar{x})$ satisfies a Hölder condition at \bar{x} .

THEOREM 3. *Let $V(\bar{x}) \geq 0$ satisfy a uniform Hölder condition at all \bar{x} not in a regular set S . Then $q(\bar{x}, s)$ is the unique solution of equation (1.14), satisfying the following conditions:*

- (1) $q(\bar{x}, s) \rightarrow 0$ as $|\bar{x}| \rightarrow \infty$.
- (2) $\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}| \leq \epsilon} (\partial q / \partial n) ds = -2$.
- (3) $\partial q / \partial x_k$ ($k=1, \dots, n$) are continuous at all $\bar{x} \neq \bar{0}$. $\partial^2 q / \partial x_k^2$ ($k=1, \dots, n$) are continuous at all $\bar{x} \neq \bar{0}$ not in S .

We obtain the integral equation

$$(4.3) \quad q(s, \bar{x}) + n \int_{|\bar{x}-\bar{\xi}| \leq b} \psi(s, |\bar{x}-\bar{\xi}|) V(\bar{\xi}) q(s, \bar{\xi}) d\bar{\xi} \\ = \psi(s, |\bar{x}|) - \frac{1}{2} \int_{|\bar{x}-\bar{\xi}|=b} q(s, \bar{\xi}) \frac{\partial \psi(s, |\bar{x}-\bar{\xi}|)}{\partial n} ds$$

by Laplace transforming equation (3.4) with respect to t . $\partial q / \partial x_k$ ($k=1, \dots, n$) exist and are continuous at all $\bar{x} \neq \bar{0}$ as can be seen by differentiating equation (4.3). Hence $q(s, \bar{x}) V(\bar{x})$ satisfies a Hölder condition at $\bar{x} \neq \bar{0}$ when $V(\bar{x})$ does. Lemma 5 implies that $q(s, \bar{x})$ satisfies equation (1.14) at all $\bar{x} \neq \bar{0}$ not in S . Equations (4.2) and (4.3) imply that

$$\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}|=\epsilon} \frac{\partial q}{\partial n} ds = -2.$$

Now

$$0 \leq q(s, \bar{x}) \leq \frac{K_{(n-2)/2}((2s)^{1/2}r)}{((2s)^{1/2}r)^{(n-2)/2}}$$

since $V(\bar{x}) \geq 0$. Hence

$$q(s, \bar{x}) \rightarrow 0 \quad \text{as} \quad |\bar{x}| \rightarrow \infty.$$

The proof of the continuity of $\partial^2 q / \partial x_k^2$ ($k=1, \dots, n$) proceeds as in Lemma 4. The uniqueness argument for $q(s, \bar{x})$ satisfying conditions (1) to (3) is analogous to the uniqueness argument for $F(t, \bar{x})$ carried out in §3.

5. The integral equation.

THEOREM 4. Let $V(t, \bar{x})$ be bounded. Then $\sigma(\bar{x}, \alpha, t)$ satisfies the following integral equation

$$(5.1) \quad \begin{aligned} C(\bar{0}, 0; \bar{x}, t) \int_y^{y+\lambda} (J(\alpha) - \sigma(\bar{x}, \alpha, t)) d\alpha \\ = \int_0^t \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) C(\bar{0}, 0; \bar{\xi}, \tau) V(\tau, \bar{\xi}) \{ \sigma(\bar{\xi}, y + \lambda, \tau) \\ - \sigma(\bar{\xi}, y, \tau) \} d\bar{\xi} d\tau \end{aligned}$$

where

$$J(x) = \frac{1 + \operatorname{sgn} x}{2}.$$

It is clear that

$$F(t, \bar{x})/C(\bar{0}, 0; \bar{x}, t) = \int_{-\infty}^{\infty} e^{i\alpha u} d_{\alpha} \sigma(\bar{x}, \alpha, t).$$

Since $V(t, \bar{x})$ is bounded, on letting $a \rightarrow \infty$ in equation (3.6) we obtain

$$(5.2) \quad F(t, \bar{x}) - C(\bar{0}, 0; \bar{x}, t) - iu \int_0^t \int_{-\infty}^{\infty} C(\bar{\xi}, \tau; \bar{x}, t) F(\tau, \bar{\xi}) V(\tau, \bar{\xi}) d\bar{\xi} d\tau = 0.$$

The boundedness of $V(t, \bar{x})$ implies that

$$\int_{-\infty}^{\infty} \alpha d_{\alpha} \sigma(\bar{x}, \alpha, t)$$

exists. The modified form of the P. Lévy inversion formula [5] used requires the existence of the first moment. Multiply equation (5.1) by $(1 - e^{-i\lambda u}/(iu)^2) \cdot (1/2\pi) e^{-iuy}$ and integrate with respect to y from $-T$ to T . The interchange of order of integration goes through readily. In the limit as $T \rightarrow \infty$ we obtain equation (5.1).

Interest in the integral equation (5.1) arises for several reasons. The equation holds without any strong regularity conditions on $V(t, \bar{x})$. Moreover, the density

$$\Pr \{ y(t) = y, \bar{x}(t) = \bar{x} \} = \frac{\partial \sigma}{\partial y}(\bar{x}, y, t)$$

need not exist even when very strong regularity conditions are imposed on $V(t, \bar{x})$ so that it makes no sense to speak of the density satisfying the differential equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \Delta P - V(t, \bar{x}) \frac{\partial P}{\partial y}.$$

In particular this is so when $V(t, \bar{x}) \equiv 1$.

6. An example. We illustrate the theory in n dimensions by considering the function

$$V(\bar{x}) = \begin{cases} 1, & |\bar{x}| \geq b, \\ 0, & |\bar{x}| < b. \end{cases}$$

The invariance proof for the distributions of the Wiener functionals $\int_0^t V(\bar{x}(\tau)) d\tau$ considered closely parallels that given in [2] for $V(x) = (1 + \operatorname{sgn} x)/2$. The computations for the case $n=1$ have been carried out in [2] and elsewhere. We solve the equation

$$\Delta q - 2(s + uV(r))q = 0, \quad V(r) = \begin{cases} 0, & r < b, \\ 1, & r \geq b, \end{cases}$$

where $r = |\bar{x}|$. The solution of the differential equation is given in terms of the Bessel functions $I_{(n-2)/2}$, $K_{(n-2)/2}$ by

$$q(s, r) = \begin{cases} \frac{1}{((2s)^{1/2}r)^{(n-2)/2}} (\alpha K_{(n-2)/2}((2s)^{1/2}r) + \beta I_{(n-2)/2}((2s)^{1/2}r)), & r < b, \\ \frac{\gamma}{((2(s+u))^{1/2}r)^{(n-2)/2}} K_{(n-2)/2}((2(s+u))^{1/2}r), & r \geq b, \end{cases}$$

where α, β, γ do not depend on r . We evaluate α, β, γ by making use of the auxiliary conditions. The continuity of q and $\partial q / \partial r$ at b implies that

$$\begin{aligned} \alpha K_{(n-2)/2}((2s)^{1/2}b) + \beta I_{(n-2)/2}((2s)^{1/2}b) \\ = \gamma \left(\frac{s}{s+u} \right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b), \\ -\alpha K_{n/2}((2s)^{1/2}b) + \beta I_{n/2}((2s)^{1/2}b) = -\gamma \left(\frac{s}{s+u} \right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b). \end{aligned}$$

We solve the equations above and find that

$$\beta = \frac{\begin{vmatrix} -K_{(n-2)/2}((2s)^{1/2}b) & -\left(\frac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b) \\ K_{n/2}((2s)^{1/2}b) & \left(\frac{s}{s+u}\right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b) \end{vmatrix}}{D}$$

and

$$\gamma = \frac{\alpha}{(2s)^{1/2}bD}$$

where

$$D = \begin{vmatrix} I_{(n-2)/2}((2s)^{1/2}b) & - \left(\frac{s}{s+u}\right)^{(n-2)/4} K_{(n-2)/2}((2(s+u))^{1/2}b) \\ I_{n/2}((2s)^{1/2}b) & \left(\frac{s}{s+u}\right)^{(n-4)/4} K_{n/2}((2(s+u))^{1/2}b) \end{vmatrix}.$$

The condition

$$\lim_{\epsilon \rightarrow 0} \int_{|\bar{x}|=\epsilon} \frac{\partial q}{\partial n} ds = -2$$

implies that $\alpha=1$. On letting $u \rightarrow \infty$, $q(s, r)$ tends to limit $\psi(s, r)$ given by (4.1) for $r < b$. $q(s, r)$ tends to zero for $r \geq b$. The limiting expression is the Laplace transform of the probability density of diffusion from $\bar{0}$ to a point \bar{x} , r units away from $\bar{0}$, when there is an absorbing barrier at the sphere of radius b about $\bar{0}$. We invert

$$I_{(n-2)/2}((2s)^{1/2}r)/I_{(n-2)/2}((2s)^{1/2}b)$$

and obtain $v_n(r, t)$ given by (3.6). Hence the probability density of the diffusion is $f(\bar{0}, 0; \bar{x}, t)$ for $|\bar{x}| \leq b$ and zero for $|\bar{x}| \geq b$.

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