HIGHER-DIMENSIONAL HEREDITARILY INDECOMPOSABLE CONTINUA

BY

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1. **Introduction and definitions.** A continuum is *indecomposable* if it is not the sum of two proper subcontinua. It is *hereditarily indecomposable* if each of its subcontinua is indecomposable. Knaster gave [5](1) an example of a 1-dimensional hereditarily indecomposable continuum. Moise described [6] a plane hereditarily indecomposable continuum (it was topologically equivalent [2] to the one described by Knaster) which was topologically equivalent to each of its nondegenerate subcontinua. I showed [1] that the same continuum was homogeneous. Recently, I gave [2] examples of many topologically different hereditarily indecomposable continua. However, each of these continua was 1-dimensional. A point is a 0-dimensional hereditarily indecomposable continuum. (In this paper we define a continuum to be a compact closed connected set.)

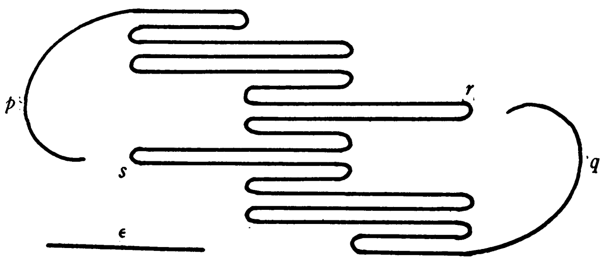

FIG. 1

Kelly has shown [4] that if there is a hereditarily indecomposable continuum of dimension more than one, there is one of infinite dimension. In this paper we show that there are hereditarily indecomposable continua of all dimensions. In describing these continua, we make use of the following term.

ε-crooked. An arc is *ε-crooked* if for each pair of its points p and q there are points r and s between p and q on the arc such that r lies between p and s, $\rho(p, s)<\epsilon$, and $\rho(r, q)<\epsilon$ where $\rho(x, y)$ denotes the distance between x and y. See Fig. 1. It may be noted that if $\epsilon'>\epsilon$, an arc is ϵ'-crooked if it is ϵ-crooked. An arc intersecting sets H and K is called ε-crooked with respect to them if each of its subarcs pq with ends p and q on H and K respectively contains

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(1) Numbers in brackets refer to the references cited at the end of the paper.

points r and s such that r lies between p and s on pq, $\rho(s, H)<\epsilon$, and $\rho(r, K)<\epsilon$.

We use E_n to denote an *n-dimensional Euclidean space* and I_ω to denote a *Hilbert cube*. The points of a Hilbert cube are countable sequences $(x_1, x_2, \cdots)$ such that $|x_i| \leqq 1/i$ and the distance between two points is given by $\rho(x, y) = \sum |x_i - y_i|/2^i$. We make use of the facts that I_ω is compact and any separable metrizable space can be topologically imbedded in I_ω [3, p. 64]. Throughout this paper we restrict ourselves to metric spaces.

2. **Description of a 2-dimensional hereditarily indecomposable continuum.** We shall show that if b and c are two points of E_3, there is a sequence $S_1, S_2, \cdots$ of bounded connected domains satisfying the following conditions:

(1) S_i contains $\overline{S}_{i+1}$.

(2) S_i separates b from c.

(3) $E_3 - S_i$ has only two components and no point of S_i is farther than $1/i$ from either of them.

(4) Each arc in S_i is $1/i$-crooked.

The intersection S_0 of $S_1, S_2, \cdots$ is a 2-dimensional hereditarily indecomposable continuum which has exactly two complementary domains and which is irreducible with respect to separating E_3.

We may take S_1 to be the set of all points p of E_3 such that minimum $(1/4, \rho(b, c)/3) < \rho(b, p) <$ minimum $(1/2, 2\rho(b, c)/3)$. That $S_2, S_3, \cdots$ may be chosen so as to satisfy conditions (1), (2), (3), and (4) follows from Theorem 2.

Since, by condition (1), $S_0 = \overline{S}_1 \cdot \overline{S}_2 \cdot \cdots$, it is a continuum. We find from condition (2) that S_0 separates b from c and from condition (3) that S_0 has only two complementary domains and is irreducible with respect to separating E_3. That S_0 is 2-dimensional follows from the fact that it is irreducible with respect to separating E_3.

Condition (4) insures that S_0 is hereditarily indecomposable. To see that this is true, assume that H and K are two intersecting continua in S_0 such that neither H nor K contains the other. There would be a positive integer n and points p and q of H and K respectively such that $\rho(p, K) > 1/n$ and $\rho(q, H) > 1/n$. Let O_H and O_K be connected open subsets of S_n containing H and K respectively such that $\rho(p, O_K) > 1/n$ and $\rho(q, O_H) > 1/n$. Let pxq be an arc from p to q in $O_H + O_K$ such that O_H and O_K contain px and xq respectively. Since $\rho(p, xq) > 1/n$ and $\rho(px, q) > 1/n$, pxq is not $1/n$-crooked. The assumption that S_0 is not hereditarily indecomposable has led to the contradiction that an arc in S_n is not $1/n$-crooked.

3. **Crooked domains.** The following two theorems hold in either E_n or in I_ω.

THEOREM 1. *Suppose D is a bounded domain that separates the point b from the point c, H and K are two point sets, and ϵ is a positive number. Then there* *is a domain E in D that separates b from c such that each arc in E from a point of H to a point of K is ε-crooked with respect to H and K.*

Proof. Let $K' = \{x | x$ element of $K,\ \rho(x, H) \geqq \epsilon\}$ where the symbol $\{x | \mathrm{P}\}$ is used to denote the collection of all points x such that x satisfies condition P. We suppose that neither K' nor H is empty because if either is, we can let E be D.

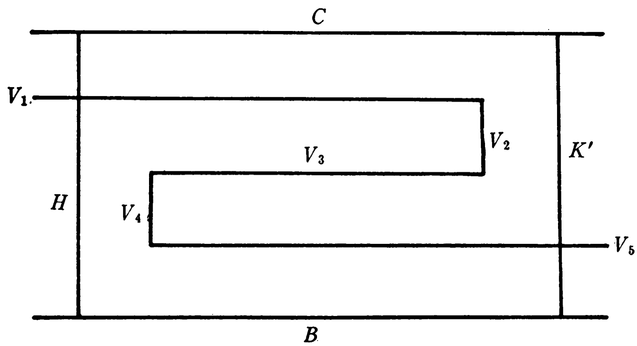

FIG. 2

Suppose the complement of D is the sum of the mutually exclusive closed sets B and C which contain b and c respectively, and that δ is a positive number less than the distance from any point of B to any point of C. We define V to be the sum of the following five sets $V_1, V_2, \cdots, V_5$.

$$V_1 = \{x \mid \rho(x, B) = 3\delta/4,\ \rho(x, K') \geqq \epsilon/2\}.$$
$$V_2 = \{x \mid \delta/2 \leqq \rho(x, B) \leqq 3\delta/4,\ \rho(x, K') = \epsilon/2\}.$$
$$V_3 = \{x \mid \rho(x, B) = \delta/2,\ \rho(x, H + K') \geqq \epsilon/2\}.$$
$$V_4 = \{x \mid \delta/4 \leqq \rho(x, B) \leqq \delta/2,\ \rho(x, H) = \epsilon/2\}.$$
$$V_5 = \{x \mid \rho(x, B) = \delta/4,\ \rho(x, H) \geqq \epsilon/2\}.$$

We find that V separates B from C (see Fig. 2) because it contains the boundary of the closed set

$$\{x \mid \rho(x, B) \leqq \delta/4\} + \{x \mid \delta/4 \leqq \rho(x, B) \leqq \delta/2,\ \rho(x, H) \leqq \epsilon/2\} + \{x \mid \delta/2 \leqq \rho(x, B) \leqq 3\delta/4,\ \rho(x, K') \geqq \epsilon/2\}$$

which contains B but no point of C.

Let $E, E_1, E_2, \cdots, E_5$ be γ neighborhoods of $V, V_1, V_2, \cdots, V_5$ respectively where γ is the minimum of $\epsilon/4$ and $\delta/16$. We note that $\overline{E}$ lies in D. Also, E separates b from c because it contains V. No point of E_4 is as far as ϵ from H and no point of E_2 is as far as ϵ from K.

We shall show that if pq is an arc in E from a point p of H to a point q of

K, there are points r and s on pq between p and q such that r is between p and s, $\rho(s, H)<\epsilon$, and $\rho(r, K)<\epsilon$. If q is nearer than ϵ to H, r and s may be found near q. If q is a point of K', q belongs to E_5 and p belongs to E_1. In Fig. 2, E would be a small neighborhood of V. Since E_2 separates $E \cdot H$ from E_4 and E_4 separates E_2 from $E \cdot K'$, there are points r and s of pq belonging to E_2 and E_4 respectively such that r is between p and s on pq. Hence, any arc in E from a point of H to a point of K is ϵ-crooked with respect to H and K.

THEOREM 2. *If D is a bounded domain that separates the point b from the point c and ϵ is a positive number, there is a connected domain E satisfying the following conditions:*

(1) *D contains $\overline{E}$.*

(2) *E separates b from c.*

(3) *The complement of E has exactly 2 components and no point of E is farther than ϵ from either of them.*

(4) *Each arc in E is ϵ-crooked.*

Proof. Suppose G is a finite covering of $D=D_0$ by sets of diameter less than $\epsilon/2$. Let (g_1, g_1'), (g_2, g_2'), $\cdots$, (g_n, g_n') be the finite collection of pairs of elements of G. By repeated applications of Theorem 1 we find that there is a domain D_i $(i=1, 2, \cdots, n)$ such that $\overline{D}_i$ lies in D_{i-1}, D_i separates b from c, each arc in D_i from g_i to g_i' is $\epsilon/2$-crooked with respect to g_i and g_i'. The domain D_n satisfies conditions (1), (2), and (4). Since E_n and I_ω are unicoherent, one component of D_n separates b from c. By subtracting the sum of a collection of arcs from this component we may obtain a connected domain E satisfying conditions (1), (2), (3), and (4).

4. **Higher-dimensional hereditarily indecomposable continua.** Using the same scheme used in §2 to define S_0 in E_3, we can define hereditarily indecomposable continua in E_n and in I_ω. The following result holds in either E_n or I_ω.

THEOREM 3. *If H and K are two mutually exclusive continua, there is a hereditarily indecomposable continuum which has exactly two complementary domains and which is irreducible with respect to separating H from K.*

Using Theorem 3, we can get the following result which shows that there are hereditarily indecomposable continua of all dimensions.

THEOREM 4. *There are infinite-dimensional hereditarily indecomposable continua in I_ω and there are n-dimensional hereditarily indecomposable continua in E_{n+1}.*

THEOREM 5. *Each $n+1$-dimensional continuum contains an n-dimensional hereditarily indecomposable continuum.*

Proof. If M is an $n+1$-dimensional continuum, it contains two mutually exclusive closed subsets B and C such that no closed subset of M of dimension less than n separates B from C [3, p. 26]. We find from Theorem 6 of the next section that there is a closed subset H of M which separates B from C such that each component of H is hereditarily indecomposable. One component C of H is not of dimension less than n [3, p. 94]. There is an n-dimensional subcontinuum of C and this subcontinuum is hereditarily indecomposable because C is.

5. Separating with collections of hereditarily indecomposable continua. In this section we consider separations by closed point sets such that the components of these closed point sets are hereditarily indecomposable.

THEOREM 6. *If B and C are two mutually exclusive closed subsets of a continuum M, there is a closed point set H in M separating B from C in M such that each component of H is hereditarily indecomposable.*

Proof. There is a continuous transformation T of M into I_ω such that $T(B)$ is a point, $T(C)$ is another point, and T is 1-1 on $M-(B+C)$. We find from Theorem 3 that there is a hereditarily indecomposable continuum K in I_ω which separates $T(B)$ from $T(C)$. Then $H=T^{-1}[K\cdot T(M)]$.

The following result can be obtained in a similar fashion.

THEOREM 7. *If B and C are two mutually exclusive bounded closed subsets of E_n, there is a closed bounded point set H separating B from C such that each component of H is hereditarily indecomposable.*

THEOREM 8. *If p and q are two points of E_n, there is a bounded closed point set H in E_n separating p from q such that each continuum in E_n containing $p+q$ contains a component of H.*

Before proving Theorem 8 we state the following interesting corollary of it.

COROLLARY. *For each pair of points p, q of E_n there is a bounded closed set H in $E_n-(p+q)$ such that each arc from p to q in E_n contains a degenerate component of H.*

Proof of Theorem 8. Suppose E_n is a plane in E_{n+1}. Then H is the intersection of E_n with a hereditarily indecomposable continuum K in E_{n+1} which separates p from q in E_{n+1}.

We shall show that the assumption that there is a continuum C in E_n which contains $p+q$ but no component of H leads to the contradiction that K contains a decomposable continuum.

Suppose there is such a continuum C in E_n. We now show that there is a unicoherent locally connected continuum M in E_{n+1} such that $M\cdot E_n=C$. Let P be a compact locally connected continuum in E_n containing C and r be a point whose distance from E_n is greater than the diameter of P. We define $M=\{x \mid x$ lies on closed interval between r and a point $p(x)$ of $P, \rho[x, p(x)]$

$\geqq \rho[p(x), C]\}$.

If M' is the cone with vertex r and base P, we may think of M as the image of M' under a homeomorphism that moves toward r those points of the cone M' which are not between r and C. Since M' is a cone whose base is a locally connected continuum, M' is locally connected and unicoherent. Hence, M is also.

Since M is a locally connected continuum and $M \cdot K$ separates p from q in M, some closed subset S_1 of $M \cdot K$ is irreducible with respect to separating p from q. Since M is unicoherent, S_1 is connected. In order for S_1 to separate p from q in M it must intersect C but must not be a subset of E_n. Let s be a point of $S_1 \cdot E_n$ and S_2 be the component of H that contains s. Now S_2 does not lie in S_1 because C contains no component of H and S_1 does not lie in S_2 because it is not a subset of E_n. Hence the assumption that there is a continuum C in E_n which contains $p+q$ but no component of H has led to the contradiction that the hereditarily indecomposable continuum K contains the decomposable continuum S_1+S_2.

THEOREM 9. *For each pair of points p, q of a continuum M there is a closed set H in $M-(p+q)$ such that each subcontinuum of M containing $p+q$ contains a component of H.*

Proof. The proof is similar to that used in Theorem 8. We suppose M lies in a hyperplane in I_ω and let H be the intersection of M with a hereditarily indecomposable continuum in I_ω that separates p from q.

6. **Homogeneity.** A set M is homogeneous if for each pair of its points p, q there is a homeomorphism of M into itself that carries p into q. I used [1] a pseudo-arc [6] (a nondegenerate hereditarily indecomposable continuum that can be chained [2]) to show that the simple closed curve is not the only type of nondegenerate homogeneous plane continuum. However, we find from the following result that if n is an integer greater than one, no n-dimensional hereditarily indecomposable continuum is homogeneous.

THEOREM 10. *If M is an n-dimensional hereditarily indecomposable continuum, there is a point p such that each nondegenerate subcontinuum of M containing p is n-dimensional.*

Proof. Suppose M is the sum of a finite number of closed sets each of diameter less than 1. One of the closed sets had dimension n and one of its components has dimension n. Hence, M has an n-dimensional subcontinuum M_1 of diameter less than 1. Similarly, M_1 has an n-dimensional subcontinuum M_2 of diameter less than $1/2$, M_2 has an n-dimensional subcontinuum M_3 of diameter less than $1/3, \cdots$. If $p = M_1 \cdot M_2 \cdot \cdots$, each nondegenerate subcontinuum C of M containing p is n-dimensional because it contains one of the M_i's. If M_i does not contain C, C contains M_i because $C+M_i$ is hereditarily indecomposable.

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