CURVATURE OF CLOSED HYPERSURFACES AND NON-EXISTENCE OF CLOSED MINIMAL HYPERSURFACES

BY

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1. **Introduction.** It is known that (1) a closed (sufficiently differentiable) surface in euclidean space E_3 has a point of positive Gaussian curvature, (2) a minimal surface in E_3 has nonpositive Gaussian curvature at every point(1), and (3) hence no closed minimal surface exists in E_3.

The object of this paper is to generalize these results to higher dimensions and noneuclidean Riemannian manifolds. The principal results appear as Theorems 1, 2 of §3 and Theorems 3, 4 of §4.

2. **Riemannian manifolds in polar coordinates.** Let R_n be a Riemannian manifold of dimension $n \geqq 2$ and class C^r(2). Let O be any point of R_n, let $(v^1, \cdots, v^n)$ be normal coordinates with respect to a unit orthogonal frame at O, so that the metric tensor in the coordinate system (v) has the form

$$ds^2 = a_{pq}(v)dv^p dv^q \text{(3)},$$

(1) $$a_{pq}(0) = \delta_{pq}, \qquad \frac{\partial a_{pq}}{\partial v^r}(0) = (0), \qquad p, q, r = 1, \cdots, n.$$

Let P_0 be any point of R_n which can be joined to O by a geodesic g_0 on which there is no conjugate point to O. Then the normal coordinates of a point P near P_0 are $v^p = \eta^p r$, where $(\eta^1, \cdots, \eta^n)$ are the components in (v) of the unit tangent vector at O to the unique geodesic g from O to P whose initial direction and length r are near those of g_0. Let r_0 be the length of g_0 and $(\eta_0) = (\eta_0^1, \cdots, \eta_0^n)$ the components of the unit tangent vector to g_0 at O.

Let $\eta^p = \eta^p(\lambda^2, \cdots, \lambda^n)$ be a nonsingular parametrization of the unit sphere $\sum \eta^p \eta^p = 1$ in (η)-space, neighboring $(\eta) = (\eta_0)$, and suppose $\eta^p(\lambda_0^2, \cdots, \lambda_0^n) = \eta_0^p$. Then we can use (r, λ) as geodesic polar coordinates [4; 6](4) "near g_0"; that is, over a domain $0 \leqq r < r_0 + \epsilon, \lambda_0^\alpha - \epsilon < \lambda^\alpha < \lambda_0^\alpha + \epsilon$. Since geodesic spheres about O are orthogonal to the geodesics issuing from O, and since r is length from O along these geodesics, the metric tensor in the system (r, λ) has the form

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(1) (1) is essentially a classical theorem of S. Bernstein, Math. Zeit. vol. 26 (1927) pp. 151–158. (2) follows from the fact that at every point of a minimal surface the sum of the principal normal curvatures is zero, hence their product is nonpositive.

(2) $r \geqq 4$ is sufficient for all purposes in this paper.

(3) The summation convention of tensor analysis is used throughout.

(4) Numbers in brackets refer to the list of references at the end of the paper.

(2) $$ds^2 = dr^2 + h_{\alpha\beta}(r, \lambda)d\lambda^\alpha d\lambda^\beta, \qquad \alpha, \beta = 2, \cdots, n.$$

Using (1), we see that

$$h_{\alpha\beta}(r, \lambda) = r^2 a_{pq}(r\eta(\lambda)) \frac{\partial \eta^p}{\partial \lambda^\alpha} \frac{\partial \eta^q}{\partial \lambda^\beta}.$$

Differentiating, and using (1), we find

(3) $$h_{\alpha\beta}(0, \lambda) = (0), \qquad \frac{\partial h_{\alpha\beta}}{\partial r}(0, \lambda) = 0,$$
$$\frac{\partial^2 h_{\alpha\beta}}{\partial r^2}(0, \lambda) = 2\sum_p \frac{\partial \eta^p}{\partial \lambda^\alpha} \frac{\partial \eta^p}{\partial \lambda^\beta}.$$

Since

$$\sum_p d\eta^p d\eta^p = \sum_p \frac{\partial \eta^p}{\partial \lambda^\alpha} \frac{\partial \eta^p}{\partial \lambda^\beta} d\lambda^\alpha d\lambda^\beta$$

it follows that

(4) $$\frac{\partial^2 h_{\alpha\beta}}{\partial r^2}(0, \lambda)T^\alpha T^\beta > 0 \qquad \text{if } (T) \neq (O).$$

Denoting $(\partial h_{\alpha\beta}/\partial r)T^\alpha V^\beta$ by $H(T, V)$, we get from (3) and (4) and the compactness of spheres the following lemma.

LEMMA 1. *Let O be a point of a Riemannian manifold R_n. There exists a neighborhood $N(O)$ such that if $ds^2 = dr^2 + h_{\alpha\beta}(r, \lambda)d\lambda^\alpha d\lambda^\beta$ is the metric in a polar coordinate system based on O, then $H(T, T) = (\partial h_{\alpha\beta}/\partial r)T^\alpha T^\beta$ is positive definite in $N(O) - O$.*

Now let R_n be a Riemannian manifold of nowhere positive sectional curvature. There exists no pair of conjugate points in R_n. Let P_0 be a point which can be joined to O by a geodesic g_0, and let (r, λ) be a polar coordinate system near g_0 based on O. Let $\overline{R}_{pqrs}$ be the Riemann tensor in this coordinate system; then

$$\overline{R}_{1\beta 1\delta} = -\frac{1}{2}\frac{\partial^2 h_{\beta\delta}}{\partial r^2} + \frac{1}{4}h^{\alpha\gamma}\frac{\partial h_{\alpha\beta}}{\partial r}\frac{\partial h_{\gamma\delta}}{\partial r}, \qquad \beta, \delta = 2, \cdots, n.$$

Hence the sectional curvature of R_n with respect to a 2-plane determined by a tangent vector to g_0 at P and a unit vector (T) orthogonal to g_0 at P is

(5) $$\overline{K}(T) = -\frac{1}{2}\frac{\partial^2 h_{\beta\delta}}{\partial r^2}T^\beta T^\delta + \frac{1}{4}h^{\alpha\gamma}\left(\frac{\partial h_{\alpha\beta}}{\partial r}T^\beta\right)\left(\frac{\partial h_{\gamma\delta}}{\partial r}T^\delta\right).$$

Then $H(T, T)$ is positive definite at P_0; for if $H(T_0, T_0) \leqq 0$ at P_0 for some

$(T_0)\neq(O)$, then by (3) and (4) there must be on g_0 a point at which $(dH(T_0, T_0)/dr)=0$ and $H(T_0, T_0)>0$, which by (5) contradicts the non-positiveness of $\overline{K}(T)$. Hence we have the following lemma.

LEMMA 2. *Let O be a point of a Riemannian manifold R_n of nowhere positive sectional curvature. If $P_0\in R_n$ can be joined to O by a geodesic and if $ds^2=dr^2+h_{\alpha\beta}(r, \lambda)d\lambda^\alpha d\lambda^\beta$ is the metric near P_0 in a polar coordinate system based on O, then $H(T, T)$ is positive definite at P_0.*

Little can be said about the size of $N(O)$ in the case of variable positive curvature. However, in case of constant positive curvature $\overline{K}$, no geodesic of length less than $\pi/\overline{K}^{1/2}$ contains a pair of conjugate points, and also we can choose polar coordinates so that

$$h_{\alpha\beta} = \frac{\sin^2 (\overline{K}^{1/2}r)}{\overline{K}} b_{\alpha\beta}(\lambda),$$

$$\frac{\partial h_{\alpha\beta}}{\partial r} = \frac{1}{\overline{K}^{1/2}} \sin (2\overline{K}^{1/2}r) b_{\alpha\beta}(\lambda)$$

where $b_{\alpha\beta}$ is positive definite. Hence we have the following lemma.

LEMMA 3. *Let O be a point of a Riemannian manifold R_n of constant positive curvature $\overline{K}$. If $P_0\in R_n$ can be joined to O by a geodesic g_0 of length less than $\pi/2(\overline{K})^{1/2}$ (less than or equal to $\pi/2(\overline{K})^{1/2}$), and if $ds^2=dr^2+h_{\alpha\beta}(r, \lambda)d\lambda^\alpha d\lambda^\beta$ is the metric near g_0 in any polar coordinate system based on O, then $H(T, T)$ is positive definite (positive semi-definite) at P_0.*

3. **Curvature of closed hypersurfaces.** Let R_{n-1} be a hypersurface of R_n, imbedded in R_n differentiably of class C^{r+1}, with local imbedding equations $x^p=x^p(u)$ in terms of coordinate systems (x) on R_n and $(u)=(u^1, \cdots, u^{n-1})$ on R_{n-1}. It is known [2] that the following relation holds between the Riemann tensors on R_{n-1} and R_n:

$$(6)\quad R_{ijkl} - \overline{R}_{pqrs}\frac{\partial x^p}{\partial u^i}\frac{\partial x^q}{\partial u^j}\frac{\partial x^r}{\partial u^k}\frac{\partial x^s}{\partial u^l} = b_{ik}b_{jl} - b_{il}b_{jk},\quad i, j, k, l = 1, \cdots, n-1,$$

where b_{ij} is the second fundamental form of R_{n-1}(5). Also, b_{ij} satisfies

$$(7)\qquad \frac{\partial^2 x^p}{\partial u^i\partial u^j} + \overline{\Gamma}^p_{qr}\frac{\partial x^q}{\partial u^i}\frac{\partial x^r}{\partial u^j} - \Gamma^k_{ij}\frac{\partial x^p}{\partial u^k} = b_{ij}\xi^p$$

where (ξ) is the normal to R_{n-1}.

Now suppose R_n is complete [3; 4], so that every pair of points A, B of R_n can be joined by a geodesic which is a shortest curve joining A to B; the

(5) I am indebted to the referee for the suggestion that the Gauss equation (6) be used. My original proof of the theorems to follow was longer, and used specialized coordinate systems on R_{n-1}.

distance between A and B is defined to be the length of a shortest geodesic joining them. Let O be any point of R_n, and let M be the minimum point locus [5; 6] with respect to O. Then $R_n = C_n \cup M$, where C_n is an open n-cell with M as its singular boundary; M is at most $(n-1)$-dimensional. Every point P of C_n can be joined to O by a unique geodesic lying in C_n, and the length of such a geodesic measures the distance OP. The distance OP is a function of class C^{r-2} of P for $P \neq 0$ in C_n.

Assume next that R_{n-1} is closed(6). Let P_0 be a point of R_{n-1} whose distance r_0 in R_n from O is maximum. Suppose P_0 lies in C_n(7); we can use the polar coordinates (r, λ) of the previous section about the geodesic OP_0 in C_n, and the length r of geodesics in C_n from O to points P near P_0 measures the distance from O to P. Then at P_0 we have $\partial r/\partial u^i = 0$, so that (7) becomes for $p=1$

$$\frac{\partial^2 r}{\partial u^i \partial u^j} - \frac{1}{2} \frac{\partial h_{\alpha\beta}}{\partial r} \frac{\partial \lambda^\alpha}{\partial u^i} \frac{\partial \lambda^\beta}{\partial u^j} = b_{ij}. \tag{8}$$

Now at P_0, $\partial^2 r/\partial u^i \partial u^j$ is negative semi-definite, and if $H(T, T)$ is positive definite at P_0, then b_{ij} is negative definite there. This implies that $(b_{ik}b_{jl} - b_{il}b_{jk})\eta^i\eta^k\zeta^j\zeta^l > 0$ for all (η), (ζ) neither of which is (O). Let $T^\alpha = (\partial\lambda^\alpha/\partial u^i)\eta^i$, $V^\alpha = (\partial\lambda^\alpha/\partial u^i)\zeta^i$. Then by (6) the relative curvature of R_{n-1} at P_0 (defined as the difference $K(\eta, \zeta) - \overline{K}(T, V)$ between corresponding sectional curvatures of R_{n-1} and R_n at P_0) is positive. Hence using Lemmas 1, 2, 3 we have the following theorem.

THEOREM 1. *Every point O of R_n has a neighborhood $N(O)$ such that every closed R_{n-1} in $N(O)$ has a point P_0 at which all relative curvatures of R_{n-1} are positive. If R_n is complete and has everywhere nonpositive sectional curvature, and if a closed R_{n-1} in R_n has the property that there exists a point O of R_n such that some point P_0 of R_{n-1} furthest from O is not on the minimum point locus with respect to O, then at P_0 all relative curvatures of R_{n-1} are positive. If R_n is complete and has constant positive curvature K, and if a closed R_{n-1} has the property that there exists a point O of R_n such that all points of R_{n-1} have distance less than $\pi/2(K)^{1/2}$ from O and some point P_0 of R_{n-1} furthest from O is not on the minimum point locus with respect to O, then at P_0 all relative curvatures of R_{n-1} are positive.*

An example of a closed hypersurface R_{n-1} in an R_n of nonpositive curvature with the striking property that, no matter what point O of R_n is chosen, all points of R_{n-1} furthest from O are on the minimum point locus with respect to O is furnished by the flat 2-torus R_2 naturally imbedded in the flat

(6) Closed = compact.

(7) This assumption and the assumption of completeness of R_n are not needed if R_{n-1} lies in a sufficiently small neighborhood of O.

3-torus R_3, the direct product of the flat 2-torus and the circle; here the relative curvature of R_2 is zero. This example is best visualized as the square with opposite sides identified imbedded in the cube with opposite faces identified.

An example to show the need for the assumption about "less than $\pi/2(K)^{1/2}$" in the case of constant positive curvature is furnished by the flat 2-torus imbedded in the 3-sphere in E_4 by means of $x=\cos\phi$, $y=\sin\phi$, $z=\cos\theta$, $w=\sin\theta$. This flat torus in E_4 also serves to show that Theorem 1 breaks down if R_{n-1} is replaced by R_m with $m<n-1$.

If R_n is *simply-connected* and has everywhere nonpositive sectional curvature, it is known [5; 1] that it is homeomorphic to E_n and the minimum point locus is absent. If R_n is simply-connected and has constant positive curvature, it is known to be isometric to a sphere [7], and the minimum point locus with respect to any point O is simply the antipodal point. Hence we have the following theorem.

THEOREM 2. *Let R_n be complete and simply-connected, and have everywhere nonpositive sectional curvature. Let $O\in R_n$, and let the hypersurface R_{n-1} be closed. Then all relative curvatures of R_{n-1} at every point P_0 furthest from O are positive. If R_n is a sphere, and if R_{n-1} lies in an open hemisphere of R_n, then at every point of R_{n-1} furthest from the pole of the hemisphere all relative curvatures of R_{n-1} are positive.*

Theorems 1 and 2 can be strengthened slightly by everywhere replacing the phrase "relative curvatures of R_{n-1} are positive" by "sectional curvatures of R_{n-1} are greater than or equal to the corresponding sectional curvatures of the geodesic hypersphere about O through P_0, which in turn exceed the corresponding sectional curvatures of R_n."

For at P_0 the sectional curvatures of the geodesic hypersphere in question are

$$K_G(T, V) = \overline{K}(T, V) + [H(T, T)H(V, V) - H^2(T, V)]/4 \tag{9}$$

so that by (6) and (8) we have, denoting $\rho(\eta, \zeta)=(\partial^2 r/\partial u^j\partial u^i)\eta^i\zeta^j$,

$$\begin{aligned} K(\eta, \zeta) &= K_G(T, V) + \rho(\eta, \eta)\rho(\zeta, \zeta) - \rho^2(\eta, \zeta) \\ &\quad - \frac{1}{2}[\rho(\eta, \eta)H(V, V) + \rho(\zeta, \zeta)H(T, T) - 2\rho(\eta, \zeta)H(T, V)] \\ &= K_G(T, V) + \rho(\eta, \eta)\rho(\zeta, \zeta) - \rho^2(\eta, \zeta) \\ &\quad - \frac{1}{2\rho(\eta, \eta)}[H(A, A) + (\rho(\eta, \eta)\rho(\zeta, \zeta) - \rho^2(\eta, \zeta))H(T, T)] \end{aligned} \tag{10}$$

where $A^\alpha=\rho(\eta, \eta)V^\alpha-\rho(\eta, \zeta)T^\alpha$. The required results follow from Lemmas 1, 2, 3, and equations (10) and (9).

This implies, for example, that a closed R_{n-1} in hyperbolic n-space H_n has a point where all sectional curvatures are positive.

4. **Minimal varieties.** An m-dimensional minimal variety in a Riemannian manifold R_n is by definition an extremal m-manifold in the calculus of variations problem associated with the multiple integral for m-dimensional area in R_n. Let R_m be such a minimal variety in R_n imbedded differentiably of class C^{r+1}. Let $x^p = x^p(u^1, \cdots, u^m)$, $p = 1, \cdots, n$, be the equations of this imbedding in terms of a local coordinate system (x) in R_n and a local coordinate system (u) in R_m.

Now since R_m is a sub-manifold of R_n, we have at every point P of R_m [2]

$$R_{ijkl} = \overline{R}_{pqrs}\frac{\partial x^p}{\partial u^i}\frac{\partial x^q}{\partial u^j}\frac{\partial x^r}{\partial u^k}\frac{\partial x^s}{\partial u^l} + \sum_{\sigma=m+1}^{n} b_{ik\sigma}b_{jl\sigma} - b_{il\sigma}b_{jk\sigma}, \tag{11}$$

$$i, j, k, l = 1, \cdots, m; \; p, q, r, s = 1, \cdots, n,$$

where $b_{ij\sigma}$ is the σth "second fundamental form" of R_m. If (η), (ζ) are two unit orthogonal vectors tangent to R_m at P, we have

$$K(\eta, \zeta) = \overline{K}(T, V) + \sum_{\sigma=m+1}^{n} (b_{ik\sigma}b_{jl\sigma} - b_{il\sigma}b_{jk\sigma})\eta^i\eta^k\zeta^j\zeta^l.$$

Keeping (η) fixed, we allow (ζ) to range over an $(m-1)$-dimensional unit orthogonal frame tangent to R_m and orthogonal to (η), and sum. We obtain

$$k(\eta) = \bar{k}(T) + g^{jl}\sum_{\sigma} (b_{ik\sigma}b_{jl\sigma} - b_{il\sigma}b_{jk\sigma})\eta^i\eta^k$$

where $k(\eta)$ is the Ricci curvature of R_m in the direction (η) and $\bar{k}(T)$ is the Ricci curvature in the same direction of the geodesic m-manifold tangent to R_m at P. But since R_m is a minimal variety, we know [2]

$$g^{jl}b_{jl\sigma} = 0, \qquad \sigma = m+1, \cdots, n,$$

so that

$$k(\eta) = \bar{k}(T) - g^{jl}\sum_{\sigma} A_{j\sigma}A_{l\sigma}.$$

For each σ, $g^{jl}A_{j\sigma}A_{l\sigma} \geqq 0$, by the positive definiteness of g^{ij}. Hence we have the following theorem.

THEOREM 3. *At each point P, the Ricci curvature of a minimal variety R_m (for arbitrary $2 \leqq m < n$) imbedded in a Riemannian manifold R_n does not exceed the corresponding Ricci curvature of the geodesic m-manifold tangent to R_m at P.*

This combines with Theorems 1 and 2 to produce the following theorem.

THEOREM 4. *If there exists a closed minimal hypersurface R_{n-1} in a complete Riemannian manifold R_n of nonpositive curvature, then all points of R_{n-1} furthest from any fixed point O of R_n are on the minimum point locus with re-*

spect to O; hence there exists no closed minimal hypersurface in a complete simply-connected Riemannian manifold of nonpositive curvature, in particular in E_n or H_n. If there exists a closed minimal hypersurface R_{n-1} in a complete Riemannian manifold R_n of constant curvature $K>0$, then every point of R_{n-1} furthest from any fixed point O of R_n is either at least $\pi/2(K)^{1/2}$ distant from O or is on the minimum point locus with respect to O; hence there exists no closed minimal hypersurface in an open hemisphere.

The flat 2-torus T_2 is a closed minimal hypersurface in the flat 3-torus T_3; the equatorial 2-sphere S_2 in the 3-sphere S_3 is a closed minimal hypersurface. Both these examples illustrate the theorem.

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