SUBRINGS OF SIMPLE RINGS WITH MINIMAL IDEALS(1)

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0. **Introduction.** In the classical theory of finite-dimensional central simple algebras a large part of the work deals with a study of the simple subalgebras of such an algebra. In particular, the commutators of simple subalgebras as well as pairs of isomorphic subalgebras have been studied by A. A. Albert and E. Noether. For a modern presentation of their principal results we refer to [3, pp. 101–104](2). It is the purpose of this paper to study extensions of these results to simple rings which possess a minimal one-sided ideal (S.M.I. rings). Throughout the work we shall use the words simple ring to mean a ring which has no radical and no proper two-sided ideals. It should be noted that it is not known whether a non-nilpotent ring with no proper two-sided ideals may be a radical ring or not. The words radical and semi-simple will be used in the sense of Jacobson [4].

The structure theory of S.M.I. rings has been given by Dieudonné [1] and Jacobson [5], and will be briefly summarized here. With every S.M.I. ring A there is associated a pair of dual vector spaces $\mathfrak{M}$, $\mathfrak{N}$ over a division ring D. $\mathfrak{M}$ and $\mathfrak{N}$ are linked by an inner product (x, f), x in $\mathfrak{M}$, f in $\mathfrak{N}$, which is a nondegenerate bilinear function from $\mathfrak{M}\times\mathfrak{N}$ to D. A linear transformation (l.t.) a on $\mathfrak{M}$ is said to have an adjoint a^* on $\mathfrak{N}$ if there is a l.t. a^* on $\mathfrak{N}$ such that $(xa, f)=(x, a^*f)$. The l.t. which possess adjoints are often called continuous. We shall call a l.t. a finite-valued if the space $\mathfrak{M}a$ is finite-dimensional. It may then be shown that any continuous finite-valued linear transformation (f.v.l.t.) a on $\mathfrak{M}$ has the form $za=\sum_i(z, f_i)x_i$, where z, x_i are in $\mathfrak{M}$, and the f_i are in $\mathfrak{N}$. The ring A may then be regarded as the ring $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ of all f.v.l.t. on $\mathfrak{M}$ which have adjoints on $\mathfrak{N}$. We shall use the notation $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ for the ring of *all* continuous l.t. on $\mathfrak{M}$. Furthermore $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ is a dense ring of f.v.l.t. on $\mathfrak{M}$. That is, for any n linearly independent vectors $x_1, x_2, \cdots, x_n$ of $\mathfrak{M}$ and any n arbitrary vectors $y_1, y_2, \cdots, y_n$ of $\mathfrak{M}$, there is a l.t. a in A such that $x_ia=y_i$. Conversely any dense ring of f.v.l.t. on $\mathfrak{M}$ is an S.M.I. ring.

If $\mathfrak{K}$ is a subspace of $\mathfrak{M}$, the annihilator of $\mathfrak{K}$ in $\mathfrak{N}$ is the set of all vectors f of $\mathfrak{N}$ such that $(\mathfrak{K}, f)=0$. The annihilator is a subspace of $\mathfrak{N}$ and will be

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denoted by $\mathfrak{K}'$. A similar definition holds for the annihilator $\mathfrak{L}'$ in $\mathfrak{M}$ of a subspace $\mathfrak{L}$ of $\mathfrak{N}$.

In §1 conditions under which a semi-simple ring is a direct sum of S.M.I. rings are obtained, and a suitable hypothesis is given which assures that a simple subring of an S.M.I. ring is again an S.M.I. ring. The next section deals with the commutator of a simple subring of an S.M.I. ring. It is seen that if the subring is a simple ring not satisfying the descending chain condition on one-sided ideals, the commutator coincides with the two-sided annihilator of the subring. This annihilator will in general be a ring with a nilpotent radical, and will be an S.M.I. ring modulo its radical. In §4 we investigate conditions under which a given isomorphism of two simple subrings of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ may be extended to an inner automorphism of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$. In the course of this investigation we study the relation between the modules $\mathfrak{M}B$ and $B^*\mathfrak{N}$, where B is an S.M.I. subring of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. It turns out that $\mathfrak{M}B$ and $B^*\mathfrak{N}$ split into equipotent families of orthogonal irreducible submodules (Theorem 6). We then go on to a study of the maximal one-sided ideals of an S.M.I. ring $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. These turn out to be of two kinds, those that are again S.M.I. rings and those that possess a radical. In the course of this work we give an example of a pair of dual spaces $\mathfrak{M}$, $\mathfrak{N}$ with $\dim \mathfrak{M} = \dim \mathfrak{N} > \aleph_0$ which do not admit biorthogonal bases. That is, there are no bases $\{x_i\}$ of $\mathfrak{M}$, $\{f_j\}$ of $\mathfrak{N}$ such that $(x_i, f_j) = \delta_{ij}$ (§5). Finally, attention is restricted to the case where both $\mathfrak{M}$ and $\mathfrak{N}$ are of countably infinite dimension over D. In that case, using an idea of Kaplansky's [9, Theorem 6], it is possible to give a complete classification of certain subrings of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ under the natural concept of equivalence; two subrings B and C are equivalent in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ if there is an invertible l.t. P in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ such that $B = P^{-1}CP$.

As will be seen the main new feature of the infinite case is the fact that the images of vectors of $\mathfrak{M}$ under l.t. belonging to the subring B of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ need not fill $\mathfrak{M}$ completely. In fact, there is a complement of the range of B which contains the subspace of all vectors annihilated by every l.t. of B, but may also contain other vectors. It is the existence of such a "tail" that leads to most of the interesting new features of the infinite case.

In conclusion I should like to express my warmest thanks to Professor I. Kaplansky for his constant advice and encouragement in the preparation of this paper. I should also like to thank Professor O. F. G. Schilling for several conversations concerning the material in §§2 to 4.

1. **Subring of rings atomic modulo the radical.** If B is a simple subring of an S.M.I. ring A, we wish to show that B is also an S.M.I. ring. In order to prove this however, we have to impose a hypothesis on B which will guarantee the existence of idempotents. Following Kaplansky we shall say that a ring B is a Zorn ring [12] if:

Every non-nil left ideal of B contains a nonzero idempotent(3).

(3) For a discussion of associative Zorn rings cf. [8, p. 63].

This hypothesis is satisfied for algebraic algebras, and indeed algebraic algebras provide a motivation for Theorem 1.

We recall that Dieudonné [1] has called the union of the minimal left ideals of a ring its left socle. Since we shall be exclusively concerned with semi-simple rings where the right and left socles coincide, we shall henceforth speak simply of the socle of a ring. Jacobson has called a semi-simple ring atomic if it is equal to its own socle and has shown [4, Theorem 33] that it then is a direct sum of S.M.I. rings. We are now in a position to state a theorem which gives the desired result as an immediate corollary:

THEOREM 1. *Let A be a ring whose radical is a nil ideal and such that A is atomic modulo its radical. Then every Zorn subring is also atomic modulo its radical.*

That the theorem is not valid for arbitrary subrings of A is apparent in the case where A is the field of rational numbers and the subring is the ring of integers. Here the subring is semi-simple but not atomic.

It should be noted that the hypotheses on A are strong enough to ensure that A is a Zorn ring too. The proof of Theorem 1 will now be carried out in a series of lemmas.

LEMMA 1. *If A is an S.M.I. ring and e is an idempotent of A, eAe is a simple ring with the descending chain condition (d.c.c.).*

Proof. The result is easily established by noting that by Theorem 9 of [7] the ring eAe is a subring of a simple ring with the d.c.c., and as such it must satisfy the d.c.c. itself(4).

An immediate consequence of this is that if A is semi-simple and atomic, eAe is a semi-simple ring with the d.c.c. for all idempotents e of A. Now it is readily seen that if K is any ring with radical $R(K)$ and e is an idempotent of K, $eKe-eR(K)e$ is isomorphic to $\bar{e}(K-R(K))\bar{e}$ where $\bar{e}$ is the residue class of e modulo $R(K)$. Thus if A is atomic modulo its radical, eAe will be a semi-simple ring with d.c.c. modulo its radical $R(eAe)=eR(A)e$. It then follows that eAe cannot contain an infinity of orthogonal idempotents. For it is easily verified that two orthogonal idempotents cannot be congruent modulo the radical, so that an infinity of orthogonal idempotents in eAe would give rise to an infinity of orthogonal idempotents in $eAe-R(eAe)$.

LEMMA 2. *Let A be a ring which is atomic modulo its radical, and let B be a Zorn subring of A. Then every idempotent e of B maps onto an idempotent of the socle of $B-R(B)$.*

Proof. The ring eBe is a Zorn ring. For let L be a non-nil left ideal of eBe. Then BL is a non-nil left ideal of B and so contains a nonzero idempotent

(4) This latter result may be immediately deduced from Theorem 2.1 of [8], although it can also be proved directly.

$f=fe$. It is readily seen that ef is a nonzero idempotent of $eBe\cap BL=L$. Now $eBe\subset eAe$ and so by the remarks subsequent to Lemma 1 it does not contain an infinity of orthogonal idempotents. Hence [8, Theorem 2.1] $eBe-eR(B)e$ satisfies the d.c.c. Thus if we let $S=B-R(B)$ and $\bar{e}$ the residue class of e modulo $R(B)$, $\bar{e}(B-R(B))\bar{e}=\bar{e}S\bar{e}$ is a semi-simple ring with the d.c.c. But then $\bar{e}S\bar{e}$ may be written as a direct sum of a finite number of minimal left ideals; $\bar{e}S\bar{e}=\sum(\bar{e}S\bar{e})\bar{e}_i$ where $\bar{e}_i(\bar{e}S\bar{e})\bar{e}_i$ is a division ring. However $\bar{e}_i\bar{e}=\bar{e}\bar{e}_i=\bar{e}_i$, so that $\bar{e}_iS\bar{e}_i$ is a division ring, $S\bar{e}_i$ is a minimal left ideal of S [6, p. 65] and, since $\bar{e}$ is in $\sum S\bar{e}_i$, $\bar{e}$ is in the socle of S.

LEMMA 3. *Let B be a Zorn subring of an atomic semi-simple ring A. Then every element b of B may be written as*

$$b = be + b'e' + \cdots + r,$$

with $e, e', \cdots$ idempotents of B, $b, b', \cdots$ in B, and r in $R(B)$.

Proof. Let A be a direct sum of S.M.I. rings A_i, where the A_i are taken to be rings $\mathcal{J}(\mathfrak{M}_i, \mathfrak{N}_i)$, for dual spaces $\mathfrak{M}_i$, $\mathfrak{N}_i$ over division rings D_i. If b is an element of B not in $R(B)$, the left ideal generated by b in B is non-nil and so contains a nonzero idempotent $e=cb+nb$, c in B, n an integer. We may write $b=\sum b_i$, $e=\sum e_i$ where b_i, e_i are in A_i. Then $e_i=(c+n)b_i$ so that $e_i^2=(c+n)b_ie_i=e_i$, thus $b_ie_i\neq 0$ provided $e_i\neq 0$, and $\mathfrak{M}_ie_i\subset\mathfrak{M}_ib_i$. Hence

$$\mathfrak{M}_ib_i = \mathfrak{M}_ib_ie_i \oplus \mathfrak{M}_ib_i(1 - e_i)$$

and since the b_i are f.v.l.t. we have, when $e_i\neq 0$,

$$\dim \mathfrak{M}_ib_i(1 - e_i) < \dim \mathfrak{M}_ib_i.$$

If $b-be\neq 0$ and is not in $R(B)$, we set $b'=b-be$ and repeat a similar argument for b'. We are thus led to an idempotent e' of B such that $b_i'e_i'\neq 0$ if $e_i'\neq 0$, and spaces $\mathfrak{M}_ib_i'\subset\mathfrak{M}_ib_i$ with $\dim \mathfrak{M}_ib_i'(1-e_i')<\dim \mathfrak{M}_ib_i'<\dim \mathfrak{M}_ib_i$. Since the spaces $\mathfrak{M}_ib_i$ are finite-dimensional and at least one nonzero e_i occurs at each step, the process must terminate, and we end up with the desired representation.

The proof of Theorem 1 is then completed as follows: Let A be a ring whose radical is a nil ideal, and which is atomic modulo its radical. Let B be a Zorn subring of A. We now consider $\overline{B}=B-B\cap R(A)$, the image of B in $A-R(A)$. The ring $\overline{B}$ is also a Zorn ring, for if $\overline{L}$ is a non-nil left ideal in $\overline{B}$, its inverse image in B contains a nonzero idempotent. Since the radical of a ring contains no idempotents except zero, this idempotent maps on a nonzero idempotent of $\overline{L}$. Hence by Lemma 3 every $\bar{b}$ in $\overline{B}$ may be written as $\bar{b}\bar{e}+\bar{b}'\bar{e}'+\cdots+\bar{r}$, where $\bar{r}$ is in $R(\overline{B})$. We now show that every element in the residue class $\bar{r}$ is in $R(B)$. The left ideal $\overline{M}$ generated by $\bar{r}$ in $\overline{B}$ is nil since $R(\overline{B})$, the radical of a Zorn ring, is a nil ideal. Thus some power of every element of M, the inverse image of $\overline{M}$ in B, lies in $B\cap R(A)$. But since the latter is a nil ideal of B, $B\cap R(A)\subset R(B)$. Since B is a Zorn ring, $R(B)$ is a nil ideal, and therefore M is also nil and so lies in $R(B)$.

Thus if b is any element in the residue class $\bar{b}$, we may write $b=be+b'e'+\cdots+r+t$, where r is in $R(B)$, t is in $B\cap R(A)\subset R(B)$, and $e, e', \cdots$ are idempotents of B. This last fact follows since the usual trick of building idempotents modulo the radical in the presence of the d.c.c. works just as well modulo any nil ideal. But by Lemma 2 it is known that $e, e', \cdots$ map into the socle of $B-R(B)$. Thus since the socle is a two-sided ideal, b maps into the socle of $B-R(B)$. Hence $B-R(B)$ is equal to its own socle and so is atomic.

COROLLARY 1. *Let A be an S.M.I. ring, and let B be a simple Zorn subring of A. Then B is an S.M.I. ring.*

2. **Commutators of subrings. The finite-dimensional case.** We now study the commutator subring of a simple Zorn subring of an S.M.I. ring $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. We denote by $A(B)$ the subring of A consisting of all those elements of A which commute with every element of the subring B. Let a be an element of $A(B)$; then

$$\mathfrak{M}aB = \mathfrak{M}Ba.$$

But $\mathfrak{M}a$ is a finite-dimensional subspace, so that B induces a ring of l.t. on the finite-dimensional subspace $\mathfrak{M}a$ to which it is homomorphic. However, B is simple so that either $aB=Ba=0$ for all a in $A(B)$, or B is isomorphic to a simple Zorn subring of the ring of all l.t. on a finite-dimensional vector space over a division ring. It is then easily seen that in this case B satisfies the d.c.c. [8, Theorem 2.1]. Thus we have proved the following theorem.

THEOREM 2. *If A is an S.M.I. ring and B is a simple Zorn subring, then either B satisfies the d.c.c. or $A(B)$ is the two-sided annihilator of B in A.*

Unfortunately we cannot obtain general results in the first of these alternatives, but have to make assumptions that will guarantee that if B satisfies the d.c.c., it is also finite-dimensional over its center. In order to ensure this we shall assume that A is an algebraic S.M.I. algebra over an algebraically closed field Φ. It then follows that the division ring D is isomorphic to Φ, and that if B is a simple subalgebra satisfying the d.c.c., it is of finite dimension over Φ. Thus B has a unit u and we proceed to show that

$$A(B) = uAu(B) \oplus (1-u)A(1-u). \tag{2.1}$$

It is readily verified that $uAu(B)$ and $(1-u)A(1-u)$ are two-sided ideals in $A(B)$ with zero intersection. Now for every element a in $A(B)$, $ua=au$, so that $a=uau+(1-u)a(1-u)$. Furthermore $buau=uaub$, so that (2.1) holds.

Now uAu and B are finite-dimensional simple algebras over Φ with the same unit and $B\subset uAu$. Since the center of uAu is an algebraic extension field of Φu, uAu is a central simple algebra. Hence $uAu(B)$ is again simple [3, p. 104].

Now let c be an element of $A(A(B))$. Then since u is in $A(B)$, $uc=cu$, so that $c=ucu+(1-u)c(1-u)$. A simple computation shows that ucu is in $uAu(uAu(B))$, and that $(1-u)c(1-u)$ lies in the center Z of $(1-u)A(1-u)$. But from (2.1) it is clear that $uAu(uAu(B))$ and Z are contained in $A(A(B))$, and since they are obviously orthogonal,

$$A(A(B)) = uAu(uAu(B)) \oplus Z.$$

Quoting [3, p. 104] again we finally obtain

$$A(A(B)) = B \oplus Z. \tag{2.2}$$

It is well known that $(1-u)A(1-u)$ is simple with A and so its center is zero or a field [5, Theorem 16]. But all the subalgebras of an algebraic algebra are Zorn algebras and so, by Corollary 1, $(1-u)A(1-u)$ is an S.M.I. algebra. Thus if $Z\neq 0$, $(1-u)A(1-u)$ has a unit and so satisfies the d.c.c. [5, p. 243]. Now $(1-u)A(1-u)$ is isomorphic to the ring of l.t. it induces on $\mathfrak{M}(1-u)$, and it is easily verified that the latter is a dense ring of f.v.l.t. But then if $(1-u)A(1-u)$ has a unit, $\mathfrak{M}(1-u)$ must be of finite dimension [5, Theorem 3]. Since $\mathfrak{M}u$ is also finite-dimensional, this implies that $\mathfrak{M}$ is of finite dimension over Φ. Hence $Z\neq 0$ if and only if A is of finite dimension over Φ and the unit of B does not coincide with that of A, in which case $Z\cong\Phi$. We have thus proved the following theorem.

THEOREM 3. *Let A be an algebraic S.M.I. algebra over an algebraically closed field Φ. Let B be a simple subalgebra satisfying the d.c.c. with unit u. Then*

(i) *$A(B)$ is a direct sum of two simple algebras, namely, the commutator of B in uAu and the two-sided annihilator of B in A.*

(ii) *If A has infinite dimension over Φ, $A(A(B))=B$.*

(iii) *If A is of finite dimension over Φ but u is not the unit of A, then $A(A(B))=B\oplus Z$, $Z\cong\Phi$.*

3. **The infinite case.** We now discuss the second of the alternatives of Theorem 2 assuming only that A is an S.M.I. ring and that B is a simple Zorn subring of A. We proceed to investigate the embedding of B in A. It is known that for a given finite set of elements of an S.M.I. ring there exists a unit element in the ring for this set [7, Theorem 9]. We shall call such a unit element a "local unit" for the set. Let $\mathfrak{M}B$ denote the set of all vectors of $\mathfrak{M}$ which are images under l.t. of B. $\mathfrak{M}B$ is a subspace of $\mathfrak{M}$: for if xb, yc are two elements of $\mathfrak{M}B$, $xb-yc=(xb-yc)e$, where e is a local unit for b and c; furthermore for all α in D, $(\alpha x)b=\alpha(xb)$. This also shows that our definition of $\mathfrak{M}B$ coincides with the usual one in which $\mathfrak{M}B$ consists of all finite sums of terms xb, x in $\mathfrak{M}$, b in B. We denote by $\mathfrak{T}(B)$ the subspace of all vectors annihilated by all l.t. in B.

LEMMA 4. $\mathfrak{M}B \cap \mathfrak{T}(B) = 0$.

Proof. Let x lie in $\mathfrak{M}B \cap \mathfrak{T}(B)$. Then $x = yb$ for some y in $\mathfrak{M}$ and b in B. Thus if e is a local unit for b, $x = xe = 0$.

Hence we may write

$$\mathfrak{M} = \mathfrak{M}B \oplus \mathfrak{T}(B) \oplus \mathfrak{B} \tag{3.1}$$

where $\mathfrak{B}$ is some complement of $\mathfrak{M}B \oplus \mathfrak{T}(B)$ in $\mathfrak{M}$. There is of course nothing unique about $\mathfrak{B}$. If B does not satisfy the d.c.c., a nonzero $\mathfrak{B}$ may very well occur as will be shown by an example. The existence of such a "tail" is in marked contrast to the case where B has the d.c.c. There B has a unit u and so $\mathfrak{M} = \mathfrak{M}u \oplus \mathfrak{M}(1-u)$, with $\mathfrak{M}B = \mathfrak{M}u$ and $\mathfrak{T}(B) = \mathfrak{M}(1-u)$. In fact, as will be seen in the subsequent work, most of the difficulties of the infinite case stem precisely from the existence of a nonzero tail.

We now make the following definition.

DEFINITION. *If a simple Zorn subring B of an S.M.I. ring $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$, $\mathfrak{M}$, $\mathfrak{N}$ dual spaces over a division ring D, is embedded in A in such a way that $\mathfrak{M}B \oplus \mathfrak{T}(B) \neq \mathfrak{M}$, B is said to be caudal in A. If on the other hand $\mathfrak{M}B \oplus \mathfrak{T}(B) = \mathfrak{M}$, B is said to be acaudal in A.*

We now give an example of a simple caudal subring. Let (x_n) be a countably infinite set of vectors over a field Φ, and let w and z be two additional vectors such that (x_n), w, and z are linearly independent over Φ. Let $\mathfrak{M} = \{x_n, w, z\}$, the space spanned by the (x_n), w, and z over Φ. Consider the set of matrix units e_{ij} defined as f.v.l.t. on $\mathfrak{M}$ by

$$x_n e_{ij} = \delta_{ni} x_j, \qquad w e_{ij} = 0, \qquad z e_{ij} = x_j, \qquad i, j = 1, 2, \cdots. \tag{3.2}$$

With these definitions it is easily verified that the l.t. e_{ij} satisfy the usual law of matrix unit multiplication $e_{ij}e_{kh} = \delta_{jk}e_{ih}$. Now let B be the algebra spanned over Φ by the e_{ij}. From our definitions it is clear that B consists entirely of f.v.l.t. on $\mathfrak{M}$. Hence if A denotes the algebra of all f.v.l.t. on $\mathfrak{M}$, B is a subalgebra of A. B is clearly isomorphic to the algebra of l.t. it induces on the space spanned by the x_n only. The latter is easily seen to be an algebraic dense algebra of f.v.l.t. on $\{x_n\}$, the space spanned by the x_n over Φ, and so B is a simple Zorn algebra [5, Theorem 9]. Furthermore it is readily verified that $\mathfrak{M}B = \{x_n\}$, and that $\mathfrak{T}(B) = \{w\}$. Thus z is not in $\mathfrak{M}B \oplus \mathfrak{T}(B)$ and B is caudal in A.

We are now in a position to determine the structure of $A(B)$ in case B does not satisfy the d.c.c. $A(B)$, the two-sided annihilator of B in A, may then also be characterized as the subring of A consisting of all those l.t. of A which have range in $\mathfrak{T}(B)$ and annihilate $\mathfrak{M}B$. Now $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$ and we may further characterize $A(B)$ as the ring of all those l.t. in A mapping $\mathfrak{M}$ into $\mathfrak{T}(B)$ and whose adjoints map $\mathfrak{N}$ into $(\mathfrak{M}B)'$. For if a is in $A(B)$, $\mathfrak{M}a \subset \mathfrak{T}(B)$, and since $Ba = 0$, $(\mathfrak{M}B, a^*\mathfrak{N}) = 0$. Conversely if a is a l.t. with these properties, $0=(\mathfrak{M}B, a^*\mathfrak{N})=(\mathfrak{M}Ba, \mathfrak{N})$. Hence $\mathfrak{M}Ba=0$ since $\mathfrak{M}$ and $\mathfrak{N}$ are dual spaces.

Analogously to the splitting (3.1) of $\mathfrak{M}$ there is one of $\mathfrak{N}$

$$\mathfrak{N} = B^*\mathfrak{N} \oplus \mathfrak{T}(B^*) \oplus \mathfrak{W}$$

where B^* is the ring of adjoints of B. We verify immediately that $(\mathfrak{M}B)' =\mathfrak{T}(B^*)$, $(B^*\mathfrak{N})'=\mathfrak{T}(B)$. Thus $A(B)$ may finally be characterized as the subring of $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ consisting of all those l.t. of A whose range is in $\mathfrak{T}(B)$ and whose adjoints have range in $\mathfrak{T}(B^*)$. In order to determine the structure of $A(B)$ we make a study of the following situation:

Let $\mathfrak{K}\subset\mathfrak{M}$, $\mathfrak{L}\subset\mathfrak{N}$ be subspaces of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. We consider the set $S=S(\mathfrak{K}, \mathfrak{L})$ of all l.t. of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ which have range in $\mathfrak{K}$ and whose adjoints map $\mathfrak{N}$ into $\mathfrak{L}$. Since if x is in $\mathfrak{K}$ and f is in $\mathfrak{L}$, the l.t. $z\rightarrow(z, f)x$ lies in S, S maps $\mathfrak{M}$ onto $\mathfrak{K}$ and S^* maps $\mathfrak{N}$ onto $\mathfrak{L}$. It is easily verified that S is a ring and in this notation $A(B)=S(\mathfrak{T}(B), \mathfrak{T}(B^*))$. If x is in $\mathfrak{L}'\cap\mathfrak{K}$, $xS=0$; for if s is in S, $(xs, \mathfrak{N})=(x, s^*\mathfrak{N})=0$. But $\mathfrak{M}$ and $\mathfrak{N}$ are dual spaces so that $xs=0$ for all s in S. Similarly we see that $S^*(\mathfrak{K}'\cap\mathfrak{L})=0$. Conversely if for x in $\mathfrak{K}$, $xS=0$, $0=(xS, \mathfrak{N})=(x, S^*\mathfrak{N})=(x, \mathfrak{L})$, so that x is in $\mathfrak{L}'\cap\mathfrak{K}$.

We may now consider the factor spaces $\bar{\mathfrak{K}}=\mathfrak{K}/\mathfrak{L}'\cap\mathfrak{K}$, $\bar{\mathfrak{L}}=\mathfrak{L}/\mathfrak{K}'\cap\mathfrak{L}$. It is easy to see that $\bar{\mathfrak{K}}$ and $\bar{\mathfrak{L}}$ are again linked by a nondegenerate inner product, for we have but to define $(\bar{x}, \bar{f})=(x, f)$ where x and f are any elements in the cosets $\bar{x}$ and $\bar{f}$. If we now define $\bar{x}s=xs$, $s^*\bar{f}=s^*f$, for representatives x and f of the cosets $\bar{x}$ and $\bar{f}$, it follows by the remarks of the preceding paragraph that S induces a ring $\bar{S}$ of f.v.l.t. on $\bar{\mathfrak{K}}$ with adjoints in $\bar{\mathfrak{L}}$. Moreover it is clear that all f.v.l.t. on $\bar{\mathfrak{K}}$ with adjoints on $\bar{\mathfrak{L}}$ are obtained in this fashion; hence [1, Proposition 1] $\bar{S}$ is an S.M.I. ring.

The mapping of S onto $\bar{S}$ obtained by sending every l.t. of S onto the one it induces on $\bar{\mathfrak{K}}$ is a homomorphism with kernel R say, $S-R\cong\mathcal{J}(\bar{\mathfrak{K}}, \bar{\mathfrak{L}})$. Since $\bar{S}$ as an S.M.I. ring is semi-simple, the radical of S is contained in R. Now R may be characterized as the set of all l.t. of S which map $\mathfrak{K}$ into $\mathfrak{L}'\cap\mathfrak{K}$. It then follows automatically that R^* will map $\mathfrak{L}$ onto $\mathfrak{K}'\cap\mathfrak{L}$. For if r is in R, $0=(\mathfrak{K}r, \mathfrak{L})=(\mathfrak{K}, r^*\mathfrak{L})$. Now let s and t be arbitrary elements of S and let r be in R, then $(\mathfrak{M}srt, \mathfrak{N})=(\mathfrak{M}sr, t^*\mathfrak{N})=0$, since $\mathfrak{M}sr\subset\mathfrak{K}r\subset\mathfrak{L}'\cap\mathfrak{K}$, and $t^*\mathfrak{N}\subset\mathfrak{L}$. Thus $srt=0$ and it follows that $SRS=0$ so that $R^3=0$. Hence R is a nilpotent ideal of S and so is contained in the radical. Combining this with our previous statement we see that R is the radical of S.

We note that R might have been defined as the set of all r in S such that $SrS=0$, for then $0=(\mathfrak{M}Sr, S^*\mathfrak{N})=(\mathfrak{K}r, \mathfrak{L})$ so that $\mathfrak{K}r\subset\mathfrak{L}'\cap\mathfrak{K}$. Furthermore if the inclusions

$$(3.3) \qquad 0 \subset \mathfrak{L}' \cap \mathfrak{K} \subset \mathfrak{K}, \qquad 0 \subset \mathfrak{K}' \cap \mathfrak{L} \subset \mathfrak{L}$$

are all proper, the index of nilpotency of the radical is actually three. For we may then pick the following nonzero elements: x in $\mathfrak{L}'\cap\mathfrak{K}$, f in $\mathfrak{L}$ but not in

$\mathfrak{K}'\cap\mathfrak{L}$, y in $\mathfrak{K}$ such that $(y, f)\neq 0$, and g in $\mathfrak{K}'\cap\mathfrak{L}$. The l.t. r and r' defined by

$$zr = (z, f)x, \qquad zr' = (z, g)y, \qquad\qquad z \text{ in } \mathfrak{M},$$

are in R and $zr'r=(z, g)(y, f)x$ so that $r'r\neq 0$. However if any one of the inclusion relations (3.3) fails to be proper, $R^2=0$. For example, suppose that $\mathfrak{L}'\cap\mathfrak{K}=\mathfrak{K}$, then if r, r' are any two elements of R, $(\mathfrak{M}rr', \mathfrak{N})=(\mathfrak{M}r, r'^*\mathfrak{N})=0$ so that $R^2=0$. If $\mathfrak{L}'\cap\mathfrak{K}=\mathfrak{K}'\cap\mathfrak{L}=0$, every r in R maps $\mathfrak{K}$ into 0; but then $0=(\mathfrak{K}r, \mathfrak{N})=(\mathfrak{K}, r^*\mathfrak{N})$ and r^* has range in $\mathfrak{L}$, so $r^*\mathfrak{N}=0$, $r=0$. Hence $R=0$ and $S\cong\bar{S}$ is an S.M.I. ring. Conversely suppose that $R=0$. Then if $x\neq 0$ were in $\mathfrak{L}'\cap\mathfrak{K}$ or $g\neq 0$ were in $\mathfrak{K}'\cap\mathfrak{L}$, the l.t.

$$z \to (z, f)x, \quad f \neq 0 \text{ in } \mathfrak{L}, \qquad z \to (z, g)y, \quad y \neq 0 \text{ in } \mathfrak{K}$$

would be nonzero elements of the radical. We have thus proved the following lemma.

LEMMA 5. *The ring $S=S(\mathfrak{K}, \mathfrak{L})$ is a ring with radical R consisting of all elements r of S such that $SrS=0$, and $S-R$ is an S.M.I. ring. The index of nilpotency of R is three if and only if all the inclusion relations* (3.3) *are proper. The radical is zero if and only if $\mathfrak{L}'\cap\mathfrak{K}=\mathfrak{K}'\cap\mathfrak{L}=0$ and then S is isomorphic to $\mathcal{J}(\mathfrak{K}, \mathfrak{L})$.*

It is to be noted that if in the proof of Lemma 5 we had taken for A the ring $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ of *all* continuous l.t. on $\mathfrak{M}$, and for $S(\mathfrak{K}, \mathfrak{L})$ the subring consisting of *all* those l.t. whose range is in $\mathfrak{K}$ and whose adjoints have range in $\mathfrak{L}$, the analogous result would have been proved. However in that situation $S-R$ would have been a primitive ring with minimal ideals rather than an S.M.I. ring. We also note that the rings $S(\mathfrak{K}, \mathfrak{N})$ and $S(\mathfrak{M}, \mathfrak{L})$ are left and right ideals in either case. If $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$, all one-sided ideals are of this form [6, p. 15].

We shall now give an example to show that the ring $A(B)=S(\mathfrak{X}(B), \mathfrak{X}(B^*))$ may have a radical with index of nilpotency three. We define dual spaces $\mathfrak{M}$ and $\mathfrak{N}$ over a division ring D as follows: $\mathfrak{M}$ has a basis of vectors $\{x_i, y_j\}$ and $\mathfrak{N}$ has a basis of vectors $\{f_k, g_h\}$ where the subscripts i, j, k, h run over the positive integers. The inner product is given by the relations

$$(x_i, f_k) = \delta_{ik}, \qquad (y_j, f_k) = 0, \qquad (x_i, g_h) = 0, \qquad (y_j, g_h) = \delta_{jh}.$$

It is then readily verified that $\mathfrak{M}$ and $\mathfrak{N}$ are dual spaces. Let

$$\mathfrak{K} = \{x_{2i-1} - x_{2i+1}, y_{2j-1} - y_{2j+1}\}, \qquad i = 1, 2, \cdots; j = 2, 3, \cdots,$$
$$\mathfrak{L} = \{f_{2k-1} - f_{2k+1}, g_{2h-1} - g_{2h+1}\}, \qquad k = 2, 3, \cdots; h = 1, 2, \cdots,$$

where the vectors in braces are a basis for the subspace in question. It is then readily verified that

$$\mathfrak{K}' = \{f_{2k}, g_1, g_{2h}\}, \qquad\qquad k, h = 1, 2, \cdots,$$

$$\mathfrak{K}'' = \{x_{2i-1}, y_{2j-1}\}, \qquad i = 1, 2, \cdots; j = 2, 3, \cdots,$$
$$\mathfrak{L}' = \{x_1, x_{2i}, y_{2j}\}, \qquad i, j = 1, 2, \cdots,$$
$$\mathfrak{L}'' = \{f_{2k-1}, g_{2h-1}\}, \qquad k = 2, 3, \cdots; h = 1, 2, \cdots.$$

We thus see that

$$\mathfrak{K}' \cap \mathfrak{L} = 0, \qquad \mathfrak{K} \cap \mathfrak{L}' = 0,$$
$$\mathfrak{K}' \cap \mathfrak{L}'' = \{g_1\}, \qquad \mathfrak{K}'' \cap \mathfrak{L}' = \{x_1\}.$$

Now if $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$, the subring $B = S(\mathfrak{K}, \mathfrak{L})$ is an S.M.I. ring by Lemma 5. Since both $\mathfrak{K}$ and $\mathfrak{L}$ are of infinite dimension over D, B does not satisfy the d.c.c. [5, Theorem 3], so that $A(B) = S(\mathfrak{T}(B), \mathfrak{T}(B^*)) = S(\mathfrak{L}', \mathfrak{K}')$. But for $A(B)$ the inclusion relations (3.3) are all proper so that the radical of $A(B)$ has index of nilpotency three.

If, however, B is acaudal in A and B^* is acaudal in A^*, a simplification results: For then it is easily seen that $\mathfrak{T}(B^*)' = \mathfrak{M}B$ and that $\mathfrak{T}(B)' = B^*\mathfrak{N}$. Then Lemma 4 yields $\mathfrak{T}(B^*)' \cap \mathfrak{T}(B) = \mathfrak{T}(B)' \cap \mathfrak{T}(B^*) = 0$, so that $A(B) = S(\mathfrak{T}(B), \mathfrak{T}(B^*))$ is an S.M.I. ring isomorphic to $\mathcal{J}(\mathfrak{T}(B), \mathfrak{T}(B^*))$. We thus have the following theorem.

THEOREM 4. *Let A be an S.M.I. ring, B a simple Zorn subring which does not satisfy the d.c.c. Then $A(B)$, the two-sided annihilator of B in A, is a ring with a nilpotent radical R, $R^3 = 0$, and $A(B) - R$ is an S.M.I. ring. If B is acaudal in A and B^* is acaudal in A^*, $A(B)$ is an S.M.I. ring and is isomorphic to $\mathcal{J}(\mathfrak{T}(B), \mathfrak{T}(B^*))$.*

However, $A(B)$ may be an S.M.I. ring without B being acaudal in A. For example it may happen that $\mathfrak{T}(B) = 0$ while B is still caudal in A (delete w in the first example of this section) and so $A(B) = 0$ and thus is an S.M.I. ring.

In general it will not be true that $A(A(B)) = B$. To obtain an example let us assume that B and B^* are acaudal in A and A^* respectively, and that the dimensions of $\mathfrak{T}(B)$ and $\mathfrak{T}(B^*)$ are both infinite. Then $A(B)$ is an S.M.I. subring of A not satisfying the d.c.c. and so $A(A(B))$ is isomorphic to $S(\mathfrak{M}B, B^*\mathfrak{N})$. Thus, unless B induces a dense ring of f.v.l.t. on $\mathfrak{M}B$, $A(A(B))$ will contain B properly. It will become apparent in the next section that B need not induce a dense ring of f.v.l.t. on $\mathfrak{M}B$. However, we shall also see that if B does induce a dense ring of f.v.l.t. on $\mathfrak{M}B$, $B = S(\mathfrak{M}B, B^*\mathfrak{N})$. Thus if B satisfies the hypotheses made at the beginning of this paragraph and induces a dense ring of f.v.l.t. on $\mathfrak{M}B$, $A(A(B)) = B$.

We now give a slight generalization of the results in the acaudal case, to the situation where A is taken to be the algebra of *all* l.t. (not necessarily finite-valued) on a vector space $\mathfrak{M}$ over a field Φ. Following Dieudonné [2] we call such an algebra a central completely primitive one. We again let B be a simple subalgebra containing nonzero f.v.l.t. Then since the set of all f.v.l.t. in B forms a two-sided ideal, B consists entirely of f.v.l.t. Hence by Theorem 18 of [5] B is an algebraic algebra, and so is a Zorn algebra. Thus B is a simple Zorn subalgebra of the S.M.I. algebra of all f.v.l.t. on $\mathfrak{M}$. Hence, by Corollary 1, B is also an S.M.I. algebra. Assuming now that $\mathfrak{M}=\mathfrak{M}B \oplus \mathfrak{T}(B)$, we can determine the structure of $A(B)$. We denote by J the subalgebra of $A(B)$ which annihilates $\mathfrak{M}B$, and by I the subalgebra which annihilates $\mathfrak{T}(B)$. Then we have:

LEMMA 6. $A(B)=I\oplus J$.

Since the proofs of this lemma and the subsequent theorem are fairly easily reconstructed, they will not be given here. Our results in this case are then summed up in the following theorem.

THEOREM 5. *Let A be the algebra of all l.t. on a vector space $\mathfrak{M}$ over a field Φ, B a simple subalgebra containing nonzero f.v.l.t. If $\mathfrak{M}=\mathfrak{M}B\oplus\mathfrak{T}(B)$, then $A(B)$, the commutator of B in A, is the direct sum of a central completely primitive algebra over Φ and an algebra which is the set of all l.t. on a vector space over a division ring.*

4. **Isomorphic subalgebras.** We now wish to investigate isomorphic simple subrings of an S.M.I. ring A. If A is a finite-dimensional central simple algebra, it is known that every isomorphism between two simple subalgebras containing the unit of A can be extended to an inner automorphism of A [3, p. 101, Theorem 15]. Now if $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ is an S.M.I. ring not satisfying the d.c.c., A contains no invertible elements, and so we cannot hope to extend an isomorphism of subrings to an inner automorphism of A.

Now it is possible to generalize the concept of inner automorphism to a ring without a unit as follows: As usual let $x \circ y=x+y+xy$ for any two x, y in the ring A. Suppose that there is an element x' in A such that $x \circ x' = x' \circ x=0$. Then the mapping $a\rightarrow x' \circ a \circ x$ is called a quasi-inner automorphism of A (Malcev). Since we can write $x' \circ a \circ x=(1+x')a(1+x)$, where of course the unit element is used purely as shorthand, it is readily verified that the mapping is actually a ring automorphism. If now $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$, $\mathfrak{M}$, $\mathfrak{N}$ dual spaces over a field Φ, it is clear that all quasi-inner automorphisms of A are mappings of the form $a\rightarrow P^{-1}aP$, with $P=I+f$ in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$. Here I is the unit of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$, and f lies in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. It then follows that not even quasi-inner automorphisms are enough for a generalization of the classical theorem. For let Q be an invertible element of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ not of the form $\alpha(I+f)$, where α is in Φ, and f is in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. Then the automorphism $a\rightarrow Q^{-1}aQ$ is not quasi-inner. Indeed if it were, there would exist an element f in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ such that $Q^{-1}(I+f)$ would commute with every element of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. Thus $Q^{-1}(I+f)=\alpha I$, where α is in Φ, a contradiction.

However, we shall be able to prove that under suitable hypotheses an isomorphism between two simple subrings of A can be extended to an inner automorphism of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$.

We specialize to the case where $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$, with $\mathfrak{M}$, $\mathfrak{N}$ dual spaces over an algebraically closed field Φ(5). In a natural manner A is then an algebra over Φ. Suppose now that B and C are simple subalgebras of A isomorphic under a mapping σ. Since they are algebraic [5, Theorem 18], B and C are Zorn algebras, and so B and C are S.M.I. algebras. We wish now to find conditions under which $\sigma(b) = P^{-1}bP$ for P in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$. It is to be noted that since σ is an algebra isomorphism, $\sigma(\alpha b) = \alpha\sigma(b)$ for all α in Φ.

If σ is of the desired type, it is clear that $A(B)$ and $A(C)$ must also be isomorphic. For if $ab = ba$, then

$$P^{-1}aPP^{-1}bP = P^{-1}bPP^{-1}aP, \qquad P^{-1}aP\sigma(b) = \sigma(b)P^{-1}aP,$$

so that $A(B)$ and $A(C)$ are also isomorphic. In the finite-dimensional case this isomorphism is automatically ensured by supposing that B and C contain the unit element of A. However, in the infinite-dimensional case even if $\mathfrak{M}B$ and $\mathfrak{M}C$ have the same dimension, $A(B)$ and $A(C)$ may fail to be isomorphic. As an example let A and B be as in the first example of §3, and let C coincide with B on the space spanned by the x_n, but let $wC = zC = 0$. Then obviously both B and C are isomorphic to the algebra of l.t. which they induce on $\{x_n\}$, and so B is isomorphic to C. However, C is acaudal in A and $A(C)$ can be seen to be an S.M.I. ring, in fact it is isomorphic to Φ_2, while the l.t. a defined by $x_n a = wa = 0$, $za = w$ clearly lies in the radical of $A(B)$, so that $A(B)$ and $A(C)$ are not isomorphic in this case. Furthermore we know that the modules $\mathfrak{M}B$ and $\mathfrak{M}C$ are fully reducible as right B and C-modules respectively [1, Théorème 1 and p. 54]. Now if $P^{-1}BP = C$, $\mathfrak{M}B$ and $\mathfrak{M}C$ must split into equipotent families of irreducible submodules. That this is not implied by the isomorphism between B and C and that of their commutators in A can easily be seen by means of examples. It is clear, therefore, that in order to arrive at the desired result we shall have to assume in addition to the isomorphism between B and C at least that $A(B)$ is isomorphic to $A(C)$, and that $\mathfrak{M}B$ and $\mathfrak{M}C$ split into equipotent families of irreducible submodules.

Before proving our main theorem we make a study of the relation between $\mathfrak{M}B$ and $B^*\mathfrak{N}$. We know that $\mathfrak{M}B$ may be written as a direct sum of irreducible right B-modules, $\mathfrak{U}_i$ say. Thus $\mathfrak{M}B = \sum_i \mathfrak{U}_i$(6), where the cardinality of the index set (i) is uniquely determined and is $\mathfrak{c}$, say. Following Dieudonné [2, p. 160] we call $\mathfrak{c}$ the height of B in A. We then prove the following theorem.

THEOREM 6. *Let $\mathfrak{M}$, $\mathfrak{N}$ be dual spaces over an algebraically closed field Φ. Let B be a simple subalgebra of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. Then to every splitting of $\mathfrak{M}B$ into*

(5) This may be realized, for example, by letting A be an algebraic S.M.I. algebra over Φ.

(6) Throughout the proofs of Theorems 6 and 7 the plain $\sum$ sign denotes a direct sum.

irreducible right B-modules, $\mathfrak{M}B=\sum_i\mathfrak{U}_i$, *there is a corresponding splitting of* $B^*\mathfrak{N}$ *into irreducible left* B^**-modules*, $B^*\mathfrak{N}=\sum_i\mathfrak{V}_i$. *The* $\mathfrak{U}_i$ *and* $\mathfrak{V}_i$ *are subspaces of* $\mathfrak{M}$ *and* $\mathfrak{N}$ *respectively with the property that* $(\mathfrak{U}_i, \mathfrak{V}_j)=0, i\neq j$. *Furthermore* $\mathfrak{U}_i$ *and* $\mathfrak{V}_i$ *are dual spaces over* Φ, *and* B_i, *the algebra of l.t. which B induces on* $\mathfrak{U}_i$, *is precisely* $\mathcal{J}(\mathfrak{U}_i, \mathfrak{V}_i)$.

Proof. The proof will be carried out in a number of steps. For the relevant theorems on modules that are used cf. [1].

(a) *The module* $\mathfrak{U}_i$ *is a subspace of* $\mathfrak{M}$ *on which B induces a dense algebra of f.v.l.t.*

If x is in $\mathfrak{U}_i$, there is an element e in B such that $xe=x$. Hence for all α in Φ, $\alpha x=\alpha(xe)=x\alpha e$ is in $\mathfrak{U}_i$. Since $\mathfrak{U}_i$ is an irreducible B-module, B induces a dense ring of f.v.l.t. on $\mathfrak{U}_i$ as a vector space over the division ring of endomorphisms of $\mathfrak{U}_i$ commuting with B [5, Theorems 6 and 9]. But we just saw that this division ring contains Φ, and since it is known to be isomorphic to a ring of the form eBe, e an idempotent of B, it is algebraic over Φ. Thus this division ring coincides with Φ.

In particular since a dense ring of f.v.l.t. contains projections onto arbitrary subspaces there is, for every x in $\mathfrak{U}_i$, an e in B such that $\mathfrak{U}_ie=\{x\}$. Now if b is in B and $\{x_1, x_2, \cdots, x_n\}$ is a basis of $\mathfrak{M}b$, we know that for any z in $\mathfrak{M}$, $zb=\sum_j(z, f_j)x_j, f_j$ in $B^*\mathfrak{N}$. We may then break up the x_j according to the decomposition $\sum_i\mathfrak{U}_i$ of $\mathfrak{M}B$ and thus write $zb=\sum_k\sum_j(z, f_{jk})x_{jk}$, where the x_{jk} are in $\mathfrak{U}_k$, and the f_{jk} are in $B^*\mathfrak{N}$. By choosing a basis of $\mathfrak{M}e$ which contains the vector x and writing the x_{ik} as linear combinations of a set of independent vectors which includes x, we arrive at the following representation of e: For all z in $\mathfrak{M}B$ we may write $z=\sum_kz_k$, with z_k in $\mathfrak{U}_k$. Since the $\mathfrak{U}_k$ are invariant under the l.t. of B, it follows that for all z in $\mathfrak{M}B$, $ze=(z, f)x+u, f$ in $B^*\mathfrak{N}$, u in $\sum_{i\neq k}\mathfrak{U}_k$.

For the rest of the proof we restrict attention to the dual spaces $\mathfrak{M}B$ and $B^*\mathfrak{N}$(7). We shall consider annihilators of subspaces of $\mathfrak{M}B$ ($B^*\mathfrak{N}$) in $B^*\mathfrak{N}$ ($\mathfrak{M}B$) only. However we shall still use the standard notations, $\mathfrak{K}'$ is the annihilator in $B^*\mathfrak{N}$ of a subspace $\mathfrak{K}$ of $\mathfrak{M}B$. We recall that Mackey [10] has called a subspace $\mathfrak{K}$ closed if $\mathfrak{K}''=\mathfrak{K}$.

(b) *Any direct sum of irreducible right B-submodules of* $\mathfrak{M}B$ *is closed.*

Let $\mathfrak{M}B=\sum_i\mathfrak{U}_i$ be a splitting of $\mathfrak{M}B$ into irreducible submodules which contains the given direct sum. We first show that $\sum_{i\neq j}\mathfrak{U}_j$ is closed. To do this it is clearly sufficient to show that for a given x in $\mathfrak{U}_i$ there exists a vector f in $B^*\mathfrak{N}$ such that $(x, f)\neq 0$, but $(\sum_{i\neq j}\mathfrak{U}_j, f)=0$. But from (a) we have an idempotent e in B such that for all z in $\mathfrak{M}B$, $ze=(z, f)x+u$. Since $xe=x$, $(x, f)\neq 0$, and since $(\sum_{i\neq j}\mathfrak{U}_j)e\subset\sum_{i\neq j}\mathfrak{U}_j$, $(\sum_{i\neq j}\mathfrak{U}_j, f)=0$. Thus $\sum_{i\neq j}\mathfrak{U}_j$ is closed. But any direct sum of the $\mathfrak{U}_i$ can be obtained as an intersection of

(7) To see that $\mathfrak{M}B$ and $B^*\mathfrak{N}$ are dual spaces we note that $(\mathfrak{M}B)'=\mathfrak{T}(B^*)$, $(B^*\mathfrak{N})'=\mathfrak{T}(B)$, and apply Lemma 4.

these subspaces and so is closed too [10, Theorem III-1]. Of course this result is also true for direct sums of irreducible left B^*-submodules of $B^*\mathfrak{N}$.

We now define $\mathfrak{V}_i = (\sum_{i\neq j} \mathfrak{U}_j)'$. It is clear that the $\mathfrak{V}_i$ are subspaces of $B^*\mathfrak{N}$ and that they also are left B^*-modules. Furthermore if f is in $\mathfrak{V}_i \cap \sum_{i\neq j} \mathfrak{V}_j$, f annihilates $\mathfrak{M}B$ and so $f=0$. Thus the $\mathfrak{V}_i$ are independent.

(c) *The $\mathfrak{V}_i$ are irreducible left B^*-modules.*

Suppose that $\mathfrak{W}$ is an irreducible nonzero submodule of $\mathfrak{V}_i$. Then $\mathfrak{W}' \supset \mathfrak{V}_i' = \sum_{i\neq j} \mathfrak{U}_j$. Now $\mathfrak{M}B/\sum_{i\neq j} \mathfrak{U}_j$ is irreducible, so that if the inclusion relation were proper, $\mathfrak{W}'/\sum_{i\neq j} \mathfrak{U}_j = \mathfrak{M}B/\sum_{i\neq j} \mathfrak{U}_j$. But $\mathfrak{W}' \subset \mathfrak{M}B$ so that this would imply $\mathfrak{M}B = \mathfrak{W}'$. But then, since $\mathfrak{W}$ is closed, $\mathfrak{W}=0$. Thus $\mathfrak{W}' = \sum_{i\neq j} \mathfrak{U}_j$, and $\mathfrak{W}'' = \mathfrak{W} = \mathfrak{V}_i$.

(d) $B^*\mathfrak{N} = \sum_i \mathfrak{V}_i$.

Suppose that $\sum_i \mathfrak{V}_i$ did not fill $B^*\mathfrak{N}$. Then, since $\sum_i \mathfrak{V}_i$ is closed, $(\sum_i \mathfrak{V}_i)'$ would not be zero. But if $(x, \sum_i \mathfrak{V}_i) = 0$, $(x, \mathfrak{V}_i) = 0$ for all i. Thus $x=0$, a contradiction.

(e) *The spaces $\mathfrak{U}_i$ and $\mathfrak{V}_i$ are dual and $B_i = \mathcal{J}(\mathfrak{U}_i, \mathfrak{V}_i)$.*

If, for f in $\mathfrak{V}_i$, $(\mathfrak{U}_i, f) = 0$, then $(\mathfrak{M}B, f) = 0$, and so $f=0$. If, for x in $\mathfrak{U}_i$, $(x, \mathfrak{V}_i) = 0$, x is in $(\sum_{i\neq j} \mathfrak{U}_j)'' = \sum_{i\neq j} \mathfrak{U}_j$ and so $x=0$. As we already saw B_i is generated by l.t. of the form $z \rightarrow (z, f)x$, z, x in $\mathfrak{U}_i$, f in $\mathfrak{V}_i$, so that $B_i \subset \mathcal{J}(\mathfrak{U}_i, \mathfrak{V}_i)$. Now for any b in B, $zb = \sum_k \sum_j (z, f_{kj}) x_{kj}$ where the x_{kj} are in $\mathfrak{U}_k$, and the f_{kj} are in $B^*\mathfrak{N}$. Since $\mathfrak{U}_k B = \mathfrak{U}_k$, it is clear that the f_{kj} are in $\mathfrak{V}_k$, so that $B_i \supset \mathcal{J}(\mathfrak{U}_i, \mathfrak{V}_i)$.

This theorem depends heavily on the fact that B consists of finite-valued l.t. For if it held for a general S.M.I. algebra of l.t. on $\mathfrak{M}$, it would hold in particular for $B = \Phi$. This would imply the existence of bases $\{x_i\}$, $\{f_j\}$ for $\mathfrak{M}$ and $\mathfrak{N}$ such that $(x_i, f_j) = \delta_{ij}$. That this is not the case in general was shown in [1, p. 75].

Suppose now that $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$, $\mathfrak{M}$, $\mathfrak{N}$ dual spaces over a division ring D, and that B is an S.M.I. subring which induces a dense ring of f.v.l.t. on the subspace $\mathfrak{M}B$ over D. The proof of Theorem 6 is applicable here and shows that $B^*\mathfrak{N}$ is also irreducible and since $\mathfrak{M}B$ and $B^*\mathfrak{N}$ are dual spaces over D, we have in our previous notation $B = S(\mathfrak{M}B, B^*\mathfrak{N}) \cong \mathcal{J}(\mathfrak{M}B, B^*\mathfrak{N})$.

We can now state and prove the main theorem of this section.

THEOREM 7. *Let $\mathfrak{M}$, $\mathfrak{N}$ be dual spaces over an algebraically closed field Φ. Let $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$ and suppose that B and C are isomorphic simple subalgebras of A. Then the isomorphism between B and C can be extended to an inner automorphism of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ if:*

(i) *$A(B)$ and $A(C)$ are isomorphic;*

(ii) *The heights of B and C in A are equal;*

(iii) *B and C are acaudal in A, B^* and C^* are acaudal in A^**(8).

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(8) It will be seen in §5 that B may be caudal in A, while B^* is acaudal in A^*.

Proof. If B and C are of infinite dimension, it follows at once from Theorem 4 that $A(B)$ induces the algebra $\overline{A(B)}=\mathcal{J}(\mathfrak{T}(B), \mathfrak{T}(B^*))$ on $\mathfrak{T}(B)$ and that $A(C)$ induces the algebra $\overline{A(C)}=\mathcal{J}(\mathfrak{T}(C), \mathfrak{T}(C^*))$ on $\mathfrak{T}(C)$. Since $A(B)$ and $A(C)$ are isomorphic under a mapping ρ, say, this is also true for $\overline{A(B)}$ and $\overline{A(C)}$. Then an application of Théorème 5 and §20 of [1] combined with Theorem 19 of [5] yields an invertible linear transformation Q from $\mathfrak{T}(B)$ onto $\mathfrak{T}(C)$ with an adjoint Q^* from $\mathfrak{T}(C^*)$ to $\mathfrak{T}(B^*)$ such that

$$\rho(\bar{a}) = Q^{-1}\bar{a}Q \quad \text{and} \quad (yQ, g) = (y, Q^*g)$$

where $\bar{a}$ is the l.t. induced by a in $A(B)$ on $\mathfrak{T}(B)$, y is in $\mathfrak{T}(B)$, and g is in $\mathfrak{T}(C^*)$. Let the isomorphism between B and C be denoted by σ. Suppose that $\mathfrak{M}B=\sum_i\mathfrak{U}_i$, $B^*\mathfrak{N}=\sum_i\mathfrak{V}_i$; $\mathfrak{M}C=\sum_i\mathfrak{W}_i$, $C^*\mathfrak{N}=\sum_i\mathfrak{X}_i$, where i runs through the same index set for all four sums, and $\mathfrak{U}_i$, $\mathfrak{V}_i$ and $\mathfrak{W}_i$, $\mathfrak{X}_i$ have the properties described in Theorem 6. The indexing thus produces a pairing off of the irreducible submodules of $\mathfrak{M}B$ and $\mathfrak{M}C$. If B_i and C_i denote the algebras of l.t. that B and C induce on $\mathfrak{U}_i$ and $\mathfrak{W}_i$ respectively, B_i and C_i are also isomorphic under σ. For if b induces a l.t. b_i on $\mathfrak{U}_i$ and c induces a l.t. c_i on $\mathfrak{W}_i$, we simply define $\sigma(b_i)=[\sigma(b)]_i$. Just as before we obtain an invertible l.t. P_i from $\mathfrak{U}_i$ to $\mathfrak{W}_i$ with an adjoint P_i^* from $\mathfrak{X}_i$ onto $\mathfrak{V}_i$ such that

$$\sigma(b_i) = P_i^{-1}b_iP_i \quad \text{and} \quad (x_iP_i, f_i) = (x_i, P_i^*f_i)$$

where x_i is in $\mathfrak{U}_i$ and f_i is in $\mathfrak{X}_i$. Since $\mathfrak{M}=\mathfrak{M}B\oplus\mathfrak{T}(B)$, $\mathfrak{N}=C^*\mathfrak{N}\oplus\mathfrak{T}(C^*)$, any vector x in $\mathfrak{M}$ and any vector f in $\mathfrak{N}$ may be written as

$$x = \sum x_i + y, \quad x_i \text{ in } \mathfrak{U}_i,\ y \text{ in } \mathfrak{T}(B), \qquad f = \sum f_i + g,\ f_i \text{ in } \mathfrak{X}_i,\ g \text{ in } \mathfrak{T}(C^*).$$

We then define invertible l.t. P, P^* on $\mathfrak{M}$ and $\mathfrak{N}$ by

$$xP = \sum x_iP_i + yQ, \qquad P^*f = \sum P_i^*f_i + Q^*g.$$

Keeping in mind that $(\mathfrak{W}_i, \mathfrak{X}_j)=0$ for $i\neq j$ and that $(\mathfrak{M}C, \mathfrak{T}(C^*))=(\mathfrak{T}(C), C^*\mathfrak{N})=0$, we have $(xP, f)=(x, P^*f)$, so that P is in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$. Furthermore for all x in $\mathfrak{M}$ and b in B, $xP\sigma(b)P^{-1}=\sum x_iP_i\sigma(b_i)P_i^{-1}=xb$ so that $\sigma(b)=P^{-1}bP$. If a is in $A(B)$, $xP\rho(a)P^{-1}=yQ\rho(\bar{a})Q^{-1}=xa$, so that $P^{-1}aP=\rho(a)$.

If B and C are of finite dimension over Φ, $A(B)$ and $A(C)$ are larger than the annihilators of B and C in A. However, it can easily be seen from Theorem 3 that, in this case, the isomorphism between $A(B)$ and $A(C)$ induces an isomorphism of the corresponding annihilators. Thus the same proof is valid also in this case.

Now A is the socle of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ and so consists of all f.v.l.t. in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$, thus $PAP^{-1}\subset A$. But it is easily verified that any f.v.l.t. of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ lies in PAP^{-1} so that $PAP^{-1}=A$.

That restrictions as to the caudality of the subalgebras are indeed necessary for the validity of Theorem 7 will be seen from the following example:

Let $\mathfrak{M}=\{x_1\}\oplus\{x_2\}\oplus\cdots\oplus\{z\}$ over an algebraically closed field Φ.

Let A and B be defined as in the first example of §3 (with w deleted). Let C be the algebra of f.v.l.t. with a basis f_{ij} over Φ which coincides with the e_{ij} on $\{x_n\}$, but such that

$$zf_{ij} = 0 \text{ if } i \text{ is even}, \qquad zf_{ij} = x_j \text{ if } i \text{ is odd}.$$

Then it is easily verified that the f_{ij} also satisfy the matrix unit law of multiplication on z; and thus we see just as before that B and C are caudal in A with a tail of dimension one in both cases. Clearly B and C are isomorphic under the mapping $\sigma(e_{ij}) = f_{ij}$, and $\mathfrak{T}(B) = \mathfrak{T}(C) = 0$ so that $A(B) = A(C) = 0$. Moreover both B and C are isomorphic to the dense algebra of f.v.l.t. they induce on $\{x_n\}$ so that they are algebraic S.M.I. algebras, and the modules $\mathfrak{M}B$ and $\mathfrak{M}C$ are irreducible. Hence in this case all the conditions of Theorem 7 except (iii) are satisfied. Suppose however that there existed an invertible l.t. P on $\mathfrak{M}$ such that $\sigma(b) = P^{-1}bP$. Then

$$x_nP = x_ne_{nn}P = x_nPf_{nn} = \eta_nx_n \quad \text{where } \eta_n \neq 0 \text{ is in } \Phi.$$

Now let $zP^{-1} = \sum_1^m \alpha_ix_i + \alpha z$, α_i, α in Φ. Since P^{-1} maps $\{x_n\}$ onto $\{x_n\}$, and is an onto mapping of $\mathfrak{M}$, $\alpha \neq 0$. But if j is an even integer greater than m,

$$zP^{-1}e_{jj}P = \alpha x_jP = \alpha\eta_jx_j \neq 0 \text{ whereas } z\sigma(e_{jj}) = zf_{jj} = 0,$$

a contradiction. Thus no P of the desired kind exists in this case.

It should be noted that there is no overlapping between our theorem and Dieudonné's analogous results [2, Théorème 8]. For there the subalgebras are assumed to contain the center of the ring of all l.t. on $\mathfrak{M}$, and so if they are simple, the only f.v.l.t. they contain is the zero transformation.

In the finite-dimensional analogues of the theorems of §2–5 the subalgebras are always assumed to contain the unit of the larger algebra. If this requirement be dropped in this case, the only additional phenomenon that occurs is the two-sided annihilator, which as we have seen is fairly easy to handle and does not introduce anything strikingly different. The situation is quite the reverse in the infinite case. One might be tempted to substitute for the assumption about the unit the assumption that $\mathfrak{M}B = \mathfrak{M}$; but this would exclude the occurrence of a tail which gives rise to several new phenomena.

5. **Maximal ideals in S.M.I. rings.** If A is a simple ring with the d.c.c., it is well known that all the maximal right ideals of A are the subrings of A annihilating a given vector in the representation of A as the full ring of l.t. on a finite-dimensional vector space. We now wish to study the maximal one-sided ideals in an arbitrary S.M.I. ring $A = \mathcal{J}(\mathfrak{M}, \mathfrak{N})$, $\mathfrak{M}$, $\mathfrak{N}$ dual spaces over a division ring D. In that case as we saw in §3 all the left (right) ideals are of the form $S(\mathfrak{K}, \mathfrak{N})$ ($S(\mathfrak{M}, \mathfrak{L})$), and there is a lattice isomorphism between the subspaces $\mathfrak{K}$ ($\mathfrak{L}$) and the left (right) ideals [6, p. 15]. Thus the maximal left ideals of A will be all the subrings of the form $S(\mathfrak{K}, \mathfrak{N})$ where $\mathfrak{K}$ is a maximal subspace of $\mathfrak{M}$, i.e., one with a one-dimensional complement in $\mathfrak{M}$. Clearly

there are two cases to be considered, $\mathfrak{K}'\neq 0$ and $\mathfrak{K}'=0$.

Suppose first of all that $\mathfrak{K}'\neq 0$, then $\mathfrak{K}''=\mathfrak{K}$. For if it were larger than $\mathfrak{K}$, it would be all of $\mathfrak{M}$, and so $\mathfrak{M}$ and $\mathfrak{N}$ could not be dual spaces. Hence $\mathfrak{K}$ is closed in the sense of Mackey [10]. Theorem III-2 of [10] then yields the fact that $\mathfrak{K}'$ is one-dimensional, $\mathfrak{K}'=\{g\}$ say. Thus $\mathfrak{K}'\cap\mathfrak{N}=\{g\}\neq 0$, and by Lemma 5 the left ideal $J=S(\mathfrak{K}, \mathfrak{N})$ is a ring with radical R which is a zero ring. The ring $J-R$ is an S.M.I. ring isomorphic to $\mathcal{J}(\mathfrak{K}, \mathfrak{N}/\{g\})$. It is also easily verified that J^* can be characterized as the subring of A^* consisting of all l.t. which annihilate g, by using the fact that J consists of all l.t. of A whose range is in $\mathfrak{K}=\{g\}'$. Moreover, these maximal left ideals are the regular ones. We recall that a left ideal J is said to be regular if there is a right unit modulo J, i.e., an element e such that $ae-a$ is in J for all a. In fact we have the following lemma.

LEMMA 7. *The maximal left ideal $J=S(\mathfrak{K}, \mathfrak{N})$ is regular if and only if $\mathfrak{K}'\neq 0$.*

Proof. Suppose that $J=S(\mathfrak{K}, \mathfrak{N})$ is regular and let $\mathfrak{M}=\mathfrak{K}\oplus\{y\}$. Then since e cannot be in J, $ze=(z, g)y+x$, where x is in $\mathfrak{K}$. But, for all z in $\mathfrak{M}$ and a in A, $zae-za$ must lie in $\mathfrak{K}$. In particular if a is in J, $zae-za=(za, g)y+x'$, x' in $\mathfrak{K}$, must lie in $\mathfrak{K}$. But as z ranges over all of $\mathfrak{M}$ and a ranges over all of J, za ranges over all of $\mathfrak{K}$ so that $(\mathfrak{K}, g)=0$.

Conversely suppose that there is a g in $\mathfrak{N}$ such that $(\mathfrak{K}, g)=0$. If $\mathfrak{M}=\mathfrak{K}\oplus\{y\}$, define a l.t. e of A by $ze=(z, g)y/(y, g)$. Since $\mathfrak{K}$ is closed, $(y, g)\neq 0$ and so e is well defined. For a in A let $za=\alpha y+x$, x in $\mathfrak{K}$, then

$$z(ae-a) = (za, g)y/(y, g) - \alpha y - x = -x$$

so that $ae-a$ is in J for all a of A [9].

COROLLARY 2. *If every maximal left ideal of A is regular, $\mathfrak{N}=\mathfrak{M}^*$, the space of all linear functionals on $\mathfrak{M}$, and so A is the ring of all f.v.l.t. on $\mathfrak{M}$.*

Proof. Let f be an arbitrary functional on $\mathfrak{M}$. Then if $\mathfrak{K}$ is the null-space of f, $\mathfrak{K}$ is a maximal subspace of $\mathfrak{M}$, and so there is a regular maximal left ideal $J=S(\mathfrak{K}, \mathfrak{N})$ corresponding to it. But then there exists a vector g in $\mathfrak{N}$ such that $(\mathfrak{K}, g)=0$. Hence f is a scalar multiple of g and so is in $\mathfrak{N}$, $\mathfrak{N}=\mathfrak{M}^*$.

These results are fairly well known. However, they have only appeared as exercises in lecture notes by Jacobson and Kaplansky, so that it was thought worthwhile to reproduce them here for the sake of completeness.

We thus see that if A is not the complete ring of f.v.l.t. on $\mathfrak{M}$, maximal ideals $S(\mathfrak{K}, \mathfrak{N})$ with $\mathfrak{K}'=0$ must also exist. By Lemma 5 it follows immediately that these are S.M.I. rings isomorphic to the ring $\mathcal{J}(\mathfrak{K}, \mathfrak{N})$. Such ideals furnish examples of S.M.I. subrings B with B caudal in A but B^* acaudal in

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(9) Added in proof June 27, 1952. It can be shown that an arbitrary left ideal $S(\mathfrak{K}, \mathfrak{N})$ is regular if and only if $\mathfrak{K}$ is closed and the dimension of $\mathfrak{M}/\mathfrak{K}$ is finite.

A^*. For since $J^*\mathfrak{N}=\mathfrak{N}$, $\mathfrak{T}(J)=0$ and so any complement of $\mathfrak{K}$ is a tail for J in A. In case dim $\mathfrak{M}=\dim \mathfrak{N}=\aleph_0$ it is known [10, Lemma p. 171] that there is a basis $\{x_i\}$ for $\mathfrak{M}$ and a basis $\{f_j\}$ for $\mathfrak{N}$ such that $(x_i, f_j)=\delta_{ij}$; following Jacobson we shall refer to such a pair of bases as biorthogonal bases. Then if $\mathfrak{K}$ is a maximal nonclosed subspace of $\mathfrak{M}$, it follows by the same lemma that $\mathfrak{K}$ and $\mathfrak{N}$ have biorthogonal bases. Hence in this case all the nonregular maximal left ideals are isomorphic to A (10). We proceed now to give an example of a ring $\mathcal{J}(\mathfrak{M}, \mathfrak{N})=A$ where not all the nonregular maximal ideals are isomorphic. However we first prove the following lemma.

LEMMA 8. *Let $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{X}$, $\mathfrak{Y}$ be two pairs of dual spaces over the same division ring D, and let all four spaces have the same dimensions. Suppose that $\{x_i\}$ is a basis of $\mathfrak{M}$, $\{f_j\}$ a basis of $\mathfrak{N}$ such that $(x_i, f_j)=\gamma_{ij}$ in D. Then $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ is isomorphic to $\mathcal{J}(\mathfrak{X}, \mathfrak{Y})$ if and only if there exists a basis $\{x_i'\}$ of $\mathfrak{X}$ and a basis $\{f_j'\}$ of $\mathfrak{Y}$ such that $(x_i', f_j')=\gamma_{ij}$.*

Proof. Suppose that $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ is isomorphic to $\mathcal{J}(\mathfrak{X}, \mathfrak{Y})$. Then there exists an invertible l.t. P from $\mathfrak{M}$ to $\mathfrak{X}$ with an adjoint P^* from $\mathfrak{Y}$ to $\mathfrak{N}$ such that $P\mathcal{J}(\mathfrak{X}, \mathfrak{Y})P^{-1}=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. But then $\gamma_{ij}=(x_i, f_j)=(x_iP, P^{*-1}f_j)$ so that $x_i'=x_iP$, $f_j'=P^{*-1}f_j$ furnish a basis of the desired type. Conversely if such x_i' and f_j' exist, define P by $x_iP=x_i'$ and P^* by $P^*f_j'=f_j$, then $(x_iP, f_j')=(x_i, P^*f_j')$. But it is easily verified under those circumstances that $P\mathcal{J}(\mathfrak{X}, \mathfrak{Y})P^{-1}=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ so that the two rings are indeed isomorphic.

To obtain an example of two nonregular maximal left ideals which are not isomorphic we consider the following situation: Let $\mathfrak{M}=\{x_i\}$, $\mathfrak{N}=\{f_j\}$ be dual spaces over a division ring D with dim $\mathfrak{M}=\dim \mathfrak{N}>\aleph_0$ and $(x_i, f_j)=\delta_{ij}$. We well-order the index set (i) and we shall keep the same ordering throughout the discussion.

Consider the subspace $\mathfrak{K}$ of $\mathfrak{M}$ spanned by x_1-x_2, $x_2-x_3, \cdots, x_\omega$, $x_{\omega+1}, \cdots$, where ω is the first limit ordinal. Clearly $\mathfrak{M}=\mathfrak{K}\oplus\{x_1\}$, and if $(\mathfrak{K}, f)=0$ for $f=\sum f_i\alpha_i$, then $\alpha_i=0$ for $i\geqq\omega$ and $\alpha_1=\alpha_i$ for $i<\omega$. Since only a finite number of α_i are nonzero, $f=0$ and $\mathfrak{K}$ and $\mathfrak{N}$ are dual spaces. Thus $S(\mathfrak{K}, \mathfrak{N})$ is a maximal nonregular left ideal isomorphic to $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$, for $\{x_1-x_2, x_2-x_3, \cdots, x_\omega, x_{\omega+1}, \cdots\}$ and $\{f_1, f_1+f_2, f_1+f_2+f_3, \cdots, f_\omega, f_{\omega+1}, \cdots\}$ are biorthogonal bases for $\mathfrak{K}$ and $\mathfrak{N}$.

Now let $\mathfrak{K}_1$ be the subspace of $\mathfrak{M}$ spanned by all the vectors x_1-x_i, $i\neq 1$. Then $\mathfrak{M}=\mathfrak{K}_1\oplus\{x_1\}$. Now suppose that $(\mathfrak{K}_1, f)=0$ with $f=\sum f_i\alpha_i$, then $\alpha_1=\alpha_i$. Since the index set (i) is infinite, $f=0$. Thus $S(\mathfrak{K}_1, \mathfrak{N})$ is another nonregular maximal left ideal. By Lemma 8 those two left ideals are isomorphic if and only if there exist biorthogonal bases for $\mathfrak{K}_1$ and $\mathfrak{N}$. We shall show that this is impossible and so we have nonisomorphic ideals in this case.

Suppose there exist a basis $\{u_i\}$ of $\mathfrak{K}_1$ and a basis $\{g_j\}$ of $\mathfrak{N}$ such that $(u_i, g_j)=\delta_{ij}$. Then

(10) Cf. Lemma 8.

$$u_i = \sum_j \alpha_{ij}(x_1 - x_j) \quad \text{and} \quad f_j = \sum_h g_h\beta_{hj}$$

where the coefficients are in D. Since $(u_i, g_j)=\delta_{ij}$, we obtain the following relations by evaluating (u_i, f_j):

$$\text{(5.1)} \qquad \text{for } j = 1, \quad \sum_h \alpha_{ih} = \beta_{i1}; \qquad \text{for } j \neq 1, \quad \alpha_{ji} = -\beta_{ij}.$$

But for a fixed j only a finite number of β_{hj} are different from zero, so that there are only a finite number of vectors u_i with $\sum_j\alpha_{ij}\neq 0$, $u_{i_1}, u_{i_2}, \cdots, u_{i_m}$ say. Also for each x_1-x_j there are only a finite number of vectors u_i involving x_1-x_j with a nonzero coefficient when written in terms of the basis $\{x_1-x_j\}$ of $\mathfrak{K}_1$. We now show that these facts lead to a contradiction.

For any u_i, $i\neq i_1, \cdots, i_m$, i.e., a u_i such that $\sum_j \alpha_{ij}=0$, we pick the maximum h such that $\alpha_{ih}\neq 0$. Then $\alpha_{ih}=-\sum_{j<h}\alpha_{ij}$ and

$$u_i = \sum_j \alpha_{ij}(x_1 - x_j) = \sum_{j<h} \alpha_{ij}(x_h - x_j).$$

Thus all the u_i except a finite number lie in a subspace $\mathfrak{W}$ of $\mathfrak{K}_1$, spanned by certain of the vectors x_h-x_j, $h\neq 1, j\neq 1$. Now each x_1-x_j occurs in only a finite number of u_i by (5.1). Therefore for every given index j there are only a *finite* number of vectors x_h-x_j in $\mathfrak{W}$. Suppose now that we adjoin a vector $x_1-x_{h_0}$ to $\mathfrak{W}$. Then any vector x_1-x_j lying in $\mathfrak{W}+\{x_1-x_{h_0}\}$ must be of the form $x_1-x_j=x_1-x_{h_0}+\sum\pi_{ik}(x_i-x_k)$, where the sum runs over n terms, say, and at least one of the terms in it is of the type $x_{h_0}-x_k$. Since there are only a *finite* number of possibilities for such a term, for each n only a finite number of vectors x_1-x_j lies in $\mathfrak{W}+\{x_1-x_{h_0}\}$. Thus only a countable number of vectors x_1-x_j lie in $\mathfrak{W}+\{x_1-x_{h_0}\}$. Similarly if we adjoin a finite number of $x_1-x_{h_r}$ to $\mathfrak{W}$, all we get is a countable number of vectors x_1-x_j lying in the extended space. But since only a finite number of u_i do not lie in $\mathfrak{W}$, we have to get all of $\mathfrak{K}_1$ by adjoining only a finite number of vectors $x_1-x_{h_r}$. Hence $\mathfrak{K}_1$ would contain only a countable infinity of vectors x_1-x_j, a contradiction.

It is hardly necessary to state that all these results can be carried over to right ideals of A by passing to A^*.

6. **Equivalent subrings in the doubly countable case.** We shall call two subrings B and C of a ring $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ equivalent in A if there is an invertible element P in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ such that $P^{-1}BP=C$. We then have the following lemma.

LEMMA 9. *The rings $B=S(\mathfrak{K}, \mathfrak{L})$ and $C=S(\mathfrak{K}_1, \mathfrak{L}_1)$ are equivalent in A if and only if there is an invertible l.t. P in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ such that $\mathfrak{K}P=\mathfrak{K}_1$, $P^*\mathfrak{L}_1=\mathfrak{L}$.*

Proof. If the rings are equivalent in A, $\mathfrak{K}_1=\mathfrak{M}C=\mathfrak{M}P^{-1}BP=\mathfrak{K}P$ and $\mathfrak{L}=B^*\mathfrak{N}=P^*C^*P^{*-1}\mathfrak{N}=P^*\mathfrak{L}_1$. Conversely if such a l.t. exists and b is in B,

$\mathfrak{M}P^{-1}bP\subset\mathfrak{K}_1$, $P^{*-1}b^*P^*\mathfrak{N}\subset\mathfrak{L}_1$ so that $P^{-1}BP\subset C$. Similarly $PCP^{-1}\subset B$ and so $P^{-1}BP=C$.

Unfortunately the only case in which any headway has been made in determining for which pairs of subspaces such P exist is the one where both $\mathfrak{M}$ and $\mathfrak{N}$ are of countable dimension over the division ring D, and it is to this case that we shall restrict ourselves from now on.

THEOREM 8. *Let $\mathfrak{M}$, $\mathfrak{N}$ be a pair of dual spaces over a division ring D, with* $\dim \mathfrak{M}=\dim \mathfrak{N}=\aleph_0$. *Let $\mathfrak{K}$, $\mathfrak{K}_1$ be two subspaces of $\mathfrak{M}$ and $\mathfrak{L}$ a subspace of $\mathfrak{N}$ such that $\mathfrak{K}'=\mathfrak{K}_1'=\mathfrak{L}'=0$, and suppose* $\dim \mathfrak{M}/\mathfrak{K}=\dim \mathfrak{M}/\mathfrak{K}_1$. *Then there exists an invertible linear transformation P in $\mathcal{L}(\mathfrak{M},\mathfrak{N})$ such that $\mathfrak{K}P=\mathfrak{K}_1$, and $P^*\mathfrak{L}=\mathfrak{L}$. Conversely, of course, if such a l.t. exists and $\mathfrak{K}'=\mathfrak{K}_1'=0$,* $\dim \mathfrak{M}/\mathfrak{K}$ $=\dim \mathfrak{M}/\mathfrak{K}_1$.

Proof. The last statement is clear. We shall show how to build P by a stepwise process; for the idea of the proof cf. [9, Theorem 6].

We first note the following: If $\mathfrak{F}$ is a finite-dimensional subspace of $\mathfrak{M}$ of dimension n, it is closed and so $\mathfrak{F}'$ has deficiency n in $\mathfrak{N}$ [10, p. 163 and Theorem III-2]. Hence for a given basis $\{x_1, x_2, \cdots, x_n\}$ of $\mathfrak{F}$, we may write $\mathfrak{N}=\mathfrak{F}'\oplus\{f_1, f_2, \cdots, f_n\}$, $(x_i, f_j)=\delta_{ij}$. Here $\{f_j\}$ is a basis for any complement $\overline{\mathfrak{F}}$ of $\mathfrak{F}'$ in $\mathfrak{N}$. The well known theorem that finite-dimensional dual spaces admit biorthogonal bases is used here.

Now let $\mathfrak{F}$ and $\mathfrak{F}_1$ be two n-dimensional subspaces of $\mathfrak{M}$, n a finite integer. Suppose that the desired mapping has already been achieved for $\mathfrak{F}$ and $\mathfrak{F}_1$, and a pair of complements $\overline{\mathfrak{F}}$, $\overline{\mathfrak{F}}_1$ of $\mathfrak{F}'$ and $\mathfrak{F}_1'$ in $\mathfrak{N}$. That is, if $\mathfrak{F}=\{x_1, x_2, \cdots, x_n\}$, $\overline{\mathfrak{F}}=\{f_1, f_2, \cdots, f_n\}$, $(x_i, f_j)=\delta_{ij}$, there is a one-to-one l.t. P from $\mathfrak{F}$ onto $\mathfrak{F}_1$ with an adjoint P^* from $\overline{\mathfrak{F}}_1$ onto $\overline{\mathfrak{F}}$. The l.t. P carries $\mathfrak{K}\cap\mathfrak{F}$ onto $\mathfrak{K}_1\cap\mathfrak{F}_1$ and the l.t. P^* carries $\mathfrak{L}\cap\overline{\mathfrak{F}}_1$ onto $\mathfrak{L}\cap\overline{\mathfrak{F}}$. Let $y_i=x_iP$, $g_j=P^{*-1}f_j$ be bases of $\mathfrak{F}_1$ and $\overline{\mathfrak{F}}_1$; then clearly $(y_i, g_j)=\delta_{ij}$ also. We now extend the mapping by a process alternating between $\mathfrak{F}$ and $\mathfrak{F}_1$. We shall describe the process of adding one element of $\mathfrak{F}$ and one to $\overline{\mathfrak{F}}_1$. Let $\{z_t\}$ be a basis of $\mathfrak{M}$ and $\{h_s\}$ a basis of $\mathfrak{N}$ which contains a basis of $\mathfrak{L}$ as a subset. Suppose that z_m is the first basis element not in $\mathfrak{F}$, and h_k is the first basis element not in $\overline{\mathfrak{F}}_1$. We must then pick a vector w in $\mathfrak{M}$ and not in $\mathfrak{F}_1$ and a vector r not in $\overline{\mathfrak{F}}$, but in $\mathfrak{L}$ if h_k is in $\mathfrak{L}$, and not in $\mathfrak{L}$ if h_k is not in $\mathfrak{L}$ such that

$$(6.1)\qquad \begin{gathered}(w, g_j) = (z_m, f_j),\quad j = 1, \cdots, n,\qquad (y_i, h_k) = (x_i, r),\quad i = 1, \cdots, n,\\ (w, h_k) = (z_m, r).\end{gathered}$$

For if this is done, $z_mP=w$, $P^*h_k=r$ will clearly be an extension of our mapping with the desired properties. There are two cases to be considered.

Case I. The vector z_m is not in $\mathfrak{K}+\mathfrak{F}$. Then the w we pick must not be in $\mathfrak{K}_1+\mathfrak{F}_1$. But then $\mathfrak{K}+\mathfrak{F}\neq\mathfrak{M}$ and so

$$\dim (\mathfrak{K} + \mathfrak{F})/\mathfrak{K} = \dim \mathfrak{F}/\mathfrak{K} \cap \mathfrak{F} < \dim \mathfrak{M}/\mathfrak{K}.$$

But dim $\mathfrak{M}/\mathfrak{K} = $ dim $\mathfrak{M}/\mathfrak{K}_1$ and dim $\mathfrak{F}_1/\mathfrak{K}_1 \cap \mathfrak{F}_1 = $ dim $\mathfrak{F}/\mathfrak{K} \cap \mathfrak{F}$ and so

$$\dim (\mathfrak{K}_1 + \mathfrak{F}_1)/\mathfrak{K} = \dim \mathfrak{F}_1/\mathfrak{K}_1 \cap \mathfrak{F}_1 < \dim \mathfrak{M}/\mathfrak{K}_1,$$

so that $\mathfrak{K}_1 + \mathfrak{F}_1$ is a proper subspace of $\mathfrak{M}$ too.

We may then write $\mathfrak{M} = \mathfrak{Z} \oplus \mathfrak{F}_1$ where for all z in $\mathfrak{Z}$, $(z, g_j) = 0 = (z, \overline{\mathfrak{F}}_1)$. For if x is in $\mathfrak{M}$ and $(x, g_j) = \alpha_j$, α_j in D, then $(x - \sum \alpha_i y_i, g_j) = 0$. If $(z_m, f_j) = \beta_j$ let $w = \sum \beta_j y_j + z$ for z in $\mathfrak{Z}$. Then $(w, g_j) = (z_m, f_j)$ and we can certainly find a z such that w is not in $\mathfrak{K}_1 + \mathfrak{F}_1$. For if this were not the case, $\mathfrak{Z} \subset \mathfrak{K}_1 + \mathfrak{F}_1$ and so $\mathfrak{M} = \mathfrak{K}_1 + \mathfrak{F}_1$.

We now show how r may be picked. Suppose first that h_k is not in $\mathfrak{L}$, so that we wish to obtain an r that is not in $\mathfrak{L}$ either. If $(y_i, h_k) = \gamma_i$, $(x_i, \sum f_j \gamma_j) = \gamma_i$. But $\mathfrak{F}'' = \mathfrak{F}$ so that there is a vector t in $\mathfrak{F}'$ such that $(z_m, t) \neq 0$. Setting $(z_m, \sum f_j \gamma_j + t\xi) = (w, h_k)$, we can solve for ξ since $(z_m, t) \neq 0$. If $(z_m, \sum f_j \gamma_j) \neq (w, h_k)$, $\sum f_j \gamma_j + t\xi = r_0$ will be a vector fulfilling the conditions of r in (6.1), but not in $\mathfrak{F}$. If $(z_m, \sum f_j \gamma_j) = (w, h_k)$, since $\mathfrak{F} + \{z_m\}$ has dimension $n+1$, there is an s in $\mathfrak{N}$ such that $(\mathfrak{F} + \{z_m\}, s) = 0$, and then $\sum f_j \gamma_j + s = r_0$ will have the same properties as the first r_0. If r_0 does not happen to lie in $\mathfrak{L}$, we let $r = r_0$.

Suppose now that r_0 were in $\mathfrak{L}$. We claim that there is a vector u in the annihilator of $\mathfrak{F} + \{z_m\}$ in $\mathfrak{N}$ which is not in $\mathfrak{L}$. Once this is established $r = r_0 + u$ will fulfill all our requirements for r. Now suppose $\mathfrak{H}$ were the annihilator of $\mathfrak{F} + \{z_m\}$ both in $\mathfrak{N}$ and in $\mathfrak{L}$. Since $\mathfrak{L}$ is dual to $\mathfrak{M}$, dim $\mathfrak{L}/\mathfrak{H} = $ dim $\mathfrak{N}/\mathfrak{H} = n+1$. But then dim $\mathfrak{N}/\mathfrak{H} \div$ dim $\mathfrak{L}/\mathfrak{H} = 1$ and so $\mathfrak{L}$ would not be a proper subspace of $\mathfrak{N}$. Thus r can always be picked to fulfill our requirements.

If h_k is in $\mathfrak{L}$, however, since $\mathfrak{M}$ and $\mathfrak{L}$ are dual spaces, the whole argument leading to r_0 in $\mathfrak{N}$ can be repeated with $\mathfrak{N}$ replaced by $\mathfrak{L}$. (That is, there are vectors f_j' in $\mathfrak{L}$ such that $(x_i, f_j') = \delta_{ij}$ and $\{f_j'\} \supset \mathfrak{L} \cap \overline{\mathfrak{F}}$, etc.) We are thus led to an r_0 in $\mathfrak{L}$ but not in $\{f_j'\}$, and so not in $\overline{\mathfrak{F}}$, which we choose as r.

Case II. The vector z_n is in $\mathfrak{K} + \mathfrak{F}$. We must now choose w to lie in $\mathfrak{K}_1$ but not in $\mathfrak{K}_1 \cap \mathfrak{F}_1$. But $\mathfrak{K}_1$ and $\mathfrak{N}$ are dual spaces and the g_j are n linearly independent forms, hence by a well known theorem of linear algebra there is a vector v in $\mathfrak{K}_1$ such that

$$(v, g_j) = (z_m, f_j), \qquad j = 1, \cdots, n.$$

If v is not in $\mathfrak{F}_1$, we let $v = w$. Suppose, however, that v lies in $\mathfrak{F}_1$. As we shall presently see, $\mathfrak{K}_1 \cap \mathfrak{Z} \neq 0$, so that there is a $z \neq 0$ in $\mathfrak{K}_1 \cap \mathfrak{Z}$. Then $(v+z, g_j) = (v, g_j)$ and since $\mathfrak{Z} \cap \mathfrak{F}_1 = 0$, $w = v + z$ is in $\mathfrak{K}_1$ but not in $\mathfrak{K}_1 \cap \mathfrak{F}_1$. It remains to show that $\mathfrak{K}_1 \cap \mathfrak{Z} \neq 0$. If $\mathfrak{K}_1 \cap \mathfrak{Z}$ were zero, dim $\mathfrak{K}_1 \leqq$ dim $\mathfrak{M}/\mathfrak{Z}$. Since dim $\mathfrak{M}/\mathfrak{Z} = $ dim $\mathfrak{F}_1 < \infty$, the dimension of $\mathfrak{K}_1$ would then also be finite which contradicts the fact that $\mathfrak{K}_1' = 0$.

For this case the vector r is picked just as before.

Now, in either case, for the induction to go through we must also be sure that the spaces $\mathfrak{F}_1 + \{w\}$ and $\overline{\mathfrak{F}}_1 + \{h_k\}$ are dual. A straightforward computation shows that they will fail to do so if and only if $(w-\sum(w, g_i)y_i, h_k)=0$. If this condition is fulfilled, we modify w as follows: There is a vector q in $\mathfrak{M}$ such that $(q, g_i)=0$, $i=1, 2, \cdots, n$, but $(q, h_k)\neq 0$. If $w+q$ lies in $\mathfrak{F}_1$ we add a vector annihilating all the g_i's and h_k to it to obtain our new w. This new w has the following properties: the spaces $\mathfrak{F}_1+\{w\}$ and $\overline{\mathfrak{F}}_1+\{h_k\}$ are dual, it does not lie in $\mathfrak{F}_1$, and $(w, g_j)=(z_m, f_j)$. Using this modified w, r is then picked as before. Thus the pairs of spaces $\mathfrak{F}_1+\{w\}$ and $\overline{\mathfrak{F}}_1+\{h_k\}$; $\mathfrak{F}+\{z_m\}$ and $\overline{\mathfrak{F}}+\{r\}$ are dual.

In either case let $\mathfrak{G}$ be spanned by $\mathfrak{F}$ and z_m, $\mathfrak{G}_1$ by $\mathfrak{F}_1$ and w. Then P extends to a one-to-one mapping of $\mathfrak{G}$ onto $\mathfrak{G}_1$ mapping $\mathfrak{G}\cap\mathfrak{K}$ onto $\mathfrak{G}_1\cap\mathfrak{K}_1$, with an adjoint on $\overline{\mathfrak{F}}_1+\{h_k\}$ to $\overline{\mathfrak{F}}+\{r\}$ mapping the respective intersections with $\mathfrak{L}$ onto each other. In this manner by reversing the role of $\mathfrak{F}$ and $\mathfrak{F}_1$ at the next step we build the required l.t. P. The same theorem of course holds for two subspaces of $\mathfrak{N}$ and one subspace of $\mathfrak{M}$.

COROLLARY 3. *Let* $\mathfrak{K}$, $\mathfrak{K}_1$ *be subspaces of* $\mathfrak{M}$, $\mathfrak{L}$, $\mathfrak{L}_1$ *be subspaces of* $\mathfrak{N}$, *such that* $\mathfrak{K}'=\mathfrak{K}_1'=\mathfrak{L}'=\mathfrak{L}_1'=0$. *Suppose further that* dim $\mathfrak{M}/\mathfrak{K}=$ dim $\mathfrak{M}/\mathfrak{K}_1$ *and that* dim $\mathfrak{N}/\mathfrak{L}=$ dim $\mathfrak{N}/\mathfrak{L}_1$. *Then there exists an invertible l.t. P in* $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ *such that* $\mathfrak{K}P=\mathfrak{K}_1$, $P^*\mathfrak{L}_1=\mathfrak{L}$. *Conversely if such a P exists and* $\mathfrak{K}_1'=\mathfrak{K}'=\mathfrak{L}'=\mathfrak{L}_1'=0$, dim $\mathfrak{M}/\mathfrak{K}=$ dim $\mathfrak{M}/\mathfrak{K}_1$ *and* dim $\mathfrak{N}/\mathfrak{L}=$ dim $\mathfrak{N}/\mathfrak{L}_1$.

Proof. By Theorem 8 we can find invertible l.t. Q and R in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ such that $\mathfrak{K}Q=\mathfrak{K}_1$, $Q^*\mathfrak{L}=\mathfrak{L}$; $\mathfrak{K}_1R=\mathfrak{K}_1$, $R^*\mathfrak{L}_1=\mathfrak{L}$. But then $P=QR$ is the required l.t.

We are now in a position to classify any two subspaces of $\mathfrak{M}$, or $\mathfrak{N}$. In fact if $\mathfrak{K}$ is any subspace of $\mathfrak{M}$, and we call the three cardinals

$$\dim \mathfrak{K}, \qquad \dim \mathfrak{K}''/\mathfrak{K}, \qquad \dim \mathfrak{K}'$$

the *invariants* of $\mathfrak{K}$ in $\mathfrak{M}$, we have the following corollary.

COROLLARY 4. *If* $\mathfrak{K}$, $\mathfrak{K}_1$ *are two subspaces of* $\mathfrak{M}$, *a necessary and sufficient condition for an invertible l.t. in* $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ *carrying* $\mathfrak{K}$ *onto* $\mathfrak{K}_1$ *to exist is that the invariants of* $\mathfrak{K}$ *and* $\mathfrak{K}_1$ *in* $\mathfrak{M}$ *be the same.*

It should be noted that if $\mathfrak{K}'=0$, dim $\mathfrak{K}=\aleph_0$ and dim $\mathfrak{K}''/\mathfrak{K}=$ dim $\mathfrak{M}/\mathfrak{K}$, so that in this case there is only the one invariant, dim $\mathfrak{M}/\mathfrak{K}$, of $\mathfrak{K}$ in $\mathfrak{M}$.

Proof of Corollary 4. The necessity of the condition is obvious. For the proof of the sufficiency we proceed as follows. By Lemma 1 of [11] we may write

$$\mathfrak{M} = \mathfrak{K}'' \oplus \mathfrak{W} = \mathfrak{K}_1'' \oplus \mathfrak{W}_1, \qquad \mathfrak{N} = \mathfrak{K}' \oplus \mathfrak{W}' = \mathfrak{K}_1' \oplus \mathfrak{W}_1'.$$

It is easily seen that $\mathfrak{K}''$, $\mathfrak{W}'$; $\mathfrak{K}_1''$, $\mathfrak{W}_1'$; $\mathfrak{W}_1$, $\mathfrak{K}_1'$; and $\mathfrak{W}$, $\mathfrak{K}'$ are pairs of dual spaces. Now if $(\mathfrak{K}, h)=0$, h is in $\mathfrak{K}'$ and so $\mathfrak{K}$ has no annihilator in $\mathfrak{W}'$. Since dim $\mathfrak{K}''/\mathfrak{K}=$ dim $\mathfrak{K}_1''/\mathfrak{K}_1$ by Theorem 8 (with the dual spaces $\mathfrak{K}''$, $\mathfrak{W}'$ and $\mathfrak{K}_1''$, $\mathfrak{W}_1'$ identified) there is an invertible l.t. R of $\mathfrak{K}''$ onto $\mathfrak{K}_1''$ carrying $\mathfrak{K}$

onto $\mathfrak{K}_1$ with an adjoint R^* mapping $\mathfrak{W}_1'$ onto $\mathfrak{W}'$. Since dim $\mathfrak{K}'=$dim $\mathfrak{M}/\mathfrak{K}''$ $=$dim $\mathfrak{W}=$dim $\mathfrak{K}_1'=$dim $\mathfrak{W}_1=1, 2, \cdots, \aleph_0$, we can find biorthogonal bases of the same cardinality for $\mathfrak{W}$ and $\mathfrak{K}'$ and for $\mathfrak{W}_1$ and $\mathfrak{K}_1'$. Thus by mapping these onto each other we get an invertible l.t. Q from $\mathfrak{W}$ onto $\mathfrak{W}_1$ with an adjoint Q^* from $\mathfrak{K}_1'$ onto $\mathfrak{K}'$. Now for any x in $\mathfrak{M}$ and any f in $\mathfrak{N}$ we may write:

$$x = x_1 + x_2,\ x_1 \text{ in } \mathfrak{K}'',\ x_2 \text{ in } \mathfrak{W}; \qquad f = f_1 + f_2,\ f_1 \text{ in } \mathfrak{W}_1',\ f_2 \text{ in } \mathfrak{K}_1'.$$

If we then define P and P^* by $xP=x_1R+x_2Q$, $P^*f=R^*f_1+Q^*f_2$, it is readily verified that $(xP, f)=(x, P^*f)$, so that P is an invertible l.t. in $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$ carrying $\mathfrak{K}$ onto $\mathfrak{K}_1$. Of course the same result holds for subspaces of $\mathfrak{N}$.

That the countability assumptions of Theorem 8 cannot be dropped may be seen from the example in §5, for there the subspaces satisfy all the requirements of Theorem 8 but cannot be carried into each other by an invertible element of $\mathcal{L}(\mathfrak{M}, \mathfrak{N})$.

These last results may be restated in terms of ideals and subrings of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. Since we know that all left (right) ideals of $A=\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ are of the form $S(\mathfrak{K}, \mathfrak{N})$ ($S(\mathfrak{M}, \mathfrak{L})$), Lemma 9 in conjunction with Corollary 4 yields the following theorem.

THEOREM 9. *Two left (right) ideals in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$, with $\mathfrak{M}$ and $\mathfrak{N}$ dual spaces of countably infinite dimension over the division ring D, are equivalent in $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$ if and only if their ranges in $\mathfrak{M}$ (in $\mathfrak{N}$) have the same invariants.*

Corollary 3 yields information on the equivalence of subrings of the form $S(\mathfrak{K}, \mathfrak{L})$ with $\mathfrak{K}'=\mathfrak{L}'=0$. Since then $B=S(\mathfrak{K}, \mathfrak{L})$ is an S.M.I. subring with $\mathfrak{T}(B)=\mathfrak{T}(B^*)=0$, the remarks subsequent to Theorem 6 combined with the results of this section give us the following theorem.

THEOREM 10. *Let $\mathfrak{M}$, $\mathfrak{N}$ be dual spaces over a division ring D, with* dim $\mathfrak{M}$ $=$dim $\mathfrak{N}=\aleph_0$. *Let B and C be two S.M.I. subrings of $\mathcal{J}(\mathfrak{M}, \mathfrak{N})$. Suppose that B and C induce dense rings of f.v.l.t. on their respective ranges and that $\mathfrak{T}(B)$ $=\mathfrak{T}(B^*)=\mathfrak{T}(C)=\mathfrak{T}(C^*)=0$. Then B and C are equivalent in A if and only if the invariants of $\mathfrak{M}B$ and $\mathfrak{M}C$ in $\mathfrak{M}$ are the same and the invariants of $B^*\mathfrak{N}$ and $C^*\mathfrak{N}$ in $\mathfrak{N}$ are the same.*

It is easy to construct examples where $\mathfrak{T}(B)=0$, $\mathfrak{T}(B^*)\neq 0$, so that these assumptions are independent.

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