# PAIRS OF MATRICES WITH PROPERTY L[1]

BY

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1. It was proved by Frobenius [1] that any function $f(A, B)$ of two commutative $n \times n$ square matrices $A$, $B$ has as characteristic roots the numbers $f(\lambda_i, \mu_i)$ where $\lambda_i$ are the characteristic roots of $A$ and $\mu_i$ the characteristic roots of $B$, both taken in a special ordering independent of the function $f$. However, commutativity of $A$ and $B$ is known not to be a necessary condition [2]. Matrices which have the above property are said to have property P. Recently M. Kac suggested a study of matrices $A$, $B$ of a less restricted nature, namely those for which any linear combination $\alpha A+\beta B$ has as characteristic roots the numbers $\alpha\lambda_i+\beta\mu_i$. In general such matrices do not have property P. This paper is concerned with the study of such matrices which will be called matrices with property L (for linear).

It will be shown, confirming a conjecture of M. Kac, that hermitian matrices with property L are commutative. If $n=2$, property L implies property P.

The paper, being concerned with pairs of matrices, concludes with an enumeration of seven properties of pairs of matrices. Each of these properties is implied by its successor. It is shown that already for matrices with $n=3$ these properties are in general nonequivalent.

Wherever there is no mention of hermitian matrices the results hold for matrices whose elements instead of being complex numbers belong to an algebraically closed field with arbitrary characteristic.

2. Theorem 1. *Let the n-rowed square matrices A and B have property* L. *Let t be the number of different characteristic roots of A and assume that all the characteristic roots $\lambda_i$ of A are arranged in sets of equal ones. Let $m_i$ be the multiplicity of the characteristic root $\lambda_i$ of A and assume that there are $m_i$ independent characteristic vectors corresponding to each $\lambda_i$. Let $\mu_i$ be the corresponding characteristic roots of B. Let $B^*=P^{-1}BP$ where $A^*=P^{-1}AP$ is in Jordan normal form. Then*

$$B^* = \begin{pmatrix} B_{11} & B_{12} \cdots B_{1t} \\ B_{21} & B_{22} \cdots B_{2t} \\ \cdot \; \cdot \; \cdot & \cdot \; \cdot \; \cdot \; \cdot \\ B_{t1} & B_{t2} \cdots B_{tt} \end{pmatrix}$$

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*where $B_{ii}$ is an $m_i$-rowed square matrix $(i=1, \cdots, t)$ and*

$$(1) \qquad |xI - B^*| = \prod_{i=1}^{t} |xI - B_{ii}|.$$

*Furthermore we have*

$$(2) \qquad \sum b_{ik}^* b_{ki}^* = 0,$$

*where the summation is over all $i<k$ with $(i, k)$ outside of every $B_{jj}$ $(j=1, \cdots, t)$.*

**Proof.** To prove (1) assume $\lambda_1=0$ because property L is not affected by a translation. Write then

$$A^* = \begin{pmatrix} 0 & 0 \\ 0 & A_{11} \end{pmatrix}, \qquad B^* = \begin{pmatrix} B_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where $A_{11}$ is nonsingular and has $n-m_1$ rows and $B_{11}$ is an $m_1$-rowed square matrix. By property L it follows that

$$|xI - \alpha A^* - B^*| = \prod_{i=1}^{m_1} (x - \mu_i) \prod_{j=m_1+1}^{n} (x - \alpha\lambda_j - \mu_j)$$

for all $\alpha$. Equating the coefficients of the $(n-m_1)$th powers of $\alpha$ on the two sides of this equation gives

$$|xI - B_{11}|\,|A_1| = \prod_{i=1}^{m_1} (x - \mu_i) \prod_{j=m_1+1}^{n} \lambda_j.$$

Since $A_1$ is nonsingular,

$$|xI - B_{11}| = \prod_{i=1}^{m_1} (x - \mu_i).$$

Applying this argument to each $\lambda_i$ in turn, we obtain (1).

Statement (2) follows by equating the coefficients of $x^{n-2}$ on the two sides of (1). On the left-hand side, this coefficient is the sum of all the principal two-rowed minors of $B^*$. On the right-hand side $x^{n-2}$ is obtained in two ways. We may suppress two factors $x$ in the same determinant $|xI-B_{ii}|$ or we may suppress one factor $x$ in each of two different determinants $|xI-B_{ii}|$ and $|xI-B_{kk}|$ $(i\neq k)$. In the first case we obtain the sum of all the principal two-rowed minors of $B_{ii}$. In the second case we obtain the sum of all products of a diagonal element of $B_{ii}$ by a diagonal element of $B_{kk}$ $(i\neq k)$. Thus the coefficient of $x^{n-2}$ on the right in (1) agrees with that on the left except for the sum whose vanishing is asserted by (2).

THEOREM 2. *If $A$ and $B$ are hermitian and have property* L, *then $AB=BA$.*

**Proof.** We may take $P$ unitary so that $B^*$ is also hermitian. Further it is well known that an hermitian matrix can be transformed to diagonal form and hence each $\lambda_i$ has $m_i$ corresponding independent characteristic vectors. If we use (2), it follows that the matrices $B_{ik}$ $(i\neq k)$ are zero. Thus $A^*B^* = B^*A^*$, and hence also $AB=BA$.

THEOREM 3. *Let $n=2$ and $A$ and $B$ have property* L. *Then $A$ and $B$ have property* P.

**Proof.** There are three different cases. (i) $A$ is scalar. In this case $AB=BA$. (ii) $A$ has two simple roots. In this case it follows from Theorem 1 that $b_{12}^*b_{21}^*=0$ and hence that $B^*$ is triangular.

(iii)
$$A^* = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Since the matrix $(xI - A^* - \beta B^*)$ has the characteristic root $\alpha\lambda + \beta\mu_1$, it follows that

$$\begin{vmatrix} \beta(b_{11}^* - \mu_1) & \alpha + \beta b_{12}^* \\ \beta b_{21}^* & b_{22}^* - \mu_1 \end{vmatrix} = 0$$

for all values of $\alpha$, hence $b_{21}^*=0$. In all three cases it follows that $A$ and $B$ can be transformed simultaneously to upper triangular form, hence they have property P (see [2]).

Theorem 1 does not apply to matrices with property L if $A$ cannot be transformed to diagonal form. As an example consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma & 1 \\ 0 & 0 & 1-\gamma \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \gamma^{-1}-1 & 0 & 0 \end{pmatrix}, \qquad \gamma\neq 1.$$

They have property L, but the submatrix of $B$ which corresponds to $\lambda_2=\lambda_3 = 1-\gamma$ does not have the roots $\mu_2=\gamma^{1/2}$, $\mu_3=-\gamma^{1/2}$. Examples for $n>3$ are obtained by adjoining zeros.

3. **Seven properties of pairs of matrices.** In the previous sections certain properties of pairs of matrices have been linked together. It may be of interest to study the logical dependency of these and some other properties of pairs of matrices.

1. $A$, $B$ are 1-linear if at least one of the characteristic roots of $\alpha A+\beta B$ ($\alpha$, $\beta$ any constants) is of the form $\alpha\lambda_i+\beta\mu_i$ for a fixed value of $i$.
2. Property L.
3. Property P. This is equivalent to the property that $A$, $B$ can be transformed simultaneously to triangular form by a similarity transformation (see [2]). Such matrices will be called co-triangular.

4. Quasi-commutativity (see [3]), or even more general $k$-commutativity (see [4]).

5. Commutativity.

6. $A$, $B$ are co-functional, i.e., they are both functions of a third matrix(²).

7. $A$, $B$ are co-canonical, i.e., they can be transformed by the same similarity transformation, into Jordan normal forms with 1's at the same places above the main diagonal.

It is well known that in this sequence of properties each implies the preceding one, except perhaps in the case of properties 6 and 7, where it will now be proved.

We shall establish that two co-canonical matrices $A$, $B$ are both functions of the same matrix. Construct a matrix $M$ in normal form with 1's in the same places as the normal forms of $A$ and $B$. Assume that each coherent block in $M$ has different characteristic roots from all the others. It will now be shown that the normal form of $A$ (and $B$) is a polynomial in $M$. For each block $P_i$ of $M$ construct a polynomial $p_i$ such that $p_i(M)$ gives the unit matrix for the block $P_i$ and the zero matrix for all the other blocks. The polynomial $p_i(x)$ must be divisible by the product of the characteristic equations of all the blocks not equal to $P_i$, however $p_i(x)-1$ must be divisible by the characteristic equation of $P_i$. Such a polynomial exists since the block $P_i$ has no characteristic root in common with the other blocks.

Further it is easily seen that the two matrices

$$\begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \cdot & & \\ & & & \cdot & 1 \\ & & & & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & 1 & & & \\ & b & 1 & & \\ & & \cdot & & \\ & & & \cdot & 1 \\ & & & & b \end{pmatrix}$$

are linear functions one of the other. Let then $k_i(x)$ be the linear polynomial which transforms the block $P_i$ of $M$ into the corresponding block in the normal form of $A$. It follows that the normal form of $A$ can be expressed in the form $\sum k_i(M)p_i(M)$.

7. If $A$ and $B$ are both hermitian, then property L implies all succeeding properties. This follows from Theorem 2 (see also [5]).

If $n=2$, then property 1 implies 3 (by Theorem 3), but not 4. Property 4 implies 7. We next show:

THEOREM 4. *For $n=3$ no two of the seven properties are in general equivalent.*

To show that property 1 does not imply property 2 consider the two matrices

(²) There is no difference in the scope of this definition if "function" is interpreted as polynomial, rational function, or power series.

$$A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & b_3 & 0 \end{pmatrix}, \qquad a_1a_2a_3 \neq 0,\ b_1b_2b_3 \neq 0.$$

It is clear that $\alpha A + \beta B$ has $\alpha a_1 + \beta b_1$ as one of its roots. However, the matrices

$$\begin{pmatrix} a_2 & 0 \\ 0 & a_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}$$

do not possess property L, hence $A$ and $B$ cannot have it either.

Property 2 does not imply property 3 as can be seen from the following example:

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

That these matrices do not possess property P can be checked easily since the characteristic roots of $B$ lie on the main diagonal of $B$: the matrix $AB$ does not in this case have the characteristic roots $\lambda_i\mu_i$. The reason for this occurrence lies in the fact that, in general, two matrices with property L and $n > 2$ cannot be transformed to triangular form simultaneously. This can be shown in the following way. Assume $A$ to have all its roots simple and to be transformed to canonical form. Assume all the elements of $B$ to be different from zero—that this can happen for $n > 2$ is clear from the example mentioned above. For characteristic 2 choose

$$A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & \epsilon \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ \epsilon & \epsilon^2 & 1 \end{pmatrix}, \qquad \epsilon + \epsilon^2 = 1.$$

Let $S$ be a matrix which transforms both $A$ and $B$ to upper (lower) triangular form. We shall show that $S$ is the product of an upper (lower) triangular and a permutation matrix. However, this leads to a contradiction with the assumption that $B$ has all its elements not equal to 0. Let $S = (s_{ik})$ and $S^{-1}AS = T$ where $T = (t_{ik})$ and $t_{ik} = 0$ if $i > k$. Since $A$ is a principal diagonal matrix

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

we have

$$s_{nk}\lambda_k = t_{nn}s_{nk}, \qquad k = 1, \cdots, n.$$

From the fact that the $\lambda_i$ are all different, it follows that $S_{nk_0} \neq 0$ for some values $k_0$, but $s_{nk}=0$ for all $k \neq k_0$. Similarly

$$S_{n-1,k}\lambda_k = t_{n-1,n-1}S_{n-1,k} + t_{n-1,n}S_{nk}, \qquad k = 1, \cdots, n.$$

Hence $s_{n-1,k_1} \neq 0$ for some value $k_1$, but $S_{n-1,k}=0$ for all $k \neq k_1$ with the possible exception of $k_0$. Continuing this argument it follows that $S$ is the product of an upper triangular and a permutation matrix.

Another example, valid for every characteristic, is found in [6]. There it is noted that the matrices

$$\alpha A + \beta B, \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

have all their characteristic roots 0. However

$$AB = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

has nonzero roots.

Property L does, however, imply a certain part of property P, namely, that for nonsingular $A$ the matrix $A^{-1} B$ has as characteristic roots $\lambda_i^{-1}\mu_i$. This follows if we consider the matrix $\lambda_i^{-1}\mu_i A - B$. By the property L this matrix has 0 as a characteristic root which means that the determinant

$$|\lambda_i^{-1}\mu_i A - B| = |A| \, |\lambda_i^{-1}\mu_i I - A^{-1}B| = 0.$$

Likewise $B^{-1}A$ has as characteristic roots $\mu_i^{-1}\lambda_i$ if $B$ is nonsingular.

Property 3 does not imply property 4 even for $n=2$.

Property 4 does not imply property 5 for $n>2$ (see [3]).

That property 5 does not imply property 6 is well known, see e.g. [7].

It will now be proved that property 6 does not imply property 7; consider the matrices

$$A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad B = A^2 = \begin{pmatrix} a^2 & 2a & 1 \\ 0 & a^2 & 2a \\ 0 & 0 & a^2 \end{pmatrix}.$$

They are evidently co-functional, $A^2$ being a function of $A$. The matrix $A$ is already in Jordan normal form, $A^2$ is not. Any matrix $S$ which transforms $A$ and $A^2$ into normal form simultaneously would have to leave $A$ unaltered, i.e., $A$ and $S$ are commutative. In this case $A^2$ and $S$ will be commutative and so $S$ would not transform $A^2$ into canonical form.

## Bibliography

1. G. Frobenius, *Über vertauschbare Matrizen*, Preuss. Akad. Wiss. Sitzungsber. (1896) pp. 601–604.

2. N. H. McCoy, *On the characteristic roots of matric polynomials*, Bull. Amer. Math. Soc. (1936) pp. 592–600.

3. N. H. McCoy, *On quasi-commutative matrices*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 327–380.

4. W. E. Roth, *On k-commutative matrices*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 483–495.

5. M. P. Drazin, J. W. Dungey, and K. W. Gruenberg, *On commutative matrices*, J. London Math. Soc. vol. 26 (1951) pp. 221–228.

6. H. Wielandt, *Lineare Scharen von Matrizen mit reellen Eigenwerten*, Math. Zeit. vol. 53 (1950) pp. 219–225.

7. H. B. Phillips, *Functions of matrices*, Amer. J. Math. vol. 41 (1919) pp. 266–278.

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