GALOIS THEORY OF SIMPLE RINGS

BY

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Introduction. The outer Galois theory, started by Jacobson [8], has been developed rather thoroughly [2; 3; 6; 12; 16]. The general Galois theory, dealing with general groups of automorphisms (with some restrictions though), has been established by Cartan [5] and Jacobson [9] in case of sfields. The purpose of the present paper is to offer a similar theory for simple rings with minimum condition(1). The same has been given in fact in Hochschild [7] for simple algebras (finite over their centers). But the method breaks down in the case of general simple rings, infinite over their centers, and a new approach is necessary(2). The writer [14] has recently shown that if A is a simple ring and C is a weakly normal (cf. §1 below) simple subring of A, then the A-left-, C-right-module A is fully reducible, and he has applied this fact, together with some methods in Dieudonné [6], to obtain a theorem of extension for isomorphisms in simple rings. It turns out that this full reducibility of A, with respect to the left-multiplication of A and the right-multiplication of C, and some crossed product theorems, proved and used in the older papers by Azumaya and the writer [3; 12; 13; 16], are appropriate means for establishing the Galois theory(3) for simple rings. In fact, if A is in particular a sfield, then A is clearly minimal (=irreducible) with respect to any operator domain containing the left-multiplication ring of A, and this fact underlies the Galois theory, as well as many other theories, for sfields. It is replaced, when A is a simple ring, by our A-C-full reducibility.

The first section of the present paper gives some preliminary, though fundamental, lemmas on weakly normal simple subrings of a simple ring. In §2 we introduce regular groups of automorphisms of a simple ring, which are the class of automorphism groups employed in our Galois theory, and consider their invariant systems. Conversely, we consider in §3 the group of automorphisms leaving a subring, of a certain type, elementwise fixed. The Galois theory follows then in §4. Although our method is rather different, we follow there the pattern of the algebra case in Hochschild [7]. Our theory can

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(1) A (non-nilpotent) simple ring with minimum condition will be called in the present paper merely a simple ring, in a somewhat old-fashioned way.

(2) It is true that the method of combining "inner" and "outer" Galois theories applies, in a sense, even in our general case. However, there the outer Galois theory should be that of a certain subring other than the center. The relationship between the automorphisms of the whole ring and such a subring is exactly what produces the difficulty in our general case.

(3) In the present paper the term "Galois theory" is taken in the strict sense as a theory which deals with automorphism groups and their invariant systems.

readily be transferred to the so-called complete primitive rings, and this fact and some other remarks are given in §5.

1. Lemmas. By a *ring* we mean, throughout this paper, a ring with unit element. By a *subring* we mean one which contains this unit element. We shall deal only with those modules for which the unit element of our ring is the identity operator. By a *simple ring* we shall mean a simple ring (with unit element and) *with minimum condition*. Let A be such a ring, and $\mathfrak{r}$ be its minimal right-ideal, unique up to isomorphism. A is decomposed into a direct sum $A=\mathfrak{r}_1\oplus\mathfrak{r}_2\oplus\cdots\oplus\mathfrak{r}_k$ of minimal right-ideals $\mathfrak{r}_i$ all isomorphic to $\mathfrak{r}$, and the number k is the so-called *capacity* of A. Any right-module $\mathfrak{m}$ of A is a direct sum of (perhaps infinitely many) minimal submodules isomorphic to $\mathfrak{r}$. By the A-(right-) rank $[\mathfrak{m}:A]_r$ of $\mathfrak{m}$ we mean the number of minimal components in such a decomposition (i.e., the A-length of $\mathfrak{m}$) divided by the capacity k. For a left-module $\mathfrak{m}$ we introduce its left-rank $[\mathfrak{m}:A]_l$ in a similar fashion. If $\mathfrak{m}$ is a two-sided module and if it happens that $[\mathfrak{m}:A]_r=[\mathfrak{m}:A]_l$, then we shall denote the common value by $[\mathfrak{m}:A]$.

Consider a simple subring C of our simple ring A. Let C_r be the right-multiplication ring of C upon A, and $V_{\mathfrak{A}}(C_r)$ be its commutator in the absolute endomorphism ring $\mathfrak{A}$ of A as module. $V_{\mathfrak{A}}(C_r)$ is thus nothing but the C_r-endomorphism ring of A. We have $V_{\mathfrak{A}}(V_{\mathfrak{A}}(C_r))=C_r$. Further, $V_{\mathfrak{A}}(C_r)$ naturally contains the left-multiplication ring A_l of A (on A), since A_l is the commutator in $\mathfrak{A}$ of the right-multiplication ring A_r and clearly $A_r\supseteq C_r$. And $[V_{\mathfrak{A}}(C_r):A_l]_r=[V_{\mathfrak{A}}(C_r):V_{\mathfrak{A}}(A_r)]_r=[A:C]_r$[4]. Now, if $V_{\mathfrak{A}}(C_r)$ is generated over A_l by a certain number of A_l-semilinear endomorphisms of A, then we say that C is *weakly normal* in A. (The notion was defined in [6; 14] in a little different way, referring to $\mathfrak{r}$, but the two definitions are equivalent, as we saw in [15].) For a nonzero A_l-semilinear endomorphism γ of A the submodule γA_l $(=A_l\gamma)$ of $\mathfrak{A}$ is a minimal A_l-two-sided module. The product of two A_l-semilinear endomorphisms of A is naturally again an A_l-semilinear endomorphism of A. Thus our $V_{\mathfrak{A}}(C_r)$ is, when C is weakly normal, a direct sum $\sum^\circ \gamma A_l$ with a certain family $\{\gamma\}$ of A_l-semilinear endomorphisms γ of A.

Another remark is that if here C is strongly normal in A in the sense that $V_{\mathfrak{A}}(C_r)$ is generated over A_l by A_l-semilinear module-automorphisms of A, then $V_{\mathfrak{A}}(C_r)$ is generated over A_l by ring-automorphisms of A (which are naturally A_l-semilinear). For, if γ is an A_l-semilinear endomorphism of A and θ is the associated (ring-) automorphism of A_l, then $(xy)^\gamma=y^{x_l\gamma}=y^{\gamma x^{\theta}_l}$ $(x,y\in A)$, x_l denoting the left-multiplication of x; or, if we consider θ also as an automorphism of A, then $(xy)^\gamma=x^\theta y^\gamma$. In particular $x^\gamma=x^\theta 1^\gamma$. Suppose here that γ is a (module-) automorphism. Then 1^γ must be a regular element of A, since $x^\gamma=x^\theta 1^\gamma$ vanishes for no $x\neq 0$ (i.e., for no $x^\theta\neq 0$) (or, since $x^\theta 1^\gamma=1$ for some x such that $x^\gamma=1$). Thus $(1^\gamma)^{-1}x^\gamma=(1^\gamma)^{-1}x^\theta 1^\gamma$, and $\gamma(1^\gamma)_l^{-1}$ is the product

(4) Provided that we do not distinguish between two infinite ranks.

of θ and the inner automorphism of A induced by the regular element 1^γ. So $\gamma_1=\gamma(1^\gamma)_l^{-1}$ is a ring-automorphism of A and $\gamma_1 A_l=\gamma A_l$. Taking γ_1 for each γ, we have $V_{\mathfrak{A}}(C_r)=\sum\gamma_1 A_l$, which proves the remark. Furthermore, the relation $C_r=V_{\mathfrak{A}}(V_{\mathfrak{A}}(C_r))=V_{\mathfrak{A}}(\sum\gamma_1 A_l)$ shows that C is the invariant system of the group generated by these (ring-) automorphisms γ_1 of A.

Although our main concern in our Galois theory will be the strongly normal case, the general weakly normal case is in a sense more natural and we shall continue in this section dealing with general weakly normal subrings. Now, a statement equivalent to the following lemma was proved in [14]. However, since it is rather fundamental in our theory, we shall repeat its proof very briefly in a fashion adapted to our present formulation.

LEMMA 1.1. *If A is a simple ring and C a weakly normal simple subring, A is fully reducible as A-left-, C-right-module, that is, as A_lC_r-module, and is in fact a direct sum of minimal A_lC_r-modules which are mutually A_l-semilinearly and C_r-linearly isomorphic.*

Proof. Let $\mathfrak{m}$ be a minimal A_lC_r-submodule of A. Let γ be an A_l-semilinear endomorphism of A contained in $V_{\mathfrak{A}}(C_r)$. $\mathfrak{m}^\gamma$ is also an A_lC_r-module. In fact $u\rightarrow u^\gamma$ $(u\in\mathfrak{m})$ gives an A_l-semilinear and C_r-linear mapping of $\mathfrak{m}$ onto $\mathfrak{m}^\gamma$. Since $\mathfrak{m}$ is A_lC_r-minimal, the same is the case for $\mathfrak{m}^\gamma$ and the mapping is an (A_l-semilinear and C_r-linear) isomorphism, unless $\mathfrak{m}^\gamma=0$. The sum of all submodules $\mathfrak{m}^\gamma$, γ running over all A_l-semilinear endomorphisms of A in $V_{\mathfrak{A}}(C_r)$ (or, over our family $\{\gamma\}$ only), is a (nonzero) $V_{\mathfrak{A}}(C_r)C_r$-module. On the other hand, A is homogeneously fully reducible with respect to C_r, in the sense that it is a direct sum of mutually isomorphic minimal C_r-modules. It follows (cf. [14]) that A is $V_{\mathfrak{A}}(C_r)C_r$-minimal. Thus our sum $\sum\mathfrak{m}^\gamma$ coincides with A, which proves the lemma.

Consider a second simple subring B of A which contains C: $A\supseteq B\supseteq C$. We have:

LEMMA 1.2. *B is weakly normal in A together with C.*

Proof. $V_{\mathfrak{A}}(B_r)$ is an A_l-two-sided submodule of $V_{\mathfrak{A}}(C_r)=\sum^\circ\gamma A_l$, and is thus, by the general theory of fully reducible modules, a direct sum of submodules (A_l-two-sided) isomorphic to some of γA_l. Such a submodule has a form δA_l, with an A_l-semilinear endomorphism δ of A.

LEMMA 1.3. *Let A, C be as in Lemma* 1.1, *and, as in Lemma* 1.2, *let B be a second (necessarily weakly normal) simple subring of A which contains C. Let β be an isomorphism of B into A over C (i.e., leaving C elementwise fixed). Suppose that the commutator rings $V_A(B)$, $V_A(B^\beta)$ of B, B^β in A are simple. Then β can be extended to an automorphism of A.*

Proof (cf. [14]). B^β is also a simple subring of A containing C, and it is, therefore, weakly normal in A too. Applying Lemma 1.1 to B and B^β, in place of C, we have

$$(1)\qquad A = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_n = \bar{\mathfrak{n}}_1 \oplus \bar{\mathfrak{n}}_2 \oplus \cdots \oplus \bar{\mathfrak{n}}_{\bar{n}}$$

where the $\mathfrak{n}_p$ are mutually A_l-semilinearly and B_r-linearly isomorphic minimal A_lB_r-modules and the $\bar{\mathfrak{n}}_q$ are mutually A_l-semilinearly and B_r^β-linearly isomorphic minimal $A_lB_r^\beta$-modules.

On the other hand, since B and B^β are isomorphic simple rings, minimal B_r-(i.e., B-right-) submodules of A are $(B\text{-}B^\beta,\ \beta)$-semilinearly isomorphic to minimal B_r^β-submodules of A; in fact, the same is the case with any minimal B-right- and B^β-right-modules. Hence there exists certainly a nonzero $(B\text{-}B^\beta,\ \beta)$-semilinear endomorphism of A. Let $\mathfrak{M}$ be the totality of elements in $\mathfrak{A}$ which are $(B_r\text{-}B_r^\beta,\ \beta)$-semilinear endomorphisms of A. Thus $\mathfrak{M}\neq 0$. It is contained in $V_{\mathfrak{A}}(C_r)$, as β is the identity on C. $\mathfrak{M}$ is, further, A_l-two-sided allowable. Thus $\mathfrak{M}$ is an A_l-two-sided submodule of $V_{\mathfrak{A}}(C_r)$, and as such is a direct sum $\sum^\circ \mu A_l$, where the μ are A_l-semilinear. Thus each μ is A_l-semilinear and $(B_r\text{-}B_r^\beta,\ \beta)$-semilinear. There exists, hence, at least one nonzero A_l-semilinear and $(B_r\text{-}B_r^\beta,\ \beta)$-semilinear endomorphism of A. It follows, by the theorem of composition series, that one of the $\mathfrak{n}_p$ is A_l-semilinearly and $(B_r\text{-}B_r^\beta,\ \beta)$-semilinearly isomorphic to one of the $\bar{\mathfrak{n}}_q$. Then any of $\mathfrak{n}_p$ is isomorphic to any of $\bar{\mathfrak{n}}_q$ in the same sense (with, perhaps, a different automorphism of A_l). It follows then in particular that $n=\bar{n}$ in (1).

$V_A(B)_r = V_{A_r}(B_r)$ is the A_lB_r-endomorphism ring of A. Since A is fully reducible with respect to A_lB_r (and the length n is finite), $V_A(B)_r$ is a semisimple ring (with minimum condition). The same is the case with $V_A(B^\beta)$. Now we use, for the first time, our assumption that $V_A(B)$, $V_A(B^\beta)$ are simple. Then $V_A(B)_r$, $V_A(B^\beta)_r$ are simple, which implies that $\mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_n$ are mutually A_lB_r-isomorphic (not only semilinearly but properly) and $\bar{\mathfrak{n}}_1, \bar{\mathfrak{n}}_2, \cdots, \bar{\mathfrak{n}}_n$ are mutually $A_lB_r^\beta$-isomorphic. On extending an A_l-semilinear and $(B_r\text{-}B_r^\beta,\ \beta)$-semilinear isomorphism of $\mathfrak{n}_1$ and $\bar{\mathfrak{n}}_1$, say, we may readily obtain an A_l-semilinear and $(B_r\text{-}B_r^\beta,\ \beta)$-semilinear module-automorphism of A, say μ. It induces a ring-automorphism α of A according to the relation $u^\mu a^\alpha = (ua)^\mu$ $(u,\ a\in A)$ (i.e., $a_r^\alpha = \mu^{-1}a\mu$); observe that $A_r = V_{\mathfrak{A}}(A_l)$. This α is an extension of β, and the lemma is thus proved.

Remark. When B is a weakly normal simple subring of a simple ring A, $V_A(B)$ is automatically semisimple as the endomorphism ring of a fully reducible module, as was observed in our proof. Hence, we need, in Lemma 1.3, simply to assume that $V_A(B)$ is directly indecomposable, or merely simple modulo radical, and similarly for $V_A(B^\beta)$; then they are automatically simple.

Also in connection with Lemma 1.3, we have:

Lemma 1.4. *Let B be a weakly normal simple subring of a simple ring A and let the commutator $V_A(B)$ in A be simple* (*or merely simple modulo radical*);

then B is strongly normal and the B_r-endomorphism ring $V_{\mathfrak{A}}(B_r)$ of A is generated over A_l by (ring-) automorphisms of A, and B is the invariant system, in A, of the group generated by those automorphisms of A.

Proof. The A_lB_r-module A is homogeneously fully reducible. Let γ be an arbitrary nonzero A_l-semilinear endomorphism of A contained in $V_{\mathfrak{A}}(B_r)$ and let θ be the associated automorphism of A_l. Consider a minimal A_lB_r-submodule $\mathfrak{m}$ of A such that $\mathfrak{m}^\gamma \neq 0$. Then γ gives, as in Lemma 1.1, an A_l-semilinear and B_r-linear isomorphism of $\mathfrak{m}$ and $\mathfrak{m}^\gamma$, with θ as its associated automorphism of A_l. Since A is homogeneously fully reducible with respect to A_lB_r, we may extend this isomorphism to an A_l-semilinear and B_r-linear module-automorphism of A, say ν, associated with the automorphism θ of A_l. ν and ν^{-1} are in $V_{\mathfrak{A}}(B_r)$, since they are B_r-linear. Thus $\nu^{-1}\gamma$ is in $V_{\mathfrak{A}}(B_r)$ and it is A_l-linear (and B_l-linear). Thus $\nu^{-1}\gamma \in V_{\mathfrak{A}}(B_r) \cap A_r = V_A(B)_r$, and $\gamma \in \nu V_A(B)_r$. Since $V_A(B)$ is a simple ring, it is generated by its regular elements. For a regular element a in $V_A(B)$, νa_r is naturally an (A_l-semilinear) module-automorphism of A. Since this is the case with every A_l-semilinear endomorphism γ in $V_{\mathfrak{A}}(B_r)$, it follows that $V_{\mathfrak{A}}(B_r)$ $(=\sum \gamma A_l)$ has an A_l-basis consisting of A_l-semilinear module-automorphisms of A. Now our lemma follows from our remark concerning strongly normal subrings.

We have further:

Lemma 1.5. *Let A, C be as in Lemma 1.1, and let $V_{\mathfrak{A}}(C_r) = \sum \gamma A_l$ with A_l-semilinear γ. Let α be an automorphism of A over C. Then α belongs to the same automorphism-class in A as the automorphism associated with one of γ, considered as an automorphism of A rather than of A_l.*

Proof. Since α leaves C elementwise fixed, $\alpha \in V_{\mathfrak{A}}(C_r)$ and $\alpha A_l \subseteq V_{\mathfrak{A}}(C_r)$. So $\alpha A_l (= A_l\alpha)$ is a minimal A_l-two-sided submodule of $V_{\mathfrak{A}}(C_r)$. As such, it is (A_l-two-sided) isomorphic to a γA_l. If θ is the automorphism of A_l associated with γ, and if we consider it as an automorphism of A, then α belongs to the same automorphism-class in A as θ. Or, what amounts to the same, if we consider α as an automorphism of A_l, then it belongs to the same automorphism-class, in A_l, as the automorphism θ of A_l. For, if α corresponds to γa_l $(a \in A)$ in an isomorphism of αA_l and γA_l, then a is regular, since $\alpha A_l = A_l\alpha$ is mapped onto $\gamma A_l = A_l\gamma$ whence[5] $a_lA_l = A_l$, $A_la_l^{\theta^{-1}} = A_l$, i.e., $A_la_l = A_l$, and $x_l\alpha = \alpha x_l^\alpha$ corresponds to $x_l\gamma a_l = \gamma x_l^\theta a_l = \gamma a_l a_l^{-1} x_l^\theta a_l$ as well as to $\gamma a_l x_l^\alpha$, whence $x_l^\alpha = a_l^{-1}x_l^\theta a_l$ $(x \in A)$.

The following special case is well known (cf. [1]):

Corollary. *Let A be a simple ring and Z be its center. If T is a simple subring of A containing Z and finite over Z, then every automorphism of A leaving the commutator $C = V_A(T)$ of T in A elementwise fixed is an inner automorphism.*

(5) Naturally either one of $a_lA_l = A_l$, $A_la_l = A_l$ is enough.

For, $C_r = V_{A_r}(T_r) = V_{\mathfrak{A}}(T_r A_l)$. $T_r A_l$ is a simple ring; observe that the direct product $T_r \times A_l$ over $Z_r = Z_l$ is simple and $T_r A_l$ is homomorphic, whence isomorphic, to it. It follows that $V_{\mathfrak{A}}(C_r) = T_r A_l$. Here the elements of T_r are A_l-linear endomorphisms of A, that is, A_l-semilinear endomorphisms of A associated with the identity automorphism. Now the corollary follows from our lemma.

2. **Regular groups of automorphisms.** Let A be a simple ring. With a group Φ of (ring-) automorphisms of A, denote by T_Φ the ring generated by all regular elements in A which effect inner automorphisms of A contained in Φ. We introduce the following:

Definition. A group Φ of automorphisms of A is called *complete* if Φ contains all inner automorphisms of A induced by the regular elements of T_Φ.

With any group Φ of automorphisms of A, which is not necessarily complete, the group generated by Φ and the totality of inner automorphisms of A induced by the regular elements of T_Φ is a complete group and is in fact the smallest complete group containing Φ. The totality of automorphisms of A leaving a certain subset of A elementwise fixed forms always a complete group. Further, if U is any subring of A containing T_Φ, the group generated by Φ and all inner automorphisms induced by U is complete.

Definition. If Φ is complete, if the ring T_Φ is a simple ring finite over the center Z of A, and if, moreover, the (invariant) subgroup Φ_0 of Φ consisting of all inner automorphisms of A contained in Φ (which is also the totality of inner automorphisms of A induced by the regular elements of T_Φ since Φ is assumed to be complete) has a finite index $(\Phi:\Phi_0)$ in Φ, then we say that Φ is a *regular* group of automorphisms of A and $(\Phi:\Phi_0)[T_\Phi:Z]$ is its *reduced order*.

Regular groups are the class of groups of automorphisms with which we want to develop our Galois theory. Needless to say, if A is in particular a sfield, then the requirement that T_Φ be a simple ring is automatically satisfied and our condition amounts to the completeness plus the finiteness of $(\Phi:\Phi_0)[T_\Phi:Z]$.

We begin with the following lemma.

Lemma 2.1. *Let Φ be regular. The ring ΦA_l $(=A_l\Phi)$[6] generated by Φ and the left-multiplication ring A_l (in $\mathfrak{A}$) is a simple ring.*

Proof (cf. [12; 13]). Let Φ_0 and T_Φ be as above, and denote T_Φ simply by T. Then $\Phi_0 A_l = T_r A_l$, and this is a simple ring (isomorphic to the direct product $T_r \times A_l$ over $Z_r = Z_l$). Let $\rho_1, \rho_2, \cdots, \rho_g$ $(g = (\Phi:\Phi_0))$ be a representative system of Φ/Φ_0. Then

$$\Phi A_l = \rho_1 \Phi_0 A_l \oplus \rho_2 \Phi_0 A_l \oplus \cdots \oplus \rho_g \Phi_0 A_l \tag{2}$$
$$= \rho_1 T_r A_l \oplus \rho_2 T_r A_l \oplus \cdots \oplus \rho_g T_r A_l.$$

(6) Here ΦA_l (resp. $A_l\Phi$) means (not the mere product but) the product-module.

The summations are necessarily direct, because $\rho_1, \rho_2, \cdots, \rho_g$ all belong to different automorphism-classes of (A and) A_l and, therefore, the A_l-two-sided modules $\rho_1 T_r A_l, \rho_2 T_r A_l, \cdots, \rho_g T_r A_l$ have no isomorphic composition residue-modules. From this last it follows also that every A_l-two-sided submodule of ΦA_l is a (direct) sum of some submodules of $\rho_i T_r A_l$. In particular, a two-sided ideal $\mathfrak{a}$ of ΦA_l has this property. If $\mathfrak{a}\neq 0$, then $\mathfrak{a}$ contains a certain nonzero element in one $\rho_i T_r A_l$. Observing that $T_r A_l$ is simple, we see readily that $\mathfrak{a}=\Phi A_l$. The minimum condition in ΦA_l is clear; it even satisfies the A_l-(right-, or left-) minimum condition.

Let $I=I(\Phi)$ be the invariant system in A of our regular automorphism group Φ, that is, the totality of elements in A left invariant by Φ. Then $I_r=A_r\cap V_{\mathfrak{A}}(\Phi)=V_{\mathfrak{A}}(\Phi A_l)$, where $\mathfrak{A}$ denotes, as in §1, the absolute endomorphism ring of A as module. Since ΦA_l is a simple ring, as we have just seen, A is homogeneously fully reducible and of finite length with respect to ΦA_l and therefore I_r is a simple ring, and so is I. Another consequence of the simplicity of ΦA_l is

$$V_{\mathfrak{A}}(I_r) = \Phi A_l. \tag{3}$$

(2) shows then that I is weakly normal in A; in fact, it is strongly normal. Further, $[A:I]_r(=[A_r:I_r]_r)=[V_{\mathfrak{A}}(I_r):V_{\mathfrak{A}}(A_r)]_r=[\Phi A_l:A_l]_r$ and this is equal to $g[T_rA_l:A_l]_r=g[T:Z]$.

THEOREM 1. *Let Φ be a regular group of automorphisms of a simple ring A, and $I=I(\Phi)$ be its invariant system. Then I is a simple ring and A has an independent right- (resp. left-) basis over I consisting of as many terms as the reduced order $(\Phi:\Phi_0)[T_\Phi:Z]$ of Φ. Φ exhausts the automorphisms of A leaving I elementwise fixed. The commutator $V_A(I)$ of I in A coincides with $T=T_\Phi$.*

Proof. The first half has been seen; observe that $[A:I]_l=g[T:Z]$ too, by symmetry. To prove the second half, let α be an automorphism of A over I. By Lemma 1.5, together with (3) (and (2)), we see that α has a form $\rho_i\phi_0\cdot$(an inner automorphism of A) ($\phi_0\in\Phi_0$). This inner automorphism of A must leave I elementwise fixed, since α, ρ_i, and ϕ_0 do, and is then induced by an element in $V_A(I)$. Here, as asserted in our theorem,

$$V_A(I) = T(= T_\Phi), \tag{4}$$

since $V_{A_r}(I_r)=A_r\cap V_{\mathfrak{A}}(I_r)=A_r\cap\Phi A_l=T_r$; observe that the sum $\sum\rho_i A_r A_l$ is, as (2), direct and that the product A_rA_l in $\mathfrak{A}$ is direct over $Z_r=Z_l$. Thus our inner automorphism belongs to Φ_0. Hence $\alpha\in\rho_i\Phi_0$, and α belongs to Φ.

Now, the invariant system, in A, of Φ_0 is nothing but the commutator $V_A(T)$ of $T=T_\Phi$. Put $S=V_A(T)$. As a special case of our theorem (applied to Φ_0 instead of Φ), and as is well known, S is a simple ring and

$$[A:S](=[A:S]_r=[A:S]_l)=[T:Z]. \tag{5}$$

The following theorem is of particular significance in the special case $U=T$ (whence $R=S$):

THEOREM 2. *Let Φ be a regular group of automorphisms of a simple ring A and Φ_0 be its (invariant) subgroup of inner automorphisms. Let U be a simple subring containing $T(=T_\Phi)$ and finite over the center Z of A. Assume that U is setwise invariant under Φ, that is, $U^\Phi=U$. Then Φ/Φ_0 is an outer group of automorphisms of the commutator $R=V_A(U)$. Its invariant system $I\cap R$ in R is a simple ring, $[R:I\cap R]=(\Phi:\Phi_0)$, and Φ/Φ_0 exhausts the automorphisms of R over $I\cap R$. The product-module RI (resp. IR) is a ring and coincides indeed with $S=V_A(T)$.*

Proof. Since $R\subseteq S$, clearly Φ_0 induces on R the identity automorphism. Suppose that $\phi\in\Phi$ induces an inner automorphism on R, induced by an element a of R. Denote by α the inner automorphism of A induced by this element a. Then $\phi\alpha^{-1}$ is an automorphism of A leaving R elementwise invariant. As such, it is an inner automorphism of A, by the corollary to Lemma 1.5. Then ϕ must be an inner automorphism of A too; $\phi\in\Phi_0$. Thus Φ/Φ_0 is an outer group of automorphisms of R.

Φ/Φ_0 is in particular a regular automorphism group of R. Its invariant system in R is evidently the intersection $I\cap R$ of R and the invariant system I in A of Φ. On applying Theorem 1 to R and Φ/Φ_0 (instead of A and Φ) we see that $[R:I\cap R]_r=[R:I\cap R]_l=(\Phi:\Phi_0)$ and Φ/Φ_0 exhausts all automorphisms of R leaving $I\cap R$ elementwise fixed.

Clearly $V_A(T)=S$ contains both I and R, hence RI. Let $(w_1, w_2, \cdots, w_g)$ $(g=(\Phi:\Phi_0))$ be an independent right-basis of R over $I\cap R$ (which exists by the outer special case of Theorem 1). Then the matrix.

$$\begin{pmatrix} w_1^{\rho_1} & w_2^{\rho_1} & \cdots & w_g^{\rho_1} \\ \cdot & \cdot & \cdot & \cdot \\ w_1^{\rho_g} & w_2^{\rho_g} & \cdots & w_g^{\rho_g} \end{pmatrix}$$

is regular, where $\rho_1, \rho_2, \cdots, \rho_g$ form, as in (2), a representative system of Φ/Φ_0. For, if $(x_1, x_2, \cdots, x_g)$ is a vector in R such that $(x_1, x_2, \cdots, x_g)$ $(w_j^{\rho_i})=0$, then $\sum_i x_i y^{\rho_i}=0$ for every element y in R $(=w_1(I\cap R)\oplus w_2(I\cap R)$ $\oplus \cdots \oplus w_g(I\cap R))$. But $\rho_1, \rho_2, \cdots, \rho_g$ are right-independent over the left-multiplication ring of R (on R), again by the special case $A=R$ of the directness of (2). Hence necessarily $x_1=x_2=\cdots=x_g=0$, which proves that our matrix is regular. Then $w_1, w_2, \cdots, w_g$ are right-independent over I too. For, if $w_1a_1+w_2a_2+\cdots+w_ga_g=0$ with $a_j\in I$, then $w_1^{\rho_i}a_1+w_2^{\rho_i}a_2+\cdots+w_g^{\rho_i}a_g$ $=0$ for every $i=1, 2, \cdots, g$, that is, $(w_j^{\rho_i})(a_j)=0$. Hence $a_1=a_2=\cdots=a_g$ $=0$ necessarily. Thus $RI=w_1I\oplus w_2I\oplus\cdots\oplus w_gI$ and $[RI:I]_r=g$. But $[S:I]_r=g$ too, according to the special case $R=S$ of the already established part of our theorem. So $S=RI$. Similarly we have $S=IR$.

Our theorem is somewhat complicated. But it reveals how the invariant system I of a regular group is situated in A. In particular, it shows how I is related to R which is (the commutator of U and is) of purely inner character in A, while $I \cap R$ is of purely outer character in R. We repeat that the case $U=T$ (whence $R=S$) is of particular significance, whereas if $[A:Z]<\infty$, then we may take A itself as U letting thus R be the center Z of A.

In connection with our theorem we may also note that every automorphism of R over $I \cap R$ is (in Φ, or, more precisely, in Φ/Φ_0, and therefore, evidently) extended to an automorphism of $S=RI=IR$ over I, and indeed in a unique manner. Further, $[S:I]=g=[R:I\cap R]$ and

$$[S:I \cap R](=[S:I][I:I \cap R]) = [R:I \cap R][I:I \cap R]. \tag{6}$$

For, (6) is certainly true if we restrict ourselves either to right- or to left-ranks. But $I\cap R$ is the invariant system, in A, of the (regular) group generated by Φ and the inner automorphisms effected by U. Hence $[A:I\cap R]_r = [A:I\cap R]_l$ and therefore $[S:I\cap R]_r=[S:I\cap R]_l$, since $[A:S]_r=[A:S]_l$. Similarly $[I:I\cap R]_r=[I:I\cap R]_l$.

3. **Subrings with simple commutators.** In connection with Theorem 1, as well as Lemmas 1.3 and 1.4, we are led to consider simple subrings of A whose commutators, in A, are also simple.

THEOREM 3. *Let B be a simple subring of a simple ring A such that $[A:B]_r<\infty$. If the commutator $V_A(B)$ of B in A is simple, then the group Φ of all automorphisms of A over B is regular. If (and only if) B is weakly normal in A, the invariant system I of Φ coincides with B.*

Proof. Set $T=V_A(B)$. The subgroup Φ_0 of all inner automorphisms of A contained in Φ is clearly the totality of inner automorphisms induced by the regular elements of T, and thus $T=T_\Phi$. Here T is simple, by assumption. Moreover, $[T:Z]=[T_rA_l:A_l]$, since the product A_rA_l is direct over $Z_r=Z_l$. Here $T_rA_l\subseteq V_\mathfrak{A}(B_r)$ and $[V_\mathfrak{A}(B_r):A_l]_r=[V_\mathfrak{A}(B_r):V_\mathfrak{A}(A_r)]_r=[A_r:B_r]_r=[A:B]_r<\infty$. Thus $[T:Z]<\infty$. Further, if $\rho_1, \rho_2, \cdots$ are representatives of Φ/Φ_0, then they all belong to different automorphism-classes and thus are right-independent over A_l, by the argument used in proving the directness of (2). The relation $[V_\mathfrak{A}(B_r):A_l]_r<\infty$ shows then also that $(\Phi:\Phi_0)<\infty$. Thus Φ is regular.

The second half of the theorem, which is in fact far deeper than the first half, follows from Lemma 1.4 (the "only if" part being clear from Theorem 1).

Given a regular group Φ of automorphisms, Theorems 1 and 3 suffice to establish 1-1 Galois correspondence between regular subgroups of Φ and simple subrings, containing the invariant system I of Φ, with simple commutators. However, postponing the statement until the next section, we consider the case where a subring B is not known to contain the invariant system of a regular group and is not known to be weakly normal.

THEOREM 4. *Let B be a simple subring of a simple ring A such that $[A:B]_r < \infty$ and the commutator $T = V_A(B)$ is simple. Let Φ be the group of automorphisms of A over B, and let U be a simple subring of A containing T and finite over the center Z of A (or, what is the same, $[U:T]_r < \infty$) such that $U^\Phi = U$, and put $R = V_A(U)$. Let W be the invariant system in R of Φ. Then the ring generated by W and B coincides with the invariant system of Φ in A.*

(Observe that Φ induces an outer automorphism group on R as in Theorem 2, and thus our theorem reduces, in a sense, the problem of the invariant system to purely outer and purely inner situations. Again the case $U = T$ (whence $R = S$ $(= V_A(T))$) is of importance.)

Proof. The group Φ_1 generated by Φ and the totality of inner automorphisms of A induced by the regular elements of U is a regular automorphism group. Its invariant system in A is nothing but W. Now, let Q be the subring of A generated by B and W. Since B, $W \subseteq V_A(T)$, we have $Q \subseteq V_A(T)$ and $V_A(Q) \supseteq T$. On the other hand, $V_A(Q) \subseteq V_A(B) = T$. Hence $V_A(Q) = T$ and this is simple. As $Q \supseteq W$ and W is weakly (in fact, strongly) normal in A, Q is weakly normal in A, because of Lemma 1.2. By Theorem 3, Q is then the invariant system of a certain regular automorphism group of A. This group is, however, clearly our Φ, and the theorem is proved.

4. Galois theory.

THEOREM 5. *Let Φ be a regular automorphism group of a simple ring A, and $I = I(\Phi)$ be its invariant system. Then there is a 1-1 dual correspondence between regular subgroups of Φ and simple subrings of A containing I and possessing simple*(7) *commutators in A, in the usual sense of Galois theory.*

Proof. Theorems 1, 3, applied to subgroups and subrings, give the desired Galois correspondence.

THEOREM 6. *Let A, Φ, I be as in Theorem 5. If B, B^β are two (not necessarily distinct) simple subrings of A containing I and possessing simple commutators, and if there exists an isomorphism β of B and B^β leaving I elementwise fixed, then the isomorphism β can be extended to an automorphism of A which is necessarily contained in Φ.*

Proof. Immediate from Lemma 1.3.

Thus we have been able to establish for A, Φ the two main features in Galois theory, i.e., the Galois correspondence and the extension of isomorphisms over the invariant system. Next consider a simple subring B of A, with simple commutator, containing $I = I(\Phi)$ and consider the subgroup Ψ of Φ corresponding to B in the sense of Galois correspondence; thus Ψ is the group of all automorphisms (necessarily contained in Φ) of A over B. Then Ψ is a normal subgroup of Φ if and only if $B^\Phi = B$. The invariant sys-

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(7) Or, simple modulo radical. Cf. §1 or §4, (ii).

tem in B of the automorphism group Φ/Ψ of B is then exactly I. However, we are interested in a little more general situation and want to obtain a condition in order that I be the invariant system in B of Λ/Ψ, where Λ is the totality of elements of Φ which leave B setwise invariant(8). We begin with the following lemma.

LEMMA 4.1. *Let A, Φ, I be as in Theorem 5. Let B be a simple subring of A containing I and possessing a simple commutator $D=V_A(B)$. Let Λ be the totality of elements of Φ which map B into itself*(9), *and let V be the ring generated by all the regular elements of A which induce inner automorphisms of A belonging to Λ*(10). *If $V_T(D)=T\cap V_A(D)$ is contained in V*(11), *and if the product, in A, of the centers Y, Z_T of D and T is semisimple, then V is semisimple too.*

Proof. Put $\Lambda_0=\Lambda\cap\Phi_0$. V is nothing but the subring of A generated by the regular elements inducing the elements of Λ_0. Now Λ_0 leaves B, hence $D=V_A(B)$, setwise fixed. Let Γ be the totality of elements of Λ_0 which induce on D inner automorphisms (of D). Thus Γ is nothing but the totality of elements of Λ_0 leaving the center Y of D elementwise fixed; observe that, since $[D:Y]$ $(\leqq[D:Z]\leqq[T:Z])<\infty$, every automorphism of D leaving Y elementwise fixed is inner. Thus Λ_0/Γ is an automorphism group of Y. It leaves Z elementwise fixed. Since $[Y:Z](\leqq[D:Z])<\infty$, $(\Lambda_0:\Gamma)$ is finite. Let $\beta_1, \beta_2, \cdots, \beta_m$ be a representative system of Λ_0/Γ, and $b_1, b_2, \cdots, b_m$ be the regular elements of T (in fact, of V) inducing $\beta_1, \beta_2, \cdots, \beta_m$. Since for each pair i, j we have $b_ib_j=b_kV_T(D)D$ with some k,

$$b_1V_T(D)D+b_2V_T(D)D+\cdots+b_mV_T(D)D \tag{7}$$

is a ring. In fact, it coincides with ring V because of our assumption $V_T(D)\subseteq V$. Since $\beta_1, \beta_2, \cdots, \beta_m$ induce automorphisms of D all in different automorphism-classes, the minimal D-two-sided modules $b_1D, b_2D, \cdots, b_mD$ are all nonisomorphic. Hence no two of $b_1V_T(D)D, b_2V_T(D)D, \cdots, b_mV_T(D)D$ have isomorphic composition residue-modules. Thus the sum (7) must be direct and, moreover, any D-two-sided submodule of the sum, i.e., V, is a sum of submodules of the summands. In particular this last is the case for any two-sided ideal of V. It follows readily that if N is the radical of $V_T(D)D$, then $b_1N\oplus b_2N\oplus\cdots\oplus b_mN$ is the radical of V; by the way, $V_T(D)D$ $(\subseteq T)$ and V are finite algebras over Z, say. Here $V_T(D)=V_T(Z_TD)$ and $V_T(D)D=V_T(Z_TD)Z_TD$. Since $Y\subseteq T$, Z_TY is commutative (and finite over Z). Suppose now, as was stated in the lemma, that Z_TY is semisimple (hence is a direct sum of a finite number of mutually

(8) In other words, Λ is the normalizor of Ψ in Φ.

(9) Then they map B onto itself isomorphically; observe that $[A:B]<\infty$.

(10) Thus V is nothing but T_Λ.

(11) Instead of assuming this, we may assume that $(V\cap V_T(D))D$ is semisimple, as our proof will show.

orthogonal fields). Then $Z_T D$ is a semisimple algebra over Z, say; it is in fact the direct product $Z_T Y \times D$ over Y. Then $V_T(Z_T D)$ is also a semisimple algebra and so is $V_T(Z_T D)Z_T D$, too, $Z_T Y$ being their common center (cf. [17] e.g.). Hence $N=0$ and V is semisimple, as was observed above, which proves the lemma.

Now we have:

THEOREM 7. *Let A, Φ, I be as in Theorem 5. Let B be a simple subring of A containing I and possessing a simple commutator in A. Let Λ be the totality of elements of Φ which map B into itself. Assume that T_Λ contains $T_\Phi \cap V_A(V_A(B))$*(12)*. In order that the invariant system of Λ in B coincide with I it is necessary and sufficient that Φ be the smallest complete group of automorphisms of A containing Λ.*

Proof. Suppose that the last condition is satisfied. Then the invariant system of Λ in A coincides with that of Φ, that is, I. Furthermore, the invariant system of Λ in B is I.

Suppose conversely that the invariant system of Λ in B coincides with I. Then the invariant system of Λ in the center Y of the (simple) commutator $D=V_A(B)$ of B is clearly $I\cap Y$. On the other hand, $Y \subseteq D = V_A(B) \subseteq V_A(I)$. Hence $I\cap Y$ is contained in the center $Z_I = I \cap V_A(I)$ of I. Thus $Z_I \cap Y$ (contains, hence) coincides with the invariant system $I \cap Y$ of Λ in Y. Therefore Y is (finite and) separable over $Z_I \cap Y$; for the finiteness observe that $(Z:I\cap Z) \leqq (\Phi:\Phi_0) < \infty$, $(Y:Z) < \infty$.

On the other hand, $Z_T \cap Y = V_A(T) \cap T \cap Y = V_A(T) \cap Y = S \cap Y \supseteq I \cap Y \supseteq Z_I \cap Y$, where Z_T denotes, as in Lemma 4.1, the center of $T=T_\Phi$ and S is $V_A(T)$. Hence Y is (finite and) separable over $Z_T \cap Y$ too. Then the direct product $Z_T \times Y$ over $Z_T \cap Y$ is semisimple, and so is the product $Z_T Y$ in A. Hence, if we denote T_Λ by V, as in Lemma 4.1, then V is semisimple, by the same lemma.

Thus, the smallest complete group containing Λ has the following property which is very close to regularity: it is complete, its subgroup of inner automorphisms has a finite index, and the ring V, generated by the regular elements inducing the inner automorphisms in it, is semisimple (and finite over Z, naturally), instead of being simple. However, as we shall observe in the next section, the last two statements in Theorem 1, in particular, are valid also for such a group. Now, on the other hand, the invariant system of Λ in A is I; observe that Λ contains all automorphisms of A over B and there-

(12) This condition is very awkward, but it is automatically satisfied in case A is finite over Z; see §5, iv. Further, the sufficiency assertion in our theorem is independent of this assumption.

This assumption may be replaced by the requirement that $(T_\Lambda \cap V_A(V_A(B))V_A(B)$ be semisimple, which is automatically the case certainly when A (or T_Φ) is a sfield; cf. footnote 11.

fore the invariant system of Λ in A is in any event contained in B. Our smallest complete group containing Λ has also the invariant system I in A. On assuming the above mentioned (as yet unverified) generalization of the exhaustion statement of Theorem 1, we see then that it is the totality of automorphisms of A over I and is thus nothing but Φ, which proves the theorem.

REMARK. Since the completion of Λ turns out to be identical with Φ and therefore regular, the ring $V=T_\Lambda$ is, in case of Theorem 7, nothing but $T=T_\Phi$ and, in particular, simple. The "reason" for this is that, in the notations of the proof of Lemma 4.1, Λ_0 is big enough so that a simple component in the semisimple ring $V_T(D)D$ is carried to any other simple component by one of $\beta_1, \beta_2, \cdots, \beta_m$, making thus (7), which is V, simple (in spite of the fact that $V_T(D)D$ may not be simple). Observe also that we did not assume the semisimplicity of $V_T(D)D$ (nor of Z_TY), but proved it.

5. **Supplementary remarks.** (i) *Semi-regular groups.* In our proof of Theorem 7 we were led to consider a generalization of regularity for automorphism groups. Thus we want to call an automorphism group Φ of a simple ring A semi-regular if Φ is complete, $(\Phi:\Phi_0)<\infty$, and if the ring T_Φ is semisimple and finite over Z, the center of A. In fact, some of our results concerning regular groups remain valid for semi-regular groups. Let, namely, Φ be semi-regular, and consider ΦA_l. A modification of our proof of Lemma 2.1 shows that ΦA_l is a semisimple ring, with minimum condition; consider either the radicals of $T=T_\Phi$ and ΦA_l or the lattices of two-sided ideals in T and ΦA_l. The invariant system $I=I(\Phi)$ of Φ is then a semisimple ring, with minimum condition, too, and (3) and (4) remain valid. If we define, as we in fact did in [14], the weak normality for arbitrary, not necessarily simple, subrings of A in exactly the same manner as we did for simple rings, in §1, then Lemma 1.5 is true for any such arbitrary weakly normal subring C of A. (Its corollary is the case for any semisimple (or, more generally, uni-serial, for instance) subring T finite over Z.) Taking this into account, we see readily that Φ, a *semi-regular automorphism group of A, exhausts the automorphisms of A over $I=I(\Phi)$*. The last statement of Theorem 1 remains true too. It is also possible to obtain some statements which may be considered as a generalization of our rank relation, but they are rather complicated and clumsy. (An exact transfer of the rank relation can be made if the T_rA_l-submodules of A corresponding to different simple subrings of (semisimple) T_rA_l all have equal ranks with regard to respective simple subrings of T_rA_l $(T=T_\Phi)$, and this is in fact the case which has been considered in [13] under a general setting that A be T_rA_l-"regular.") Also the first statement of Theorem 2 is true for a semi-regular group Φ, and in fact for a semisimple U. In dealing with semi-regular automorphism groups, we have naturally to consider not only simple, but semisimple subrings of A, so the present supplementary remark (i) is closely related to the latter part of the succeeding remark (ii).

(ii) *The condition of simple commutators.* We considered, in our Galois theory, those subrings of A whose commutators in A are simple. In general, there are simple subrings of A containing the invariant system I of a regular group Φ whose commutators are not simple and which in fact are not invariant systems of automorphism groups; see [18] or [7, p. 298]. We note however that if in particular $\Phi_0=\Phi$ or $=1$, i.e., if Φ is either an inner or an outer automorphism group, then the condition is satisfied automatically, that is, every simple subring of A containing I has a simple commutator and is the invariant system of the corresponding subgroup of Φ.

As we observed in a remark in connection with Lemma 1.3, a superfluous weakening of the requirement can be made by demanding that the commutator be simple modulo radical. In regard to (Lemma 1.3 and) Theorem 6, it is also possible to weaken the condition somewhat in a more essential manner by assuming, for instance, that the commutator is semisimple and the capacities of its simple components are all equal, together with some further requirements in connection with Φ. However, the condition is needed more definitely in regard to (Lemma 1.4 and) the Galois correspondence; cf. the example of Teichmüller alluded to above.

(iii) *The condition of simple commutators* (*continued*). It might be of some interest and use to observe that instead of demanding directly the simpleness of the commutator $V_A(B)$ of B, where B is a subring of A containing the invariant system $I=I(\Phi)$ of a regular automorphism group Φ, we may demand that the commutator $V_A(RB)$ of the product (or, of the ring generated by R, B) be simple, where R is any subring of $S=V_A(T_\Phi)$. For, since $V_A(R)\supseteq V_A(S)=T_\Phi=V_A(I)\supseteq V_A(B)$, we have $V_A(RB)=V_A(B)$.

(iv) *Algebra case.* In the case where our simple ring A is finite over its center Z, every automorphism of A leaving Z elementwise fixed is inner, and the Galois theory of A is, roughly speaking, a combination of the theory of inner automorphisms of A and the Galois theory of the (commutative) field Z (see Hochschild [7]; cf. also Baer [4]). In fact, our U in Theorem 2 may be chosen to be A itself (in case $[A:Z]<\infty$), giving $R=Z$. Thus the theorem and a remark which accompanied it, at the end of §2, show that if I is the invariant system of a regular automorphism group, ZI is simple, $[Z:Z\cap I]$ is finite, and the product ZI is direct over $Z\cap I$, showing that I is regular in the sense of [7, Definition 2.3]. So, our Theorems 1, 2, together with the accompanying remark, generalize [7, Theorem 2.1]. On the other hand, (if $[A:Z]$ $<\infty$ and) if ZB is simple, then the commutator $V_A(B)$ is simple too. Thus an almost regular subring B in the sense of [7, Definition 2.3] is nothing but a simple subring with simple commutator, and our Theorems 3, 4 form a generalization of [7, Theorem 2.2]. (We could also apply our above remark (iii) to $R=Z$.) Needless to say, our Theorem 6 specializes to [7, Theorem 2.4] in case of an algebra A. Further, our Theorem 7 generalizes Theorem 2.5 in [7]. Indeed, our side condition $T_\Phi\cap V_A(V_A(B))\subseteq T_\Lambda$ is automatically satisfied in case $[A:Z]<\infty$. Observe, to see this, that $V_A(V_A(B))=ZB$ (in case

$[A:Z]<\infty$). Taking a B-basis of ZB consisting of elements of Z and observing that $T_\Phi=V_A(I)$ $(I\subseteq B)$ we see readily that $T_\Phi\cap ZB=Z(T_\Phi\cap B)$, and this last product is contained in T_Λ since both Z and $T_\Phi\cap B(=V_B(I))$ are contained in T_Λ. We note also that our Theorem 7 improves [7, Theorem 2.4] in the algebra case by showing that the assumption of the simpleness of $I\cdot$(center of B)$\cdot Z$ ($=LPC$ in the notation of [7]) is rather unnecessary.

(v) *Sfield case*. Concerning the case of a sfield A, we merely mention the following facts. Firstly, and methodologically, A is trivially minimal (hence homogeneously fully reducible) with respect to any operator domain containing A_l, say. Secondly, and with respect to the results formulated, every subring possesses a simple (in fact, sfield) commutator, and T_Φ is, again trivially, a simple ring (in fact a sfield) for any automorphism group Φ.

(vi) *Complete primitive rings*. Our theory can easily be extended to the case of a complete primitive ring (cf. [6; 10; 11; 16]; they were called closed irreducible in [11; 16]). It is in fact possible to transfer our arguments step by step to this case. However, without doing so, let us observe that Galois theory of such a ring, with respect to a regular automorphism group (of finite reduced order), can be reduced to that of a simple ring. Let, namely, A be a (right-) complete primitive ring, and Φ be a regular automorphism group of A, defined exactly in the same manner as in §2. Thus T_Φ is a simple ring finite over the center Z of A. We consider

$$\Phi A_r = \rho_1 T_l A_r + \rho_2 T_l A_r + \cdots + \rho_g T_l A_r \tag{8}$$

(rather than ΦA_l), where $T=T_\Phi$ and $\rho_1, \rho_2, \cdots, \rho_g$ form a representative system of Φ/Φ_0. Here T_lA_r ($\simeq T_l\times A_r$ over $Z_l=Z_r$) is a complete primitive ring too. Let $\mathfrak{z}$ be the (unique) smallest two-sided ideal of A. Minimal A_r-two-sided modules $\rho_1\mathfrak{z}_r, \rho_2\mathfrak{z}_r, \cdots, \rho_g\mathfrak{z}_r$ are all mutually nonisomorphic, the proof being similar to the one in [16, Lemma 7]. Hence no two of the A_r-two-sided modules $\rho_1T_l\mathfrak{z}_r, \rho_2T_l\mathfrak{z}_r, \cdots, \rho_gT_l\mathfrak{z}_r$ have mutually isomorphic composition residue-modules. A fortiori, they have no mutually isomorphic composition residue-modules as T_lA_r-two-sided modules. Here $T_l\mathfrak{z}_r$ is the smallest two-sided ideal of T_lA_r, and it follows that the automorphisms of T_lA_r induced by $\rho_1, \rho_2, \cdots, \rho_g$ all belong to different automorphism-classes (of T_lA_r). By [16, Theorem 14] our ΦA_r is a primitive ring with minimal right- (or left-) ideals, the directness of the summation in (8) being immediate. Let $\mathfrak{n}$ be a minimal right-ideal of ΦA_r, and denote by $\mathfrak{N}$ the absolute endomorphism ring of $\mathfrak{n}$. Then $\mathfrak{n}$ is a direct sum of a finite number of faithful minimal T_lA_r-(right-) modules, as is seen from the proof of Theorem 14 in [16]. Also, $\mathfrak{n}$ is a direct sum of a finite number of faithful minimal A_r-modules. Thus the A_r-endomorphism ring $V_\mathfrak{N}(A_r)$ of $\mathfrak{n}$ is a simple ring (with minimum condition). Further[13], $V_\mathfrak{N}(V_\mathfrak{N}(A_r))=A_r$. For each $\rho\in\Phi$ we have $\rho^{-1}V_\mathfrak{N}(A_r)\rho$

[13] Here, and in the following, A_r, Φ, T_r are all considered as operator domains of (the ΦA_r-module) $\mathfrak{n}$.

$=V_{\mathfrak{R}}(A_r)$, since $\rho^{-1}A_r\rho=A_r$, and each ρ induces thus an automorphism of $V_{\mathfrak{R}}(A_r)$. By [16, Theorem 5] it is readily seen that $\rho_1, \rho_2, \cdots, \rho_g$ induce automorphisms of $V_{\mathfrak{R}}(A_r)$ all belonging to distinct automorphism-classes of $V_{\mathfrak{R}}(A_r)$. $T_rV_{\mathfrak{R}}(A_r)$ $(\simeq T_r\times_{z_r}V_{\mathfrak{R}}(A_r))$ is a simple ring and

$$\rho_1T_rV_{\mathfrak{R}}(A_r)+\rho_2T_rV_{\mathfrak{R}}(A_r)+\cdots+\rho_gT_rV_{\mathfrak{R}}(A_r) \tag{9}$$

is also a simple ring, as we readily see, either as above or as in Lemma 2.1 (together with the directness of the sum); for the ring property of (9) observe that $V_{\mathfrak{R}}(A_r)\supseteq T_l$ $(\subseteq\Phi A_r)$ and thus $\Phi_0\subseteq T_rT_l\subseteq T_rV_{\mathfrak{R}}(A_r)$. $\mathfrak{n}$ is, hence, decomposed into a direct sum of (perhaps infinitely many) submodules minimal with respect to (9). Let

$$\mathfrak{n}=\cdots\oplus\mathfrak{n}_\mu\oplus\cdots \tag{10}$$

be such a decomposition of $\mathfrak{n}$. All $\mathfrak{n}_\mu$ are isomorphic with respect to (9), and a fortiori with respect to $V_{\mathfrak{R}}(A_r)$. Let $\{\epsilon_{\mu\nu}\}$ be a system of matric unit endomorphisms of $\mathfrak{n}$ with respect to this homogeneous decomposition (10). The $\epsilon_{\mu\nu}$ are in A_r $(=V_{\mathfrak{R}}(V_{\mathfrak{R}}(A_r)))$. They also commute with every element of (9) and thus are invariant under Φ. Then $\epsilon_{11}A_r\epsilon_{11}$ is setwise invariant under Φ, and Φ can be considered as its automorphism group. Now $\epsilon_{11}A_r\epsilon_{11}$ is a simple ring (with minimum condition), since $\mathfrak{n}_1$ (and in fact every $\mathfrak{n}_\mu$) is a direct sum of a finite number of minimal $V_{\mathfrak{R}}(A_r)$-modules, and Φ is regular as its automorphism group, as we see readily. The Galois theory of A with respect to Φ can now be reduced to that of the simple ring $\epsilon_{11}A_r\epsilon_{11}$, with respect to Φ; the argument is parallel with [16, §7].

(vii) *Regular groups of infinite reduced orders.* The first step in generalizing our theory to an automorphism group of infinite reduced order (in the natural sense) is to allow either $(\Phi:\Phi_0)$ or $(T_\Phi:Z)$ to be infinite and to restrict the other to be finite. Each case produces difficulties. With respect to the first, Jacobson has recently established an elegant infinite outer Galois theory $(\Phi_0=1)$ for a sfield A. The writer has collaborated with him in extending the theory to the case $\Phi_0\neq 1$ under more restrictions than $(T_\Phi:Z)<\infty$. But the last result still seems indecisive.

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