REMARKS ON SOME MODULAR IDENTITIES

BY

MORRIS NEWMAN

Introduction. We shall consider a certain class of functions invariant with respect to the substitutions of the congruence subgroup $\Gamma_0(p)$ of the modular group Γ. By specializing these functions, we shall obtain classical identities in the analytical theory of numbers: E.g., the Ramanujan identities for partitions modulo 5, 7 and Mordell's identity for $\tau(n)$. We shall also derive some new identities.

These functions bear some resemblance to those considered by Rademacher in his paper [1](1) to prove the Ramanujan identities, certain modular equations, etc. The type of function considered, however, seems first to have been studied by Watson in his paper [2].

1. Definitions, notations.

(1.1) Γ is the full modular group; i.e., the group of 2×2 matrices of determinant 1 with rational integral elements.

(1.2) $\Gamma_0(m)$ is the subgroup of Γ characterized as follows: The element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of Γ belongs to $\Gamma_0(m)$ if and only if $c\equiv 0\ (m)$.

$$(1.3)\qquad \begin{aligned} \eta(\tau) &= \exp \pi i\tau/12\cdot \prod (1 - x^n) \\ &= \exp \pi i\tau/12\cdot \left\{\sum p(n)x^n\right\}^{-1}, \qquad x = \exp 2\pi i\tau,\ im\tau > 0. \end{aligned}$$

Here, as in the sequel, all products will be extended from 1 to ∞ and all sums from 0 to ∞, unless otherwise indicated.

A few words about $\eta(\tau)$ (the Dedekind η-function) are in order. In the interior of the upper half-plane, $\eta(\tau)$ is free from poles and zeros. At $\tau=i\infty$ and at $\tau=0$, $\eta(\tau)$ is zero. $\eta(\tau)$ is a modular form of dimension $-1/2$, and satisfies the following transformation formula, which will be used extensively:

(1.4) If

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad \text{and} \quad c > 0,$$

then

$$\eta(V\tau) = \left\{-i(c\tau + d)\right\}^{1/2} \exp -\pi i N\cdot \eta(\tau),$$

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(1) Numbers in brackets refer to bibliography at end of paper.

where $N = s(a, c) - (a+d)/12c$. Also, $\eta(\tau+1) = \exp \pi i/12 \cdot \eta(\tau)$.

$s(a, c)$ is a "Dedekind sum" and is defined as follows: If we set

$$((x)) = \begin{cases} x - [x] - 1/2 & x \text{ nonintegral} \\ 0 & x \text{ integral,} \end{cases}$$

then $s(a, c) = \sum ((r/c))((ar/c))$, where r runs over a complete set of residues modulo c in the summation.

We shall put

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is well known that S and T are generators of Γ. We observe that

$$S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

$$W^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad T^2 = -I.$$

Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)/(c\tau + d) = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \tau,$$

$T^2\tau = \tau$.

We shall also put

$$z_p = \exp -2\pi i/p\tau, \qquad \prod (1 - x^n)^r = \sum p_r(n) x^n.$$

We shall be concerned with series whose coefficients are $p_r(n)$'s.

Notice that $p_{-1}(n) = p(n)$ in this notation.

Some remarkable congruence properties of the $p_r(n)$'s modulo 5, 7 have been found by Ramanathan in his paper [6].

2. **Functions on** $\Gamma_0(p)$. In what follows, p will always be a prime greater than 3. The fundamental region Q_p of $\Gamma_0(p)$ has only the parabolic points $i\infty$, 0. A function on $\Gamma_0(p)$ (invariant with respect to the substitutions of $\Gamma_0(p)$) which is regular in the interior of Q_p and bounded at $i\infty$, 0 accordingly is constant. For this reason functions on $\Gamma_0(p)$ are amenable to numerical calculation. We need the following elementary theorem, the proof of which we omit:

(2.1) $\Gamma_0(p^2)$ is of index p in $\Gamma_0(p)$, and a set of right representatives for $\Gamma_0(p^2)$ in $\Gamma_0(p)$ is given by:

$$R_k = W^{-pk}, \qquad k = 0, 1, \cdots, p - 1.$$

We also need the following:

(2.2) THEOREM. *Let* Γ_0, Γ_1 *be subgroups of* Γ, $\Gamma_0 \supseteq \Gamma_1$, $(\Gamma_0 : \Gamma_1) = \mu < \infty$. *Let* $R_0, R_1, \cdots, R_{\mu-1}$ *be a set of right representatives for* Γ_1 *in* Γ_0. *Let* $g(\tau)$ *be a function on* Γ_1, *and let* $F(x_0, x_1, \cdots, x_{\mu-1})$ *be a symmetric function of its variables. Put* $f(\tau) = F(g(R_0\tau), g(R_1\tau), \cdots, g(R_{\mu-1}\tau))$. *Then* $f(\tau)$ *is a function on* Γ_0.

Proof. Let $M \in \Gamma_0$.

R_iM may be written as M_iR_i', where $M_i \in \Gamma_1$, and R_i' is a representative. Also, if $R_iM = M_iR_i'$, $R_jM = M_jR_j'$, then if $R_i' = R_j'$, we have $R_iR_j^{-1} = M_iM_j^{-1} \in \Gamma_1$, so that $R_i = R_j$. That is, as R_i runs through a complete set of representatives, so does R_i'.

Hence

$$\begin{aligned} f(M\tau) &= F(g(R_0M\tau), g(R_1M\tau), \cdots, g(R_{\mu-1}M\tau)) \\ &= F(g(M_0R_0'\tau), g(M_1R_1'\tau), \cdots, g(M_{\mu-1}R_{\mu-1}'\tau)) \\ &= F(g(R_0'\tau), g(R_1'\tau), \cdots, g(R_{\mu-1}'\tau)) \\ &= f(\tau). \end{aligned}$$

The function $\eta(p^2\tau)/\eta(\tau) = x^\nu \prod \{(1-x^{p^2n})/(1-x^n)\}$, where $\nu = (p^2-1)/24$ (which is an integer since $(p, 6) = 1$), may easily be shown to be a function on $\Gamma_0(p^2)$ by use of the transformation formula (1.4) and some theorems about $s(a, c)$ which may be found in [1] or in [3]. We shall omit the proof. We put

(2.3.1) $$h = h(\tau) = \eta(p^2\tau)/\eta(\tau),$$

(2.3.2) $$S_r = S_r(\tau) = \sum_{n=0}^{p-1} h^r(R_n\tau), \ r \text{ integral.}$$

By (2.1) and (2.2), S_r is a function on $\Gamma_0(p)$. The S_r's are the functions we shall consider. Clearly, S_r is regular and bounded in the interior of the upper half-plane (see the remark to (1.3)). Hence to determine the behaviour of S_r completely, we need only know its behaviour at the parabolic points $i\infty$, 0 of Q_p. It will be our purpose in general to compare S_r with

(2.3.3) $$g = g(\tau) = \{\eta(p\tau)/\eta(\tau)\}^s,$$

where $s = s(p)$ is the least positive even integer such that $s(p-1) \equiv 0$ (24). g is also a function on $\Gamma_0(p)$ (see [1]). g has a zero of order $s(p-1)/24$ at $\tau = i\infty$ in x, and a pole of order $s(p-1)/24$ at $\tau = 0$ in z_p. For

$$\begin{array}{ccc} p = 5 & p = 7 & p = 13 \\ s = 6, & s = 4, & s = 2, \end{array}$$

g is a "Hauptmodul" for $\Gamma_0(p)$: I.e., any function on $\Gamma_0(p)$ with polar singularities at most in appropriate uniformizing variables is a rational function of g.

We shall rewrite S_r. We have

$h(R_n\tau) = \eta(p^2R_n\tau)/\eta(R_n\tau) = \eta(p^2W^{-pn}\tau)/\eta(W^{-pn}\tau) = \eta(pW^{-n}p\tau)/\eta(W^{-pn}\tau).$

For $(n, p)=1$ define n' as the least positive solution of the congruence $nx\equiv 1\ (p)$, and for $p|n$ define $n'=0$.

If $(n, p)=1$, we may rewrite $pW^{-n}p\tau$ as follows:

$$pW^{-n}p\tau = M_1(\tau - n'/p), \qquad \text{where } M_1 = \begin{pmatrix} -p & -n' \\ n & (nn'-1)/p \end{pmatrix}.$$

Hence $h(R_n\tau)=\eta(M_1(\tau-n'/p))/\eta(W^{-pn}\tau)$. Making use of (1.4), we obtain $h(R_n\tau)=p^{-1/2}\exp -\pi i N_1\cdot\eta(\tau-n'/p)/\eta(\tau)$, where

$$N_1 = \{s(-p, n) - (-n + (nn'-1)/p)/12n\} - \{s(-1, pn) + 1/6pn\}.$$

It may further be shown by theorems from [1] and [3] that $\exp -\pi i N_1 = \exp -\pi i(p-1)/4\cdot\exp \pi i n'p/12\cdot(n'/p)$, where (n'/p) denotes the Legendre-Jacobi symbol of quadratic reciprocity. Hence

$$h(R_n\tau) = p^{-1/2}\exp -\pi i(p-1)/4\cdot\exp \pi i n'p/12\cdot(n'/p)\cdot\eta(\tau - n'/p)/\eta(\tau),$$

and so

$$(2.4)\qquad \begin{aligned} S_r &= \sum_{n=0}^{p-1} h^r(R_n\tau) = h^r(\tau) + \sum_{n=1}^{p-1} h^r(R_n\tau) \\ &= h^r(\tau) + c^r\eta^{-r}(\tau)\sum_{n=1}^{p-1}(n'/p)^r\cdot\exp \pi i n'pr/12\cdot\eta^r(\tau - n'/p), \end{aligned}$$

where we have put $c=p^{-1/2}\exp -\pi i(p-1)/4$.

Distinguishing cases r even and r odd, and using the well known Gaussian sum formula

$$\sum_{n=1}^{p-1}(n/p)\exp 2\pi i n a/p = \exp \pi i(p-1)^2/8\cdot p^{1/2}(a/p),$$

we find

(2.5.1) For r even,

$$S_r = -c^r + h^r(\tau) + pc^r\prod(1-x^n)^{-r}\sum_{\lambda\geqq 0,\lambda\equiv r\nu(p)} p_r(\lambda)x^\lambda.$$

(2.5.2) For r odd,

$$S_r = h^r(\tau) + \exp \pi i(p-1)^2/8\cdot p^{1/2}c^r\prod(1-x^n)^{-r}\sum_{\lambda\geqq 0}((r\nu-\lambda)/p)p_r(\lambda)x^\lambda.$$

The formulae (2.5.1), (2.5.2) furnish the desired information as to the behaviour of the S_r's at $i\infty$. To study the behaviour of the S_r's at 0, we subject them to the transformation T and study the TS_r's at $i\infty$ [2]. Proceeding

[2] A device employed by Rademacher in [1].

as before, we have for $h(R_nT\tau)$, $(n, p)=1$:

$$h(R_nT\tau) = c(n'/p) \exp \pi i n'p/12 \cdot \eta(T\tau - n'/p)/\eta(T\tau).$$

We may rewrite $T\tau - n'/p$ as follows:

$$T\tau - n'/p = M_2(\tau + np)/p^2, \qquad \text{where } M_2 = \begin{pmatrix} -n' & (nn'-1)/p \\ p & -n \end{pmatrix}.$$

Making use of (1.4) again,

$$\begin{aligned} h(R_nT\tau) &= c(n'/p) \exp \pi i n'p/12 \cdot \eta(M_2(\tau + np)/p^2)/\eta(T\tau) \\ &= p^{-1/2}c(n'/p) \exp \pi i n'p/12 \cdot \exp -\pi i N_2 \cdot \eta((\tau + np)/p^2)/\eta(\tau), \end{aligned}$$

where $N_2 = s(-n', p) + (n+n')/12\,p$.

After the necessary simplifications, we obtain

$$h(R_nT\tau) = p^{-1} \exp -\pi i n p/12 \cdot \eta\left(\frac{\tau + np}{p^2}\right)/\eta(\tau).$$

If we note here that $h(R_nT\tau)$ agrees formally with $h(T\tau)$ for $n=0$, we have

$$\begin{aligned} TS_r &= \sum_{n=0}^{p-1} h^r(R_nT\tau) \\ &= p^{-r}\eta^{-r}(\tau) \sum_{n=0}^{p-1} \exp -\pi i n p r/12 \cdot \eta^r\left(\frac{\tau + np}{p^2}\right)/\eta(\tau). \end{aligned} \tag{2.6}$$

Expanding into a power series, we have

$$TS_r = p^{-r+1} \prod (1 - x^n)^{-r} \sum_{\lambda \geqq 0, \lambda \equiv r\nu(p)} p_r(\lambda) x^{(\lambda - r\nu)/p^2}. \tag{2.7}$$

A study of (2.5.1), (2.5.2), and (2.7) leads us to the construction of the following table:

(2.8) (a) $r > 0$.

	$i\infty$	0
S_r:	Zero-free and pole-free in x unless r is odd and divisible by p, in which case zero of order 1 in x.	Pole of order $[r\nu/p]$ in z_p(3).
$S_r + c^r$, r even:	Zero of order $r\nu - p[r\nu/p]$ in x(3).	Pole of order $[r\nu/p]$ in z_p(3).

(b) $r < 0$. Put $r = -r_1$, $r_1 > 0$. We then have:

	$i\infty$	0
S_r:	Pole of order $r_1\nu$ in x.	Zero of order $1 + [r_1\nu/p]$ in z_p.

3. **Applications.**

(3) Provided that $p_r(r\nu - p[r\nu/p]) \neq 0$.

(3.1) $r=-1,\ p=5$.

Then S_{-1} has a pole of order 1 in x at $\tau=i\infty$, and a zero of order 1 in z_5 at $\tau=0$. This implies that the product gS_{-1} is constant, or that

$$S_{-1} = K_0 g^{-1}.$$

If we replace τ by $-1/5\tau$ and evaluate K_0, we obtain the first of Ramanujan's identities:

$$\sum p(5n+4)x^n = 5\prod (1-x^{5n})^5(1-x^n)^{-6}.$$

(3.2) $r=-1,\ p=7$.

Then S_{-1} has a pole of order 2 in x at $\tau=i\infty$, and a zero of order 1 in z_7 at $\tau=0$. This implies that g^2S_{-1} is linear in g, or that

$$S_{-1} = K_0 g^{-2} + K_1 g^{-1}.$$

If we replace τ by $-1/7\tau$ and evaluate K_0, K_1, we obtain the second of Ramanujan's identities:

$$\sum p(7n+5)x^n = 7\prod (1-x^{7n})^3(1-x^n)^{-4} + 49x\prod (1-x^{7n})^7(1-x^n)^{-8}.$$

(3.3) $r=-1,\ p=13$.

Then S_{-1} has a pole of order 7 in x at $\tau=i\infty$, and a zero of order 1 in z_{13} at $\tau=0$. This implies that g^7S_{-1} is a polynomial in g of degree 6, or that

$$S_{-1} = K_0 g^{-7} + K_1 g^{-6} + \cdots + K_6 g^{-1}.$$

If we replace τ by $-1/13\tau$ and evaluate $K_0, K_1, \cdots, K_6$, we obtain an identity of Zuckerman (see [3]).

(3.4) $r=24$.

Then $S_{24}+c^{24}$ has a zero of order $p-1$ at least in x at $\tau=i\infty$, and a pole of order $p-1$ at most in z_p at $\tau=0$. This implies that the quotient $(S_{24}+c^{24})/\{\eta(p\tau)/\eta(\tau)\}^{24}$ is bounded, which in turn implies that

$$S_{24} = -c^{24} + K_0\{\eta(p\tau)/\eta(\tau)\}^{24}.$$

If we replace τ by $-1/p\tau$, evaluate K_0, and set $\tau(n)=p_{24}(n-1)$, we obtain Mordell's identity for $\tau(n)$(4):

$$\sum \tau(np+p)x^n = \tau(p)\prod (1-x^n)^{24} - p^{11}x^{p-1}\prod (1-x^{np})^{24}.$$

(3.5) We can easily generalize Mordell's identity as follows: Choose r even, $0<r\leqq 24$, $r(p-1)\equiv 0$ (24). Put $\delta=r(p-1)/24$. Then we have for S_r+c^r:

	$i\infty$	0
S_r+c^r:	Zero of order δ at least in x.	Pole of order δ at most in z_p.

This implies that the quotient $(S_r+c^r)/\{\eta(p\tau)/\eta(\tau)\}^r$ is bounded which

(4) See [5].

in turn implies that

$$S_r = -c^r + K_0\{\eta(p\tau)/\eta(\tau)\}^r.$$

If we replace τ by $-1/p\tau$ and evaluate K_0, we obtain

$$\sum p_r(np+\delta)x^n = p_r(\delta)\prod(1-x^n)^r - p^{r/2-1}x^\delta\prod(1-x^{np})^r.$$

(3.6) $p=5$.

We have the following table:

	$i\infty$	0
S_1:	Zero-free and pole-free in x.	Zero-free and pole-free in z_5.
S_2:	Zero-free and pole-free in x.	Zero-free and pole-free in z_5.
S_3:	Zero-free and pole-free in x.	Zero-free and pole-free in z_5.
S_4:	Zero-free and pole-free in x.	Zero-free and pole-free in z_5.
S_5:	Zero of order 1 in x.	Pole of order 1 in z_5.

These imply that S_1, S_2, S_3, S_4 are constant, while S_5 is proportional to $B=\{\eta(5\tau)/\eta(\tau)\}^6$. Hence in the polynomial $\prod_{i=0}^4(u-h(R_i\tau))=u^5-c_1u^4+c_2u^3-c_3u^2+c_4u-c_5$, the coefficients c_1, c_2, c_3, c_4 must also be constant, while c_5 is linear in B(5).

Putting $u=h(\tau)=\eta(25\tau)/\eta(\tau)=A$, we find that B is a polynomial of degree 5 in A. The actual polynomial turns out to be

$$B = 25A^5 + 25A^4 + 15A^3 + 5A^2 + A. \tag{3.6.1}$$

If we set $A_0=5^{1/2}A$, $B_0=B/A$, then (3.6.1) reads

$$B_0 = A_0^4 + 5^{1/2}A_0^3 + 3A_0^2 + 5^{1/2}A_0 + 1. \tag{3.6.2}$$

(3.6.2) shows the reciprocal nature of the modular equation.

The reason there is an identity (3.6.1) is that $\Gamma_0(25)$ is of genus zero (as are $\Gamma_0(5)$, $\Gamma_0(7)$, $\Gamma_0(13)$) and so a Hauptmodul exists for $\Gamma_0(25)$. (3.6.1) indicates that we may choose $\eta(25\tau)/\eta(\tau)$ for this Hauptmodul(6). The genus of $\Gamma_0(49)$, however, turns out to be 1, so no analogous identity exists for $\{\eta(7\tau)/\eta(\tau)\}^4$ in terms of $\eta(49\tau)/\eta(\tau)$. The polynomial relationship between these functions is *quadratic* in $\{\eta(7\tau)/\eta(\tau)\}^4$.

Some new identities (of the many which are possible) obtained by specializing the S_r's follow, without comment. Of particular interest are those for which p differs from 5,7,13 (since then the subgroups in question are not of genus zero). These identities are isolated instances, not at all characteristic

(5) The c's are the elementary symmetric functions on the roots, and the result follows from Newton's formulae relating the c's and the S's.

(6) $\eta(25\tau)/\eta(\tau)$ has a zero of order 1 in x at $\tau=i\infty$, a pole of order 1 in z_{25} at $\tau=0$, and is finite and different from zero at the other parabolic points $\pm 1/5$, $\pm 2/5$ of Q_{25}. That is, it has precisely one pole (in the proper uniformizing variable) in Q_{25}. This guarantees that it is a Hauptmodul for $\Gamma_0(25)$.

of the true situation.

1. $r=5,\ p=5.$

$$\sum p_5(5n)x^n = \prod (1 - x^n)^6(1 - x^{5n})^{-1}\text{(}^7\text{)}.$$

2. $r=5,\ p=7.$

$$\sum p_5(7n + 3)x^n = 10\prod (1 - x^{7n})(1 - x^n)^4 + 49x\prod (1 - x^{7n})^5.$$

3. $r=7,\ p=7.$

$$\sum p_7(7n)x^n = \prod (1 - x^n)^8(1 - x^{7n})^{-1} + 49x\prod (1 - x^n)^4(1 - x^{7n})^3\text{(}^7\text{)}.$$

4. $r=2,\ p=11.$

$$\sum p_2(11n + 10)x^n = \prod (1 - x^{11n})^2.$$

5. $r=4,\ p=11.$

$$\sum p_4(11n + 20)x^n = - 11\prod (1 - x^{11n})^4.$$

6. $r=2,\ p=17.$

$$\sum p_2(17n + 24)x^n = - \prod (1 - x^{17n})^2.$$

7. $r=6,\ p=31.$

$$\sum p_6(31n + 240)x^n = 961\prod (1 - x^{31n})^6.$$

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National Bureau of Standards,
Washington, D. C.

(7) Dr. Lehmer informs me that these identities are known to him, though he has never published them.