THE INTEGRAL GEOMETRY DEFINITION OF ARC LENGTH FOR TWO-DIMENSIONAL FINSLER SPACES

BY

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Introduction. A curve C^* in a 2-dimensional Finsler space, metrized by $ds = F(x, y, dx, dy)$, is admissible if it is a simple, closed, regular, extremal convex curve, where extremal convexity means locally C^* lies in the closure of a component (side) of any tangent geodesic.

A conjecture of Hans Lewy is that an "integral geometry" relation, proved by Legendre for euclidean space, can be generalized to the following:

THEOREM. *Specify the geodesics by suitable parameters θ, p and let D denote the cartesian region consisting of all points (θ, p) for which the corresponding geodesic intersects C^*. Then there exists a contact transformation $(x, y, dx, dy) \rightarrow (\theta, p, d\theta, dp)$ and a density function $\sigma(\theta, p)$ such that the Finsler length of any admissible C^*, extremal convex in the large, is*

$$\oint_{C^*} F(x, y, dx, dy) = \iint_D \sigma(\theta, p) dp d\theta. \tag{1}$$

The integral equality (1) will be established for admissible C^* contained in a region U, where: (a) Certain regularity and boundedness conditions hold for F and specified partial derivatives of F on U; for example, F is to define a regular variational problem. (b) The geodesics of F satisfy a "field hypothesis," see Part II, (3.4). (c) An "embedment hypothesis" holds for C^*, see Part II, (9.1).

Assuming the geodesics are straight lines, Eberhard Hopf devised an elegant proof of (1) and with no restriction as to the sign of F. I thank Professor Hopf for allowing me to present his proof as Part I of this paper. In Part II the author resolves the technical difficulties met in generalizing the Hopf procedure and establishes the theorem and a generalization applicable to C^* not necessarily extremal convex in the large.

PART I

We shall assume that the Finsler metric, $ds = F(x, y, dx, dy)$, has straight lines for geodesics, is of class C^3 in its arguments, and that U is any convex region on which F is defined.

1. **Variation of Finsler length.** Impose the restriction, later to be removed, that C^* encloses the origin of coordinates, and let ϕ denote the positive angle formed by the directed tangent with the polar axis. Then,

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$$\frac{dx}{dt} = \cos\phi, \qquad \frac{dy}{dt} = \sin\phi, \qquad \frac{dy}{dx} = \tan\phi, \tag{1.1}$$

where $dt=((dx)^2+(dy)^2)^{1/2}$. Hence, as $F(x, y, dx, dy)$ is homogeneous of degree one in its differentials,

$$ds = F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right)dt \equiv L(x, y, \phi)dt, \tag{1.2}$$

defining $L(x, y, \phi)$.

If δ denotes the first variation, then

$$\delta dt = (\delta dx)\cos\phi + (\delta dy)\sin\phi. \tag{1.3}$$

Furthermore, since

$$\sec^2\phi\delta\phi = \frac{dx(\delta dy) - dy(\delta dx)}{(dx)^2},$$

it follows that

$$dt\delta\phi = (\delta dy)\cos\phi - (\delta dx)\sin\phi. \tag{1.4}$$

Assuming fixed end points P_0 and P_1, we now compute the variation of

$$I = \int_{P_0}^{P_1} L(x, y, \phi)dt. \tag{1.5}$$

By means of (1.3), (1.4), and the vanishing of the variations at P_0 and P_1, we find

$$\begin{aligned}\delta I &= \int_{P_0}^{P_1} [L_x dt - d(L\cos\phi) + d(L_\phi \sin\phi)]\delta x \\ &\quad + \int_{P_0}^{P_1} [L_y dt - d(L\sin\phi) - d(L_\phi\cos\phi)]\delta y.\end{aligned} \tag{1.6}$$

Therefore, if

$$X \equiv L_x dt - d[-L_\phi \sin\phi + L\cos\phi] \tag{1.7}$$

and

$$Y \equiv L_y dt - d[L_\phi\cos\phi + L\sin\phi], \tag{1.8}$$

then

$$\delta I = \int_{P_0}^{P_1} [X\delta x + Y\delta y]. \tag{1.9}$$

This result also holds for closed curves, $P_0 \equiv P_1$, as well as curvilinear segments bounded by distinct end points.

2. **Simplification of δI.** The differential forms X and Y satisfy the following relations:

$$X \cos \phi + Y \sin \phi = 0, \tag{2.1}$$

$$X \sin \phi - Y \cos \phi = S(x, y, \phi)dt + T(x, y, \phi)d\phi, \tag{2.2}$$

where

$$T(x, y, \phi) \equiv L + L_{\phi\phi} \tag{2.3}$$

and

$$S(x, y, \phi) \equiv L_{x\phi} \cos \phi + L_{y\phi} \sin \phi + L_x \sin \phi - L_y \cos \phi. \tag{2.4}$$

A consequence of (2.3) and (2.4) is

$$\frac{\partial S}{\partial \phi} = \frac{\partial T}{\partial x} \cos \phi + \frac{\partial T}{\partial y} \sin \phi. \tag{2.5}$$

Furthermore, as the extremals of the variational problem, I = minimum, are straight lines, $X = Y = 0$, whenever $d\phi = 0$, which implies

$$S(x, y, \phi) \equiv 0. \tag{2.6}$$

Thus, because of (2.1), (2.2), and (2.6),

$$X = T \sin \phi d\phi, \qquad Y = - T \cos \phi d\phi. \tag{2.7}$$

Substituting (2.7) into (1.9) gives

$$\delta I = \oint_{C^*} T[\delta x \sin \phi - \delta y \cos \phi] d\phi. \tag{2.8}$$

3. **Identity of I with double integral.** Since the closed convex curve C^* encircles the origin, there is a unique $p = p(\phi)$ such that C^* is the envelope of its tangent lines:

$$x \sin \phi - y \cos \phi = p(\phi). \tag{3.1}$$

The parametric equations of C^* are

$$\begin{aligned} x &= p \sin \phi + p' \cos \phi, \\ y &= - p \cos \phi + p' \sin \phi, \end{aligned} \qquad \left(p' \equiv \frac{dp}{d\phi}\right). \tag{3.2}$$

Thus, because of (2.5), (2.6), and (3.2),

$$\frac{\partial T}{\partial p'} = \frac{\partial T}{\partial x} \cos \phi + \frac{\partial T}{\partial y} \sin \phi = \frac{\partial S}{\partial \phi} \equiv 0, \tag{3.3}$$

so $T = T(\phi, p)$ is not a function of p'. Now, let the variation of C^* be that obtained by varying the support function p. Then,

$$\delta p = \delta x \sin \phi - \delta y \cos \phi, \tag{3.4}$$

and (2.8) becomes

$$\delta I = \int_0^{2\pi} T\delta p d\phi = \delta \iint_D T(\phi, p) dp d\phi, \tag{3.5}$$

where D consists of all points (ϕ, p) for which $[0 \leqq \phi \leqq 2\pi;\ 0 \leqq p \leqq p(\phi)]$, or all points (ϕ, p) such that the corresponding line (3.1) intersects C^*. Consequently, as the two integrals in (3.5) have equal variations and as both vanish when C^* is contracted to the origin, they are equal. That is,

$$I \equiv \oint_{C^*} L(x, y, \phi) dt = \iint_D [L + L_{\phi\phi}] dp d\phi, \tag{3.6}$$

where the integrand of the double integral depends only on the variables (ϕ, p).

Consider now the case that C^* does not enclose the origin. Let α and β be the smaller and larger, respectively, of the positive angles formed by the polar axis and the two directed tangents to C^* from the origin. The map of C^* under (3.1) will consist of two separate curvilinear segments:

$$p = p_1(\phi), \qquad p = p_2(\phi), \tag{3.7}$$

where $p_1(\phi) \leqq p_2(\phi)$ and $\alpha \leqq \phi \leqq \pi + \beta$. If D is taken as the set of points (ϕ, p) satisfying the inequalities

$$\alpha \leqq \phi \leqq \pi + \beta, \qquad p_1(\phi) \leqq p \leqq p_2(\phi),$$

then (3.6) still holds and has the same "integral geometry" interpretation.

Part II

We shall assume that the Finsler metric, $ds = F(x, y, dx, dy)$, is of class C^3 in its arguments and defines a regular variational problem on a region U which is extremal convex in the large. Furthermore, on U, a "field hypothesis," (3.4), is satisfied by the geodesics and an "embedment hypothesis," (9.1), by the admissible curves C^*.

1. **The Euler equations.** Any geodesic C: $[x(t), y(t)]$ of the space is a solution of the Euler equations:

$$F_x - \frac{dF_{\dot{x}}}{dt} = 0, \qquad F_y - \frac{dF_{\dot{y}}}{dt} = 0, \tag{1.1}$$

where $(x, y, \dot{x}, \dot{y}) \equiv (x(t), y(t), dx/dt, dy/dt)$. Furthermore, the homogeneity condition

$$F(x, y, k\dot{x}, k\dot{y}) = kF(x, y, \dot{x}, \dot{y}) \qquad (k \geqq 0) \tag{1.2}$$

implies the following:

$$F_1(x, y, \dot{x}, \dot{y}) \equiv \frac{F_{\dot{x}\dot{x}}}{\dot{y}^2} = -\frac{F_{\dot{x}\dot{y}}}{\dot{x}\dot{y}} = \frac{F_{\dot{y}\dot{y}}}{\dot{x}^2}, \tag{1.3}$$

$$F_1(x, y, k\dot{x}, k\dot{y}) = k^{-3}F_1(x, y, \dot{x}, \dot{y}). \tag{1.4}$$

Hence, for regular curves C, that is,

$$\rho \equiv (\dot{x}^2 + \dot{y}^2)^{1/2} \neq 0, \tag{1.5}$$

the Euler equations are equivalent to the single equation:

$$(\dot{x}\ddot{y} - \dot{y}\ddot{x})F_1 + F_{x\dot{y}} - F_{\dot{x}y} = 0 \qquad (F_1 > 0). \tag{1.6}$$

Thus, if

$$G(x, y, \dot{x}, \dot{y}) \equiv (F_{y\dot{x}} - F_{x\dot{y}})/F_1, \tag{1.7}$$

which implies the homogeneity relation

$$G(x, y, k\dot{x}, k\dot{y}) = k^3G(x, y, \dot{x}, \dot{y}), \tag{1.8}$$

the differential equation (1.6) is equivalent to

$$\ddot{x} = -\dot{y}G(x, y, \dot{x}, \dot{y}), \qquad \ddot{y} = +\dot{x}G(x, y, \dot{x}, \dot{y}), \tag{1.9}$$

the parametrization $\rho = 1$ being assumed.

From (1.8) follows

$$GF_{1\dot{y}} + F_{x\dot{y}\dot{y}} - F_{y\dot{x}\dot{y}} + G_{\dot{y}}F_1 = 0, \quad GF_{1\dot{x}} + F_{x\dot{x}\dot{y}} - F_{\dot{x}\dot{x}y} + G_{\dot{x}}F_1 = 0, \tag{1.10}$$

which, because of (1.4), can be written as

$$GF_{1\dot{y}} + \dot{x}^2F_{1x} + \dot{x}_{\dot{y}}F_{1y} + G\dot{y}F_1 = 0, \quad GF_{1\dot{x}} - \dot{x}\dot{y}F_{1x} - \dot{y}^2 F_{1y} + G_{\dot{x}}F_1 = 0. \tag{1.11}$$

Thus, since $\rho = 1$ by hypothesis,

$$G(\dot{x}F_{1\dot{y}} - \dot{y}F_{1\dot{x}}) + \dot{x}\, F_{1x} + \dot{y}F_{1y} + (\dot{x}G_{\dot{y}} - \dot{y}G_{\dot{x}})F_1 = 0. \tag{1.12}$$

2. **Variation of Finsler length.** For a simple, regular arc C: $[(x(t), y(t)), t_0 \leqq t \leqq t_1]$, bounded by prescribed end points $P_0 \equiv (x(t_0), y(t_0))$ and $P_1 \equiv (x(t_1), y(t_1))$, the variation of the Finsler length

$$I = \int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y})dt \tag{2.1}$$

is

$$\delta I = \int_{t_0}^{t_1} \left\{ \left[F_x - \frac{dF_{\dot{x}}}{dt} \right] \delta x + \left[F_y - \frac{dF_{\dot{y}}}{dt} \right] \delta y \right\} dt, \tag{2.2}$$

where we assume that

$$F_{\dot{x}}^2 + F_{\dot{y}}^2 < A, \tag{2.3}$$

a uniform bound for all $\rho \neq 0$. Moreover, as (2.2) also holds for closed arcs, that is, $P_0 = P_1$,

$$\delta I = \oint_C [(\dot{x}\ddot{y} - \dot{y}\ddot{x})F_1 + F_{x\dot{y}} - F_{\dot{x}y}][dy\delta x - dx\delta y] \tag{2.4}$$

for closed curves C. Now, because of (1.9), (2.4) becomes

$$\delta I = \oint_C [(\dot{x}\ddot{y} - \dot{y}\ddot{x}) - G(x, y, \dot{x}, \dot{y})]F_1(x, y, \dot{x}, \dot{y})[dy\delta x - dx\delta y]. \tag{2.5}$$

3. **Transformation of** $F_1(x, y, \dot{x}, \dot{y})$. Here a transformation of the variables $(x, y, \dot{x}, \dot{y})$, with $\dot{x}^2 + \dot{y}^2 = 1$, is defined and the altered form of $F_1(x, y, \dot{x}, \dot{y})$ is determined. The existence of the transformation will be assured by invoking a "field hypothesis," (3.4).

Denote by E an arbitrarily directed lineal element at the origin, which makes a positive angle θ with the positive X-axis, and assign to the geodesic tangent to E the orientation of E. Let this extremal be defined by the parametric equations

$$x = X(0, \theta, p), \qquad y = Y(0, \theta, p) \qquad (0 \leqq \theta < 2\pi), \tag{3.1}$$

where the parameter p is signed euclidean arc length, measured from the origin along the geodesic, the sign being positive or negative according as the sense of measurement coincides with the orientation of the extremal or not.

Consider the extremal which is transverse to the geodesic (3.1) at the point (3.1). Orient it by defining its positive sense to be from the point (3.1) to the positive side, the left, of the geodesic (3.1). This extremal will be defined by parametric equations

$$x = X(s, \theta, p), \qquad y = Y(s, \theta, p), \tag{3.2}$$

the parameter s being signed euclidean arc length, where

$$(dX/ds)^2 + (dY/ds)^2 = 1, \tag{3.3}$$

measured from the point (3.1) along the curve (3.2) and, as above, the sign being positive or negative according as the sense of measurement coincides with the orientation of the extremal or not.

The need for certain field properties, holding in the large for the geodesics (3.2), forces us to postulate the following:

(3.4) FIELD HYPOTHESIS. *There is a neighborhood, U, of the origin which possesses the following properties*: (a) *U is extremal convex in the large.* (b) *Any direct lineal element of U is tangent to a unique equally directed extremal* (3.2). (c) *The jacobians*

$$J_1 = J_1\left(\frac{X, Y, X_s}{s, \theta, p}\right), \qquad J_2 = J_2\left(\frac{X, Y, Y_s}{s, \theta, p}\right)$$

never vanish simultaneously on U.

Hence, a unique triplet (s, θ, p) is associated with any given directed lineal element $(x, y, \dot{x}, \dot{y})$ of U, and

$$(3.5)\qquad \begin{aligned} x &= X(s, \theta, p) & \dot{x}/\rho &= X_s(s, \theta, p), \\ y &= Y(s, \theta, p) & \dot{y}/\rho &= Y_s(s, \theta, p), \end{aligned} \qquad \rho = (x^2 + y^2)^{1/2}.$$

Consequently,

$$(3.6)\qquad F_1 = F_1(X, Y, X_s, Y_s)$$

is a function of (s, θ, p) which when partially differentiated with respect to s yields—recalling (1.9)—

$$(3.7)\qquad \frac{dF_1}{ds} = X_s F_{1x} + Y_s F_{1y} + G[X_s F_{1\dot{y}} - Y_s F_{1\dot{x}}].$$

Therefore, equation (1.12) becomes

$$(3.8)\qquad \frac{dF_1}{ds} + (X_s G_{\dot{y}} - Y_s G_{\dot{x}}) F_1 = 0,$$

yielding

$$F_1(X, Y, X_s, Y_s) = M(\theta, p) \exp\left[-\int_0^s (X_s G_{\dot{y}} - Y_s G_{\dot{x}}) ds\right],$$

for suitably chosen $M = M(\theta, p)$. Finally, because of (1.4),

$$(3.9)\qquad F_1(x, y, \dot{x}, \dot{y}) = \frac{M(\theta, p)}{\rho^3} \exp\left[-\int_0^s (X_s G_{\dot{y}} - Y_s G_{\dot{x}}) ds\right]$$

for arbitrary positive ρ.

4. Evaluations for J_1 and J_2. By reason of (1.9) the geodesics (3.2) are integral curves for the following differential equations:

$$(4.1)\qquad X_{ss} = -Y_s G(X, Y, X_s, Y_s), \qquad Y_{ss} = X_s G(X, Y, X_s, Y_s).$$

It follows that

$$(4.2)\qquad \begin{aligned} J_1\left(\frac{X, Y, X_s}{s, \theta, p}\right) &= Y_s G[X_p Y_\theta - X_\theta Y_p] + X_{s\theta}[X_p Y_s - X_s Y_p] \\ &\quad + X_{sp}[X_s Y_\theta - X_\theta Y_s], \end{aligned}$$

$$(4.3)\qquad \begin{aligned} J_2\left(\frac{X, Y, Y_s}{s, \theta, p}\right) &= X_s G[Y_p X_\theta - X_p Y_\theta] + Y_{s\theta}[X_p Y_s - X_s Y_p] \\ &\quad + Y_{sp}[X_s Y_\theta - X_\theta Y_s]. \end{aligned}$$

We now deduce from (3.3) that

$$X_sX_{sp} + Y_sY_{sp} = 0 \tag{4.4}$$

and

$$X_sX_{s\theta} + Y_sY_{s\theta} = 0, \tag{4.5}$$

which imply

$$X_{sp}Y_{s\theta} - X_{s\theta}Y_{sp} = 0. \tag{4.6}$$

These above equations immediately yield

$$X_sJ_1 + Y_sJ_2 = 0. \tag{4.7}$$

We shall now show that $(Y_sJ_1 - X_sJ_2)$ is the solution of a certain differential equation. A rather tedious direct calculation shows that

$$\begin{aligned}\frac{\partial J_1}{\partial s} = (X_sG_{\dot y} - Y_sG_{\dot x})J_1 + X_sG(X_pX_{s\theta} + Y_pY_{s\theta} - Y_\theta Y_{sp} - X_\theta X_{sp}) \\ + X_sG^2(X_pY_\theta - X_\theta Y_p),\end{aligned} \tag{4.8}$$

where the simplification only involves using (4.4), (4.5), and (4.6). Another like calculation shows that

$$\begin{aligned}\frac{\partial J_2}{\partial s} = (X_sG_{\dot y} - Y_sG_{\dot x})J_2 + Y_sG(Y_pY_{s\theta} + X_pX_{s\theta} - Y_\theta Y_{sp} - X_\theta X_{sp}) \\ - Y_sG^2(Y_pX_\theta - X_pY_\theta).\end{aligned} \tag{4.9}$$

We now infer from (4.8) and (4.9) the relation

$$Y_s\frac{\partial J_1}{\partial s} - X_s\frac{\partial J_2}{\partial s} = (X_sG_{\dot y} - Y_sG_{\dot x})(Y_sJ_1 - X_sJ_2), \tag{4.10}$$

and from (4.1) and (4.7)

$$\frac{\partial}{\partial s}(Y_sJ_1 - X_sJ_2) = Y_s\frac{\partial J_1}{\partial s} - X_s\frac{\partial J_2}{\partial s}. \tag{4.11}$$

Thus, because of (4.10) and (4.11),

$$\frac{\partial}{\partial s}(Y_sJ_1 - X_sJ_2) = (X_sG_{\dot y} - Y_sG_{\dot x})(Y_sJ_1 - X_sJ_2), \tag{4.12}$$

which implies

$$Y_sJ_1 - X_sJ_2 = N(\theta, p)\exp\left[\int_0^s (X_sG_{\dot y} - Y_sG_{\dot x})ds\right], \tag{4.13}$$

for suitably chosen $N = N(\theta, p)$.

An important relation immediately derivable from (3.9) and (4.13) is

$$F_1(x, y, \dot{x}, \dot{y}) = \frac{M(\theta, p)N(\theta, p)}{\rho^3[Y_s J_1 - X_s J_2]} \cdot \tag{4.14}$$

Since, by hypothesis, $F_1(x, y, \dot{x}, \dot{y}) > 0$ and $(Y_s J_1 - X_s J_2) \neq 0$, (4.14) implies that the density $MN \neq 0$.

5. **A contact transformation.** Henceforth, the simple, closed, regular curve C: $[(x(t), y(t)), t_0 \leqq t \leqq t_1]$ will be traced in the positive sense, that is, C is so traversed, for increasing values of t, that its finite component is to the left. Because of the "field hypothesis," associated with $(x, y, \dot{x}, \dot{y})$ is a unique triplet (s, θ, p), and

$$\begin{aligned} x &= X(s, \theta, p), & \dot{x} &= \rho X_s(s, \theta, p), \\ y &= Y(s, \theta, p), & \dot{y} &= \rho Y_s(s, \theta, p), \end{aligned} \qquad \rho = (x^2 + y^2)^{1/2}. \tag{5.1}$$

An easily established consequence of (5.1) is

$$\dot{\theta} = \tau[X_s Y_p - X_p Y_s], \qquad \dot{p} = \tau[X_\theta Y_s - X_s Y_\theta], \tag{5.2}$$

where

$$\tau = \frac{(\rho - \dot{s})}{[X_\theta Y_p - Y_\theta X_p]} \cdot \tag{5.3}$$

Equations (5.1) and (5.2) define a contact transformation of directed lineal elements $(x, y, \dot{x}, \dot{y}) \subset U$ into lineal elements $(\theta, p, \dot{\theta}, \dot{p})$. Furthermore, as seen in §8, any regular, extremal convex curve C is mapped by these equations into a regular curve F^*: $[(\theta(t), p(t)), t_0 \leqq t \leqq t_1]$ in the cartesian (θ, p)-plane.

6. **Transformation of δI.** The altered form of δI, (2.5), under the contact transformation will now be computed.

Relations (1.9), (5.1), and (5.2) imply

$$\begin{aligned} (\dot{x}\ddot{y} - \dot{y}\ddot{x}) = {} & \rho^2 \dot{s} G(X, Y, X_s, Y_s) \\ & + \rho^2 \tau[X_{sp}(Y_\theta Y_s X_s - Y_s^2 X_\theta) + X_{s\theta}(X_p Y_s^2 - X_s Y_p Y_s)] \\ & + \rho^2 \tau[Y_{sp}(Y_s X_\theta X_s - Y_\theta X_s^2) + Y_{s\theta}(X_s^2 Y_p - X_p Y_s X_s)]. \end{aligned} \tag{6.1}$$

Now, computing the value of $X_s J_2 - Y_s J_1$ and using it in the simplification of (6.1) yields

$$(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = \rho^2[\dot{s} + \tau(X_\theta Y_p - X_p Y_\theta)]G + \tau\rho^2[Y_s J_1 - X_s J_2]. \tag{6.2}$$

Finally, by reason of (1.8) and (5.3), (6.2) reduces to

$$(\dot{x}\ddot{y} - \dot{y}\ddot{x}) = G(x, y, \dot{x}, \dot{y}) + \tau\rho^2[Y_s J_1 - X_s J_2]. \tag{6.3}$$

Consequently, by reason of (2.5), (4.14), (5.1), and (6.3), the variation

δI can be expressed as

$$\delta I = \int_{F^*} M(\theta, p)N(\theta, p)\tau\left[\frac{Y_s\delta x - X_s\delta y}{(\dot\theta^2 + \dot p^2)^{1/2}}\right]((d\theta)^2 + (dp)^2)^{1/2}. \tag{6.4}$$

7. **Map of admissible curves** C^*. If the admissible curve $C^* \subset U$ encloses the origin, a restriction which is removed in §9, then, as we now show, the map of C^*, under the contact transformation, is a regular arc F^*: $[(\theta(t), p(t)), t_0 \leqq t \leqq t_1]$, in the cartesian (θ, p)-plane, bounded by end points $Q_0 \equiv (\theta(t_0), p(t_0))$, and $Q_1 \equiv (\theta(t_1), p(t_1)) = (\theta(t_0) + 2\pi, p(t_0))$.

The end point condition for F^* is automatically fulfilled, because of the nature of the contact transformation. Furthermore, the regularity of the arc F^* follows from (5.2) if it can be asserted that

$$[X_sY_p - X_pY_s]^2 + [X_\theta Y_s - X_sY_\theta]^2 \neq 0 \tag{7.1}$$

and

$$\tau \neq 0. \tag{7.2}$$

The negation of (7.1) gives

$$X_sY_\theta - X_\theta Y_s = 0, \qquad X_sY_p - X_pY_s = 0, \tag{7.3}$$

which, in turn, implies

$$X_\theta Y_p - Y_\theta X_p = 0, \tag{7.4}$$

since $X_s^2 + Y_s^2 = 1$. Therefore, J_1 and J_2, see (4.2) and (4.3), both vanish, which contradicts the "field hypothesis."

We now establish (7.2). Extremal convexity of C^* implies that on C^*

$$\frac{\dot x\ddot y - \dot y\ddot x}{\rho^3} - G\left(x, y, \frac{\dot x}{\rho}, \frac{\dot y}{\rho}\right) \neq 0.$$

Consequently, because of (6.3), $\tau \neq 0$ on C^*.

Furthermore, if C^* is extremal convex in the large, then the image F^* is a simple arc and has the unique representation $p = p(\theta)$.

8. **Double integral with variation** δI. Let

$$K(D^*) \equiv -\iint_\Omega n(\theta, p, D^*)M(\theta, p)N(\theta, p)dpd\theta, \tag{8.1}$$

where Ω is the entire cartesian (θ, p)-space and $n(\theta, p, D^*)$ is the topological order of the point (θ, p) with respect to the closed oriented curve D^*. The existence of a $D^* \supset F^*$ will be established for which the variation of $I(C^*)$ is identical with the induced variation of $K(D^*)$.

D^* is defined as follows The contact transformation maps C^* into F^* with an orientation induced by that of C^*. The projections of the end points of F^*, Q_0 and Q_1, on the θ-axis will be denoted by $R_0 \equiv (\theta(t_0), 0)$ and $R_1 \equiv (\theta(t_1), 0)$. Then, D^* is the oriented curvilinear polygon consisting of the directed line-segments Q_1R_1, R_1R_0, R_0Q_0 and the directed arc F^*.

The verification of $\delta I(C^*) = \delta K(D^*)$ is now considered. Embed $F^* \equiv F_0^*$ in a 1-parameter family of curves

$$(8.2) \qquad F_k^*: \quad \begin{array}{lll} \theta = \theta(t, k); & t_0 \leqq t \leqq t_1; & \theta(t_0, k) = \theta(t_0), \\ p = p(t, k); & 0 \leqq k \leqq k_0; & \theta(t_1, k) = \theta(t_1), \end{array}$$

for sufficiently small k_0. Introduce the notation $K(k) \equiv K(D_k^*)$, where D_k^* and F_k^* are related in the same manner as D^* and F^*, and observe that since D_0^* has at most a finite number of singularities,

$$(8.3) \qquad \lim_{\epsilon \to 0} \left[n(\theta, p, D_\epsilon^*) - n(\theta, p, D_0^*) \right] = \operatorname{sign} \left[\delta p \theta_t - \delta \theta p_t \right],$$

where $(\delta\theta, \delta p) \equiv (\theta_k(t, 0), p_k(t, 0))$. Consequently,

$$(8.4) \qquad \begin{aligned} \delta K(D_0^*) &\equiv \lim_{\epsilon \to 0} \frac{K(\epsilon) - K(0)}{\epsilon} \\ &= - \int_{F_0^*} M(\theta, p) N(\theta, p) \left[\frac{\delta p \theta_t - \delta \theta p_t}{(\dot\theta^2 + \dot p^2)^{1/2}} \right] ((d\theta)^2 + (dp)^2)^{1/2}. \end{aligned}$$

Therefore, by reason of (6.4) and (8.4), $\delta I(C^*) = \delta K(D^*)$ if

$$(8.5) \qquad \tau \left[\frac{Y_s \delta x - X_s \delta y}{(\dot\theta^2 + \dot p^2)^{1/2}} \right] = \left[\frac{\delta\theta p_t - \delta p \theta_t}{(\dot\theta^2 + \dot p^2)^{1/2}} \right],$$

where $(\delta\theta, \delta p)$ is the variation induced in $(\theta(t), p(t))$ by the variation $(\delta x, \delta y)$ of $(x(t), y(t))$. We now consider the proof of (8.5). Since

$$(8.6) \qquad \begin{aligned} \delta x &= \delta s X_s + \delta\theta X_\theta + \delta p X_p \quad \text{and} \\ \delta y &= \delta s Y_s + \delta\theta Y_\theta + \delta p Y_p, \end{aligned}$$

it follows that

$$(8.7) \qquad Y_s \delta x - X_s \delta y = (Y_s X_p - X_s Y_p) \delta p + (Y_s X_\theta - X_s Y_\theta) \delta\theta.$$

Thus, because of (5.2),

$$(8.8) \qquad Y_s \delta x - X_s \delta y = \left[\frac{\delta\theta p_t - \delta p \theta_t}{\tau} \right],$$

which completes the verification of (8.5).

9. **The identity** $I(C^*) = K(D^*)$. We now impose the following condition:

(9.1) EMBEDMENT HYPOTHESIS. *The admissible curves C^* can be embedded in a 1-parameter family of closed curves C_k^*:* $[(x(t, k), y(t, k))$; $t_0 \leqq t \leqq t_1$,

$0<k\leqq 1$] *which possess the following properties*: (a) $C_1^* \equiv C^*$ (b) *The* C_k^* *are admissible curves which are sufficiently differentiable in the variables* (t, k). (c) C_k^* *tends uniformly to the origin as* k *tends to zero.* (d) *The map* F_k^* *of* C_k^*, *under the contact transformation, is such that* $\theta(t_0, k)$ *and* $\theta(t_1, k)$ *are independent of* k.

The identity will now be proved. Since the curves (3.2) were so defined that $X(0, \theta, 0) = Y(0, \theta, 0) = 0$, the "field and embedment hypothesis" imply that D_k^* tends uniformly to a finite segment on the θ-axis, as k tends to zero. Therefore, since $\delta I = \delta K$,

$$I(C^*) = \lim_{k\to 0}\int_k^1 \delta I dk = \lim_{k\to 0}\int_k^1 \delta K dk = K(D^*), \tag{9.2}$$

which is the same as

$$\oint_{C^*} F(x, y, dx, dy) = -\iint_\Omega n(\theta, p, D^*)M(\theta, p)N(\theta, p)dpd\theta; \tag{9.3}$$

and, if C^* is extremal convex in the large, (9.3) reduces to

$$\oint_{C^*} F(x, y, dx, dy) = -\iint_D M(\theta, p)N(\theta, p)dpd\theta, \tag{9.4}$$

since F^* now admits the unique, continuously differentiable, nonparametric representation $p = p(\theta)$.

If C^* does not enclose the origin, C^* will lie in a curvilinear sector defined by two tangent geodesics (3.1) having minimum and maximum values of θ. If an "embedment hypothesis" can be invoked which guarantees the contraction of C^* in the sector to the origin by means of admissible curves whose end points lie on the bounding tangents, then the validity of (9.3) and (9.4) is assured.

10. Results for special metrics. If the geodesics of the metric are straight lines, then

$$\begin{aligned} x &= X(s, \theta, p) = p\cos\theta - s\sin\theta, \\ y &= Y(s, \theta, p) = p\sin\theta + s\cos\theta, \end{aligned} \tag{10.1}$$

and

$$\oint_{C^*} F(x, y, dx, dy) = \iint_D \rho^3 F_1(x, y, \dot{x}, \dot{y})dpd\theta. \tag{10.2}$$

Let

$$2u = (1/p + p), \qquad 2v = (1/p - p),$$

and $C^* \subset U$: $x^2 + y^2 < 1$. Then, for the hyperbolic metric,

$$
(10.3) \qquad
\begin{aligned}
x &= X(s,\theta,p) = u\cos\theta - v\cos(s/v-\theta),\\
v &= Y(s,\theta,p) = u\sin\theta + v\sin(s/v-\theta),
\end{aligned}
$$

and

$$
(10.4) \qquad \oint_{C^*} \frac{((dx)^2+(dy)^2)^{1/2}}{1-x^2-y^2} = \iint_D \frac{1+p^2}{(1-p^2)^2}\,dp\,d\theta;
$$

while in the case of the spherical metric,

$$
(10.5) \qquad
\begin{aligned}
x &= X(s,\theta,p) = -v\cos\theta + u\cos(s/u+\theta),\\
y &= Y(s,\theta,p) = -v\sin\theta + u\sin(s/u+\theta),
\end{aligned}
$$

and

$$
(10.6) \qquad \oint_{C^*} \frac{((dx)^2+(dy)^2)^{1/2}}{1+x^2+y^2} = \iint_D \frac{1-p^2}{(1+p^2)^2}\,dp\,d\theta.
$$

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