# REPRESENTATIONS OF PRIME RINGS

BY

R. E. JOHNSON

This paper is a continuation of the study of prime rings started in [2]. We recall that a prime ring is a ring having its zero ideal as a prime ideal.

A right (left) ideal $I$ of a prime ring $R$ is called *prime* if $\mathfrak{a}\mathfrak{b}\subseteq I$ implies that $\mathfrak{a}\subseteq I$ ($\mathfrak{b}\subseteq I$), $\mathfrak{a}$ and $\mathfrak{b}$ right (left) ideals of $R$ with $\mathfrak{b}\neq 0$ ($\mathfrak{a}\neq 0$). We denote by $\mathfrak{P}_r$ ($\mathfrak{P}_l$) the set of all prime right (left) ideals of $R$. For any subset $A$ of $R$, $A^r$ ($A^l$) denotes the right (left) annihilator of $A$; $A^r$ ($A^l$) is a right (left) *annihilator ideal* of $R$. The set of all right (left) annihilator ideals of $R$ is denoted by $\mathfrak{A}_r$ ($\mathfrak{A}_l$).

For the prime rings $R$ studied in [2], it was assumed that there existed a mapping $I\rightarrow I^*$ of the set of all right (left) ideals of $R$ onto a subset $\mathfrak{R}$ ($\mathfrak{L}$) of $\mathfrak{P}_r$ ($\mathfrak{P}_l$) having the following seven properties:

(P1) $\quad I^* \supseteq I.$ $\qquad$ (P2) $\quad I^{**} = I^*.$

(P3) $\quad$ If $I \supseteq I'$, then $I^* \supseteq I'^*.$ $\qquad$ (P4) $\quad 0^* = 0.$

(P5) $\quad$ If $I \cap I' = 0$, then $I^* \cap I'^* = 0.$

(P6) $\quad aI^* \subseteq (aI)^* \quad (I^*a \subseteq (Ia)^*), \quad a \in R.$

(P7) $\quad \mathfrak{R}$ ($\mathfrak{L}$) is atomic.

That the above properties arise naturally may be seen by letting $I^*=p(I)$, the least prime right (left) ideal of $R$ containing $I$. Then properties (P1)–(P6) are known to hold [2]. Thus (P1)–(P7) hold for any ring having minimal prime right (left) ideals. In particular, these properties hold for a primitive ring with minimal right ideals.

A subset $\mathfrak{R}$ ($\mathfrak{L}$) of $\mathfrak{P}_r$ ($\mathfrak{P}_l$) satisfying (P1)–(P7) will be called a *right structure* (*left structure*) of $R$. A right (left) structure $\mathfrak{R}$ ($\mathfrak{L}$) of $R$ may be made into a lattice in the usual way. Thus for any $I$, $I'$ in $\mathfrak{R}$ ($\mathfrak{L}$), define $I\cap I'$ as the intersection of these ideals and $I\cup I'$ as $(I+I')^*$. It follows from [2] that $\mathfrak{R}$ ($\mathfrak{L}$) is a modular lattice under these operations. A consequence of [2, p. 803] is that $\mathfrak{A}_r\subseteq\mathfrak{R}$ ($\mathfrak{A}_l\subseteq\mathfrak{L}$). Since $(I+I')^l=(I\cup I')^l$ by (P6), it is evident that $(I\cup I')^l=I^l\cap I'^l$ for any $I$, $I'$ in $\mathfrak{R}$, and similarly for $\mathfrak{L}$.

It is assumed in this paper that the prime ring $R$ has both a right and a left structure. Some properties of structures, in addition to those given in [2], are developed in the first section. Next, atoms of these structures are used for dual representation spaces of $R$. It is shown that these structures in $R$ have isomorphic structures in their dual representation spaces. Finally, the

Presented to the Society, September 5, 1952; received by the editors May 24, 1952.

given ring is shown to be an $n$-fold transitive ring of transformations on these spaces in a certain restricted sense.

1. **Right-left structure relations.** We assume that the prime ring $R$ has both a right structure $\mathfrak{R}$ and a left structure $\mathfrak{L}$. Each of the results of this section has a dual obtained by interchanging the roles of $\mathfrak{R}$ and $\mathfrak{L}$.

1.1 LEMMA. *If $I$ is an atom of $\mathfrak{R}$ and $x$ is any nonzero element of $I$, then $(Rx)^*$ is an atom of $\mathfrak{L}$.*

To prove this, let $L$ be any atom of $\mathfrak{L}$. The primeness of $R$ implies that $L\cap xR\neq 0$. Select $xa\in L\cap xR$, $xa\neq 0$; since $L$ is an atom, $(xa)^l$ is a maximal element of $\mathfrak{L}$ by [2, 4.11]. Now $I\cap (xa)^{lr}\neq 0$, and therefore $I\subseteq (xa)^{lr}$. Thus $(x)^l=(xa)^l$ and $(Rx)^*$ is an atom of $\mathfrak{L}$ by [2, 4.11].

The ring union of all atoms of $\mathfrak{R}$ is shown in [2, 4.2] to be an ideal of $R$. The above lemma shows that this ideal is also the ring union of all atoms of $\mathfrak{L}$.

1.2 THEOREM. *If $I$ is an atom of $\mathfrak{R}$, then $I^l$ is a maximal element of $\mathfrak{L}$, while if $I$ is a maximal element of $\mathfrak{R}$ for which $I^l\neq 0$, then $I^l$ is an atom of $\mathfrak{L}$.*

If $I$ is an atom of $\mathfrak{R}$, then $I^l=(x)^l$ for any nonzero $x$ in $I$, and hence $I^l$ is maximal in $\mathfrak{L}$ by the proof of the above lemma.

On the other hand, if $I$ is maximal in $\mathfrak{R}$ and $I^l\neq 0$, then $(x)^r=I$ for any nonzero $x$ in $I^l$. Thus $(xR)^*$ is an atom of $\mathfrak{R}$ by [2, 4.11]. Since $x$ is in $(xR)^*$ [2, 1.2], we have by 1.1 that $(Rx)^*$ is an atom of $\mathfrak{L}$ for every nonzero $x$ in $I^l$. If $I^l$ is not an atom of $\mathfrak{L}$, it must contain atoms $L_1$ and $L_2$ such that $L_1\cap L_2=0$ [2, 4.3]. Let $x_1$ be any nonzero element of $L_1$. Since $(x_1)^l L\neq 0$ due to the primeness of $R$, there must exist a nonzero element $x_2$ in $L_2$ such that $(x_1)^l\neq (x_2)^l$. Then $R(x_1+x_2)\cap Rx_i\neq 0$, $i=1, 2$, and therefore $(R(x_1+x_2))^*=(Rx_1)^*=(Rx_2)^*$. This contradicts the assumption that $L_1\cap L_2=0$, and proves 1.2.

It is a corollary of 1.2 that the atoms of $\mathfrak{R}$ ($\mathfrak{L}$) are contained in $\mathfrak{A}_r$ ($\mathfrak{A}_l$).

1.3 THEOREM. *If $I$ is an atom of $\mathfrak{R}$ and $I'$ is any element of $\mathfrak{A}_r$, then $I\cup I'$ also is in $\mathfrak{A}_r$.*

Since $(I\cup I')^{lr}=(I^l\cap I'^l)^r$, what we wish to prove is that $(I^l\cap I'^l)^r=I\cup I'$. Clearly $I\cup I'\subseteq (I^l\cap I'^l)^r$, so that we need only prove that $(I^l\cap I'^l)^r\subseteq I\cup I'$. In view of [2, 4.3], this can be accomplished by showing that every atom $I_1$ of $\mathfrak{R}$ contained in $(I^l\cap I'^l)^r$ is also contained in $I\cup I'$.

So let us assume that $I_1\subseteq (I^l\cap I'^l)^r$, $I_1$ an atom of $\mathfrak{R}$. If either $I_1=I$ or $I_1\subseteq I'$, nothing remains to be proved; henceforth we shall assume that $I_1\neq I$ and $I_1\not\subseteq I'$. Then necessarily $I\not\subseteq I'$ and $I'^l\not\subseteq I^l$. Hence there exists an atom $L$ of $\mathfrak{L}$ [2, 4.3] such that $L\subseteq I'^l$, $L\cap I^l=0$. Since $I^l$ is maximal in $\mathfrak{L}$ by 1.2, evidently $L\cup I^l=R$. From the modularity of $\mathfrak{L}$ we see that $L\cup (I^l\cap I'^l)=I'^l$, and therefore that $L^r\cap (I^l\cap I'^l)^r=I'$. Since $L^r$ is a maximal element of $\mathfrak{R}$ and $I\not\subseteq L^r$, clearly $I\cup L^r=R$. Hence it follows from (P5) that $I_1\cap (I+L^r)\neq 0$, and therefore that $(I_1+I)\cap L^r\neq 0$. Since also $I_1+I\subseteq (I^l\cap I'^l)^r$, it follows that $(I_1+I)\cap I' \neq 0$. Thus $I_1\cap(I+I')\neq 0$ and $I_1\subseteq I\cup I'$. This proves the theorem.

1.4 COROLLARY. *If $I_1, \cdots, I_n$ are atoms of $\mathfrak{R}$, then $I_1\cup \cdots \cup I_n$ is in $\mathfrak{A}_r$.*

The corollary follows by mathematical induction.

2. ***R*-modules.** If $M$ is a right (left) $R$-module and $A$ is a subset of $M$, we shall again use the notation $A^r$ $(A^l)$ to denote the annihilator of $A$ in $R$. A right (left) $R$-module $M$ is called *prime* if $A^r=0$ $(A^l=0)$ for every nonzero submodule $A$ of $M$. A submodule $M'$ of $M$ is called a *prime submodule* of $M$ if $M-M'$ is a prime module. If the ring $R$ has a right (left) structure $\mathfrak{R}$ $(\mathfrak{L})$, then a right (left) $R$-module $M$ is called *admissible* relative to $\mathfrak{R}$ $(\mathfrak{L})$ if $M$ is prime and $(x)^r\in\mathfrak{R}$ $((x)^l\in\mathfrak{L})$ for every $x\in M$. For any $I$ in $\mathfrak{R}$ $(\mathfrak{L})$, both $I$ and $R-I$ are examples of admissible right (left) $R$-modules.

It is shown in [2, p. 804] that an admissible right $R$-module $M$ has a structure much the same as $R$ does. For any submodule $N$ of $M$, define

$$N^* = \{x;\, x\in M,\ [(N:x)]^* = R\}.$$

Here $(N:x)$ denotes the annihilator in $R$ of the element $x+N$ in $M-N$. Then the set $\mathfrak{M}$ of all submodules $N^*$ of $M$ is a structure of $M$ in that it possesses the properties analogous to (P1)–(P7). Naturally, similar remarks hold for admissible left $R$-modules.

Let us assume now that $R$ is a ring with a right structure $\mathfrak{R}$ and a left structure $\mathfrak{L}$, and that $N$ is a fixed atom of $\mathfrak{R}$. Select an atom $M$ of $\mathfrak{L}$ so that

$$M\cdot N \neq 0.$$

Such an $M$ must exist, since the ring union $S$ of all atoms of $\mathfrak{L}$ is an ideal of $R$ [2, 4.2], and $S\cdot N\neq 0$ due to the primeness of $R$. Let

$$K = M\cap N,$$

a nonzero subring of $R$. If we consider the rings $K$, $M$, and $N$ as modules, it is evident that $K$ is an $(N, M)$-module, that $M$ is an $(R, K)$-module, and that $N$ is an $(K, R)$-module. Clearly $N\cdot M\subseteq K$.

2.1 LEMMA. *For $x$ in $M$ and $y$ in $N$, $xy=0$ if and only if $x=0$ or $y=0$.*

If $x\neq 0$, then $(x)^r$ is an element of $\mathfrak{R}$ and therefore either $(x)^r\cap N=0$, in which case the desired conclusion follows immediately, or $N\subseteq(x)^r$. In this latter case evidently $N^l\cap M\neq 0$ and $M\subseteq N^l$, which is contrary to the choice of $M$. This proves 2.1.

An obvious corollary of this lemma is that $K$ is an integral domain.

2.2 LEMMA. *The integral domain $K$ possesses a quotient division ring $D$.*

If $x$ and $y$ are nonzero elements of $K$, then $(xN)^*=(yN)^*=N$ in view of 2.1 and the fact that $N$ is an atom. Thus $xN\cap yN\neq 0$ by (P5), and hence $xN\cap yN\cap M\neq 0$. However, $xN\cap yN\cap M\subseteq K$ so that evidently $xK\cap yK\neq 0$. This proves that $K$ has a right quotient division ring $D$. That $D$ also is the left quotient of $K$ follows by duality.

2.3 LEMMA. *If $x$ and $y$ are nonzero elements of $N$, then $Kx\cap Ky\neq 0$ if and only if $(x)^r=(y)^r$.*

If $Kx\cap Ky\neq 0$, then obviously $(x)^r=(y)^r$. Conversely, if $(x)^r=(y)^r$, then $x$ and $y$ are in $(x)^{rl}$, an atom of $\mathfrak{L}$. Now $Mx\neq 0$ and $My\neq 0$, and since both $Mx$ and $My$ are contained in the atom $(x)^{rl}$, necessarily $Mx\cap My\neq 0$ by (P5). Since $K$ is a right $M$-module, evidently $Kx\cap Ky\neq 0$ as desired.

We shall consider $K$ as having the trivial left and right structures, namely the structures consisting of the set $(0, K)$. In view of 2.2, which guarantees that (P5) holds, it is evident that these structures satisfy (P1)–(P7).

Now $N$, as an admissible $(K, R)$-module, has a left structure induced by $K$ and a right structure induced by $R$. It is clear that for any $K$-submodule $A$ of $N$, the closure $A^*$ of $A$ is defined as follows:

$$A^* = \{x;\ x\in N,\ x=0 \text{ or } Kx\cap A\neq 0\}.$$

If $\mathfrak{N}$ denotes the set of all closed $K$-submodules of $N$, then $\mathfrak{N}$ is a left structure of $N$. Since $N$ is an atom of $\mathfrak{N}$, the right structure of $N$ induced by $R$ is the trivial one.

In an analogous way, of course, $M$ has left and right structures induced by $R$ and $K$ respectively. The left structure is trivial; the right structure of $M$ induced by $K$ will be denoted by $\mathfrak{M}$.

The following results, although frequently just stated for $\mathfrak{N}$, have the obvious duals relative to $\mathfrak{M}$.

Any $A$ of $\mathfrak{N}$ is actually a left $N$-module. For if $x\in N$ and $a\in A$ with $xa\neq 0$, then $(xa)^r=(a)^r$ since both of these right ideals are maximal elements of $\mathfrak{R}$ by [2, 4.11], and $K(xa)\cap Ka\neq 0$ by 2.3. Thus $K(xa)\cap A\neq 0$ and $xa\in A$ since $A^*=A$.

If $L$ is in $\mathfrak{L}$ and $kx$ is a nonzero element of $L\cap N$, $k\in K$ and $x\in N$, then $(Rx)^*\cap L\neq 0$ and, since $(Rx)^*$ is an atom of $\mathfrak{L}$ by 1.1, evidently $(Rx)^*\subseteq L$. Thus $x$ is in $L\cap N$ and we have proved that $L\cap N$ is in $\mathfrak{N}$. Furthermore, if $L$ is an atom of $\mathfrak{L}$, then $L\cap N$ is an atom of $\mathfrak{N}$. This is so since for any nonzero elements $x$ and $y$ of $L\cap N$, $(x)^r=(y)^r=L^r$, and hence $Kx\cap Ky\neq 0$ by 2.3.

On the other hand, if $A$ is an atom of $\mathfrak{N}$, then $Kx\cap Ky\neq 0$ for any nonzero $x, y\in A$. Hence $A^{rl}=(x)^{rl}=L$, an atom of $\mathfrak{L}$, and $A=L\cap N$.

The above remarks constitute part of the proof of the following theorem.

2.4 THEOREM. *The $K$-submodule $A$ of $N$ is in $\mathfrak{N}$ if and only if $A=L\cap N$ for some $L$ in $\mathfrak{L}$.*

To complete the proof of this theorem, let $A$ be any nonzero element of $\mathfrak{N}$ and let $L=(RA)^*\in\mathfrak{L}$. We shall prove that $A=L\cap N$. If $L_1\subseteq L$, $L_1$ an atom of

$\mathfrak{L}$, then $L_1 \cap RA \neq 0$ so that $N \cdot (L_1 \cap RA) \neq 0$ and $L_1 \cap NRA \neq 0$. Since $NRA \subseteq NA \subseteq A$, we have proved that $L_1 \cap A \neq 0$. Now $L_1 \cap N$ is an atom of $\mathfrak{N}$ and therefore $L_1 \cap N \subseteq A$. It follows that $L \cap N \subseteq A$, and the proof of the theorem is completed.

2.5 THEOREM. *The lattices* $\{\mathfrak{L}; \subseteq, \cup, \cap\}$ *and* $\{\mathfrak{N}; \subseteq, \cup, \cap\}$ *are isomorphic under the correspondence* $L \rightarrow L \cap N$. *Dually, the lattices* $\{\mathfrak{R}; \subseteq, \cup, \cap\}$ *and* $\{\mathfrak{M}; \subseteq, \cup, \cap\}$ *are isomorphic under the correspondence* $I \rightarrow I \cap M$.

It is sufficient to prove that the mapping $L \rightarrow L \cap N$ of $\mathfrak{L}$ onto $\mathfrak{N}$ is a 1-1 order-preserving mapping in order to prove that these lattices are isomorphic. Clearly the mapping is order-preserving. In order to show that it is a 1-1 mapping, we need only note that if $L_1 \not\subseteq L_2$, $L_i \in \mathfrak{L}$, then there exists an atom $L$ of $\mathfrak{L}$ such that $L \subseteq L_1$, $L \cap L_2 = 0$ by [2, 4.3]. Hence $L \cap N \not\subseteq L_2 \cap N$ and therefore $L_1 \cap N \not\subseteq L_2 \cap N$. This proves 2.5.

In case $R$ is a primitive ring with nonzero socle $S$, and $N$ and $M$ are simple conjugate right and left $R$-modules respectively with common centralizer $D$, this theorem yields the well known isomorphism existing between the lattice of left (right) ideals of $S$ and the lattice of $D$-submodules of $N$ ($M$). This application to primitive rings is obtained by letting $\mathfrak{N}$ ($\mathfrak{L}$) be the set of all prime right (left) ideals of $R$.

If $A \in \mathfrak{N}$, say $A = L \cap N$ for $L \in \mathfrak{L}$, then obviously $A^r \supseteq L^r$. Since, however, $A^{rl} \in \mathfrak{L}$ and $A^{rl} \supseteq A$, evidently $L \subseteq A^{rl}$ and $A^r \subseteq L^r$. Thus $A^r = L^r$. If, in particular, $L \in \mathfrak{A}_l$, then $A = A^{rl} \cap N$. Let us denote by $\mathfrak{N}_l$ ($\mathfrak{M}_r$) the set of all $K$-submodules of $N$ ($M$) that are annihilators of right (left) ideals of $R$. Then $\mathfrak{N}_l = \{A; A \in \mathfrak{N}, A = L \cap N \text{ for some } L \in \mathfrak{A}_l\}$, and similarly for $\mathfrak{M}_r$.

In view of the isomorphism existing between $\mathfrak{L}$ and $\mathfrak{N}$, Theorem 1.3 has the following counterpart in $\mathfrak{N}$.

2.6 THEOREM. *If $A$ is an atom of $\mathfrak{N}$ and $B$ is any element of $\mathfrak{N}_l$, then $A \cup B$ also is in $\mathfrak{N}_l$.*

A corollary of this theorem is as follows (1.4):

2.7 COROLLARY. *If $A_1, \cdots, A_n$ are atoms of $\mathfrak{N}$, then $A_1 \cup \cdots \cup A_n$ is in $\mathfrak{N}_l$.*

For a primitive ring $R$, analogues of Theorem 2.6 and its corollary can be found in a recent paper by Artin [1, pp. 68, 69]. His results are more general than ours in that his ring $R$ is not assumed to have minimal right ideals. Of course, they are also less general in that they are restricted to apply to primitive rather than prime rings.

3. **Transitivity of $R$ over $N$.** As usual, the elements $x_1, \cdots, x_n$ of $N$ are called $K$-linearly independent if and only if $k_1x_1 + \cdots + k_nx_n = 0$ implies all $k_i = 0$, $k_i \in K$. An alternate lattice-theoretic definition is given by the following lemma.

3.1 LEMMA. *The elements $x_1, \cdots, x_n$ of $N$ are $K$-linearly independent if and only if*

$$(x_j)^r \not\supseteq \bigcap_{i=1, i\neq j}^{n} (x_i)^r, \qquad j = 1, \cdots, n.$$

To prove this lemma, note first that all $A_i = (x_i)^{rl} \cap N = (Kx_i)^*$ are atoms of $\mathfrak{N}$ (assuming, of course, that $x_i \neq 0$). If $k_1x_1 + \cdots + k_nx_n = 0$ with $k_j \neq 0$, then

$$A_j \subseteq \bigcup_{i=1, i\neq j}^{n} A_i,$$

and, since $A_i^r = (x_i)^r$,

$$(x_j)^r \supseteq \bigcap_{i=1, i\neq j}^{n} (x_i)^r.$$

Conversely, if the above inclusion relation holds for some $j$, then

$$A_j \subseteq \bigcup_{i=1, i\neq j}^{n} A_i,$$

$$Kx_j \cap \sum_{i=1, i\neq j}^{n} Kx_i \neq 0.$$

Thus the elements $x_1, \cdots, x_n$ are $K$-linearly dependent, and 3.1 follows.

3.2 LEMMA. *Let $I$ be any right ideal of $R$ and $K'$ be any left $N$-submodule of $K$. Then for any $x$ and $y$ in $N$ such that $xI \neq 0$ and $K'y \neq 0$, also $xI \cap K'y \neq 0$.*

To prove this lemma, let $k$ be any nonzero element of $K'$. Then, by the primeness of $N$, $xaky \neq 0$ for some $a$ in $I$. Now $x(aky) = (xak)y$ where $aky \in I$ and $xak \in K'$, and therefore the lemma is proved.

We now are in a position to prove the main result of our paper, namely that $R$ acts almost as an $n$-fold transitive ring of $K$-linear transformations on $N$ for any integer $n$ not exceeding the $K$-dimension of $N$. To be more precise, we shall prove the following theorem.

3.3 TRANSITIVITY THEOREM. *If $x_1, \cdots, x_n$ is any set of $n$ $K$-linearly independent elements of $N$ and if $y_1, \cdots, y_n$ is any set of $n$ elements of $N$, then there exist an element $a$ of $R$ and a nonzero element $k$ of $K$ such that*

$$x_ia = ky_i, \qquad i = 1, \cdots, n.$$

To aid in the proof of this theorem, let

$$I_j = \bigcap_{i=1, i\neq j}^{n} (x_i)^r.$$

In view of 3.1, evidently $I_j \not\subset (x_j)^r$ for any $j$. Hence, by 3.2, there exist elements $a_j \in I_j$ and $k_j \in K$, $k_j \neq 0$, such that $x_j a_j = k_j y_j \neq 0$ for all $j$ such that $y_j \neq 0$. If $y_j = 0$, select $a_j = 0$. Now for all $y_j \neq 0$, $k_j y_j K$ is a right ideal of $K$, and $\bigcap_j k_j y_j K \neq 0$ by 2.2. Select $k \in \bigcap_j k_j y_j K$, $k \neq 0$; $k = k_j y_j c_j$ for each $j$ such that $y_j \neq 0$. Then $x_j(a_j c_j y_j) = k y_j$, and if we let $a = a_1 c_1 y_1 + \cdots + a_n c_n y_n$, evidently $x_j a = k y_j$ as desired.

We give now an example of a prime ring of the type considered in this paper. Denote by $I$ the ring of integers and by $I_2$ the ring of all $2 \times 2$ matrices over $I$. We use the notation $E_{ij}$ for the matrix with 1 in its $(i, j)$ position and zeros elsewhere. Now denote by $R$ the set of all matrices of $I_2$ having all even or all odd integers for components. It is easily established that $R$ is a prime ring.

The right ideal $N = 2IE_{11} + 2IE_{12}$ is a minimal prime right ideal and the left ideal $M = 2IE_{11} + 2IE_{21}$ is a minimal prime left ideal. Clearly $K = M \cap N = 2IE_{11}$ is an integral domain. The sets of prime right and left ideals of $R$ form right and left structures of $R$.

As an illustration of the transitivity theorem, let $x_1 = 2E_{11}$, $x_2 = 4E_{12}$, $y_1 = 0$, and $y_2 = 2E_{11}$. Then for $a = 2E_{11}$ and $k = 4E_{11}$ we have $x_1 a = k y_1$ and $x_2 a = k y_2$. We note that there is no $a$ in $R$ such that $x_1 a = y_1$ and $x_2 a = y_2$.

In the case of a primitive ring $R$, the minimal right ideals are all isomorphic as right $R$-modules. That such is not the case in general for a prime ring follows from this example. To show this, let $N' = [I(E_{11} + E_{21}) + I(E_{12} + E_{22})] \cap R$. It is not too difficult to show that $N'$ is a minimal prime right ideal of $R$. If $N$ and $N'$ were isomorphic, then we would have $2aE_{11} \to c(E_{11} + E_{21})$, $2bE_{12} \to d(E_{12} + E_{22})$ for some integers $a$, $b$, $c$, $d$ in order for the annihilators of corresponding elements of $N$ and $N'$ to be the same. But then $c$ and $d$ would have to be even integers, and nothing in $N$ would correspond to the matrices in $N'$ having odd integers for elements.

## Bibliography

1. E. Artin, *The influence of J. H. M. Wedderburn on the development of modern algebra*, Bull. Amer. Math. Soc. vol. 56 (1950) pp. 65–72.

2. R. E. Johnson, *Prime rings*, Duke Math. J. vol. 18 (1951) pp. 799–809.

Smith College,
Northampton Mass.