FURTHER RESULTS ON ORDER TYPES AND DECOMPOSITIONS OF SETS

BY

SEYMOUR GINSBURG

In this paper the study of problems P and Q [3; 4] is continued. The reader is referred to the aforementioned two references for all unfamiliar terms and symbols.

The first three sections deal with the decomposition of a linear set into a family of pairwise disjoint sets, the order types of the sets being required to satisfy some specified conditions. §1 is concerned with the decomposition of an arbitrary linear set, of power $2^{\aleph_0}$, into a family of pairwise disjoint sets, the order types of the sets being pairwise incomparable. §2 is concerned with the decomposition of certain linear sets into families of pairwise disjoint, similar sets. §3 is concerned with the decomposition of an arbitrary linear set into families, of power $\aleph_0$ and $2^{\aleph_0}$, of pairwise disjoint sets, each set having property A.

Let $\{\alpha_\xi\}$, $\xi<\theta$, be a sequence of order types, of power $2^{\aleph_0}$ each, such that $\alpha_\xi<\lambda$ for each ξ. In §4 it is shown that

(1) problem P, as applied to $\sigma=\alpha_\xi$ and $\mu=\lambda$, admits of a solution τ_ξ such that the τ_ξ are pairwise incomparable order types (Theorem 4.1); and

(2) problem P, as applied to each $\sigma=0$ and $\mu=\alpha_\xi$, admits of a solution τ_ξ such that the τ_ξ are pairwise incomparable order types (Theorem 4.3).

1. **Decompositions into incomparable order types.** In this section the decomposition of a linear set into a finite number and into a denumerably infinite number of pairwise disjoint sets A_i, where the order types of the A_i are pairwise incomparable, is studied. The decomposition of a linear set into $2^{\aleph_0}$ pairwise disjoint sets A_i, where the order types of the A_i are pairwise incomparable, is treated in §3 (Theorem 3.4).

Definition. A linear set E, of power $2^{\aleph_0}$, will be said to have property C if each element of E is a c-condensation point of E.

For any linear set E, by $K(E)$ is meant the set of similarity transformations of E into R. By $K^*(E)$ is meant the set $K(E)-\{I\}$, where I is the identity transformation of E.

For any similarity transformation f of A into B, by f^* is meant the inverse of f.

Lemma 1.1. *Let E be a linear set which has property* C. *For each ordinal number α, where $2\leqq\alpha\leqq\omega$, E is the union of a family $\{A_i|i<\alpha\}$ of pairwise disjoint, exact sets, each of which has property* C *and is a dense subset of E. Fur-*

Presented to the Society, April 24, 1953; received by the editors February 16, 1953.

thermore, if f is any element of $K(A_i)$ for which the power of the set $E \cap f(A_i)$ is $2^{\aleph_0}$, then there are $2^{\aleph_0}$ elements x in A_i such that $f(x)$ is also in A_i.

Proof. Since E has property C, it follows that for each element f in $K^*(E)$, the power of the set, $\{x \mid f(x) \neq x, x \in E\}$, is $2^{\aleph_0}$. Well order the elements of E and $K^*(E)$ into the two sequences, $\{x_\xi\}$, $\xi < \theta$, and $\{f_\xi\}$, $\xi < \theta$, where each element in $K^*(E)$ appears $2^{\aleph_0}$ times in the latter sequence. If $\alpha = \omega$, let $\beta = \alpha$. If α is an integer let $\beta = \alpha - 1$. Suppose that for each ordinal number ξ, where $\xi < \mu < \theta$, the elements $p_\xi^{i,m}$, where $i = 1, 2, \cdots, 10$, and $m < \beta$, have been defined. Let

$$X_\mu = \{x \mid f_\mu(x) \neq x, x \in E\} \quad \text{and} \quad Y_\mu = \{p_\xi^{i,m} \mid \xi < \mu, m < \beta, 1 \leqq i \leqq 10\}.$$

Denote by $\{p_\mu^{1,m} \mid m < \beta\}$ and by $\{p_\mu^{2,m} \mid m < \beta\}$ the first $\beta 2$ elements in the set $E - Y_\mu$. Let

$$A_\mu = Y_\mu \cup \{p_\mu^{i,m} \mid i \leqq 2, m < \beta\} \quad \text{and} \quad B_\mu = X_\mu - [A_\mu \cup f_\mu^*(A_\mu)].$$

Let $p_\mu^{3,0}$ be the first element in B_μ and $p_\mu^{4,0} = f_\mu(p_\mu^{3,0})$. Since the power of X_μ is $2^{\aleph_0}$ and the power of A_μ is $< 2^{\aleph_0}$, the two distinct elements $p_\mu^{3,0}$ and $p_\mu^{4,0}$ certainly exist. In general, for $m < \beta$ let $p_\mu^{3,m}$ be the first element in the set $B_\mu - [G_\mu^m \cup f^*(G_\mu^m)]$, where

$$G_\mu^m = \{p_\mu^{3,i} \mid i < m\} \cup \{p_\mu^{4,i} \mid i < m\}.$$

Let $p_\mu^{4,m} = f_\mu(p_\mu^{3,m})$. Let

$$C_\mu = Y_\mu \cup \{p_\mu^{i,m} \mid i \leqq 4, m < \beta\} \quad \text{and} \quad D_\mu = X_\mu - [C_\mu \cup f_\mu^*(C_\mu)].$$

Let $p_\mu^{6,0}$ be the first element in D_μ and $p_\mu^{5,0} = f_\mu(p_\mu^{6,0})$. Let $p_\mu^{5,i}$ and $p_\mu^{6,i}$, $1 \leqq i < \beta$, represent no elements. Denote by Z_μ the set $Z_\mu = \{x \mid x \in E, f_\mu(x) \in E\}$. Suppose that the power of Z_μ is $2^{\aleph_0}$. Let

$$E_\mu = Y_\mu \cup \{p_\mu^{i,m} \mid i \leqq 6, m < \beta\} \quad \text{and} \quad F_\mu = Z_\mu - [E_\mu \cup f_\mu^*(E_\mu)].$$

Let $p_\mu^{7,0}$ be the first element in F_μ and $p_\mu^{9,0} = f_\mu(p_\mu^{7,0})$. In general, for $m < \beta$ let $p_\mu^{7,m}$ be the first element in the set $F_\mu - [H_\mu^m \cup f_\mu^*(H_\mu^m)]$, where

$$H_\mu^m = \{p_\mu^{7,i} \mid i < m\} \cup \{p_\mu^{9,i} \mid i < m\},$$

and $p_\mu^{9,m} = f_\mu(p_\mu^{7,m})$. Let

$$M_\mu = \{p_\mu^{7,i} \mid i < \beta\} \cup \{p_\mu^{9,i} \mid i < \beta\}.$$

Denote by $p_\mu^{8,0}$ the first element in the set $F_\mu - [M_\mu \cup f_\mu^*(M_\mu)]$, and by $p_\mu^{10,0}$, the element $f_\mu(p_\mu^{8,0})$. For $1 \leqq i < \beta$ let $p_\mu^{8,i}$ and $p_\mu^{10,i}$ represent no elements. If the power of Z_μ is $< 2^{\aleph_0}$, let $p_\mu^{i,m}$, where $7 \leqq i \leqq 10$ and $m < \beta$, represent no elements. For each $i < \beta$, let

$$A_i = E \cap \{p_\mu^{2n+1,i} \mid n \leqq 4, \mu < \theta\} \quad \text{and}$$

$$A_\omega = E \cap \{p_\mu^{2n+2,i} \mid i < \beta, n \leqq 4, \mu < \theta\}.$$

The sets A_γ are evidently pairwise disjoint. Let J be any open interval which contains a point of E. There exists an elements f_μ of $K^*(E)$ which is the identity on $E-J$ and for which $f_\mu(x) \neq x$ when x is in $J \cap E$. Consequently the elements $p_\mu^{1,m}$ and $p_\mu^{2,m}$ are in the set $J \cap E$. This shows that for each γ, A_γ is a dense subset of E. As each element in $K^*(E)$ occurs $2^{\aleph_0}$ times in the sequence $\{f_\xi\}$, $\xi < \theta$, each point of A_γ is a c-condensation point of A_γ. From the selection of the elements $p_\mu^{1,m}$ and $p_\mu^{2,m}$, it follows that the set union of the sets A_γ is E. Since A_γ is a dense subset of E and E has property C, each element f of $K^*(A_\gamma)$ can be extended to be an element of $K^*(E)$. From this and the manner in which the elements $p_\mu^{i,m}$, $3 \leqq i \leqq 6$, were chosen, it follows that $f(A_\gamma)$ is not a subset of A_γ. Therefore A_γ is exact. The manner of selection of the elements $p_\mu^{i,m}$, $i \geqq 7$, guarantees that for each element f in $K(A_\gamma)$ such that the power of the set $E \cap f(A_\gamma)$ is $2^{\aleph_0}$, there are $2^{\aleph_0}$ elements x in A_γ such that $f(x)$ is also in A_γ. Q.E.D.

Suppose that f is a similarity transformation of one of the sets A_i into one of the sets A_j, where $i \neq j$. Since $f(A_i)$ is a subset of A_j, the power of the set $f(A_i) \cap E$ is $2^{\aleph_0}$. Therefore there are $2^{\aleph_0}$ elements x in A_i such that $f(x)$ is also in A_i, thus not in A_j. Consequently f does not map A_i into A_j, i.e.,

THEOREM 1.1. *Let E and α be the same as in Lemma* 1.1. *Then E is the union of a family $\{A_i \mid i < \alpha\}$ of pairwise disjoint, exact sets, of power $2^{\aleph_0}$ each, such that the $\overline{A}_i$ are pairwise incomparable order types.*

COROLLARY. *For each ordinal number α, where $2 \leqq \alpha \leqq \omega$, R is the union of a family $\{A_i \mid i < \alpha\}$ of pairwise disjoint sets, which have the following properties:*

(a) *For each i, A_i is an exact, dense subset of R.*

(b) *Each set A_i has property* C.

(c) *The $\overline{A}_i$ are pairwise incomparable order types.*

If, in Theorem 1.1, the assumption that E has property C is removed, then some of the conclusions are no longer valid. In particular, it is no longer true that each set A_i can be made exact. To see this consider a set E which is defined as an ordered sum $E = B + C + D$(1), where C is denumerably infinite. Now let E be the union of $n+1$ disjoint sets E_i, where n is some non-negative integer. For at least one of the sets, say E_j, the set $E_j \cap C = G$ is denumerably

(1) Let $\{E_a \mid a \in A\}$ be a family of (not necessarily nonempty) pairwise disjoint simply ordered sets, where A is a nonempty simply ordered set. By the ordered sum $\sum_{a \in A} E_a$ is meant the set union of the E_a, $M = \bigcup_{a \in A} E_a$, the elements being ordered as follows: If $u < v$ in E_a, where u and v are two elements in E_a, then $u < v$ in M. If u is in E_a and v is in E_b, where $a < b$, then $u < v$ in M.

infinite. Let E_j be the ordered sum $F+G+H$. In [2, pp. 322–323] it was shown that no denumerably infinite set is exact. From the remark following Corollary 1 of Theorem 2 in [3], the set E_j is not exact. Thus we have demonstrated the following result:

THEOREM 1.2. *A necessary condition that E be the union of a finite number of disjoint exact sets is that E not be the ordered sum $B+C+D$, where C is denumerably infinite.*

As a modification of Theorem 1.1 we have

THEOREM 1.3. *Let A be any linear set of power $2^{\aleph_0}$. For each ordinal number α, where $2 \leqq \alpha \leqq \omega$, A is the union of a family $\{E_i \mid i<\alpha\}$ of pairwise disjoint sets, of power $2^{\aleph_0}$ each, such that the $\overline{E}_i$ are pairwise incomparable order types.*

Proof. Let E be the set of c-condensation points of A which belong to A. As is well known, the power of the set $A-E$ is $<2^{\aleph_0}$. The set E satisfies the assumptions of Lemma 1.1. Let $\{A_i \mid i<\alpha\}$ be the family of sets obtained from the conclusion of the lemma. Let $E_0=A_0\cup(A-E)$, and for $i\geqq 1$, let $E_i=A_i$. Suppose that $f(E_i)$ is a subset of E_j, where $i\neq j$, for some element f in $K(E_i)$. Then the power of the set $E\cap f(A_i)$ is $2^{\aleph_0}$. Since there are $2^{\aleph_0}$ elements x in A_i for which $f(x)$ is also in A_i, and since the E_i are pairwise disjoint, it follows that $f(A_i)$, thus $f(E_i)$, is not a subset of E_j. Thus the function f cannot exist. Therefore the order types of the E_i are pairwise incomparable.

COROLLARY. *In Lemma* 1.1 *let property* C *be replaced by the following property: Except for a finite number of points, each element of E is a c-condensation point of E. Then for each ordinal number α, where $2\leqq\alpha\leqq\omega$, E is the union of a family $\{A_i \mid i<\alpha\}$ of pairwise disjoint, exact sets, of power $2^{\aleph_0}$ each, such that if f is any element of $K(A_i)$ for which the power of the set $E\cap f(A_i)$ is $2^{\aleph_0}$, then there are $2^{\aleph_0}$ elements x in A_i such that $f(x)$ is also in A_i.*

REMARK. There is no difficulty in extending Lemma 1.1 and Theorems 1.1 and 1.3 to hold in the case where $2\leqq\alpha<\theta$.

THEOREM 1.4. *If $2^{\aleph_0}=\aleph_1$, then each linear set A, of power $2^{\aleph_0}$, is the union of a family $\{E_n \mid n<\omega\}$ of pairwise disjoint, exact sets, of power $2^{\aleph_0}$ each, such that the order types of the E_n are pairwise incomparable.*

Proof. Let E be the set of c-condensation points of A which belong to A. Since $2^{\aleph_0}=\aleph_1$ and the power of the set $A-E$ is $<2^{\aleph_0}$, $A-E$ is denumerable. Let $A-E=\{x_i \mid i<\mu\leqq\omega\}$. Let $\{A_i \mid i<\omega\}$ be the family of sets obtained from the conclusion of Lemma 1.1. For $i<\mu$ let $E_i=A_i\cup\{x_i\}$, and for $i\geqq\mu$ let $E_i=A_i$. Since A_i is exact, the set $A_i\cup\{x_i\}$ is exact. The pairwise incomparability of the order types of the E_i follows from an argument similar to that given in Theorem 1.3.

REMARK. The conclusion of Theorem 1.4 will be strengthened later to require that each set E_n have property A (Theorem 3.2).

We conclude this section with denumerable sets.

LEMMA 1.2. *Let S be a set such that* $\overline{S}\equiv\eta$. *If S is the union of a finite family* $\{A_i \mid i<n\}$ *of pairwise disjoint sets, then for at least one integer i,* $\overline{A}_i\equiv\eta$.

Proof. For each denumerable ordinal number ξ, $\xi\leqq\omega^\xi$ [5, p. 107]. Now ω^ξ is denumerable. For assume that ω^v is denumerable for each $v<\xi$. If $\xi=\gamma+1$, then $\omega^\xi=\omega^\gamma\omega$. If ξ is a limit number, then, by definition, $\omega^\xi=\lim_{v<\xi}\omega^v$. Being either the product of two, or the limit of a denumerable number of denumerable ordinal numbers, ω^ξ is denumerable. Let B_ξ be a subset of S of order type ω^ξ. Since S is the union of the finite number of sets A_i, one of the sets, say $A_{i(\xi)}$, contains a subset of B_ξ of order type ω^ξ. Therefore one of the sets, say A_j, has the property that for each denumerable ordinal number ξ, A_j contains a subset of order type $v\geqq\xi$. This implies that for each denumerable ordinal number ξ, A_j contains a subset of order type ξ. It is known that this is sufficient for $\overline{A}_j\equiv\eta$ to hold [6].

A consequence of the lemma is

THEOREM 1.5. *If* $\overline{S}\equiv\eta$, *then S is not the union of a finite family* $\{A_i \mid i<n\}$ *of pairwise disjoint sets such that the* $\overline{A}_i$ *are pairwise incomparable order types.*

2. **Decompositions into similar sets.** Due to the existence of sets which have property A, it is impossible to obtain results on decompositions into sets whose order types are equal as general as the results on decompositions into sets whose order types are pairwise incomparable.

Several decompositions of R into families of $\aleph_0$ and $2^{\aleph_0}$ pairwise disjoint, similar sets, of power $2^{\aleph_0}$ each, are already in the literature, e.g. [7; 12; 15]. Whether or not the sets obtained there are exact is an open problem. By Corollary 3.2 of [4], R cannot be the union of two disjoint, similar, exact sets. Whether or not R is the union of n disjoint, similar, exact sets, n being an integer >2, is unknown. In this section we shall show that R is the union of families of $\aleph_0$ and $2^{\aleph_0}$ pairwise disjoint, similar, exact sets, of power $2^{\aleph_0}$ each (Theorem 2.1 and Corollary 2 of Theorem 2.2).

LEMMA 2.1. *If* $\overline{E}=\overline{D}_0(\omega^*+\omega)$, *where* D_0 *is the union of two disjoint, exact sets (of power* $2^{\aleph_0}$ *each), B and C, then E is the union of a family of* $\aleph_0$ *pairwise disjoint, similar, exact sets (of power* $2^{\aleph_0}$ *each).*

Proof. Since $\overline{E}=\overline{D}_0(\omega^*+\omega)$, E is the ordered sum of the sets

$$E = \cdots + D_{-i} + \cdots + D_0 + D_1 + \cdots + D_j + \cdots,$$

each set D_i, $i=0, \pm 1, \pm 2, \cdots$, being similar to D_0. Let f_0 be the identity transformation of D_0 and f_i a similarity transformation of D_0 onto D_i. It follows from Theorem 2 of [3] that the family of sets

$$\{A_i = f_i(B) \cup f_{i+1}(C) \mid i = \pm n,\ n < \omega\}$$

satisfy the conclusion of the lemma.

Since $R = \{(i, i+1] \mid i = \pm n,\ n < \omega\}$, we obtain from Lemma 2.1 and the corollary to Theorem 1.3

THEOREM 2.1. *R is the union of a family $\{E_n \mid n < \omega\}$ of pairwise disjoint, similar, exact sets, of power $2^{\aleph_0}$ each.*

OPEN QUESTIONS. (1) Can the family of sets in Theorem 2.1 be chosen so that each set has property A?

(2) Let E be any linear set of power $2^{\aleph_0}$. Is R the union of a family $\{E_n \mid n < \omega\}$ of pairwise disjoint sets, each set being similar to E?

REMARK. It is not true that every linear set E, such that $\overline{E} \equiv \lambda$, is the union of a family $\{A_n \mid n < \omega\}$ of pairwise disjoint, similar, exact sets. Consider the following example. Let B be a dense subset of $(0, 1)$, which has properties A and C. The existence of such a set is guaranteed by Lemma 1 of [3] and Theorem 5 of [3]. The set E is defined to be

$$E = B \cup \{2 - 1/n \mid 1 \leqq n < \omega\} \cup (3, 4).$$

Evidently $\overline{E} \equiv \lambda$. Suppose that E is the union of a family $\{A_n \mid n < \omega\}$ of pairwise disjoint, similar sets. Since B has property A, no two of the sets, A_i and A_j, each contain $2^{\aleph_0}$ elements of B. Let $G_i = B \cap A_i$. Then $2^{\aleph_0} = \sum_{n<\omega} \text{top}(G_n)$, top (G_n) designating the power of G_n. By König's Theorem [5, p. 45], the power of one of the sets, say G_0, is $2^{\aleph_0}$. Since B has property A, at most one of the sets G_i is of power $2^{\aleph_0}$. Thus top $(G_i) < 2^{\aleph_0}$ for $i > 0$. Thus the power of the set $\bigcup_{i>0} G_i$ is $< 2^{\aleph_0}$. Now the set B has the property that if a and b are any two elements in B, then the power of the set $\{x \mid a < x < b,\ x \in B\}$ is $2^{\aleph_0}$. As $G_0 = B - \bigcup_{i>0} G_i$, it follows that G_0 has the same property, i.e., if a and b are any two elements in G_0, then the power of the set $\{x \mid a < x < b,\ x \in G_0\}$ is $2^{\aleph_0}$. Now $\overline{A}_0 = \overline{A}_i$. Thus, for each $i > 0$, if G_i had at least two elements, it would have to have $2^{\aleph_0}$ elements, a contradiction. Hence, for each $i > 0$, G_i has at most one element, say a_i. But G_0 has no first element. Therefore G_i must be empty, i.e., $G_0 = B$. Since A_0 has no first element, A_i has no first element. Therefore G_0 contains the set $\{2 - 1/n \mid 1 \leqq n < \omega\}$. By Theorem 1.2, the set A_0 cannot be exact.

We now prove a general theorem on decompositions of R.

THEOREM 2.2. *Let $\{\alpha_\xi\}$, $\xi < \theta$, be a sequence of linear order types, where $\alpha_\xi \neq 0$. Then R is the union of a family $\{E_\xi \mid \xi < \theta\}$ of pairwise disjoint sets, such that $\overline{E}_\xi = \alpha_\xi$.*

Proof. Let C be the Cantor set, i.e., all numbers of the form $x = \sum_{i=1}^{\infty} a_i/3^i$, where $a_i = 0$ or 2. For each sequence of 0's and 2's, say $t_1t_3t_5 \cdots$, the set

$$C(t_1t_3t_5 \cdots) = \{x \mid x \in C,\ a_i = t_i \text{ for all odd } i\}$$

is a perfect set. The family of all such sets is then a family of $2^{\aleph_0}$ pairwise disjoint perfect sets. Under a mapping of the form $ax+b$, $a\neq 0$, a perfect set is mapped into a perfect set. Thus each interval of R contains a family of $2^{\aleph_0}$ pairwise disjoint perfect sets. For each integer n, which is positive, negative, or 0, let $\{P_\xi^n \mid \xi<\theta\}$ and $\{Q_\xi^n \mid \xi<\theta\}$ be two families of pairwise disjoint, perfect sets such that Q_ξ^n and P_ξ^n are subsets of the open intervals $(n, n+1/2)$ and $(n+1/2, n+1)$ respectively. Furthermore, for each n

(*) let the power of the two sets,

$$(n, n+1/2) - \bigcup_{\xi<\theta} Q_\xi^n \quad \text{and} \quad (n+1/2, n+1) - \bigcup_{\xi<\theta} P_\xi^n$$

be $2^{\aleph_0}$ each.

Denote by $Z=\{Y_v \mid v<\theta\}$ the family of sets

$$Z = \{(P_\xi^{i-1} \cup Q_\xi^{i+1}) \mid i = \pm n, n < \omega, \xi < \theta\}.$$

For each $v<\theta$ define A_v and B_v to be the P-set and Q-set respectively of Y_v. For each $v<\theta$ there exist two order types σ_v and τ_v (possibly 0) such that $\alpha_v=\sigma_v+1+\tau_v$. Let C_v be a subset of A_v having order type σ_v, and D_v a subset of B_v having order type τ_v. The sets C_v and D_v exist since, P_ξ^n being a perfect set, $\overline{P_\xi^n}\equiv\lambda$. Let H be the set $R-\bigcup_{v<\theta}[C_v\cup D_v]$. By (*) the power of H is $2^{\aleph_0}$. Well order the elements of H into the sequence $\{x_\xi\}$, $\xi<\theta$. Assume that the set $E_v=C_v\cup D_v\cup\{y_v\}$ has been defined for each ordinal number v, where $v<\mu<\theta$. Denote by y_μ the first element in the set $H-\{y_v \mid v<\mu\}$ such that $u<y_\mu<v$ for each element u in A_μ and v in B_μ. From (*) and the fact that an interval $(n, n+1)$ lies between A_μ and B_μ the element y_μ exists. Denote by E_μ the set $C_\mu\cup D_\mu\cup\{y_\mu\}$. Obviously the E_μ are pairwise disjoint sets such that $\overline{E}_\mu=\alpha_\mu$. There is no difficulty in seeing that each element x_ξ is one of the y_v. In fact, consider the element x_ξ of H. Suppose that x_ξ is in the closed interval $[n, n+1]$. Let Y_v be the ξth set in Z whose P-set and Q-set are separated by $[n, n+1]$. From the selection of the points y_μ, x_ξ must be in the set $\{y_\mu \mid \mu\leqq\xi\}$. Consequently the union of the E_μ is R. Q.E.D.

Corollary 1. *For each linear order type α, $\alpha\neq 0$, R is the union of a family $\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint sets, each of order type α.*

Corollary 2. *R is the union of a family, $\{E_\xi \mid \xi<\theta\}$, of pairwise disjoint, similar, exact sets, each having property* A.

The same method of proof may be used to show

Theorem 2.3. *Let $\{\overline{G}_\xi\}$, $\xi<\theta$, be a sequence of linear order types such that for $2^{\aleph_0}$ ordinal numbers ξ, G_ξ has both a first and a last element. Then each set E for which $\overline{E}\equiv\lambda$ is the union of a family $\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint sets such that $\overline{G}_\xi=\overline{E}_\xi$ for each $\xi<\theta$.*

The problem of whether or not, for $2^{\aleph_0}$ ordinal numbers ξ, G_ξ must have both a first and a last element in order that the conclusion of Theorem 2.3 hold for every set E for which $\overline{E}\equiv\lambda$ remains unanswered. In the special case that $\overline{G}_\xi=\overline{G}_v$ for $\xi\neq v$ the answer is obviously in the affirmative (consider when E is the closed interval $[a, b]$), i.e.,

THEOREM 2.4. *Let B be a linear set. A necessary and sufficient condition that each set E, where $\overline{E}\equiv\lambda$, be the union of a family $\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint sets, each of order type $\overline{B}$, is that B have both a first and a last element.*

We now consider a general result on the decomposition of a linear set into a family of $2^{\aleph_0}$ pairwise disjoint, similar, exact sets, each set having property A.

THEOREM 2.5. *Let E be a linear set which contains a family $E=\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint, similar subsets, of power $2^{\aleph_0}$ each. Then E is the union of a family $T=\{A_\xi \mid \xi<\theta\}$ of pairwise disjoint, similar, exact sets, each of which has property* A. *Furthermore, $\overline{A}_\xi<\overline{E}_\xi$.*

In order to prove the theorem we first prove two lemmas.

LEMMA 2.2. *Let $F=\{E_\xi \mid \xi<\theta\}$ be a family of pairwise disjoint, similar, linear sets, of power $2^{\aleph_0}$ each. Then there exists a family $G=\{B_\xi \mid \xi<\theta\}$ of similar subsets, of power $2^{\aleph_0}$ each, and a closed interval $[p, q]$ of R, with the following properties:*

(a) *Each element of G is a subset of $[p, q]$.*

(b) *Each element of G is a subset of an element of F, and each element of F contains, at most, one element of G (thus the B_ξ are pairwise disjoint).*

(c) *For each $\xi<\theta$, if u and v, where $u<v$, are any two elements of B_ξ, then the power of the set $\{x \mid u<x<v,\ x\in B_\xi\}$ is $2^{\aleph_0}$.*

(d) *There are $2^{\aleph_0}$ elements x in the set $\bigcup_{\xi<\theta}E_\xi$ such that $x<p$, and there are $2^{\aleph_0}$ elements x in the set $\bigcup_{\xi<\theta}E_\xi$ such that $x>q$.*

Proof. For each $\xi<\theta$ let f_ξ be a similarity transformation of E_0 onto E_ξ. Denote by C the set of c-condensation points of E_0 which are in E_0. Let I_0 be an interval of C which has a first element a and a last element b, $a\neq b$. Let p be a c-condensation point of the set $H=\{f_\xi(a) \mid \xi<\theta\}$ which is in H, and let $K=\{v \mid f_v(a)\geqq p\}$. Let q be a c-condensation point of the set $J=\{f_v(b) \mid v\in K\}$ which is in J, and let $L=\{v \mid f_v(b)\leqq q,\ v\in K\}$. The power of L is $2^{\aleph_0}$. Denote by $G=\{B_\xi \mid \xi<\theta\}$ the family of sets $\{f_v(I) \mid v\in L\}$. The family G satisfies the conclusion of the lemma.

LEMMA 2.3. *If $G=\{B_\xi \mid \xi<\theta\}$ is a family of sets which satisfy the conclusion of Lemma 2.2, then there exists a family $M=\{D_\xi \mid \xi<\theta\}$ of similar sets, of power $2^{\aleph_0}$ each, with the following properties:*

(1) *Each element of M is a subset of an element of G (thus the D_ξ are pairwise disjoint).*

(2) *M is the union of two disjoint families, $Y=\{P_\xi \mid \xi<\theta\}$ and $Z=\{Q_v \mid v<\theta\}$, such that $x<y$ for all elements x in P_ξ and y in Q_v.*

Proof. For each $\xi<\theta$ let g_ξ be a similarity transformation of B_0 onto B_ξ. For each element x in B_0 let $W_x=\{g_\xi(x) \mid \xi<\theta\}$ and V_x be the set of c-condensation points of W_x which are in W_x. V_x being a bounded set, there exists a least upper bound to the elements of V_x, say $v(x)$. For $x\leqq y$, $v(x)\leqq v(y)$. Since V_x has property C, $y<v(x)$ for each element y in V_x. Let $v(s)$, where s is in B_0, be an element of the set $S=\{v(x) \mid x\in B_0\}$ which has the following property:

(*) For each real number $u<v(s)$, there are $2^{\aleph_0}$ elements x, $x<s$, in B_0 such that $v(x)$ is in the half open interval of R, $(u, v(s)]$.

Let y and t, where $t<s$, be elements of V_s and B_0 respectively, such that $y<v(t)\leqq v(s)$. By (*) the element t exists. Since $v(t)$ is the least upper bound of the elements of V_t, there exists an element z in V_t such that $y<z<v(t)$. Let

$$U = \{v \mid g_v(s) \leqq y\} \quad \text{and} \quad V = \{v \mid g_v(t) \geqq z\}.$$

Since $t<s$ and $y<z$, it follows that U and V are disjoint sets. Since y and z are c-condensation points of V_s and V_t respectively, the power of U and V is $2^{\aleph_0}$ each. Let $N=\{x \mid t\leqq x\leqq s, x\in B_0\}$, $Y=\{g_v(N) \mid v\in U\}$, $Z=\{g_v(N) \mid v\in V\}$, and $M=Y\cup Z$. Condition (1) is satisfied. From (c) of Lemma 2.2, the power of N is $2^{\aleph_0}$. Thus the power of each set D_ξ is $2^{\aleph_0}$. If ξ is in U and x is in N, then $p\leqq g_\xi(x)\leqq y$. If v is in V and x is in N, then $y<z\leqq g_v(x)\leqq q$. Therefore condition (2) is satisfied.

Turning to the proof of Theorem 2.5 let $Y=\{P_\xi \mid \xi<\theta\}$ and $Z=\{Q_v \mid v<\theta\}$ be two families of sets which satisfy the conclusion of Lemma 2.3. Note that $\overline{P}_0\leqq\overline{B}_0$. Let h_ξ and k_ξ be similarity transformations of P_0 onto P_ξ and Q_ξ respectively. Let F_0 be an exact subset of P_0, having property A, such that $1+\overline{F}_0+1<\overline{P}_0$. The set F_0 exists by Theorems 4 and 5 in [3]. Let $M_\xi=h_\xi(F_0)$ and $N_\xi=k_\xi(F_0)$. By means of transfinite induction we can define four disjoint sets, $J_i=\{p_\xi^i \mid \xi<\theta\}$, $i<4$, whose set union is the set $E-[\bigcup_{\xi<\theta}(M_\xi\cup N_\xi)]$, such that

(a) p_ξ^0 precedes, and p_ξ^1 follows, each element of M_ξ; and

(b) p_ξ^2 precedes, and p_ξ^3 follows, each element of N_ξ.

In view of (d) of Lemma 2.2 and (2) of Lemma 2.3, the sets J_i, $i<4$, exist. Define T to be the family of sets

$$T = \{M_\xi \cup \{p_\xi^0, p_\xi^1\} \mid \xi<\theta\} \cup \{N_\xi \cup \{p_\xi^2, p_\xi^3\} \mid \xi<\theta\}.$$

The family T satisfies the conclusion of the theorem. Q.E.D.

Using the methods of this section the following two results can be shown:

Theorem 2.6. *Let D be a denumerable set. A necessary and sufficient condition that each set E, for which $\overline{E}\equiv\eta$, be the union of a family $\{E_n \mid n<\omega\}$ of pair-*

wise disjoint sets, where $\overline{E}_n = \overline{D}$ *for each* n, *is that* D *have both a first and a last element.*

THEOREM 2.7. *For each sequence of denumerable order types* $\{\alpha_n\}_{n<\omega}$, *where* $\alpha_n \neq 0$, *the set of rational numbers is the union of a family* $\{E_n \mid n<\omega\}$ *of pairwise disjoint sets such that* $\overline{E}_n = \alpha_n$.

3. Decompositions into sets having property A.

LEMMA 3.1. *A linear set* E, *of power* $2^{\aleph_0}$, *which is the union of a family* $\{A_i \mid i \leqq n\}$ *of pairwise disjoint, similar sets, where* n *is a non-negative integer, cannot be the union of less than* $n+1$ *pairwise disjoint sets, each of which has property* A.

Proof. Assume the contrary, i.e., assume that E is the union of k disjoint sets B_i, where $k \leqq n$ and each set B_i has property A. Let f_i be a similarity transformation of A_i onto A_{i+1} for $i<n$. As the power of the set E is $2^{\aleph_0}$, the power of each set A_i must be $2^{\aleph_0}$. Therefore the power of the intersection of at least one of the sets B_i with A_0 is $2^{\aleph_0}$. For simplicity of notation we shall assume that B_0 is one such set. Denote by C_0 the set A_0 and by C_1 the set $f_0(A_0 \cap B_0)$. Suppose that for each integer i, where $1 \leqq i \leqq j$, the set C_i of power $2^{\aleph_0}$ has been defined so that $C_i = f_{i-1}(C_{i-1} \cap B_{i-1})$. Now the power of the intersection of at least one of the sets B_i with C_j is $2^{\aleph_0}$. Suppose that for some integer $i<j$ the power of the set $C_j \cap B_i$ is $2^{\aleph_0}$. Then

$$C_k \cap B_i \quad \text{and} \quad \overset{*}{f}_i \overset{*}{f}_{i+1} \cdots \overset{*}{f}_{k-1}(C_k \cap B_i)$$

are two disjoint, similar subsets of B_i, of power $2^{\aleph_0}$ each. This, however, contradicts the assumption that the set B_i has property A. Therefore, if the power of the set $C_j \cap B_i$ is $2^{\aleph_0}$, then $i \geqq j$. For simplicity we assume that the power of the set $C_j \cap B_j$ is $2^{\aleph_0}$. Denote by C_{j+1} the set $f_j(C_j \cap B_j)$. By mathematical induction, the sets C_i become defined for $i \leqq k-1$. Since $k \leqq n$, the set $C_k = f_{k-1}(C_{k-1} \cap B_{k-1})$ is well defined and of power $2^{\aleph_0}$. Therefore the power of the intersection of at least one of the sets B_i with C_k is $2^{\aleph_0}$. If B_i is such a set, where of necessity $i<k$, then

$$C_k \cap B_i \quad \text{and} \quad \overset{*}{f}_i \overset{*}{f}_{i+1} \cdots \overset{*}{f}_{k-1}(C_k \cap B_i)$$

are two disjoint, similar subsets of B_i, of power $2^{\aleph_0}$ each. This contradicts the assumption that B_i has property A. We conclude that E cannot be the union of less than $n+1$ disjoint sets, each of which has property A.

COROLLARY 1. *A linear set* E, *of power* $2^{\aleph_0}$, *whose order type is* $\sigma(n+1)$, *where* n *is a non-negative integer, cannot be the union of less than* $n+1$ *pairwise disjoint sets, each of which has property* A.

COROLLARY 2. R *is not the union of a finite number of disjoint sets, each of which has property* A.

THEOREM 3.1. *A linear set E, which contains a set D, of power $2^{\aleph_0}$, such that D is the union of a family of $n+1$ pairwise disjoint, similar sets, where n is a non-negative integer, cannot be the union of less than $n+1$ disjoint sets, each of which has property* A.

Proof. Suppose that E is the union of a family $\{B_i | i<k\}$ of pairwise disjoint sets B_i, each of which has property A, where $k<n+1$. Let $A_i = B_i \cap D$. As D is of power $2^{\aleph_0}$, at least one of the sets A_i is of power $2^{\aleph_0}$. Let C_0^*, $C_1, \cdots, C_m$, where $m<k$, be those sets A_i which are of power $2^{\aleph_0}$. As subsets, of power $2^{\aleph_0}$ each, of sets which have property A, C_0^* and C_i each has property A. Denote by C_0 the set union of C_0^* and those sets A_i which are of power $<2^{\aleph_0}$. By Lemma 3.1 of [4], C_0 has property A. This implies that D is the union of less than $n+1$ disjoint sets, each of which has property A. But this contradicts Lemma 3.1. Hence our result.

COROLLARY. *A linear set which contains a set D such that D is the union of a family $\{B_i | i<\omega\}$ of pairwise disjoint, similar sets, of power $2^{\aleph_0}$ each, cannot be the union of less than $\aleph_0$ disjoint sets, each of which has property* A.

OPEN QUESTION. Let E be a linear set of power $2^{\aleph_0}$. If k is the largest integer for which E contains a subset D, of power $2^{\aleph_0}$, such that D is the union of k pairwise disjoint, similar sets, then is E the union of k disjoint sets, each of which has property A?

LEMMA 3.2. *Let $\{A_i | i<n\}$ be a finite family of pairwise disjoint, similar, linear sets, of power $2^{\aleph_0}$ each. Then there exists a family $\{B_i | i<n\}$ of similar, exact sets, with the following properties:*

(1) *Each set has a first and a last element.*

(2) *For each $i<n$, B_i is a subset of A_i.*

(3) *For each pair of distinct sets A_i and A_j, either $x<y$ for each element x in B_i and y in B_j, or $y<x$ for each element x in B_i and y in B_j.*

(4) *B_i has property* A.

Proof. We shall first prove the lemma for $n=2$. Denote by M the set of c-condensation points of A_0 which are in A_0. The set M has property C. Let f be a similarity transformation of A_0 onto A_1. Since A_0 and A_1 are disjoint, for any element p in M, $f(p)\neq p$, say $f(p)>p$. As p is a c-condensation point of M, the power of the set $P=\{x | p<x<f(p), x\in M\}$ is $2^{\aleph_0}$. The function f being a similarity transformation of A_0 onto A_1, $f(x)>f(p)$ for each element x in P. Now P contains an exact subset B_0, having property A, with a first and a last element. Let $B_1=f(B_0)$. The two sets, B_0 and B_1, satisfy the conclusion of the lemma. An analogous argument occurs if $f(p)<p$.

We continue by induction. Assume that the lemma is true for $n\leqq k$. We now show that it is true for $n=k+1$. By our induction hypothesis there exists a family $\{C_i | i<k\}$ of similar, exact sets, having property A, which satisfies

the lemma. Without loss of generality we may assume that if x is in C_i and y is in C_j, where $i<j<k$, then $x<y$. For each $i<k$, let a_i and b_i denote the first and the last elements respectively of C_i. Let G be a subset of A_k which is similar to C_0. Consider the open intervals of R,

$$(*) \qquad (-\infty, a_0), (a_0, b_0), (b_0, a_1), \cdots, (a_{k-1}, b_{k-1}), (b_{k-1}, +\infty).$$

Since the power of G is $2^{\aleph_0}$, one of the intervals in (*) contains $2^{\aleph_0}$ elements of G.

(a) Suppose that the set $H=G\cap(a_i, b_i)$ is of power $2^{\aleph_0}$. Since the lemma is true for two sets, C_i and H contain exact subsets B_i and B_k respectively, each of power $2^{\aleph_0}$, which satisfy the conclusion of the lemma. For each $j<k$, $j\neq i$, let B_j be a subset of C_j which is similar to B_i.

(b) Suppose that the set $H=G\cap J$, where J is an interval in (*) not of the form (a_i, b_i) for some $i<k$, is of power $2^{\aleph_0}$. Let B_k be an exact subset of H, of power $2^{\aleph_0}$, which has a first and a last element. For each $i<k$, let B_i be a subset of C_i which is similar to B_k.

In each case the family of sets $\{B_i | i<k+1\}$ satisfy the conclusion of the lemma. Q.E.D.

In view of the previous lemma the open question stated prior to it may be phrased as follows: Let E be a linear set of power $2^{\aleph_0}$. If k is the largest integer for which E contains a subset D, of power $2^{\aleph_0}$, such that $\overline{D}=\sigma k$ for some order type σ, does it follow that E is the union of k disjoint sets, each of which has property A? If $\overline{E}\equiv\overline{D}$, then the answer trivially is yes.

We have seen that certain sets cannot be the union of a finite number of disjoint sets, each of which has property A. The problem arises of determining whether or not each linear set, of power $2^{\aleph_0}$, is the union of $\aleph_0$ disjoint sets, each of which has property A. The answer to this, in the affirmative provided that one assumes the continuum hypothesis, i.e., $2^{\aleph_0}=\aleph_1$ (Theorem 3.2). The problem of the decomposition of an arbitrary linear set of power $2^{\aleph_0}$, into a family of $2^{\aleph_0}$ disjoint sets, each having property A, was settled in the affirmative in Theorem 3.1 of [4].

For each linear set E, of power $2^{\aleph_0}$, denote by $V(E)$ the family of those Borel subsets of E, with respect to E, which are of power $2^{\aleph_0}$.

THEOREM 3.2. *If $2^{\aleph_0}=\aleph_1$, then each linear set E, of power $2^{\aleph_0}$, is the union of a family $\{E_n | n<\omega\}$ of pairwise disjoint, exact sets, each of which has property* A. *In addition, the order types of the E_n are pairwise incomparable. If, furthermore, E has property* C, *then each set E_n is a dense subset of E, which has property* C.

Proof. The demonstration is based, to a certain extent, on a modification of a lemma due to Banach [1].

Denote by M the set of c-condensation points of E which belong to E. Let G be the set of couples $s=(f, B)$, where f is in $K^*(B)$ and B is an element of $V(M)$ having property C. Well order the elements of G into the sequence $\{s_\xi\}$, $\xi<\theta$, where each element in G occurs $2^{\aleph_0}$ times in the sequence, and $s_\xi=(f_\xi, B_\xi)$. We further require the sequence $\{s_\xi\}$, $\xi<\theta$, to have the property that for $\xi=v+6t$, where v is either 0 or a limit number and t is a nonnegative integer,

$$s_\xi = s_{\xi+1} = s_{\xi+2} = s_{\xi+3} = s_{\xi+4} = s_{\xi+5}.$$

Such a sequence is certainly possible. Denote by J_ξ the set

$$J_\xi = \{x \mid f_\xi(x) \neq x,\ x \in B_\xi\}.$$

Since each element of B_ξ is a c-condensation point of B_ξ, the power of J_ξ is $2^{\aleph_0}$. Well order the elements of R into the sequence $\{x_\xi\}$, $\xi<\theta$.

Let $N_0^0=\{p_i^{0,0} \mid i<\omega\}$ be the set of the first ω elements in R. Let $p_0^{0,1}$ be the first element in the set $J_0-[N_0^0\cup f_0^*(N_0^0)]$, and $p_0^{0,2}=f_0(p_0^{0,1})$. In general, for each positive integer n, let $p_n^{0,1}$ be the first element in the set $J_0-[N_n^0\cup f_0^*(N_n^0)]$, where

$$N_n^0 = N_0^0 \cup \{p_i^{0,1} \mid i < n\} \cup \{p_i^{0,2} \mid i < n\}, \quad \text{and} \quad p_n^{0,2} = f_0(p_n^{0,1}).$$

Denote by f^0 the identity function and, for $n>0$, by f^n the function $f[f^{n-1}]$, and by f^{-n} the function $f^*[f^{-n+1}]$.

Suppose that for each ordinal number ξ, where $\xi<\alpha<\theta$, the set

$$P^\xi = \{p_m^{\xi,j} \mid j < 3,\ m < \omega\}$$

has been defined. For any finite number of ordinal numbers, $\alpha_1, \cdots, \alpha_k$, each smaller than α, for any integers (positive, negative, or 0), $n_1, \cdots, n_k$, and for any element x in the set $\bigcup_{\xi<\alpha}P^\xi$, consider the element

$$f_{\alpha_1}^{n_1} f_{\alpha_2}^{n_2} \cdots f_{\alpha_k}^{n_k}(x).$$

Denote by W_α the set of all such elements. Since $2^{\aleph_0}=\aleph_1$, the power of the set W_α is $\aleph_0$. Let $N_0^\alpha=\{p_i^{\alpha,0} \mid i<\omega\}$ be the set of the first ω elements in $R-W_\alpha$. Let $p_0^{\alpha,1}$ be the first element in the set $J_\alpha-[N_1^\alpha\cup f_\alpha^*(N_1^\alpha)]$, where $N_1^\alpha=N_0^\alpha\cup W_\alpha$, and $p_0^{\alpha,2}=f_\alpha(p_0^{\alpha,1})$. In general, for each positive integer n, let $p_0^{\alpha,1}$ be the first element in the set $J_\alpha-[N_n^\alpha\cup f_\alpha^*(N_n^\alpha)]$, where

$$N_n^\alpha = W_\alpha \cup N_0^\alpha \cup \{p_i^{\alpha,j} \mid j = 1, 2;\ i < n\},$$

and $p_n^{\alpha,2}=f_\alpha(p_n^{\alpha,1})$.

Now denote by H_0 the set W_1, and by H_v the set $W_{v+1}-W_v$. Following the proof of Banach, where $F=\{f_\xi \mid \xi<\theta\}$, we note that

$$P^v \subseteq H_v \quad \text{and} \quad \{p_i^{v,1} \mid i < \omega\} \subseteq H_v \cap E.$$

Since the power of W_{v+1} is $\aleph_0$, the power of H_v, and also of $H_v \cap E$, is $\aleph_0$. Furthermore, the family of sets $\{H_v | v < \theta\}$ satisfy the conclusion of the lemma in [1].

Denote by $\{y_j^v | j < q(v) \leqq \omega\}$ the elements of the denumerable set

$$(E \cap H_v) - \{p_j^{v,i} \mid i = 1, 2; j < \omega\}.$$

We shall now define three families of disjoint subsets of R, $\{S_i^* | i < \omega\}$, $\{T_i^* | i < \omega\}$, and $\{U_i^* | i < \omega\}$, by the following procedure: Let v be 0 or a limit number, and let t be a non-negative integer.

(1) If $\xi = v + 6t$, then $p_i^{\xi,1}$ is in S_i^*, $p_i^{\xi,2}$ is in T_i^*, and y_i^ξ is in U_i^*.

(2) If $\xi = v + 6t + 1$, then $p_i^{\xi,1}$ is in S_i^*, $p_i^{\xi,2}$ is in U_i^*, and y_i^ξ is in T_i^*.

(3) If $\xi = v + 6t + 2$, then $p_i^{\xi,1}$ is in T_i^*, $p_i^{\xi,2}$ is in S_i^*, and y_i^ξ is in U_i^*.

(4) If $\xi = v + 6t + 3$, then $p_i^{\xi,1}$ is in T_i^*, $p_i^{\xi,2}$ is in U_i^*, and y_i^ξ is in S_i^*.

(5) If $\xi = v + 6t + 4$, then $p_i^{\xi,1}$ is in U_i^*, $p_i^{\xi,2}$ is in S_i^*, and y_i^ξ is in T_i^*.

(6) If $\xi = v + 6t + 5$, then $p_i^{\xi,1}$ is in U_i^*, $p_i^{\xi,2}$ is in T_i^*, and y_i^ξ is in S_i^*.

For each i let $S_i = M \cap S_i^*$, $T_i = T_i^* \cap M$, and $U_i = U_i^* \cap M$.

Since the family of sets

$$\{S_i \mid i < \omega\} \cup \{T_i \mid i < \omega\} \cup \{U_i \mid i < \omega\}$$

is denumerable, its elements may be relabeled as $\{Q_n | n < \omega\}$. From the fact that M has property C, it follows that

(α) each set Q_n has property C, and

(β) each set Q_n is a dense subset of M.

For each n consider the set Q_n. Let Y and Z be any two disjoint subsets of Q_n of power $2^{\aleph_0}$ each. Now let f be any element of $K^*(Y)$. The function f can be extended to be an element g of $K^*(B)$, where B is an element of $V(M)$. From Theorem 2 of [1], the power of the set $g(Y) \cap Z = f(Y) \cap Z$ is $< 2^{\aleph_0}$. Therefore Y is similar to no subset of Z. This shows that Q_n has property A. Combining this with (α) we conclude, in virtue of Theorem 2.3 of [4], that Q_n is exact. Since $2^{\aleph_0} = \aleph_1$, the set $E - M$ is denumerable, say $\{z_i | i < \delta \leqq \omega\}$. For $i < \delta$ let $E_i = Q_i \cup \{z_i\}$, and for $i \geqq \delta$, let $E_i = Q_i$.

From Corollary 2 of Theorem 2 of [3] and Lemma 3.1 of [4], $\{E_n | n < \omega\}$ is a family of pairwise disjoint, exact sets, each of which has property A. Therefore we only have to show that the order type of the E_n are pairwise incomparable. Suppose the contrary, i.e., for two integers i and j, $i \neq j$, there exists a similarity transformation f of E_i into E_j. We shall show that this entails a contradiction. First suppose that either $E_i = S_m \cup \{z_i\}$ or $E_i = S_m$. Thus f is a similarity transformation of S_m into E_j. From (β), the fact that $E_i \cap E_j = \varnothing$, and Lemma 1.1 of [4], it follows that f, acting on S_m, can be extended to be an element g in $K^*(B)$, where (g, B) is in G and B contains S_m. From (1), (2), and the properties of the sequence $\{s_\xi\}$, $\xi < \theta$, there are $2^{\aleph_0}$ elements x in S_m such that $g(x)$ is in T_m^*, and $2^{\aleph_0}$ elements x in S_m such that $g(x)$ is in U_m^*. This implies that $f(E_i)$ cannot be a subset of E_j. If S_m is replaced by T_m or U_m, an analogous argument is possible. Under no circumstances, therefore, can f be a similarity transformation of E_i into E_j.

The last sentence in the statement of the theorem is obvious since then $E=M$. Q.E.D.

While proving Theorem 3.2 we have also demonstrated the following result:

THEOREM 3.3. *Let E be a linear set of power $2^{\aleph_0}$, and F a family of one-to-one transformations, of power $2^{\aleph_0}$, which contains $K(E)$. If $2^{\aleph_0}=\aleph_1$, then there exists a family $T=\{H_\xi \mid \xi<\theta\}$ of pairwise disjoint sets which satisfy the following conditions*:

(1) $R=\bigcup_{\xi<\theta}H_\xi$;

(2) *for each ordinal number ξ, $\xi<\theta$, the power of each of the two sets, H_ξ and $H_\xi\cap E$, is $\aleph_0$; and*

(3) *f being any transformation belonging to F and D being the domain of f, there exists an ordinal number $\alpha<\theta$ such that $f(D\cap H_\xi)$ is a subset of H_ξ for $\xi>\alpha$.*

If, furthermore, E has property C, *then the family T can be chosen to satisfy the following additional condition*:

(4) *for each $\xi<\theta$, if p_ξ is an element of $H_\xi\cap E$, then the set $D=\{p_\xi \mid \xi<\theta\}$ is a dense subset of E having property* C.

Turning to decompositions into a family of $2^{\aleph_0}$ sets we have

THEOREM 3.4. *Each linear set E, of power $2^{\aleph_0}$, is the union of a family $H=\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint, exact sets, with the following property: If P is any subfamily of H, of power $<2^{\aleph_0}$, and if $S(P)=\bigcup_{E_\xi\in P}E_\xi$, then $S(P)$ has property* A. *Furthermore, if E has property* C, *then each set $S(P)$ is exact.*

Proof. Let B be a subset of E which has properties A and C. Let $\{x_\xi \mid \xi<\mu\leqq\theta\}$ be the set $E-B$. Let B be the union of a family $\{C_\xi \mid \xi<\theta\}$ of pairwise disjoint, nonempty sets, each of which has property C. Such a decomposition is possible. For example, let F be the family of those open intervals of R which contain at least one point (thus $2^{\aleph_0}$ points) of B. Let G be the set of those pairs (I, ξ), where I is in F and ξ is an ordinal number $<\theta$. Well order the elements of G into a sequence $\{(I_v, \xi_v)\}$, where each pair in G occurs $2^{\aleph_0}$ times in the sequence. Let $B=\{y_\xi \mid \xi<\theta\}$. Let z_0 be the first element in $I_0\cap B$. Continuing by transfinite induction, suppose that z_μ is defined for each $\mu<\gamma$. Let z_γ be the first element in $(I_\gamma\cap B)-\{z_\mu \mid \mu<\gamma\}$. For each $v<\theta$, let $C_v=\{z_\mu \mid \xi_\mu=v\}$. It is easily seen that (1) $B=\{z_\gamma \mid \gamma<\theta\}$, (2) B is the union of the disjoint sets C_v, and (3) each C_v has property C. By Theorem 2.3 of [4], each set C_ξ is exact. For each $\xi<\mu$ let $E_\xi=C_\xi\cup\{x_\xi\}$, and for $\xi\geqq\mu$ let $E_\xi=C_\xi$. Each set E_ξ is exact. By Lemma 3.1 of [4], if P is any subfamily of H, of power $<2^{\aleph_0}$, then $S(P)$ has property A.

On assuming that E has property C, B and each set E may be chosen to be a dense subset of E. Therefore each point of $S(P)$ is a c-condensation point of $S(P)$. Since $S(P)$ also has property A, by Theorem 2.3 of [4], $S(P)$ is exact.

REMARK. Let P and Q be any two disjoint subfamilies of H, of power $<2^{\aleph_0}$ each. Then $S(P\cup Q)$ has property A. Consequently $S(P)$ and $S(Q)$ have incomparable order types. Letting $P=\{E_\xi\}$ and $Q=\{E_v\}$, i.e., P and Q each consist of just one set, we see that the order types of the E_ξ are pairwise incomparable.

In view of the preceding remark we have

COROLLARY. *Each linear set E, of power $2^{\aleph_0}$, is the union of a family $H=\{E_\xi \mid \xi<\theta\}$ of pairwise disjoint, exact sets, with the following two properties:*

(1) *For each E_ξ, no two disjoint subsets, of power $2^{\aleph_0}$ each, of E_ξ are similar.*

(2) *For $\xi\neq v$, no subset of E_ξ, of power $2^{\aleph_0}$, is similar to a subset of E_v.*

The question arises as to whether or not the above corollary holds for a decomposition into a denumerable family. The answer to this will be shown to be in the negative.

THEOREM 3.5. *Let E be a linear set of power $2^{\aleph_0}$. For each ordinal number α, where $2\leqq\alpha\leqq\theta$, E is the union of a family $\{E_\gamma \mid \gamma<\alpha\}$ of pairwise disjoint sets, of power $2^{\aleph_0}$ each, with the following property: For each $\gamma<\alpha$, each subset N of power $2^{\aleph_0}$ of E_γ, and each similarity transformation f of N into R, the power of $[E\cap f(N)]-E_\gamma$ is $<2^{\aleph_0}$.*

Proof. Denote by G the set of triples $s=(f, B, v)$, where B is in $V(E)$, f is an element of $K^*(B)$ such that the power of the set $\{x \mid f(x)\neq x, x\in B\}$ is $2^{\aleph_0}$, and v is an ordinal number $<\alpha$. Well order the elements of G into a sequence $\{s_\xi\}$, $\xi<\theta$, where $s_\xi=(f_\xi, B_\xi, v_\xi)$. Repeat the procedure of Theorem 3.2, obtaining the sets W_v and H_v. Thus E is the union of the disjoint sets $H_v\cap E$, and for each v, $f_v(H_\xi\cap B_v)$ is a subset of H_ξ for $\xi>v$. Notice that since we are not assuming that $2^{\aleph_0}=\aleph_1$, we can only deduce that the powers of W_v, H_v, $H_v\cap E$, and $\bigcup_{v<\mu<\theta}H_v$ are infinite and $<2^{\aleph_0}$. For each $\gamma<\alpha$ let $T(\gamma)=\{\xi \mid \gamma=v_\xi\}$. The $T(\gamma)$ are pairwise disjoint sets, which contain $2^{\aleph_0}$ ordinal numbers. For each $\gamma<\alpha$ let $E_\gamma=\bigcup_{\xi\in T(\gamma)}(H_\xi\cap E)$. Then E is the union of the family $\{E_\gamma \mid \gamma<\alpha\}$ of pairwise disjoint sets, of power $2^{\aleph_0}$ each.

To see that the E_γ satisfy the theorem let γ be any ordinal number $<\alpha$, N a subset of power $2^{\aleph_0}$ of E_γ, and f a similarity transformation of N into R. Suppose that the power of $[E\cap f(N)]-E_\gamma$ is $2^{\aleph_0}$. By Lemma 1.1 of [4], the function f can be extended to become an element f of $K^*(B)$, for some B in $V(E_\gamma)$, which contains N. Since

$$[f(N)\cap E]-E_\gamma\subseteq\{f(x)\mid f(x)\neq x,\ x\in B\}$$

it follows from the power assumption on f that the power of $[f(N)\cap E]-E_\gamma$ is $2^{\aleph_0}$. Therefore there exists a $\delta=\delta_\gamma$ such that $f=f_\delta$ and $B=B_\delta$. Then

$$f(N) = f_\delta(N) = f_\delta\left[\bigcup_{\xi\in T(\gamma)} (H_\xi \cap N)\right] = \bigcup_{\xi\in T(\gamma)} f_\delta(H_\xi \cap N).$$

Now $f_\delta(H_\xi \cap N)$ is a subset of H_ξ for $\xi > \delta$, and $H_\xi \cap H_v = \emptyset$ for $\xi \neq v$. Hence for each ξ in $T(\gamma)$ which is $> \delta$,

$$\begin{aligned} f_\delta(H_\xi \cap N) \cap (E - E_\gamma) &= f_\delta(H_\xi \cap N) \cap \left[\bigcup_{v\notin T(\gamma)} (H_v \cap E)\right] \\ &= \bigcup_{v\notin T(\gamma)} [f_\delta(H_\xi \cap N) \cap (H_v \cap E)] = \emptyset. \end{aligned}$$

Let $S(\gamma)$ be the set of those ξ in $T(\gamma)$ which are $\leqq \delta$. Then

$$\begin{aligned} f(N) \cap (E - E_\gamma) &= f_\delta(N) \cap (E - E_\gamma) = f_\delta\left[\bigcup_{\xi\in T(\gamma)} (H_\xi \cap N)\right] \cap (E - E_\gamma) \\ &= \left[\bigcup_{\xi\in T(\gamma)} f_\delta(H_\xi \cap N) \cap (E - E_\gamma)\right] \\ &= \bigcup_{\xi\in T(\gamma)} [f_\delta(H_\xi \cap N) \cap (E - E_\gamma)] \\ &= \bigcup_{\xi\in T(\gamma)} [f_\delta(H_\xi \cap N) \cap (E - E_\gamma)]. \end{aligned}$$

Since the power of $\bigcup_{\xi\leqq\delta} H_\xi$ is $< 2^{\aleph_0}$, the power of $f(N) \cap (E - E_\gamma) = [E \cap f(N)] - E_\gamma$ is $< 2^{\aleph_0}$. From this contradiction we conclude that no such function f can exist. Q.E.D.

From Theorem 3.5 there immediately follows

COROLLARY 1. *Let E be a linear set of power $2^{\aleph_0}$. For each ordinal number α, where $2 \leqq \alpha \leqq \theta$, E is the union of a family $\{E_\xi \mid \xi < \alpha\}$ of pairwise disjoint sets, each of power $2^{\aleph_0}$, with the following property: For $\xi \neq v$, there is no subset of E_ξ, of power $2^{\aleph_0}$, which is similar to a subset of E_v.*

In Theorem 3.5 let $R = E$ and consider the sets E_v obtained. For each v let $P_v = R - E_v$. On assuming that the order type of one of the two sets E_v or P_v is $\equiv \lambda$, we see that there exists a similarity transformation f which maps the other set into it. If $\overline{E}_v \equiv \lambda$, then $f(E_\gamma) \subseteq E_v$, where $\gamma \neq v$, a contradiction of $f(E_\gamma) - E_\gamma$ being of power $< 2^{\aleph_0}$. If $\overline{P}_v \equiv \lambda$, then $f(E_v) \subseteq P_v$, a contradiction of $f(E_v) - E_v$ being of power $< 2^{\aleph_0}$. It follows that both $\overline{E}_v$ and $\overline{P}_v$ are each $< \lambda$. Therefore, both E_v and P_v, which is the complement of E_v, contain no perfect set. Another way of expressing this is to say that both E_v and its complement meet every perfect set. Since each measurable set of positive measure contains a perfect set, it follows that both P_v and E_v are nonmeasurable. Summarizing we have

COROLLARY 2. *For each ordinal number α, where $2 \leqq \alpha \leqq \theta$, R is the union of* *a family* $\{E_\xi \mid \xi < \alpha\}$ *of pairwise disjoint sets with the following property*:

For each set E_v *and each similarity transformation* f *of* E_v *into* R, *the power of the set* $f(E_v) - E_v$ *is* $< 2^{\aleph_0}$.

In any such family of sets, each set is necessarily nonmeasurable.

Since each translation of a linear set is also a similarity transformation of that set we have generalized the following two results due to Sierpiński [9, pp. 24–25; 13]:

(a) There exists a linear, nonmeasurable set, of power $2^{\aleph_0}$, which is transformed by each translation into itself, with the exception of a set of power $< 2^{\aleph_0}$.

(b) There exists a decomposition of the line into $2^{\aleph_0}$ disjoint sets, each of power $2^{\aleph_0}$, such that each translation of the line maps each of the sets into itself, with the exception for each set, of a set of power $< 2^{\aleph_0}$.

In Corollary 2 let $\alpha \leqq \omega$ and consider the sets E_i which are obtained. Let $\{f_n\}$, $n < \omega$, be a sequence of similarity transformations of R onto R. For a given integer i consider the two sets, $S = \bigcup_n [R - f_n(E_i)]$ and $T = \bigcup_n f_n(E_i)$. Suppose that $\overline{T} \equiv \lambda$. Then there exists a similarity transformation f of P_i into T, where $P_i = R - E_i$. Since $f(P_i) \subseteq T$,

$$f(P_i) \cap T = f(P_i) \cap \left[\bigcup_n f_n(E_i) \right] = \bigcup_n [f(P_i) \cap f_n(E_i)]$$

is of power $2^{\aleph_0}$. By König's theorem, for some integer, say j, the power of $f(P_i) \cap f_j(E_i)$ is $2^{\aleph_0}$. In view of f_j being a similarity transformation, the subset $N = f_j^*[f(P_i) \cap f_j(E_i)]$ of E_i is of power $2^{\aleph_0}$. Furthermore, N is similar to the subset of $f^* f_j(N)$ of P_i. This contradicts the fact, obtained from Corollary 2, that the power of $f^* f_j(N) - E_i$ is $< 2^{\aleph_0}$. Therefore $\overline{T} < \lambda$. Suppose that $\overline{S} \equiv \lambda$. There exists a similarity transformation g of E_i into S. Since $g(E_i) \subseteq S$, for some integer, say k, $g(E_i) \cap [R - f_k(E_i)]$ is of power $2^{\aleph_0}$. By Corollary 2, since $f_k^* g$ is a similarity transformation of E_i, $f_k^* g(E_i) \cap [R - E_i]$ is of power $< 2^{\aleph_0}$. Since f_k^* is a similarity transformation of R onto R and $g(E_i) \cap [R - f_k(E_i)]$ is of power $2^{\aleph_0}$, the power of $f_k^*\{g(E_i) \cap [R - f_k(E_i)]\}$ is of power $2^{\aleph_0}$. Now

$$\begin{aligned} f_k^*\{g(E_i) \cap [R - f_k(E_i)]\} &= f_k^* g(E_i) \cap f_k^*[R - f_k(E_i)] \\ &= f_k^* g(E_i) \cap [f_k^*(R) - f_k^* f_k(E_i)] \\ &= f_k^* g(E_i) \cap [R - E_i]. \end{aligned}$$

Thus the power of $f_k^* g(E_i) \cap [R - E_i]$ is $2^{\aleph_0}$, a contradiction. We conclude that $\overline{S} < \lambda$. Since $\overline{S}$ and $\overline{T}$ are each $< \lambda$, for every n, $\overline{f_n(E_i)}$ and $\overline{R - f_n(E_i)}$ are each $< \lambda$. For a linear set B, if $\overline{B} < \lambda$, then B contains no perfect set, B therefore being of interior measure zero. Hence S, T, $f_n(E_i)$, and $R - f_n(E_i)$ are each of interior measure zero. As in the proof of Corollary 2 we see that $f_n(E_i)$ and $R - f_n(E_i)$ are nonmeasurable. Therefore the exterior measures of

$f_n(E_i)$ and $R-f_n(E_i)$ are each positive. Consequently the exterior measures of S and T are each positive. From this it follows that S and T are each nonmeasurable. We have thus shown

COROLLARY 3. *For each ordinal number α, where $2 \leqq \alpha \leqq \omega$, R is the union of a family $\{E_i | i<\alpha\}$ of nonmeasurable, pairwise disjoint sets, of power $2^{\aleph_0}$ each, satisfying the following*:

For each sequence $\{f_n\}$, $n<\omega$, of similarity transformations of R onto R, each of the two sets $\bigcup_n f_n(E_i)$ and $\bigcup_n [R-f_n(E_i)]$ is nonmeasurable and of interior measure zero. Furthermore, the order types of each of the two sets are $<\lambda$.

From Corollary 3 we obtain the following result due to Ruziewicz [8] and Sierpiński [10; 11]:

"There exists a linear, nonmeasurable set C, such that for any sequence of linear sets $\{C_n\}$, $n<\omega$, each of which is similar to C by a translation, each of the two sets $\bigcup_n C_n$ and $\bigcup_n(R-C_n)$ is of interior measure zero."

Note that the cited proposition is no longer true if the words "by a translation" are removed. For if C is any linear set, then C is similar to a subset C_0 of $(0, 1)$. Then $R-C_0$ is not of interior measure zero, so that $\bigcup_n(R-C_n)$ is not of interior measure zero.

REMARKS. (1) If E has property C, then the sets E_ξ in Theorem 3.5 can be chosen so as to have property C. If, in addition, E is a dense subset of R, then each E_v can be chosen to be a dense subset of R.

(2) Theorem 3.5 cannot be extended to require that at least one of the sets E_v have property A. For example, let $R=E$ be the union of a family $\{E_v | v<\alpha\}$ of pairwise disjoint sets satisfying Theorem 3.5. Let $f(x)=x+1$. Since the power of $f(E_v)-E_v$ is $<2^{\aleph_0}$, the power of $M_v=\{x | x \in E_v, f(x) \in E_v\}$ is of power $2^{\aleph_0}$. Let z_v be a c-condensation point of M_v and let $N_v=M_v \cap (z_v-1/4, z_v+1/4)$. Then N_v and $f(N_v)$ are two disjoint, similar subsets, of power $2^{\aleph_0}$ each, of E_v.

For $\alpha=\theta$, Corollary 1 can be extended to require each of the sets E_v to have property A (Corollary of Theorem 3.4). For $\alpha \leqq \omega$, Corollary 1 cannot be extended to require at least one of the sets E_v to have property A. For example, let $R=E$. In view of Corollary 2 of Lemma 3.1, only the case where $\alpha=\omega$ needs to be examined. Suppose that R is the union of a family $\{E_i | i<\omega\}$ of pairwise disjoint sets which satisfy the conclusion of Corollary 1. For any integer i let p be an element of E_i which is a c-condensation point of E_i. Let (a, b) be an open interval of R which contains the element p, and let $G=E_i \cap (a, b)$. Since p is a c-condensation point of E_i, the power of the set G is $2^{\aleph_0}$. Let c be an element of R which is greater than b, and f a similarity transformation of G into the open interval (b, c) of R. Since the power of the set $f(G)$ is $2^{\aleph_0}$, by König's Theorem, for some j the power of the set $E_j \cap f(G)$ is $2^{\aleph_0}$. However, since the sets E_k satisfy the conclusion of Theorem 3.5, j must be i. This implies that E_i contains two similar, disjoint sets, $f(G) \cap E_i$

and $f^*(E_i \cap f(G))$, of power $2^{\aleph_0}$ each. Consequently no set E_i can have property A.

(3) Let E be a linear set, of power $2^{\aleph_0}$, such that for each element f in $K(E)$, the power of the set $f(E)-E$ is $<2^{\aleph_0}$. Now each linear set of positive inner measure contains a perfect set, so that its order type is $\equiv\lambda$. Consequently the complement of the set E cannot have positive inner measure. Thus E cannot have finite measure. In [9, pp. 22–24] the following result was proved:

"If $2^{\aleph_0}=\aleph_1$, then there exists a linear, nondenumerable set E, of measure 0, which is transformed by each translation, into itself, with the exception of at most a denumerable number of points."

From our discussion it is clear that the word "translation" cannot be replaced by the words "element of $K(E)$."

(4) The problem of whether or not in Theorem 3.5 and Corollary 1 the sets E_i can be chosen so that at least one of them contains a fixed point remains open.

4. Problem P and incomparable order types.

THEOREM 4.1. *Let* $\{L_\xi\}$, $\xi<\theta$, *be a sequence of linear sets such that* $\overline{L}_\xi<\lambda$. *Then there exists a sequence* $\{B_\xi\}$, $\xi<\theta$, *of sets with the following properties*:

(1) B_ξ *is an exact, dense subset of* R, *which has property* C.

(2) L_ξ *is a subset of* B_ξ *such that* $\overline{L}_\xi<\overline{B}_\xi<\lambda$.

(3) *If* f *is any similarity transformation of* B_ξ *into* R, *then for* $v\neq\xi$, *the power of the set* $f(B_\xi)-B_v$ *is* $2^{\aleph_0}$ (*thus the order types of the* B_ξ *are pairwise incomparable*). *If* f *is not the identity, then the power of the set* $f(B_\xi)-B_\xi$ *is* $2^{\aleph_0}$.

Proof. It follows from Theorem 2.4 of [4] that for each $\xi<\theta$, an exact, dense subset A_ξ of R, having property C and containing L_ξ, can be found. Furthermore, $\overline{L}_\xi<\overline{A}_\xi<\lambda$.

For simplicity the proof is divided into two parts. We shall first show that there exists a linear set M_0 whose order type is incomparable with each $\overline{A}_\xi$. Furthermore, the set $B_0=A_0\cup M_0$ satisfies (1) and (2) above, and also (4) below.

(4) For any similarity transformation f of B_0 into R and each $\xi\geqq 1$, the power of the set $f(B_0)-A_\xi$ is $2^{\aleph_0}$.

To see this let

$$T = \{(C, f, v) \mid v < \theta, C \in V(A_\xi), \xi < \theta, f \in K(C)\},$$
$$U = \{(R, f, v) \mid v < \theta, f \in K^*(R)\},$$

and $S=T\cup U$. The power of S is $2^{\aleph_0}$. Well order the elements of R and S into the two sequences, $\{x_\xi\}$, $\xi<\theta$, and $\{s_\xi\}$, $\xi<\theta$, where $s_\xi=(C_\xi, f_\xi, v_\xi)$. For each $v<\theta$ such that $C_v=R$, denote by J_v the set

$$J_v = \{x \mid f_v(x) \neq x,\ x \in R\}.$$

Since f_v is in $K^*(R)$, the set J_v contains an interval of R, so that $\overline{J}_v \equiv \lambda$.

Suppose that p_ξ, q_ξ, r_ξ, and t_ξ have been defined for $\xi<\gamma<\theta$. Define P_γ to be the set $\{p_\xi, q_\xi, r_\xi, t_\xi \mid \xi<\gamma\}$. Suppose that s_γ is in T. Denote by p_γ the first element in the set $E_\gamma=[R-f_\gamma(C_\gamma)]-P_\gamma$. Since $\overline{C}_\gamma<\lambda$, $\overline{f_\gamma(C_\gamma)}<\lambda$. Therefore the power of the set $R-f_\gamma(C_\gamma)$ is $2^{\aleph_0}$, by Lemma 3 of [14]. Hence the power of the set E_γ is $2^{\aleph_0}$, so that the element p_γ exists. Let q_γ and r_γ represent no elements. Denote by t_γ the first element in the set

$$D_\gamma = f_\gamma(C_\gamma) - [P_\gamma \cup \{p_\gamma\}].$$

The element t_γ certainly exists since the power of $f_\gamma(C_\gamma)$ is $2^{\aleph_0}$. Suppose that s_γ is in U. Let p_γ and t_γ represent no elements. Let r_γ be the first element in the set $[J_\gamma-f_\gamma^*(Q_\gamma)]-P_\gamma$, where $Q_\gamma=A_0\cup P_\gamma$, and let $q_\gamma=f_\gamma(r_\gamma)$. Since $\overline{A}_0<\lambda$, $\overline{Q}_\gamma<\lambda$ and $\overline{f_\gamma^*(Q_\gamma)}<\lambda$. Thus the power of the set $[J_\gamma-f_\gamma^*(Q_\gamma)]$ is $2^{\aleph_0}$, so that r_γ and q_γ exist. Note that q_γ is not an element of $Q_\gamma\cup\{r_\gamma\}$.

For each $v<\theta$ let $M_0^v=\{p_\xi, r_\xi \mid s_\xi=(C, f, v)\}$ and M_0 the set union of the M_0^v. We shall now show that $\overline{M}_0^v$ is incomparable with each $\overline{A}_\xi$. Let f be a similarity transformation of A_ξ into R. Since A_ξ is an element of $V(A_\xi)$, there exists an ordinal number β such that $s_\beta=(A_\xi, f, v)$. It follows from the definitions of the element t_β and the set M_0^v that t_β is an element of the set $f(A_\xi)-M_0^v$. Therefore $f(A_\xi)$ is not a subset of M_0^v, i.e., f is not a similarity transformation of A_ξ into M_0^v. Now suppose that f is a similarity transformation which maps M_0^v into A_ξ. The function f^* is a similarity transformation of $f(M_0^v)$ into M_0^v. By Lemma 1.1 of [4], f^* can be extended to be an element h of $K(C)$, for some element C, in $V(A_\xi)$, which contains the set $f(M_0^v)$. Thus $h[f(x)]=x$ for each element x in the set M_0^v. There exists an ordinal number β for which $s_\beta=(C, h, v)$. The element p_β is in the two sets M_0^v and $R-h(C)$. But

$$R - h(C) \subseteq R - h[f(M_0^v)] = R - M_0^v.$$

Therefore the element p_β is in the two disjoint sets, M_0^v and $R-M_0^v$. From this contradiction it follows that f does not map the set M_0^v into A_ξ. Consequently $\overline{M}_0^v$ and $\overline{A}_\xi$ are incomparable order types. A similar procedure shows that $\overline{M}_0$ and $\overline{A}_\xi$ are incomparable order types. For each $v<\theta$, $\xi\geqq 1$, and f in $K(B_0)$, $f(M_0^v)-A_\xi$ is nonempty. Thus the power of the set $f(M_0)-A_\xi$ is $2^{\aleph_0}$. Therefore the power of the set $f(B_0)-A_\xi$ is $2^{\aleph_0}$. To see that B_0 is exact it is sufficient to show that for any element h of $K^*(B_0)$ the power of the set $h(B_0)-B_0$ is $2^{\aleph_0}$. Since B_0 contains A_0 and A_0 is a dense subset of R, B_0 is a dense subset of R. Thus h can be extended to be an element f of $K^*(R)$. For each ordinal number v there exists an ordinal β_v such that $s_{\beta_v}=(R, f, v)$. Since r_{β_v} is an element of B_0, q_{β_v} is an element of the set $f(B_0)$. Since q_{β_v} is an element of $R-A_0$ which is not in M_0, q_{β_v} is not in B_0. Therefore the power of the set $f(B_0)-B_0$ is $2^{\aleph_0}$. Incidentally, the sets M_0 and M_0^v, being dense subsets of R, can be shown to be exact as above (this is not needed for our proof).

We shall now assume that for each $\xi<\mu$, where $\mu<\theta$, the linear set M_ξ

has been defined such that $\overline{M}_\xi$ is incomparable with each order type $\overline{A}_\gamma$, where $\gamma \geqq \mu$. Furthermore, if $B_\xi = A_\xi \cup M_\xi$, then

(5) for any element f in $K^*(B_\xi)$ the power of the set $f(B_\xi) - B_\xi$ is $2^{\aleph_0}$ (thus B_ξ is exact);

(6) for any element f of $K(B_\xi)$, where $\xi < \mu$, and each $\gamma \geqq \mu$, the power of the set $f(B_\xi) - A_\gamma$ is $2^{\aleph_0}$; and

(7) if $\xi < \mu$ and $v < \mu$, $\xi \neq \mu$, then for each element f in $K(B_\xi)$, the power of $f(B_\xi) - B_v$ is $2^{\aleph_0}$.

For $\mu = 1$, M_ξ has already been defined. We now define M_μ. To do this we modify the argument given above. Let

$$\begin{aligned} T &= \{(C, f, v, D, g) \mid v < \theta; C \in V(A_\xi), \xi \geqq \mu, f \in K(C); \\ &\qquad D = B_\alpha, \alpha < \mu, g \in K(D)\}, \\ U &= \{(R, f, v) \mid v < \theta; f \in K^*(R)\}, \\ V &= \{(C, f, v) \mid v < \theta; C \in V(B_\alpha), \alpha < \mu, f \in K(C)\}, \end{aligned}$$

and $S = T \cup U \cup V$(2). The power of S is $2^{\aleph_0}$. Well order the elements of S into the sequence $\{s_\xi\}$, $\xi < \theta$, where s_ξ has one of the forms $(C_\xi, f_\xi, v_\xi, D_\xi, g_\xi)$, (R, f_ξ, v_ξ), or (C_ξ, f_ξ, v_ξ).

Suppose that p_ξ, q_ξ, r_ξ, t_ξ, and u_ξ have been defined for $\xi < \gamma < \theta$. Define P_γ to be the set $\{p_\xi, q_\xi, r_\xi, t_\xi, u_\xi \mid \xi < \gamma\}$. If s_γ is in T, define p_γ, q_γ, r_γ, and t_γ as in the first case. Let u_γ be the first element in the set

$$[g_\gamma(D_\gamma) - A_\mu] - [P_\gamma \cup \{p_\gamma, q_\gamma, r_\gamma, t_\gamma\}].$$

By (6) the element u_γ exists. If s_γ is in U, define p_γ, q_γ, r_γ, and t_γ as in the first case. Let u_γ represent no element. If s_γ is in V, define p_γ, q_γ, r_γ, and t_γ as if s_γ were in T. Let u_γ represent no element.

For each $v < \theta$ let $M_\mu^v = \{p_\xi, r_\xi \mid s_\xi = (C, f, v, D, g)\}$, and M_μ the set union of the M_μ^v. The argument given in the first case can be applied to show that $\overline{M}_\mu$ is incomparable with each order type $\overline{A}_\gamma$, where $\gamma > \mu$. Furthermore, if $B_\mu = A_\mu \cup M_\mu$, then

(8) for any element f in $K^*(B_\mu)$, the power of the set $f(B_\mu) - B_\mu$ is $2^{\aleph_0}$;

(9) for any element f of $K(B_\mu)$ and each $\gamma > \mu$, the power of the set $f(B_\mu) - A_\gamma$ is $2^{\aleph_0}$; and

(10) for any element g of $K(B_\mu)$ and each $\xi < \mu$, the power of the set $g(B_\mu) - B_\xi$ is $2^{\aleph_0}$.

For each element g of $K(B_\alpha)$, where $\alpha < \mu$, consider the power of the set $g(B_\alpha) - B_\mu$. To each $v < \theta$ there corresponds an ordinal number α_v such that $s_{\alpha_v} = (A_\mu, I, v, B_\alpha, g)$, where I is the identity transformation of A_μ. Each ele-

(2) To be logically correct a superscript μ should be affixed on all the letters appearing in the μth case to distinguish them from the letters used in the αth case, $\alpha < \mu$. Since no misunderstanding will occur if the superscript is omitted, this is done. The omission of the superscript will also be done in the proof of Theorem 4.3.

ment u_{α_v} is in $g(B_\alpha)-A_\mu$. Furthermore, u_{α_v} is not in M_μ. Thus each element u_{α_v} is in $g(B_\alpha)-B_\mu$. Therefore the power of $g(B_\alpha)-B_\mu$ is $2^{\aleph_0}$.

By transfinite induction we obtain a sequence of sets $\{B_\xi\}, \xi<\theta$. This sequence of sets clearly satisfies the conclusion of the theorem. Q.E.D.

The reasoning in Theorem 4.1 has also demonstrated

THEOREM 4.2. *Let $\{L_\xi\}, \xi<\theta$, be a sequence of linear sets, of power $2^{\aleph_0}$ each, such that $\overline{L}_\xi<\lambda$. Then there exists an exact, linear set B, of power $2^{\aleph_0}$, whose order type is incomparable with each $\overline{L}_\xi$.*

A companion result to Theorem 4.1 is

THEOREM 4.3. *Let $\{L_\xi\}, \xi<\theta$, be a sequence of linear sets, of power $2^{\aleph_0}$ each. Then there exists a sequence of sets $\{B_\xi\}, \xi<\theta$, whose members have the following properties*:

(α) *B_ξ is a subset of L_ξ such that $\overline{B}_\xi<\overline{L}_\xi$;*

(β) *B_ξ is an exact set which has properties A and C (thus, if f is an element of $K^*(B_\xi)$, the power of the set $f(B_\xi)-B_\xi$ is $2^{\aleph_0}$); and*

(γ) *if f is any similarity transformation of B_ξ into R, then for $v\neq\xi$, the power of the set $f(B_\xi)-B_v$ is $2^{\aleph_0}$ (thus the $\overline{B}_\xi$ are pairwise incomparable order types).*

Proof. For each ξ let A_ξ be a subset of L_ξ, having properties A and C, such that $\overline{A}_\xi<\overline{L}_\xi$. For each ξ let A_ξ be the union of $2^{\aleph_0}$ disjoint sets A_ξ^γ, $\gamma<\theta$, and each A_ξ^γ the union of $2^{\aleph_0}$ disjoint sets $A_\xi^{\gamma v}$, $v<\theta$, the power of each set $A_\xi^{\gamma v}$ being $2^{\aleph_0}$. Let

$$Q = \{A_\xi \mid \xi < \theta\} \cup \{A_\xi^\gamma \mid \xi, \gamma<\theta\} \cup \{A_\xi^{\gamma v} \mid \xi, \gamma, v<\theta\}.$$

Well order the elements of Q into a sequence

$$T_0, T_1, \cdots, T_\xi, \cdots \qquad (\xi < \theta) \tag{1}$$

which has the following three properties:

(a) Each set A_μ^γ follows A_μ in (1).

(b) Each set $A_\mu^{\gamma v}$ follows A_μ^γ in (1).

(c) The sequence $\{A_\xi\}, \xi<\theta$, is a subsequence of $\{T_\xi\}, \xi<\theta$.

Such a well ordering is certainly possible. Note that $T_0=A_0$.

We shall first show that there exists an exact subset B_0 of A_0, having property C, such that for $\xi\geqq 1$ and f any similarity transformation of T_ξ into R, the power of the set $f(T_\xi)-B_0$ is $2^{\aleph_0}$.

To see this let $G^0=\{T_\xi \mid \xi>0\}$. Let S^0 be the set of ordered pairs, (C, f), where C is in G^0 and f is a similarity transformation of C into R. The power of S^0 is $2^{\aleph_0}$. Well order the elements of R and S^0 into the two sequences

$$x_0, x_1, \cdots, x_\xi, \cdots \qquad (\xi < \theta), \tag{2}$$

$$s_0, s_1, \cdots, s_\xi, \cdots \qquad (\xi < \theta), \tag{3}$$

where $s_\xi = (C_\xi, f_\xi)$ and each element of S^0 appears $2^{\aleph_0}$ times in (3). Suppose that p_ξ and q_ξ have been defined for $\xi < \mu < \theta$. Let

$$R_\mu = \{p_\xi, q_\xi \mid \xi < \mu\}, \; X_\mu = f_\mu(C_\mu) - A_0, \quad \text{and} \quad Y_\mu = f_\mu(C_\mu) - X_\mu.$$

Notice that Y_μ is a subset of A_0. If the power of Y_μ is $2^{\aleph_0}$, let p_μ be the first element in the set $Y_\mu - R_\mu$. If the power of Y_μ is $<2^{\aleph_0}$, then the power of X_μ is $2^{\aleph_0}$. In this case let p_μ represent no element. Suppose that C_μ is the set A_0 and that f_μ is not the identity transformation on A_0. Since A_0 has property C, the power of the set $V_\mu = \{x \mid f_\mu(x) \neq x, x \in A_0\}$ is $2^{\aleph_0}$. Let q_μ be the first element in the set $V_\mu - P_\mu$, where $P_\mu = R_\mu \cup \{p_\mu\}$. If C_μ is not the set A_0, or if C_μ is the set A_0 and f_μ is the identity transformation on A_0, let q_μ represent no element.

Denote by M^0 the set $M^0 = \{p_\xi \mid \xi < \theta\}$, by N^0 the set $N^0 = \{q_\xi \mid \xi < \theta\}$, and by B_0 the set $B_0 = A_0 - M^0$. Note that N^0 is a dense subset of A_0 which has property C. Since N^0 is a subset of B_0, it follows that B_0 is a dense subset of A_0 which has property C. Being the subset of a set which has property A, the set B_0 has property A. By Theorem 2.3 of [4], B_0 is exact. Suppose that the power of the set $f_\mu(C_\mu) - A_0$ is $<2^{\aleph_0}$. There exists an increasing sequence of ordinal numbers $\{\alpha_\xi\}$, $\xi < \theta$, such that $f_\mu(C_\mu) = f_{\alpha_\xi}(C_{\alpha_\xi})$. From the definition of P_{α_ξ}, it follows that the element p_{α_ξ} is in the set $f_{\alpha_\xi}(C_{\alpha_\xi}) - B_0$. Therefore the power of the set $f_\mu(C_\mu) - B_0$ is $2^{\aleph_0}$.

Now suppose that for each $\xi < \mu < \theta$, the set B_ξ has been defined, and has the following properties:

(d) B_ξ is an exact subset of A_ξ which has properties A and C.

(e) If f is any similarity transformation of T_v into R, where T_v follows A_ξ in (1), then the power of the set $f(T_v) - B_\xi$ is $2^{\aleph_0}$.

(f) If f is any similarity transformation of B_ξ into R, and if $\xi < \mu$ and $\gamma < \mu$, where $\xi \neq \gamma$, then the power of the set $f(B_\xi) - B_\gamma$ is $2^{\aleph_0}$.

For $\mu = 1$, B_ξ has already been defined. We now define B_μ. Suppose that the set $A_\mu^{\gamma v} - g(E)$ is empty, where E is some subset of B_ξ, $\xi < \mu$, and g is some similarity transformation of E into R. Then the set $g^*(A_\mu^{\gamma v}) - E$ is empty. Thus the set $g^*(A_\mu^{\gamma v}) - B_\xi$ is empty. Now $A_\mu^{\gamma v}$ follows A_μ^γ and A_μ^γ follows A_μ, in (1). Since $\{A_\xi\}$, $\xi < \theta$, is a subsequence of (1), A_μ follows A_ξ, in (1). This implies that $A_\mu^{\gamma v}$ follows A_ξ. From the induction hypothesis, $g^*(A_\mu^{\gamma v}) - B_\xi$ is nonempty. We conclude that the set $A_\mu^{\gamma v} - g(E)$ is nonempty. Consequently the power of the set $A_\mu^\gamma - g(E)$ is $2^{\aleph_0}$.

Denote by F^μ the family $F^\mu = \{W \mid W \in V(B_\xi), \xi < \mu\}$. Let

$$H^\mu = \{T_\xi \mid T_\xi \text{ follows } A_\mu \text{ in } (1)\}, \quad \text{and} \quad G^\mu = F^\mu \cup H^\mu.$$

The power of G^μ is $2^{\aleph_0}$. Let S^μ be the set of triples (A_μ^γ, C, f), where $\gamma < \theta$, C is in G^μ, and f is in $K(C)$. The power of S^μ is $2^{\aleph_0}$. Well order the elements of S^μ into the sequence

$$(4) \qquad s_0, s_1, \cdots, s_\xi, \cdots \qquad (\xi < \theta),$$

where $s_\xi = (D_\xi, C_\xi, f_\xi)$, and each element of S^μ appears $2^{\aleph_0}$ times in (4). Assume that p_ξ, q_ξ, and r_ξ have been defined for $\xi < \gamma < \theta$. If C_γ is an element of F^μ, let r_γ be the first element in the set, $[D_\gamma - f_\gamma(C_\gamma)] - R_\gamma$, where $R_\gamma = \{p_\xi, q_\xi, r_\xi | \xi < \gamma\}$. The element r_γ certainly exists since D_γ is one of the sets A^v_μ, and the power of the set $A^v_\mu - f_\gamma(C_\gamma)$ is $2^{\aleph_0}$. If C_γ is not in F^μ, let r_γ represent no element. Let $X_\gamma = f_\gamma(C_\gamma) - A_\mu$ and $Y_\gamma = f_\gamma(C_\gamma) - X_\gamma$. Notice that Y_γ is a subset of A_μ. If the power of Y_γ is $2^{\aleph_0}$, let p_γ be the first element in the set $Y_\gamma - [R_\gamma \cup \{r_\gamma\}]$. If the power of Y_γ is $< 2^{\aleph_0}$, then the power of the set X_γ is $2^{\aleph_0}$. In this case let p_γ represent no element. Suppose that C_γ is the set A_μ and f_γ is not the identity transformation on A_μ. Since A_μ has property C, the power of the set $V_\gamma = \{x | f_\gamma(x) \neq x, x \in A_\mu\}$ is $2^{\aleph_0}$. Let q_γ be the first element in the set $V_\gamma - P_\gamma$, where $P_\gamma = R_\gamma \cup \{p_\gamma, r_\gamma\}$. If C_γ is not the set A_μ, or if C_γ is the set A_μ, let q_γ represent no element.

For each ordinal number $\mu < \theta$ let $M^\mu = \{p_\xi | \xi < \theta\}$, $N^\mu = \{q_\xi | \xi < \theta\}$, $B_\mu = A_\mu - M^\mu$, and $B^\gamma_\mu = A^\gamma_\mu \cap B_\mu$. Note that N^μ is a dense subset of A_μ which has property C. Since N^μ is a subset of B_μ, B_μ is a dense subset of A_μ which has property C. Condition (d) is satisfied for $\xi \leqq \mu$. It is easily seen that

(5) the power of each of the two sets, B^γ_μ and $f_\gamma(C_\gamma) - B_\mu$, where C_γ is in G^μ, is $2^{\aleph_0}$. Let C_γ be an element of F^μ. Consider the set $B^v_\mu - f_\gamma(C_\gamma)$. There exists an ordinal number β such that $s_\beta = (A^v_\mu, C_\gamma, f_\gamma)$. Since r_β is an element of the two sets, $D_\beta - f_\beta(C_\beta)$ and B_μ, it follows that the set $B^v_\mu - f_\gamma(C_\gamma)$ is nonempty.

Consider the two sets B_ξ and B_μ, where $\xi < \mu$. If f is a similarity transformation of B_ξ into R, then, from (5), the power of the set $f(B_\xi) - B_\mu$ is $2^{\aleph_0}$. Now let f be any similarity transformation of B_μ into R. Let h be the function defined by $h(x) = f(x)$ for x in B^γ_μ. Suppose that $h(B^\gamma_\mu)$ is a subset of B_ξ. Then h^* can be extended to be a similarity transformation g of C into R, for some element C, in $V(B_\xi)$, which contains the set $h(B^\gamma_\mu)$. Consider the set $B^\gamma_\mu - g(C)$. There exists an ordinal number β such that $s_\beta = (A^\gamma_\mu, C, g)$. The element r_β is in $B^\gamma_\mu - g(C)$. But

$$B^\gamma_\mu - g(C) \subseteq B^\gamma_\mu - g[f(B^\gamma_\mu)] = B^\gamma_\mu - B^\gamma_\mu = \varnothing.$$

From this contradiction we see that h does not map the set B^γ_μ into B_ξ, i.e., $f(B^\gamma_\mu) - B_\xi$ is nonempty. Consequently the power of the set $f(B_\mu) - B_\xi$ is $2^{\aleph_0}$.

By transfinite induction we define a sequence of sets, $\{B_\xi\}$, $\xi < \theta$. The members of this sequence satisfy the conclusion of Theorem 4.3.

Theorem 4.2 may be sharpened as follows.

THEOREM 4.4. *Let $\{L_\xi\}$, $\xi < \theta$, be a sequence of linear sets, of power $2^{\aleph_0}$ each, such that $\overline{L}_\xi < \lambda$. Then there exists an exact set B, having property* A, *whose order type is incomparable with each $\overline{L}_\xi$.*

Proof. Let S denote the set of all 5-tuples (I, C, f, D, g), where I is any half-open interval of R, C is in $V(R)$, f is in $K(C)$, D is in $V(L_v)$, where

$v<\theta$, and g is in $K(D)$. Well order the elements of R and S into the two sequences $\{x_\xi\}$, $\xi<\theta$, and $\{s_\xi\}$, $\xi<\theta$, where $s_\xi=(I_\xi, C_\xi, f_\xi, D_\xi, g_\xi)$. Suppose that p_ξ, q_ξ, and r_ξ have been defined for each $\xi<\alpha<\theta$. Let $P_\alpha=\{p_\xi, q_\xi \mid \xi<\alpha\}$ and $Q_\alpha=P_\alpha\cup\{r_\xi\mid\xi<\alpha\}$. Denote by p_α the first element in $R-S_\alpha$, where

$$S_\alpha = Q_\alpha \cup \bigcup_{\xi<\alpha} f_\xi(P_\alpha) \cup \bigcup_{\xi<\alpha} f_\xi^*(P_\alpha).$$

Denote by q_α the first element in the set $I_\alpha-[g_\alpha(D_\alpha)\cup T_\alpha]$, where

$$T_\alpha = S_\alpha \cup \{p_\alpha\} \cup \{f_\xi(p_\alpha) \mid \xi < \alpha\} \cup \{f_\xi^*(p_\alpha) \mid \xi < \alpha\}.$$

Since $\overline{D}_\alpha<\lambda$, $\overline{g_\alpha(D_\alpha)}<\lambda$. By Lemma 3 of [14], the power of $I_\alpha-g_\alpha(D_\alpha)$ is $2^{\aleph_0}$. Consequently q_α exists. Denote by r_α the first element of $g_\alpha(D_\alpha)-[T_\alpha\cup\{q_\alpha\}]$. Let $B=\{p_\xi, q_\xi\mid\xi<\theta\}$.

To show that B has property A we modify the argument given in Theorem 5 of [3]. Suppose that F and G are two disjoint, similar subsets, of power $2^{\aleph_0}$ each, of B. Let f be a similarity transformation of F onto G. By Lemma 1.1 of [4], f may be extended to be an element f_ξ of $K(C_\xi)$, for some ordinal number ξ, where C_ξ is an element of $V(R)$ which contains the set F. For each element p_α which is in F, where $\alpha>\xi$, consider $f_\xi(p_\alpha)$. As f_ξ maps F onto G, $f_\xi(p_\alpha)\neq p_\alpha$. From the definition of q_α, $f_\xi(p_\alpha)\neq q_\alpha$. For each ordinal number $\tau>0$, $p_{\alpha+\tau}$ and $q_{\alpha+\tau}$ are not elements of $f_\xi(P_{\alpha+\tau})$. Since p_α is in $P_{\alpha+\tau}$, $f_\xi(p_\alpha)\neq p_{\alpha+\tau}$ and $f_\xi(p_\alpha)\neq q_{\alpha+\tau}$. Suppose that $f_\xi(p_\alpha)=p_\mu$ or that $f_\xi(p_\alpha)=q_\mu$, where $\mu<\alpha$. Then $f_\xi(p_\alpha)$ is an element of P_α, so that p_α is an element of $f_\xi^*(P_\alpha)$. This, however, contradicts the manner in which p_α was selected. Consequently $f_\xi(p_\alpha)\neq p_\mu$ and $f_\xi(p_\alpha)\neq q_\mu$, for $\mu<\alpha$. It follows that for $\alpha>\xi$ and p_α in F, $f_\xi(p_\alpha)$ is not in B. An analogous argument shows that for $\alpha>\xi$ and q_α in F, $f_\xi(q_\alpha)$ is not in B. This implies that the power of the set $f_\xi(F)\cap B$ is $<2^{\aleph_0}$. But

$$f_\xi(F) \cap B = G \cap B = G,$$

which is of power $2^{\aleph_0}$. From this contradiction we conclude that the two sets F and G do not exist, i.e., B has property A.

To show that $\overline{B}$ and $\overline{L}_\xi$ are incomparable order types, for each $\xi<\theta$, repeat the argument given in Theorem 4.1, using q_β and r_β for p_β and t_β respectively.

To show that B is exact we shall show that each point of B is fixed. Let y be any element of B and let I be any half-open interval of R, $(x, y]$ or $[y, z)$. For each $v<\theta$, I appears in one of the 5-tuples (I, R, f, L_v, g). Hence for suitable ordinal numbers α_ξ, $\xi<\theta$, $I=I_{\alpha_\xi}$. Then $\{p_{\alpha_\xi}\mid\xi<\theta\}$ is a subset of $B\cap I$. Therefore y is a c-condensation point of B. From Theorem 2.3 of [4], y is a fixed point of B. Q.E.D.

We conclude with a generalization of Theorem 1.1 of [4].

THEOREM 4.5. *Let B be a subset, of power $2^{\aleph_0}$, of the linear set E, such that*

$\overline{B} < \overline{E}$. Let B have the property that if $\overline{X} \leqq \overline{B}$, then the power of the set $E - X$ is $2^{\aleph_0}$. Then there exists an exact subset M of E such that

(1) *M has properties* A *and* C;

(2) *$\overline{M}$ and $\overline{B}$ are incomparable order types*; *and*

(3) *$\overline{B} < \overline{B \cup M} < \overline{E}$.*

If, furthermore, E has property C, then M is a dense subset of E. If $E = R$, then M can be defined so that the set $B \cup M$ is exact.

Proof. For each element D in $V(E)$ let $L(D)$ be the set of those similarity transformations f of D into R for which the power of the set $\{x \mid f(x) \neq x, x \in D\}$ is $2^{\aleph_0}$. Denote by P the set of couples,

$$P = \{(D, f) \mid D \in V(E), f \in L(D)\}.$$

Denote by Q the set of all triples (D, f, g), where D is in $V(B)$, f is in $K(D)$, and g is in $K(E)$. The power of the set $S = P \cup Q$ is $2^{\aleph_0}$. Well order the elements of R and S into the two series, $\{x_\xi\}$, $\xi < \theta$, and $\{s_\xi\}$, $\xi < \theta$, where each element in S appears $2^{\aleph_0}$ times in the latter sequence, and s_ξ is either of the form (D_ξ, f_ξ, g_ξ) or (D_ξ, f_ξ). Let E^* be the set of c-condensation points of E which are in E, $B^* = B \cap E^*$, and $D_\xi^* = D_\xi \cap E^*$. If $s_\xi = (D_\xi, f_\xi)$, define J_ξ to be the set $J_\xi = \{x \mid f_\xi(x) \neq x, x \in D_\xi^*\}$. Since the power of the set $D_\xi - D_\xi^*$ is $< 2^{\aleph_0}$, the power of J_ξ is $2^{\aleph_0}$.

If s_γ is an element of Q, then as shown in Theorem 1.1 of [4], the power of the set $g_\gamma(E) - f_\gamma(D_\gamma)$ is $2^{\aleph_0}$. From this it follows that the power of $g_\gamma(E^*) - f_\gamma(D_\gamma)$ is $2^{\aleph_0}$.

Suppose that s_0 is an element of P. Let p_0 be the first element in J_0, $t_0 = f_0(p_0)$, and let q_0 and r_0 represent no elements. Suppose that s_0 is an element of Q. Let p_0 and q_0 be the first two elements in the set $g_0(E^*) - f_0(D_0)$, and r_0 the first element in the set $f_0(D_0^*) - \{p_0, q_0\}$. Let t_0 represent no element. Now suppose that for each ordinal number $\xi < \mu$, where $\mu < \theta$, p_ξ, q_ξ, r_ξ, and t_ξ have been defined. If s_μ is an element of P let p_μ denote the first element in the set $J_\mu - [C_\mu \cup f_\mu^*(C_\mu)]$, where $P_\mu = \{p_\xi \mid \xi < \mu\}$ and

$$C_\mu = \{q_\xi, r_\xi, t_\xi \mid \xi < \mu\} \cup P_\mu \cup \bigcup_{\xi<\mu} f_\xi(P_\mu) \cup \bigcup_{\xi<\mu} f_\xi^*(P_\mu).$$

Let $t_\mu = f_\mu(p_\mu)$ and q_μ and r_μ represent no elements. If s_μ is an element of Q let p_μ and q_μ denote the first two elements in the set

$$g_\mu(E^*) - [f_\mu(D_\mu) \cup \{p_\xi, q_\xi, r_\xi, t_\xi \mid \xi < \mu\} \cup \bigcup_{\xi<\mu} f_\xi(P_\mu) \cup \bigcup_{\xi<\mu} f_\xi^*(P_\mu)].$$

Denote by r_μ the first element in the set

$$f_\mu(D_\mu^*) - [\{p_\xi, q_\xi, r_\xi, t_\xi \mid \xi < \mu\} \cup \{p_\mu, q_\mu\}].$$

Let t_μ represent no element.

Denote by M the set $\{p_\xi \mid \xi < \theta\} \cap E^*$. Suppose that F and G are two disjoint, similar subsets of M of power $2^{\aleph_0}$ each. Let f be a similarity transformation of F onto G. By Lemma 1.1 of [4], f can be extended to be an element g of $L(D)$, for some element D, in $V(E)$, which contains F. For some ordinal number ξ, $(B, g) = (B_\xi, f_\xi)$. The argument given in Theorem 5 of [3] can be carried over to show that the set M has property A. The argument given in Theorem 1.1 of [4] can be carried over to show that $\overline{M}$ and $\overline{B}$ are incomparable order types, and that $\overline{B} < \overline{M \cup B} < \overline{E}$. In fact, the arguments can be carried over to show that $\overline{M}$ and $\overline{B}^*$ are incomparable order types and that $\overline{B}^* < \overline{M \cup B}^* < \overline{E}^*$.

Now M is a dense subset of E^*. This is so by the following reasoning. For each element p in E^* and each open interval I of R, containing p, there exists a similarity transformation of E into R such that $f(x) = x$ for x in $E - I$ and $f(x) \neq x$ for x in $I \cap E$. Since p is a c-condensation point of E, the function f is in $L(E)$. Hence there are $2^{\aleph_0}$ elements in $E \cap I$. It follows that M is a dense subset of E which has property C. By Theorem 2.3 of [4], the set M is exact.

If $E = R$, then the above construction of M is modified slightly. The only variation from the above occurs if J_μ contains an open interval I_μ of R. In this case denote by p_μ the first element in the set

$$N_\mu = I_\mu - [C_\mu \cup f_\mu^*(C_\mu) \cup f_\mu^*(B)].$$

Since $\overline{I}_\mu \equiv \lambda$ and $\overline{f_\mu^*(B)} < \lambda$, the power of the set N_μ is $2^{\aleph_0}$. Consequently the element p_μ certainly exists. Suppose that g is a similarity transformation of $N = M \cup B$ into N, which is not the identity transformation. As M is a dense subset of R, N is a dense subset of R. Thus the function g can be extended to become a similarity transformation h of R into R. Since h is not the identity transformation, h is in $L(R)$. Thus, for some ordinal number δ, $D_\delta = R$ and $h = f_\delta$. Now $g(p_\delta) = f_\delta(p_\delta) = t_\delta$. The element t_δ is not in B and not in M, i.e., not in N. Therefore the function g does not map N into N, so that the set N is exact.

COROLLARY. *If the power of the order type α is $2^{\aleph_0}$, and $\alpha < \lambda$, then there exists an order type β with the following properties:*

(1) *β has property* A;

(2) *α and β are incomparable order types*; and

(3) $\alpha < \alpha + \beta < \lambda$.

REMARKS. (1) if $\overline{X} \leqq \overline{D}^*$, then the power of the set $E^* - X$ is $2^{\aleph_0}$. This follows from the power of the set $E - E^*$ being $< 2^{\aleph_0}$.

(2) If E has property A and B is a subset of E for which the power of $E - B$ is $2^{\aleph_0}$, then E and B satisfy the first group of hypotheses in Theorem 4.5. To see this suppose the contrary. Let X be a set such that $\overline{X} \leqq \overline{B}$ and the power of $E - X$ is $< 2^{\aleph_0}$. As $\overline{X} \leqq \overline{B}$, there exists a similarity transformation f of X into B. Define F to be the set $E \cap X$. Since the power of E is $2^{\aleph_0}$ and the power of $E - F$ is $< 2^{\aleph_0}$, the power of F is $2^{\aleph_0}$. Combining this with the

fact that the power of $E-F$ is $<2^{\aleph_0}$, we see that the power of the set $G=(E-B)\cap F$ is $2^{\aleph_0}$. Then G and $f(G)$ are two disjoint, similar subsets of E, of power $2^{\aleph_0}$. This contradicts E having property A.

(3) If E is a proper subset of R, then it may not be possible for the set M to be chosen so that $B\cup M$ is exact. An example will illustrate this point. Let C and D be dense subsets of the open intervals (0, 1/2) and (1, 2) respectively, such that $C\cup D$ has properties A and C. Let f_0 be the identity transformation on C. For each positive integer n, let f_n be a similarity transformation, of the form $ax+b$, of C into the open interval $(n/(n+1), (n+1)/(n+2))$. Let $B=\bigcup_{n<\omega}f_n(C)$, and $E=B\cup D$. The two sets, E and B, satisfy the first two groups of hypotheses of Theorem 4.5. Nevertheless, for no subset M of E is $B\cup M$ exact.

Bibliography

1. S. Banach, *Sur les transformations biunivoques*, Fund. Math. vol. 19 (1932) pp. 10–16.

2. B. Dushnik and E. W. Miller, *Concerning similarity transformations of linearly ordered sets*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 322–326.

3. S. Ginsburg, *Order types and similarity transformations*, submitted to Fund. Math.

4. ———, *Some remarks on order types and decompositions of sets*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 514–535.

5. E. Kamke, *Theory of sets*, New York, 1950.

6. G. Kurepa, *Sur les ensembles ordonnés*, Hrvatsko Prirodoslovno Društvo Glasnik Mat.-Fiz. Astr. Ser II vol. 3 (1948) p. 146.

7. S. Mazurkiewicz, *Sur la décomposition d'un segment en une infinité d'ensembles non-mesurables superposables deux à deux*, Fund. Math. vol. 2 (1921) pp. 8–14.

8. S. Ruziewicz, *Sur une propriété de la base hamelienne*, Fund. Math. vol. 26 (1936) pp. 56–58.

9. W. Sierpiński, *Sur les translations des ensembles linéaires*, Fund. Math. vol. 19 (1932) pp. 22–28.

10. ———, *Un théorème de la théorie générale des ensembles*, Fund. Math. vol. 25 (1935) p. 550.

11. ———, *Remarque sur les translations d'ensembles*, Fund. Math. vol. 26 (1936) pp. 59–60.

12. ———, *Sur les translations des ensembles linéaires*, Fund. Math. vol. 35 (1948) pp. 159–164.

13. ———, *Sur une décomposition de la droite*, Comment. Math. Helv. vol. 22 (1949) pp. 317–320.

14. ———, *Sur les types d'ordre des ensembles linéaires*, Fund. Math. vol. 37 (1950) pp. 253–264.

15. J. von Neumann, *Die Zerlegung eines Intervalles in abzählbar viele kongruente Teilmengen*, Fund. Math. vol. 11 (1928) pp. 230–238.

University of Miami,
Coral Gables, Fla.