

INTEGRAL REPRESENTATIONS OF POSITIVE DEFINITE FUNCTIONS. II

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1. Introduction. This paper may be considered as a continuation and extension of a previous paper of ours [1], which we shall refer to as IRI. Here, we partially generalize the main theorem of IRI (Theorem 5) to a theorem about the integral representations of positive definite functions whose domains are certain convex sets in a Euclidean space. We shall be more precise about these convex sets a little later. Since many of the results and methods of IRI are used in this paper, it is hoped that the reader will have some acquaintance with the first paper.

One of the theorems which we prove in this paper contains as a special case a theorem of M. Krein [1, p. 17, Theorem 1]. Further, we also obtain as a special case the correct statement of an incorrect theorem stated by M. Livshitz [1, p. 6, Theorem 3]. We shall say more about this later.

We wish to say a few words about our use of the terminology *positive definite function*. Let E_m and E_n be the Euclidean spaces of dimension m and n respectively. Let Q be a convex set in E_{m+n} ($= E_m \times E_n$) which contains the origin and such that if $(x, y) \in Q$, for $x \in E_m, y \in E_n$, then $(x, -y) \in Q$. Further let $Q/2$ be the set of all (x, y) in Q such that $(2x, 2y) \in Q$ and $f(x, y)$ a complex-valued function defined on Q . We say that f is a positive definite function if for every finite set of points $\{(x_1, y_1), \dots, (x_n, y_n)\} \subseteq Q/2$ and every finite set of complex numbers $\{\xi_1, \dots, \xi_n\}$ we have

$$\sum_{i=1}^n \sum_{j=1}^n \xi_i \bar{\xi}_j f(x_j + x_i, y_j - y_i) \geq 0.$$

The terminology *positive definite function* has been used in the literature for many different things. In particular this title has been given to continuous functions defined on the real axis for which $\sum_{i,j=1}^n \xi_i \bar{\xi}_j f(y_j - y_i) \geq 0$. These functions have also been called functions of positive type. Continuous functions defined on the half real axis for which $\sum_{i,j=1}^n \xi_i \bar{\xi}_j f(x_i + x_j) \geq 0$ have been called exponentially convex functions. Actually, any function of positive type gives rise to a group of unitary operators acting on the reproducing kernel space corresponding to it, while any exponentially convex function gives rise to a semi-group of positive semi-definite self-adjoint operators. From this point

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of view it might be really natural to call these latter type functions the positive definite functions. Or better still, in naming them it might be natural to indicate the character of the operators which arise from these functions. At any rate we do not wish to settle any questions of terminology here, and shall leave the ultimate decision to history.

The first part of this paper is concerned with a brief review of some topics which are needed later on. The second part of this paper is concerned with a slight generalization of Theorem 5 of IRI which is needed for the later work. The techniques used in this second part are essentially the same as those used in IRI and we have given a discussion of only those points which we felt might cause the reader some trouble. The third part is concerned with positive definite functions on convex sets. The last part of the paper concerns itself with a problem of moments.

2. Preliminaries. For the convenience of our readers we shall recall briefly some of the essential features of the theory which we shall use in this investigation. Let E be a set (not necessarily having any structure) and $K(x, y)$ a complex-valued function defined on the Cartesian product $E \times E$. The function $K(x, y)$ is called a reproducing kernel⁽²⁾ (r. k.) if there exists a Hilbert space \mathcal{F} , of functions defined on E , such that $K(x, y) \in \mathcal{F}$ when considered as a function of x for every y in E and for every $f(x) \in \mathcal{F}$ and every $y \in E$, $f(y) = (f(x), K(x, y))$. \mathcal{F} is called a reproducing kernel space (r. k. s.). The function $K(x, y)$ is said to be a positive matrix⁽²⁾ (p. m.) if for every finite set of complex numbers $\{\xi_1, \dots, \xi_n\}$ and points $\{y_1, \dots, y_n\} \subseteq E$, $\sum_{i,j=1}^n \xi_i \bar{\xi}_j K(y_j, y_i) \geq 0$. The following theorem gives the connection between a r. k. and a p. m.

THEOREM A⁽²⁾. (E. H. MOORE-N. ARONSZAJN). *A necessary and sufficient condition that $K(x, y)$ be a r. k. is that it be a p. m.*

For a given p. m. $K(x, y)$, the corresponding space \mathcal{F} is the completion of the set \mathcal{F}' of functions of the form

$$g(x) = \sum_{i=1}^n \xi_i K(x, y_i),$$

with norm given by

$$\|g\|^2 = \sum_{i,j=1}^n \xi_i \bar{\xi}_j K(y_j, y_i).$$

As in IRI, \mathcal{F} will always denote the r. k. s. of the p. m. which we are discussing at a given time and \mathcal{F}' will always denote the space of functions given above for that p. m.

Let now T denote an operator from \mathcal{F} to itself, \mathcal{D}_T its domain, and $M(x, y)$

⁽²⁾ N. Aronszajn [1; 2].

$\in \mathcal{F}$ for every $y \in E$. T is said to correspond to the kernel $M(x, y)$, in symbols $T \sim M(x, y)$, if for every $f \in \mathcal{D}_T$, $Tf(y) = (f(x), M(x, y))$. T is said to correspond to $M(x, y)$ in the maximal sense, in symbols $T \cong M(x, y)$, if \mathcal{D}_T consists of all the elements f in \mathcal{F} such that $(f(x), M(x, y))$ is again an element of \mathcal{F} , when considered as a function of y and for every $f \in \mathcal{D}_T$, $Tf(y) = (f(x), M(x, y))$.

Following Aronszajn we write $M_1(x, y) \ll M_2(x, y)$ or $M_2(x, y) \gg M_1(x, y)$ if $M_2(x, y) - M_1(x, y)$ is a p. m. We have the following theorem.

THEOREM B⁽³⁾. *Let $M(x, y)$ be a Hermitian symmetric function (i.e. $M(x, y) = \overline{M(y, x)}$) defined over $E \times E$ which belongs to \mathcal{F} when considered as a function of x for every $y \in E$.*

(a) *If $T \cong M(x, y)$, then T^* and T^{**} exist and $T^{**} = T$. T^* is a symmetric operator and is the closure of the restriction of T to \mathcal{F}' ; every self-adjoint operator which corresponds to $M(x, y)$ is an extension of T^* . A necessary and sufficient condition that T be self-adjoint is that the linear manifold determined by the class $\{M(x, y) + iK(x, y)\}$ is dense in \mathcal{F} .*

(b) *A necessary and sufficient condition that there exists a self-adjoint operator $H \sim M(x, y)$ which is bounded below (above) by the finite number c is that $M(x, y) \gg cK(x, y)$ ($M(x, y) \ll cK(x, y)$). A necessary and sufficient condition that there exists a self-adjoint $H \cong M(x, y)$ with lower bound $\geq \lambda > -\infty$ and upper bound $\leq \Lambda < +\infty$ is that $\lambda K(x, y) \ll M(x, y) \ll \Lambda K(x, y)$.*

Another result which we must make strong use of in this paper concerns the restriction of a r. k. $K(x, y)$, defined on $E \times E$, to a set $E_1 \times E_1 \subseteq E \times E$. The restriction $K_1(x, y)$ is a p. m. and its corresponding r. k. s. \mathcal{F}_1 is obtained from \mathcal{F} , the r. k. s. of $K(x, y)$, in the following way. Let \mathcal{F}_0 be the space of all $g \in \mathcal{F}$ such that $g(x) = 0$ for $x \in E_1$ and let \mathcal{F}_0^\perp be its orthogonal complement. Then the space \mathcal{F}_1 consists of the restrictions of the elements of \mathcal{F}_0^\perp to E_1 with the same norm. The pertinent theorem is as follows.

THEOREM C⁽⁴⁾. *If K is the reproducing kernel of the space \mathcal{F} of functions defined on the set E with norm $\| \cdot \|$, then K restricted to the subset $E_1 \times E_1 \subseteq E \times E$ is the reproducing kernel of the class \mathcal{F}_1 of all restrictions of functions of \mathcal{F} to the subset E_1 . For any such restriction, $f_1 \in \mathcal{F}_1$, the norm $\|f_1\|_1$ is the minimum of $\|f\|$ for all $f \in \mathcal{F}$ whose restrictions to E_1 is f_1 .*

More detailed information on the topics discussed above will be found in N. Aronszajn [1; 2] and A. Devinatz [1].

3. Positive definite functions on columns. Before we prove the results of this section we shall say a few words about the notation which will be used here and in the future. In general the notation will be the same as in IRI.

Vectors in an n -dimensional Euclidean space⁽⁵⁾, E_n , will usually be de-

⁽³⁾ A. Devinatz [1], N. Aronszajn [2].

⁽⁴⁾ N. Aronszajn [2].

⁽⁵⁾ For convenience we shall also include vectors which have components at $\pm \infty$.

noted by lower case latin letters and their components by the same letters with superscripts (e. g. $x = (x^{(1)}, \dots, x^{(n)})$). In case we use these letters as scalars it will be clear from the context that we are doing so or else we shall explicitly state this fact. The letters u_k shall designate the "unit" vectors, i.e. those vectors for which $u_k^{(j)} = \delta_{jk}$, where δ_{jk} is the Kronecker symbol. We shall write $x \leq y$ if each component of x is less than or equal to each component of y .

We shall also have occasion to use a certain linear subspace of the $(m+n)$ -dimensional unitary space. Let x be any vector in E_{m+n} whose last n components are zero and y any vector in E_{m+n} whose first m components are zero. The vectors $z = x + iy$, $i = (-1)^{1/2}$, form a linear subspace of the $(m+n)$ -dimensional unitary space. We shall designate this subspace by Z_{m+n} . Using the terminology of the complex number field, if $z \in Z_{m+n}$, we shall write $\bar{z} = x - iy$, $\Re(z) = x$, $\Im(z) = y$.

If $t \in E_n$, when we write $\int_a^b f(t) d\alpha(t)$ we shall mean that $d\alpha(t)$ is a non-negative measure on E_n and that the integral is a Lebesgue-Radon-Stieltjes integral over the closed interval $a \leq t \leq b$.

In the following lemmas we shall be working in the 1-dimensional case. These lemmas are proved in much the same manner as the corresponding lemmas in IRI. However, there do exist points here which require some discussion.

Lemma 1⁽⁶⁾. *Let $f(n)$ be defined for $n = 0, 1, \dots, 2m+1$. Necessary and sufficient conditions that there exists a bounded non-negative measure $d\alpha(t)$ such that*

$$f(n) = \int_0^\infty t^n d\alpha(t), \quad \text{for } 0 \leq n \leq \begin{cases} 2m & \text{if } \det |f(j+k)|_{j,k=0}^m = 0, \\ 2m+1 & \text{if } \det |f(j+k)|_{j,k=0}^m \neq 0, \end{cases}$$

are:

$$(1) \quad f(j+k) \gg 0, \quad i, k = 0, 1, \dots, m.$$

$$(2) \quad f(j+k+1) \gg 0, \quad j, k = 0, 1, \dots, m.$$

Proof. The necessity of these conditions follows immediately. For the sufficiency we have two cases to consider. First of all suppose that the determinant

$$|f(j+k)|_{j,k=0}^m = 0.$$

The r. k. s. \mathcal{F} associated with $f(j+k)$ is then generated by the elements $\{f(j+k)\}_{k=0}^{m-1}$. Let $f_1(j+k)$ be the restriction of $f(j+k)$ to $j, k = 0, 1, \dots, m-1$, and \mathcal{F}_1 its corresponding r. k. s. The space \mathcal{F} may be written as $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_0^\perp$, where \mathcal{F}_0 is the class of all elements in \mathcal{F} which are zero for j

⁽⁶⁾ This lemma slightly sharpens a theorem of S. Verblunsky [1].

$= 0, 1, \dots, m-1$, and \mathcal{F}_0^\perp is its orthogonal complement. The space \mathcal{F}_0^\perp is generated by the set of elements $\{f(j+k)\}_{k=0}^{m-1}$; but since these elements generate the whole space, \mathcal{F}_0 can contain only the zero element.

Let $M(j, k) = f(j+k+1)$ be defined for $j, k = 0, 1, \dots, m-1$. Then, for fixed k , $M(j, k) \in \mathcal{F}_1$. Since \mathcal{F}_1 is finite-dimensional, by Theorem B the operator

$$T \cong M(j, k)$$

must be bounded and therefore self-adjoint. Further by condition (2) it must be positive. For any $g_1(j) \in \mathcal{F}_1$ we have

$$Tg_1(j) = g(j+1)$$

where $g(j) \in \mathcal{F}$ and g_1 is its restriction. Let $\{E_t\}$ be the canonical resolution of the identity associated with T . Then if $f_0(j) = f_1(j+0)$ and $r+s=n$, we have

$$f(n) = (f(j+r), f(j+s)) = (T^r f_0, T^s f_0)_1 = \int_0^\infty t^n d(E_t f_0, f_0)_1.$$

Suppose now that

$$|f(j+k)|_{i,k=0}^m \neq 0.$$

Then it is clearly possible to define a number $f(2m+2)$ such that

$$|f(j+k)|_{i,k=0}^{m+1} = 0.$$

We can then proceed as before.

LEMMA 2. *Let $f(x)$ be a continuous function defined on the interval $0 \leq x \leq a$. Further, let $f(x+y) \gg 0$ for $0 \leq x, y \leq a/2$. Then there exists an analytic function $F(z)$, defined in the strip $0 < \Re(z) < a$ such that $F(z+\bar{w}) \gg 0$ for $0 < \Re(z), \Re(w) < a/2$ and $F(x) = f(x)$ for $0 < x < a$.*

With the help of Lemma 1, this lemma may be proved by the same method as Lemma 2 of IRI.

LEMMA 3. *Let $f(x)$ be as in Lemma 2 and \mathcal{F} the corresponding r. k. s. Let $0 < s < a/8$ and let I_s be the set of all x in $[0, a/2]$ such that $x+4s \in [0, a/2]$. Further, let $f_1(x+y)$ be the restriction of $f(x+y)$ to $I_s \times I_s$ and \mathcal{F}_1 the corresponding r. k. s. If $M_{x_0}(x, y)$ is defined on $I_s \times I_s$ such that for $0 \leq x_0 \leq s$, $M_{x_0}(x, y) = f(x+y+x_0)$, then the operators $T_{x_0} \cong M_{x_0}(x, y)$ are self-adjoint.*

Proof. The space \mathcal{F} may be written as the direct sum $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_0^\perp$, where \mathcal{F}_0 is the space of elements in \mathcal{F} which are zero on I_s and \mathcal{F}_0^\perp is its orthogonal complement. If \mathcal{F}_2 is the r. k. s. associated with the $F(z+\bar{w})$ of Lemma 2, then, for every $g \in \mathcal{F}_0$, there exists a $g_2 \in \mathcal{F}_2$ whose restriction to the open interval $(0, a/2)$ coincides with $g(x)$ on this open interval. Therefore, since g_2 is zero on an interval of the x -axis and is analytic, it is identically zero. Since $g(x)$

is continuous on $[0, a/2]$ it must be identically zero on the closed interval. Therefore \mathcal{F}_0 consists only of the zero element and \mathcal{F}_1 consists of the restrictions of the elements of \mathcal{F} to I_s with the same norm. Consequently, $\mathcal{D}_{T_{x_0}}$ is the set of all $g_1 \in \mathcal{F}_1$ such that if $g \in \mathcal{F}$ restricted to I_s is g_1 , then $g(x+x_0) \in \mathcal{F}_1$ for x in I_s .

If T_{x_0} is not self-adjoint, then since $T_{x_0}^* \subset T_{x_0}$ and $T_{x_0}^{**} = T_{x_0}$, $T_{x_0}^*$ is not self-adjoint and there exists a nonzero $g_1 \in \mathcal{F}_1$ such that $T_{x_0} g_1 = i g_1$, i.e. $g(x+x_0) = i g(x)$ for $x \in I_s$, where $g \in \mathcal{F}$ and g_1 is the restriction to I_s of g . Now, there exists a $g_2 \in \mathcal{F}_2$ such that $g(x)$ and $g_2(x)$ coincide on $(0, a/2)$. Since $g_2(z)$ is analytic, $g(x+x_0) = i g(x)$ for all x for which x and $x+x_0$ are in $(0, a/2)$. By continuity this must also be true for all $x+x_0$ which lie in the closed interval $[0, a/2]$. Consequently, if $x \in I_s$, $x+2x_0 \in [0, a/2]$ and $g(x+2x_0) = i g(x+x_0) = i^2 g(x)$. Repeating this argument twice more we get $g(x+4x_0) = g(x)$ for x in I_s . Since $g_2(z)$ is analytic, for $\Re(z) + 4x_0 \in (0, a/2)$ we have $g_2(z+4x_0) = g_2(z)$ and therefore $g_2(z)$ may be extended analytically over the whole complex plane. Now,

$$|g_2(z)| = |(g_2(w), F(w + \bar{z}))_2| \leq \|g_2\|_2 (f(2x))^{1/2},$$

where $x = \Re(z)$ and $(\cdot, \cdot)_2$ is the inner product of \mathcal{F}_2 . Therefore $g_2(z)$ is bounded in every strip $x_1 \leq \Re(z) \leq x_2$ and consequently by Liouville's theorem it must be a constant, which of course must be zero. This contradicts the hypothesis that $g_1 \neq 0$, which means that T_{x_0} is self-adjoint.

LEMMA 4. *Let $f(x)$ be as in the previous lemmas. Then there exists a spectral measure dE_t defined on \mathcal{F}_1 , such that*

$$f(x) = \int_0^\infty t^x d(E_t f_0, f_0)_1(t),$$

where $f_0(x) = f_1(x+0)$ and $(\cdot, \cdot)_1$ is the inner product of \mathcal{F}_1 . The measure $d\alpha(t) = d(E_t f_0, f_0)_1$ is uniquely determined. The operators T_{x_0} for $0 \leq x_0 \leq s$ are given by

$$T_{x_0} = \int_0^\infty t^{x_0} dE_t.$$

Proof. There exists a spectral measure dE_t such that the measure of the interval $[-\infty, 0)$ is zero and

$$T_s = \int_0^\infty t^s dE_t.$$

Since every positive self-adjoint operator has a unique positive self-adjoint square root we have, by Lemma 3,

$$T_{s/2^n} = \int_0^\infty t^{s/2^n} dE_t, \quad n = 0, 1, 2, \dots$$

(7) Cf. S. Bernstein [1] and D. V. Widder [2, 3].

Suppose now that m is an integer and $ms/2^n \leq a/2$. Then from the fact that the space \mathcal{F}_0 of Lemma 3 consists of the zero element only, we have

$$T_{s/2^n}^m f_0(x) = f(x + ms/2^n)$$

and

$$f((p+q)s/2^n) = (f(x + ps/2^n), f(x + qs/2^n)) = (T_{s/2^n}^p f_0(x), T_{s/2^n}^q f_0(x))_1.$$

Therefore,

$$f((p+q)s/2^n) = \int_0^\infty t^{(p+q)s/2^n} d(E_t f_0, f_0)_1.$$

If we now pass to the limit we get our result. The uniqueness of $d\alpha(t)$ may be proved by the same method as used in Lemma 1 of IRI.

The integral representation of the operators T_{x_0} may be obtained by the same method as used in Lemma 4 of IRI. However, since we shall have occasion to use this representation later we shall reproduce the proof here. Consider the operators given by

$$H_{x_0} = \int_0^\infty t^{x_0} dE_t.$$

The operators $H_{ms/2^n}$ certainly coincide with $T_{ms/2^n}$, provided $0 \leq ms/2^n \leq s$. Since $f_y(x) = f_1(x+y) \in \mathcal{D}_{T_{ms/2^n}}$, $f_y \in \mathcal{D}_{H_{x_0}}$ for every x_0 such that $0 \leq x_0 \leq s$. Therefore

$$f(x + y + ms/2^n) = (T_{ms/2^n} f_x, f_y)_1 = \int_0^\infty t^{ms/2^n} d(E_t f_x, f_y)_1.$$

If we choose a sequence of the $ms/2^n$ which approaches any $0 < x_0 \leq s$ we get

$$f(x + y + x_0) = \int_0^\infty t^{x_0} d(E_t f_x, f_y)_1 = (H_{x_0} f_x, f_y)_1.$$

Therefore, $H_{x_0} f_y(x) = f(x + y + x_0)$, which means $H_{x_0} \sim M_{x_0}(x, y) = f(x + y + x_0)$. Consequently, $H_{x_0} \subseteq T_{x_0}$ and since both of these operators are self-adjoint we must have $H_{x_0} = T_{x_0}$.

With the help of these lemmas it is now possible to prove a modification of Theorem 5 of IRI. Let P be a set of points z in Z_{m+n} such that $\mathcal{R}(z)$ runs over an interval (finite or infinite) in E_m which contains the origin and $-\infty < \Im(z) < \infty$. Let $P/2$ be the set z in P such that $2z \in P$, $0 < s^{(k)} < 1/8 \sup |z \cdot u_k|$, where the sup is taken over $z \in P$, $s = (s^{(1)}, \dots, s^{(m)}, 0, \dots, 0)$, and P_s the set of points in $P/2$ such that $z + 4s \in P/2$. Further let $a = (a^{(1)}, \dots, a^{(m)}, -c^{(m+1)}, \dots, -c^{(m+n)})$, $b = (b^{(1)}, \dots, b^{(m)}, c^{(m+1)}, \dots, c^{(m+n)})$, and $s_r = (0, \dots, 0, s^{(m+1)}, \dots, s^{(m+n)})$. The numbers $a^{(k)}$ may take on the value

$-\infty$, $b^{(k)}$ and $c^{(k)}$ the value $+\infty$, $a^{(k)} \leq b^{(k)}$ and $c^{(k)} \geq 0$.

THEOREM 1. *Let $f(z)$ be a continuous complex-valued function defined on P . Necessary and sufficient conditions that there exists a bounded non-negative measure $d\alpha(t)$, such that*

$$f(z) = \int_a^b e^{z \cdot t} d\alpha(t),$$

are:

- (1) $f(z + \bar{w}) \gg 0$ for z, w in $P/2$.
- (2) There exists a vector $s > 0$ such that for $k=1, \dots, m$, and z, w in P_s ,
 $\exp(s^{(k)} a^{(k)}) f(z + \bar{w}) \ll f(z + \bar{w} + s^{(k)} u_k) \ll \exp(s^{(k)} b^{(k)}) f(z + \bar{w})^{(8)}$.
- (3) There exists a sequence $s_r \rightarrow 0$ such that $s_r^{(k)} = 0$ if $c^{(k)} = \infty$, $s_r^{(k)} \neq 0$, $c^{(k)} < \infty$, and such that for $k=m+1, \dots, m+n$ and $r=0, 1, \dots$,
 $f(z + \bar{w} - i s_r^{(k)} u_k) + f(z + \bar{w} + i s_r^{(k)} u_k) \gg 2 \cos(s_r^{(k)} d^{(k)}) f(z + \bar{w})^{(9)}$.

Proof. The necessity of these conditions follows immediately by a direct computation. For the sufficiency let $f_1(z + \bar{w})$ be the restriction of $f(z + \bar{w})$ to $P_s \times P_s$ and \mathcal{F}_1 and \mathcal{F} their corresponding r. k. s. As in Lemma 3 we may easily check that the elements of \mathcal{F}_1 are the restrictions to P_s of the elements of \mathcal{F} with the same norm. Using this fact, if $M(z, w)$ and $N(z, w)$ are defined on $P_s \times P_s$ such that $M(z, w) = f(z + \bar{w} + x^{(k)} u_k)$ and $N(z, w) = f(z + \bar{w} - i y^{(k)} u_k)$, if we set up the operators

$$T_{x^{(k)} u_k} \cong M(z, w)$$

and

$$U_{y^{(k)} u_k} \cong N(z, w),$$

we can proceed exactly as in the proof of Theorem 5 of IRI, using the previous lemmas.

4. Positive definite functions on convex sets. Let Q be an open convex set in the complex plane⁽¹⁰⁾ which contains the origin and which is symmetric with respect to the real axis. Let $Q/2$ be the set of all z in Q for which $2z \in Q$. Further, let $d^{(k)} = \sup |z \cdot u_k|$, where the sup is taken over $z \in Q$, $0 < s^{(k)} < d^{(k)}/8$, $s = s^{(1)} + i s^{(2)}$, and Q_s the set of all z in $Q/2$ such that $z \pm 4s \in Q/2$ and $z \pm 4\bar{s} \in Q/2$. It is obvious that $Q/2$ and Q_s are convex sets in the complex plane which are open, contain the origin, and are symmetric with re-

⁽⁸⁾ In case $a^{(k)} = -\infty$, the corresponding left "inequality" is clearly redundant. In case $b^{(k)} = +\infty$, let us agree that the right "inequality" is also redundant.

⁽⁹⁾ Define $0 \cdot \infty = 0$.

⁽¹⁰⁾ In case Q reduces to a segment of the real or imaginary axis we shall mean that Q is open with respect to the topology of these straight lines.

spect to the real axis. Let $f(z)$ be a continuous complex-valued function defined on Q such that $f(z+\bar{w}) \gg 0$ for z and w in $Q/2$. Let $f_1(z+\bar{w})$ be the restriction of $f(z+\bar{w})$ to $Q_s \times Q_s$. Suppose further that \mathcal{F} is the r. k. s. associated with $f(z+\bar{w})$ and \mathcal{F}_1 the r. k. s. associated with $f_1(z+\bar{w})$ with inner product $(\ , \)_1$. If \mathcal{F}_0 is the set of elements in \mathcal{F} which are zero on Q_s and \mathcal{F}_0^\perp the orthogonal complement of \mathcal{F}_0 , then we know by Theorem C that \mathcal{F}_1 is the restriction of the elements of \mathcal{F}_0^\perp to Q_s with the same norm. We know further that all elements of the form $f(z+\bar{w})$ for $w \in Q_s$ belong to \mathcal{F}_0^\perp .

Let us consider the elements $f(z+\bar{w}_0+x)$, for x real, $w_0 \in Q_s$, and $w_0+x \in Q/2$. We may then write

$$f(z+\bar{w}_0+x) = K_0^\perp(z, w_0+x) + K_0(z, w_0+x),$$

where $K_0^\perp(z, w_0+x) \in \mathcal{F}_0^\perp$ and $K_0(z, w_0+x) \in \mathcal{F}_0$. Let us now consider the complex numbers $w(x)$, where $\Im(w(x)) = \Im(w_0)$ and $\Re(w(x))$ varies so that $w(x) \in Q/2$. Since Q_s is open these complex numbers actually trace out some line segment in Q_s which we shall designate by L_s and an extension in $Q/2$ which we shall designate by L . Let L' be the projection of L on the real axis. If we consider the restriction of $f(z+\bar{w})$ to $L \times L$, then this is exactly the same p. m. which we get if we consider the restriction of $f(z+\bar{w})$ to $L' \times L'$. Therefore this restriction is an analytic function of x (see Lemma 2) and all functions which belong to the r. k. s. of this restriction are analytic. Now, for any $z \in Q/2$ and $w(x) \in L_s$, $K_0(z, w(x)) = 0$. Therefore, since for any fixed $z \in Q/2$, $\bar{K}_0(w(x), z)$ when considered as a function of x belongs to the r. k. s. corresponding to the restriction of $f(z+\bar{w})$ to $L \times L$, and since it is zero on L_s , it must be zero for all of L . Consequently, for $w_0 \in Q_s$ and $z \in Q/2$,

$$(4.1) \quad f(z+\bar{w}_0+x) = K_0^\perp(z, w_0+x),$$

which means that $f(z+\bar{w}_0+x) \in \mathcal{F}_0^\perp$.

On the space \mathcal{F}_1 set up the operators $T_x \cong M_x(z, w) = f(z+\bar{w}+x)$ for $|x| \leq s^{(1)}$. The domain of T_x consists of those elements $g_1 \in \mathcal{F}_1$ such that $g(z+x) \in \mathcal{F}_1$, where $g \in \mathcal{F}_0^\perp$ and g_1 is the restriction of g to Q_s . These operators are self-adjoint (see Lemma 3 and IRI Lemma 6). Further, since $f(z+\bar{w}_0+x) \in \mathcal{F}_0^\perp$ for $w_0 \in Q_s$ and any x such that $w_0+x \in Q/2$, then for any integer n such that $w_0+nx \in Q/2$,

$$T_x^n f_1(z+\bar{w}_0) = f(z+\bar{w}_0+nx).$$

This means that if $g_1 \in D_{T_x^n}$, then $T_x^n g_1(w_0) = g(w_0+nx)$, where $g \in \mathcal{F}_0^\perp$ and g_1 is the restriction of g . This follows from the fact that

$$\begin{aligned} T_x^n g_1(w_0) &= (T_x^n g_1(z), f_1(z+\bar{w}_0))_1 = (g_1(z), T_x^n f_1(z+\bar{w}_0))_1 \\ &= (g(z), f(z+\bar{w}_0+nx)). \end{aligned}$$

Let us also consider the operators

$$V_y \cong N_y(z, w) = f(z + \bar{w} - iy)$$

for $|y| \leq s^{(2)}$. If $h(z) = \sum_{k=1}^n \xi_k f_1(z + \bar{w}_k)$ with $\bar{w}_k + iy \in Q_s$, then clearly

$$\|V_y h\|_1 = \|h\|_1.$$

In general, for any $g \in \mathcal{F}_1$,

$$\|V_y g\|_1 \leq \|g\|_1.$$

Consequently, V_y is a bounded operator and $V_y g_1(z) = g(z + iy)$, where $g \in \mathcal{F}_0^+$ and g_1 is the restriction of g to Q_s . From this remark it is easy to see that if each of the points $\bar{w} + iy_1$ and $\bar{w} + iy_2$ belong to Q_s then

$$(4.2) \quad V_{y_1} V_{y_2} f_1(z + \bar{w}) = V_{y_1 + y_2} f_1(z + \bar{w}) = V_{y_2} V_{y_1} f_1(z + \bar{w}).$$

Further, it is clear that $V_y g$ is a strongly continuous function of y for every $g \in \mathcal{F}_1$, $V_0 = I$, and since $V_y \cong \bar{N}_y(w, z) = \bar{f}(\bar{z} + w - iy) = f(z + \bar{w} + iy)$ (Aronszajn [2, p. 372]), we have $V_y^* = V_{-y}$.

We shall now show that each of the operators V_y permutes with each of the operators T_x . We have first of all $T_x f_1(z + \bar{w}) = f(z + \bar{w} + x)$. Since, for $z \in Q/2$, $f(z + \bar{w} + x) \in \mathcal{F}_0^+$, we have $V_y T_x f_1(z + \bar{w}) = f(z + \bar{w} + x + iy)$. On the other hand, $V_y f_1(z + \bar{w}) = f(z + \bar{w} + iy)$. Now the element $f(z + \bar{w} + iy)$ for $z \in Q_s$ is the restriction to Q_s of the function $K_0^+(z, w - iy)$ which was defined above. Therefore, $T_x V_y f_1(z + \bar{w}) = K_0^+(z + x, w - iy)$. However, by (4.1) we see that $K_0^+(z + x, w - iy) = f(z + \bar{w} + x + iy)$ for z, w in Q_s . This means that T_x and V_y permute on the set \mathcal{F}_1^+ . Now Theorem B tells us that T_x is the closure of its restriction to \mathcal{F}_1^+ . Therefore V_y permutes with T_x .

Let us now discuss the question of the extension of a complex-valued positive linear functional on a space of complex-valued functions whose field of scalars is the complex number field. The result which we state below is, we believe, well known and has been used implicitly in the literature. However, we have never seen a formal statement of this result so we think it is well, for the sake of completeness, to give it here.

LEMMA 5⁽¹¹⁾. *Let X be a complex linear space of complex-valued functions $x(t)$, which are defined over some domain D . Let S be a subspace of X and $l(x)$ a linear functional on S with range in the complex number field. Further, let us suppose that $l(x)$ is real whenever $x(t)$ is real on some fixed subdomain $D_1 \subseteq D$, $l(x) \geq 0$ whenever $x(t) \geq 0$ on D_1 , and for every $x_0 \in X$ which is real on D_1 , $\sup l(x) < \infty$ and $\inf l(x) > -\infty$, where the sup is taken over all $x \in S$ which are real on D_1 and $x(t) \leq x_0(t)$ on D_1 and the inf is taken over all $x \in S$ which are real on D_1 and such that $x(t) \geq x_0(t)$ on D_1 . Then there exists a linear functional defined on all of X which is an extension of $l(x)$ and which retains all of the properties of l .*

⁽¹¹⁾ I am indebted to Professor Harry Pollard for an enlightening discussion of this lemma.

Proof. Let X' be the set of all real elements in X and $S' = S \cap X'$. Then X' is a real linear space and S' is a linear subspace of X' . By hypothesis the restriction l' , of l to S' , is a real positive linear functional. Then, by the extension theorem for real positive linear functionals (see Achieser and Krein [1]), l' may be extended to a positive functional L' over X' . Now, it is necessary to extend L' to the complex linear manifold determined by S and X' so that L' coincides with l on S . But this is easily done. We may write every element in this manifold in the form $y = x + ax'$, where $x \in S$, $x' \in X'$, and a is a complex number. If y_1 and y_2 are any such elements let us consider $c_1y_1 + c_2y_2 = c_1x_1 + c_2x_2 + c_1a_1x'_1 + c_2a_2x'_2$, where the coefficients are complex numbers. Let us then write

$$L(c_1y_1 + c_2y_2) = c_1l(x_1) + c_2l(x_2) + c_1a_1L'(x'_1) + c_2a_2L'(x'_2).$$

This is a linear functional which coincides with l on S and L' on X' . Now, extend L to all of X in any way so as to be linear and the lemma is proved.

It should be pointed out that the condition that l be real for real elements of S is independent of the condition of positivity and is needed. For, there exist spaces in which the failure of this condition is enough to insure that there can be no positive extension. Consider, for example, S as the complex linear space which consists of polynomial functions $P(t)$, $-\infty < t < \infty$, with complex coefficients and such that $P(1) = 0$. Define $l(P) = P(0) + iP'(1)$, where $i = (-1)^{1/2}$ and the prime stands for the derivative. If $P(t) \geq 0$, then $P'(1) = 0$ and $l(P) \geq 0$. However, l is not necessarily real for real $P(t)$. Suppose now that $P(t)$ is a real polynomial such that $l(P)$ is not real, i.e. $P'(1) \neq 0$. Write $P(t) = P_+(t) - P_-(t)$, where $P_+(t) = \max [P(t), 0]$ and $P_-(t) = -\min [P(t), 0]$. Let X be the complex linear space generated by the space S and the functions P_+ and P_- . Suppose it would be possible to extend l to L defined on X such that L is a positive linear functional. Then $L(P) = l(P) = L(P_+) - L(P_-)$. But since P_+ and P_- are both positive, $l(P)$ is real which gives a contradiction.

With all of these preliminaries out of the way we may now state and prove the theorem we want.

THEOREM 2. Let $f(z)$ be a continuous complex-valued function defined on Q . Necessary and sufficient conditions that there exists a bounded non-negative measure $d\alpha(t_1, t_2)$ which is zero on the complement of the interval $a \leq t_1 \leq b$, $-c \leq t_2 \leq c$ (a may take the value $-\infty$, b and c the value $+\infty$) and such that

$$f(z) = \int_{-c}^c \int_a^b e^{x t_1 + i y t_2} d\alpha(t_1, t_2)$$

are:

- (1) $f(z + \bar{w}) \gg 0$ for $z, w \in Q/2$.
- (2) There exists a positive sequence $r_n \rightarrow 0$ such that for $n = 1, 2, \dots$,

$$e^{r_n a} f(z + \bar{w}) \ll f(z + \bar{w} + r_n) \ll e^{r_n b} f(z + \bar{w}),$$

for all z and w in $Q/2$ for which the functions are defined⁽¹²⁾.

(3) There exists a sequence $s_n \rightarrow 0$ such that for $n=1, 2, \dots$, $s_n=0$ if $c=\infty$, $s_n \neq 0$ if $c<\infty$, and

$$f(z + \bar{w} - is_n) + f(z + \bar{w} + is_n) \gg 2 \cos(s_n c) f(z + \bar{w})^{(9)},$$

for all z and w in $Q/2$ for which the functions are defined.

Before we proceed with the proof of this theorem let us notice certain things. If the set Q reduces to one half of the real axis and if $a = -\infty$, $b=0$, this is equivalent to the Hausdorff [1]—S. Bernstein [1]—Widder [1] theorem on completely monotone functions. For $a = -\infty$, $b = +\infty$, this is the S. Bernstein [1]—Widder [2; 3] theorem on exponentially convex functions. If the set Q reduces to an interval on the imaginary axis and if $c = \infty$, this is a theorem of M. Krein [1] and if Q is the whole imaginary axis this is the well known theorem of S. Bochner. Finally, for $a = -\infty$, $b=c = +\infty$, this is the correct version of an incorrect theorem stated by M. Livshitz [1, Theorem 3].

According to Livshitz, condition (1) of our Theorem 2 is necessary and sufficient that there exist a bounded non-negative measure $d\alpha(t)$, defined on the real line such that

$$f(z) = \int_{-\infty}^{\infty} e^{zt} d\alpha(t).$$

That condition (1) of Theorem 2 is not sufficient for such a representation can easily be seen by the following simple counter example. Let $d\alpha_1(t)$ be a bounded non-negative measure such that the integral

$$f(x) = \int_{-\infty}^{\infty} e^{xt} d\alpha_1(t)$$

exists for $x \geq 0$. Suppose further that the measure is normed so that $f(0) = 1$. Let $d\alpha_2(t)$ be a non-negative measure different from $d\alpha_1(t)$ and such that $\int_{-\infty}^{\infty} d\alpha_2(t) = 1$. Form the function

$$g(y) = \int_{-\infty}^{\infty} e^{iyt} d\alpha_2(t),$$

for $-\infty < y < \infty$. Then if $d\alpha_1(t_1)d\alpha_2(t_2)$ represents the product measure, the function

$$F(z) = f(x)g(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{zt_1 + iy t_2} d\alpha_1(t_1) d\alpha_2(t_2)$$

⁽¹²⁾ For some types of domains, for example an interval, the existence of only one number, r , for which the "inequalities" hold is sufficient. We have been unable to determine whether this is true for all convex sets. As before, if $b = +\infty$, consider the corresponding "inequality" as redundant.

satisfies all of the conditions of the theorem of Livshitz. Therefore if his theorem were true, there exists a bounded non-negative measure $d\alpha_3(t)$ such that

$$F(z) = \int_{-\infty}^{\infty} e^{zt} d\alpha_3(t).$$

Now, if we take $\Im(z) = 0$, we get

$$F(x) = f(x) = \int_{-\infty}^{\infty} e^{xt} d\alpha_3(t),$$

from which it follows that $d\alpha_1 = d\alpha_3$. On the other hand if $\Re(z) = 0$,

$$F(iy) = g(y) = \int_{-\infty}^{\infty} e^{iyt} d\alpha_3(t),$$

from which it follows that $d\alpha_2 = d\alpha_3$. This gives a contradiction. The theorem of Livshitz will become true if we allow $f(z)$ to be analytic instead of merely continuous.

Proof of Theorem 2. The necessity follows immediately by a direct computation. For the sufficiency let us consider a set Q_s and the set of operators $\{T_x\}$ which we have constructed previously. By the same method as used in Lemma 4 we may show the existence of a spectral measure dG_t such that for $0 \leq x \leq s^{(1)}$,

$$T_x = \int_0^{\infty} t^x dG_t.$$

However, in this case, for any x such that $|x| \leq s^{(1)}$, T_{-x} is the inverse of T_x and therefore the origin has a zero measure with respect to dG_t . Therefore, there exists a spectral measure dE_t such that for $|x| \leq s^{(1)}$,

$$T_x = \int_{-\infty}^{\infty} e^{xt} dE_t.$$

From the remarks made at the beginning of this section, for any x such that $z+x \in Q_s$ and for any $g \in \mathcal{D}_{T_x}$, where T_x is defined by the above integral if $|x| > s^{(1)}$, we have $T_x g(z) = g(z+x)$. This follows from the fact that for any x_1 and integer n such that $|x_1| \leq s^{(1)}$ and $nx_1 = x$, $T_{x_1}^n = T_x$.

There exists a countable set of mutually orthogonal subspaces, $\{\mathcal{M}_k\}_{k=1}^{\infty}$, whose direct sum is \mathcal{H}_1 , which reduces every T_x , and the restriction of T_x to \mathcal{M}_k is a bounded self-adjoint operator which we shall denote by $T_x^{(k)}$. Therefore, we may write

$$T_x^{(k)} = \int_{a_k}^{b_k} e^{xt} dE_t^{(k)},$$

where a_k and b_k are finite and $dE_t^{(k)}$ is a spectral measure which is zero outside of the interval $[a_k, b_k]$. If $f_k(z, w)$ is the r. k. of \mathcal{M}_k , then if $w+x \in Q_s$,

$$\begin{aligned} T_x^{(k)} f_k(z, w) &= (T_x^{(k)} f_k(u, w), f_k(u, z))_1 = (f_k(u, w), T_x^{(k)} f_k(u, z))_1 \\ &= \overline{T_x^{(k)} f_k(w, z)} = \overline{f_k(w+x, z)} = f_k(z, w+x). \end{aligned}$$

Let

$$A_k = \int_{a_k}^{b_k} t dE_t^{(k)}.$$

It is clear that if

$$A_\delta^{(k)} = \frac{T_\delta^{(k)} - I}{\delta},$$

then for any $g \in \mathcal{M}_k$ we have

$$A_\delta^{(k)} g \rightarrow A_k g,$$

where the bold arrow indicates convergence in the strong topology of \mathcal{H}_1 . However, if δ is sufficiently small,

$$A_\delta^{(k)} g(z) = \frac{g(z+\delta) - g(z)}{\delta}.$$

Therefore, if $x \in \mathcal{R}(z)$, since \mathcal{H}_1 is a r. k. s. we get

$$A_k g(z) = \frac{\partial g(z)}{\partial x}.$$

In \mathcal{M}_k let $\mathcal{D}_{B_k'}$ be the set of all $g(z) \in \mathcal{M}_k$ such that for $y = \mathcal{I}(z)$, $\partial g(z)/\partial y$ exists and belongs to \mathcal{M}_k . We shall then define

$$B_k' g(z) = \frac{\partial g(z)}{\partial y}.$$

The first thing that we wish to show is that $\mathcal{D}_{B_k'}$ is dense in \mathcal{M}_k . To this end let us consider the operators

$$W_\delta = \frac{V_\delta - I}{\delta}, \quad |\delta| \leq s^{(2)} \quad (V_\delta \text{ defined at beginning of §4}).$$

If P_k is the projection operator whose range is \mathcal{M}_k , then since W_δ permutes with every T_x it also permutes with P_k . Consequently, if W is the strong limit of W_δ as $\delta \rightarrow 0$, then W also permutes with P_k and therefore \mathcal{M}_k reduces W . Now, clearly if $g \in \mathcal{D}_W$, then

$$Wg = \frac{\partial g}{\partial y}.$$

Therefore, if we can show that \mathcal{D}_W is dense in \mathcal{Y}_1 , we shall have shown that $\mathcal{D}_{B'_k}$ is dense in \mathcal{M}_k .

Let us consider an element of the form

$$g(z) = \sum_{k=1}^n \xi_k f_1(z + \bar{w}_k).$$

Since Q_s is open there exists a real r such that $w_k - 2ir \in Q_s$ for $k=1, \dots, n$. If we set up the element

$$g_r(z) = \frac{1}{r} \int_0^r V_{ig}(z) dt,$$

then $g_r \in \mathcal{Y}_1$ and $g_r \rightarrow g$ as $r \rightarrow 0$. Therefore, this class of elements of the form g_r is dense in \mathcal{Y}_1 . Further, if $|\delta| \leq |r|$, using (4.2) we get

$$W_{\delta} g_r = \frac{1}{\delta r} \left[\int_r^{r+\delta} V_{ig} dt - \int_0^{\delta} V_{ig} dt \right] \rightarrow W_r g$$

as $\delta \rightarrow 0$ (13). Consequently \mathcal{D}_W is dense in \mathcal{Y}_1 .

Let us now show that for $g \in \mathcal{D}_{W^*}$,

$$W^* g = - \frac{\partial g}{\partial y}.$$

To this end let us consider the elements

$$f_r(z) = \frac{1}{r} \int_0^r V_{if_1}(z + \bar{w}) dt,$$

where w is fixed and r is taken so small that $\bar{w} + ir \in Q_s$. Then since $Wf_r(z) = W_r f_1(z + \bar{w})$ we have

$$\begin{aligned} (g(z), Wf_r(z))_1 &= (g(z), W_r f_1(z + \bar{w}))_1 = \left(g(z), \frac{f_1(z + \bar{w} + ir) - f_1(z + \bar{w})}{r} \right)_1 \\ &= (W^* g(z), f_r(z))_1. \end{aligned}$$

Therefore, since $f_r(z) \rightarrow f_1(z + \bar{w})$, we get

$$W^* g(w) = \lim_{r \rightarrow 0} (W^* g, f_r)_1 = \lim_{r \rightarrow 0} \frac{g(w - ir) - g(w)}{r} = - \frac{\partial g(w)}{\partial y}.$$

This shows that if W_k^* is the restriction of W^* to \mathcal{M}_k , then $W_k^* \subseteq -B'_k$.

(13) We have followed here a technique of N. Dunford and I. E. Segal [1].

If we consider the operator

$$B_k = -iB'_k,$$

then we have shown that

$$-iW_k \subseteq B_k \quad \text{and} \quad (-iW_k)^* \subseteq B_k.$$

Therefore

$$B_k^* \subseteq (-iW_k)^* \subseteq B_k.$$

Further, since it is clear that B_k is a closed operator we have that $\mathcal{D}_{B_k^*}$ is dense in \mathcal{M}_k and therefore B_k^* is a closed symmetric operator with $B_k^{**} = B_k$.

We wish to show now that the deficiency index of B_k^* is (m, m) and therefore it can be extended to a self-adjoint operator. Let us first show that if $h(z) \in \mathcal{F}_1$, then $\bar{h}(\bar{z}) \in \mathcal{F}_1$ and

$$(\bar{h}(\bar{z}), \bar{g}(\bar{z}))_1 = \overline{(h(z), g(z))_1}.$$

Consider first any $h \in \mathcal{F}'_1$, i.e.

$$h(z) = \sum_{k=1}^n \xi_k f_1(z + \bar{w}_k).$$

Then

$$\bar{h}(\bar{z}) = \sum_{k=1}^n \bar{\xi}_k f_1(z + w_k) \in \mathcal{F}'_1.$$

It is clear that if g is also in \mathcal{F}'_1 , then

$$(\bar{h}(\bar{z}), \bar{g}(\bar{z}))_1 = \overline{(h(z), g(z))_1}.$$

Suppose now that $h(z)$ is any element of \mathcal{F}_1 . Then there exists a sequence $\{h_n\}_{n=1}^\infty \subseteq \mathcal{F}'_1$ such that $h_n \rightarrow h$. Therefore $\|h_n(z) - h_m(z)\|_1 = \|\bar{h}_n(\bar{z}) - \bar{h}_m(\bar{z})\|_1 \rightarrow 0$ as $n, m \rightarrow \infty$. This means there exists a $k(z) \in \mathcal{F}_1$ such that $\bar{h}_n(\bar{z}) \rightarrow k(z)$. But since strong convergence in \mathcal{F}_1 implies pointwise convergence in \mathcal{F}_1 , we must have $k(z) = \bar{h}(\bar{z})$. Clearly if $\bar{g}(\bar{z}) \in \mathcal{F}_1$, then the inner product of $\bar{h}(\bar{z})$ and $\bar{g}(\bar{z})$ has the desired relation.

Let J be the conjugation operator defined by the formula

$$Jg = \bar{g}(\bar{z}).$$

It is clear that J permutes with each of the operators T_x for $|x| \leq s^{(1)}$. Therefore \mathcal{M}_k reduces J . If J_k is the restriction of J to \mathcal{M}_k , it is also clear that J_k permutes with B_k and therefore with B_k^* . This means that B_k^* is real with respect to the conjugation J_k and therefore its deficiency index is of the form (m, m) .

If S_k is the Cayley transform of B_k^* , then the deficiency spaces of B_k^* are given by $\mathcal{H}_k^+ = \mathcal{M}_k \ominus \mathcal{D}_{S_k}$ and $\mathcal{H}_k^- = \mathcal{M}_k \ominus \mathcal{R}_{S_k}$, where \mathcal{D}_{S_k} and \mathcal{R}_{S_k} are respectively the domain and range of S_k . The spaces \mathcal{H}_k^+ and \mathcal{H}_k^- are given respectively by the elements in \mathcal{M}_k which satisfy the equations

$$B_k g = ig \quad \text{and} \quad B_k h = -ih.$$

The solutions of these partial differential equations yield

$$g(z) = e^{-v}g(x) \quad \text{and} \quad h(z) = e^v h(x).$$

Furthermore, any functions of this form which belong to \mathcal{M}_k , for $x+iy \in Q_s$, are in \mathcal{H}_k^+ and \mathcal{H}_k^- respectively.

We wish to show now that if $g(z) \in \mathcal{H}_k^+$ then $\bar{g}(z) \in \mathcal{H}_k^+$ and $\|g\|_1 = \|\bar{g}\|_1$. It will follow by a similar argument that if $h(z) \in \mathcal{H}_k^-$, then $\bar{h}(z) \in \mathcal{H}_k^-$. Let $K(z, w)$ be the r. k. of \mathcal{H}_k^+ and $\{\phi_n(z)\}$ a complete orthonormal system in \mathcal{H}_k^+ . If $w = v + iv$, then since $\phi_n(z) = e^{-v}\phi_n(x)$, $K(z, w)$ has the expansion

$$K(z, w) = \sum_n e^{-v}\phi_n(x)e^{-v}\bar{\phi}_n(u) = e^{-(v+v)}K(x, u).$$

Now, \mathcal{H}_k^+ reduces the operator A_k and therefore each of the operators $T_x^{(k)}$. In order to see this we notice first that since A_k and B_k are differential operators, $\mathcal{D}_{A_k} = \mathcal{M}_k$ and every element of \mathcal{M}_k is continuous in $\mathcal{R}(z)$ and $\mathcal{I}(z)$, it follows that A_k permutes with B_k . Further, since A_k is self-adjoint this means that A_k permutes with B_k^* and therefore $A_k S_k \subseteq S_k A_k$. This means that A_k takes \mathcal{D}_{S_k} into itself. On the other hand since every element of \mathcal{H}_k^+ is the form $e^{-v}g(x)$ and every element of this form which belongs to \mathcal{M}_k is in \mathcal{H}_k^+ , it follows that A_k takes \mathcal{H}_k^+ into itself. Therefore A_k may be written as the direct sum of two operators defined respectively on \mathcal{D}_{S_k} and \mathcal{H}_k^+ . This means that if P_k^+ is the projection operator whose domain is \mathcal{M}_k and range is \mathcal{H}_k^+ , then A_k and therefore $T_x^{(k)}$ permutes with P_k^+ .

The kernel $K(z, w)$ is given by the formula

$$K(z, w) = P_k^+ f_k(z, w),$$

where, as we have noted before, $f_k(z, w)$ is the r. k. of \mathcal{M}_k . Therefore,

$$\begin{aligned} K(z, w) &= P_k^+ T_u^{(k)} f_k(x + iy, iv) = T_u^{(k)} P_k^+ f_k(x + iy, iv) \\ &= T_x^{(k)} T_u^{(k)} P_k^+ f_k(iy, iv) = T_{x+u}^{(k)} P_k^+ f_k(iy, iv). \end{aligned}$$

Therefore, $K(x, u)$ is a function of $x+u$ which means that

$$K(z, w) = K'(z + w) = e^{-(v+v)}K'(x + u).$$

This means that $K(z, w)$ is real from which our assertion follows (see N. Aronszajn [2, p. 354]).

Define a linear operator S_k' , with $\mathcal{D}_{S_k'} = \mathcal{H}_k^+$ and $\mathcal{R}_{S_k'} = \mathcal{H}_k^-$, by the formula

$$S'_k e^{-\nu} g(x) = e^{i\theta} e^{\nu} g(x),$$

where θ is any real number. The function $e^{\nu} g(x) \in \mathcal{H}_k^-$ since $J e^{-\nu} g(x) = e^{\nu} \bar{g}(x) \in \mathcal{H}_k^-$. Further, since

$$\|e^{-\nu} g(x)\|_1 = \|J e^{-\nu} g(x)\|_1 = \|e^{\nu} g(x)\|_1,$$

it follows that S'_k is an isometric operator. Consequently, $U_k = S_k \oplus S'_k$ is a unitary operator and this gives rise to a self-adjoint extension of B_k^* whose Cayley transform is U_k . Clearly if C_k is any self-adjoint extension of B_k^* , then $C_k \subseteq B_k$. If the deficiency index of B_k^* is not $(0, 0)$ there exists an infinite number of self-adjoint extensions of B_k^* of the type which we have described.

Suppose that C_k is a self-adjoint extension of B_k^* of the type we have described. In order that A_k permutes with C_k , it is sufficient that A_k permutes with U_k . However, we have already shown that $A_k S_k \subseteq S_k A_k$ and since clearly $A_k S'_k \subseteq S'_k A_k$, we have $A_k U_k = U_k A_k$. Therefore every $T_x^{(k)}$ permutes with C_k and consequently if C is the direct sum of the operators $\{C_k\}_{k=1}^\infty$, then each operator T_x permutes with the self-adjoint operator C .

If condition (3) of Theorem 2 is fulfilled with $c < \infty$, we shall show that C has the bound c . Since $s_n \rightarrow 0$, we can find an N such that for $n > N$ and for all $w \in Q$, $w \pm i s_n \in Q/2$. By Theorem B, condition (3) means that

$$\frac{1}{2} [V_{s_n} + V_{s_n}^*] \geq \cos(s_n c) I.$$

If s_n is chosen sufficiently small, then the right-hand side is positive. Consequently we get

$$\frac{1}{4} [V_{s_n} + V_{s_n}^*]^2 \geq \cos^2(s_n c) I.$$

Therefore,

$$I - \frac{1}{4} [V_{s_n} + V_{s_n}^*]^2 \leq \sin^2(s_n c) I.$$

Now, from the fact that $\|V_{s_n}\| \leq 1$ it follows that $V_{s_n} V_{s_n}^* \leq I$ and $V_{s_n}^* V_{s_n} \leq I$ and from this it follows immediately that

$$\frac{i^2}{4} [V_{s_n} - V_{s_n}^*]^2 \leq I - \frac{1}{4} [V_{s_n} + V_{s_n}^*]^2 \leq \sin^2(s_n c) I.$$

Since $i/2 [V_{s_n} - V_{s_n}^*]$ is self-adjoint it follows that

$$- \sin(s_n c) I \leq \frac{i}{2} [V_{s_n} - V_{s_n}^*] \leq \sin(s_n c) I.$$

From this we get

$$-\frac{\sin(s_n c)I}{s_n} \leq \frac{i}{2} \left[\frac{V_{s_n} - I}{s_n} + \frac{V_{-s_n} - I}{-s_n} \right] \leq \frac{\sin(s_n c)I}{s_n}.$$

From this inequality it immediately follows that

$$-cI \leq -iW \leq cI,$$

which means that $-iW$ is self-adjoint and coincides with C .

Let $\{F_t\}$ be the canonical resolution of the identity associated with the operator C . Consider the group of unitary operators

$$U_y = \int_{-\infty}^{\infty} e^{iyt} dF_t.$$

Let $g \in \mathfrak{F}_1$ for which

$$U_y g = \int_{-d}^d e^{iyt} dF_t g,$$

where $d > 0$ is a finite number. The set of such elements is dense in \mathfrak{F}_1 and it may easily be verified that

$$U_y g = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} C^n g,$$

where the convergence is taken in the strong sense.

Let us now consider the set of points $x_0 + iy \in Q_s$ for which x_0 is fixed. Consider the function

$$G_{x_0}(\zeta) = \int_{-d}^d e^{\zeta t} dF_t g(x_0 + iy),$$

where $\zeta = x + iy$. The function $G_{x_0}(\zeta)$ coincides with $g(x_0 + iy)$ for $x = 0$. Further,

$$\frac{\partial G_{x_0}(\zeta)}{\partial x} = \int_{-d}^d t e^{\zeta t} dF_t g(x_0 + iy)$$

and

$$\begin{aligned} \frac{\partial G_{x_0}(\zeta)}{\partial y} &= i \int_{-d}^d e^{\zeta t} dCF_t g(x_0 + iy) \\ &= i \int_{-d}^d t e^{\zeta t} dF_t g(x_0 + iy). \end{aligned}$$

This last formula for the partial derivative with respect to y follows immediately if we first integrate by parts, then take the partial derivative, and then integrate by parts again. This shows that $G_{x_0}(\zeta)$ is an analytic function of ζ .

Therefore if $z \pm iy \in Q_s$ we have

$$U_y g(z) = \sum_{n=0}^{\infty} \frac{\partial^n g(z)}{\partial y^n} \frac{y^n}{n!} = g(z + iy).$$

This means that since U_y is bounded and \mathcal{F}_1 is a r. k. s., for any $h \in \mathcal{F}_1$, $U_y h(z) = h(z + iy)$, provided $z \pm iy \in Q_s$. Therefore if $iy \in Q_s$ and $f_z(w) = f_1(w + \bar{z})$, then

$$U_y f_0(z) = (U_y f_0(w), f_z(w))_1 = \overline{U_{-y} f_z(0)} = \overline{f_z(-iy)} = f_1(z + iy).$$

Let us consider now the set $2Q_s$, i.e. the set of all $z \in Q$ such that $z/2 \in Q_s$. Suppose that $z \in 2Q_s$ and $x = \mathcal{R}(z)$, $y = \mathcal{I}(z)$. Let $x_1 = x/2$, $y_1 = y/2$. There exists an x_2 and an integer n such that $|x_2| \leq s^{(1)}$ and $nx_2 = x_1$. If $T_{x_1} = T_{x_2}^n$, then by condition (2) of Theorem 2, Theorem C and our previous remarks, we have

$$f(z) = (T_{x_1} U_{y_1} f_0, T_{x_1} U_{-y_1} f_0)_1 = \int_a^b \int_{-c}^c e^{x t_1 + i y t_2} d(E_{t_1} F_{t_2} f_0, f_0)_1,$$

where, as we have noted before, $\{E_{t_1}\}$ is the resolution of the identity associated with the set $\{T_x\}$ and $\{F_{t_2}\}$ is the canonical resolution of the identity of the operator C . Since E_{t_1} permutes with F_{t_2} , $\{E_{t_1} F_{t_2}\}$ is a resolution of the identity which of course gives rise to a spectral measure $dE_{t_1} F_{t_2}$.

Let us now consider the linear space S of functions of the form

$$(4.3) \quad g(t_1, t_2) = \sum_{k=1}^n \xi_k e^{x_k t_1 + i y_k t_2},$$

where $x_k = z_k + i y_k \in Q$ and $a \leq t_1 \leq b$, $-c \leq t_2 \leq c$. On S set up the linear functional

$$l(g) = \sum_{k=1}^n \xi_k f(z_k).$$

By the integral representation of $f(z)$ given above it is clear that if $g(t_1, t_2)$ is a function of the form (4.3) with $z_k \in 2Q_s$, then $g(t_1, t_2) \geq 0$ implies $l(g) \geq 0$. However, if we allow $s = s^{(1)} + i s^{(2)} \rightarrow 0$, then the set $2Q_s$ approaches the set Q and therefore $l(g)$ is a positive linear functional on all of S .

Let us extend S to the linear space X , which consists of functions of the form (4.3) where now the y_k vary between $-\infty$ and ∞ and x_k is in Q . The positive linear functional satisfies all of the conditions of Lemma 5 and therefore may be extended to a positive linear functional L , defined on all of X . If we set

$$F(z) = L(e^{x t_1 + i y t_2}),$$

it is clear that $F(z)$ coincides with $f(z)$ on Q . Further, $F(z + \bar{w}) \gg 0$ for $\mathcal{R}(z)$, $\mathcal{R}(w)$ in $Q/2$.

It remains to prove that $F(z)$ is a continuous function of z . First of all we know that for z in Q , $F(z)(=f(z))$ is continuous. Let \mathfrak{F}_2 be the r. k. s. associated with $F(z+\bar{w})$. Then if $x \in Q$ and δ_1 and δ_2 are sufficiently small,

$$\begin{aligned} & |F(x + iy + \delta_1 + i\delta_2) - F(x + iy)| \\ &= |(F(w + x/2), F(w + x/2 - iy + \delta_1 - i\delta_2) - F(w + x/2 - iy))_2| \\ &\leq \|F(w + x/2)\|_2 \|F(w + x/2 - iy + \delta_1 - i\delta_2) - F(w + x/2 - iy)\|_2 \\ &= (f(x))^{1/2} (f(x + 2\delta_1) + f(x) - 2\Re(f(x + \delta_1 + i\delta_2)))^{1/2}. \end{aligned}$$

As $\delta_1 + i\delta_2 \rightarrow 0$, the right-hand side goes to zero which proves our result.

Now, by the definition of the linear functional L , if $g(t_1, t_2) \geq 0$ for $a \leq t_1 \leq b$ and $-c \leq t_2 \leq c$, then $L(g) \geq 0$. Consequently, since for an r suitably small

$$\begin{aligned} & \sum_{j,k=1}^n \xi_j \bar{\xi}_k [F(z_k + \bar{z}_j + r) - e^{ra} F(z_k + \bar{z}_j)] \\ &= L\left((e^{rt_1} - e^{ra}) \left| \sum_{k=1}^n \xi_k e^{x_k t_1 + i y_k t_2} \right|^2\right), \end{aligned}$$

if $t_1 \geq a$, the right side and therefore the left side is positive. Similarly, one shows that

$$F(z + \bar{w} + r) \ll e^{rb} F(z + \bar{w})$$

and that there exists a sequence $s_n \rightarrow 0$ such that

$$F(z + \bar{w} - is_n) + F(z + \bar{w} + is_n) \gg 2 \cos(s_n c) F(z + \bar{w}).$$

Therefore, by Theorem 1 there exists a spectral measure dE_{t_1, t_2} such that

$$F(z) = \int_{-c}^c \int_a^b e^{x t_1 + i y t_2} d(E_{t_1, t_2} F_0, F_0).$$

This concludes the proof of our theorem.

Using the notation of §2, suppose that Q' is an open convex set in Z_{m+n} which contains the origin and with the further conditions that $\mathcal{R}(z)$ fills out an interval in E_m as z runs over Q' and $z \in Q'$ implies $z - 2iy^{(k)}u_k \in Q'$ for $k = m+1, \dots, m+n$. The set Q' is then symmetric in each component of $\mathfrak{F}(z)$. $Q'/2$ shall be the set of all z in Q' such that $2z \in Q'$. If we let $a = (a^{(1)}, \dots, a^{(m)}, -c^{(m+1)}, \dots, -c^{(m+n)})$, $b = (b^{(1)}, \dots, b^{(m)}, c^{(m+1)}, \dots, c^{(m+n)})$, $t = (t^{(1)}, \dots, t^{(m+n)})$, $s_r = (s_r^{(1)}, \dots, s_r^{(m+n)})$ where $a^{(k)} \leq b^{(k)}$, $a^{(k)}$ may take on the value $-\infty$, $b^{(k)}$ the value $+\infty$, and $c^{(k)} \geq 0$ and is finite, then we have the following theorem.

THEOREM 3. *Let $f(z)$ be a continuous complex-valued function defined on Q' . Necessary and sufficient conditions that there exists a bounded non-negative measure $d\alpha(t)$ such that*

$$f(z) = \int_a^b e^{z \cdot t} d\alpha(t)$$

are:

(1) $f(z + \bar{w}) \gg 0$ for $z, w \in Q'/2$.

There exists a sequence $s_r \rightarrow 0$ such that:

(2) For $k=1, \dots, m$ and $r=1, 2, \dots, s_r^{(k)} > 0$ and

$$\exp(s_r^{(k)} a^{(k)}) f(z + \bar{w}) \ll f(z + \bar{w} + s_r^{(k)} u_k) \ll \exp(s_r^{(k)} b^{(k)}) f(z + \bar{w})$$

for all z and w in $Q'/2$ for which the functions are defined.

(3) For $k=m+1, \dots, m+n$ and $r=1, 2, \dots, s_r^{(k)} \neq 0$ and

$$f(z + \bar{w} - i s_r^{(k)} u_k) + f(z + \bar{w} + i s_r^{(k)} u_k) \gg 2 \cos(s_r^{(k)} c^{(k)}) f(z + \bar{w})$$

for all z and w in $Q'/2$ for which the functions are defined.

The necessity of these conditions is immediate. The proof of sufficiency will proceed exactly along the same lines as the proof of Theorem 2. However, since here $c^{(k)} < \infty$, condition (3) tells us that all of the differential operators

$$C_{u_k} g(z) = -i \frac{\partial g(z)}{\partial y^{(k)}}, \quad k = m+1, \dots, m+n,$$

are bounded and therefore permute with one another. In the case where these operators are unbounded we have been unable as yet to prove the permutability. However, we think it is very likely that Theorem 3 is true even when some of the $c^{(k)}$ are unbounded.

5. A problem of moments. By making use of methods similar to some of the methods initiated in the last section we can prove the following theorem.

THEOREM 4. Let $\{\mu_{m,n}\}$ be a sequence of numbers defined for $m=0, 1, 2, \dots$ and $n=0, \pm 1, \pm 2, \dots$. A necessary and sufficient condition that there exists a non-negative bounded measure $d\alpha(t_1, t_2)$ which is zero outside the strip $-\infty < t_1 < \infty, -\pi \leq t_2 \leq \pi$ and such that

$$\mu_{m,n} = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} t_1^m e^{i n t_2} d\alpha(t_1, t_2)$$

is

$$\mu_{m_1+m_2, n_1-n_2} \gg 0.$$

In order that the measure $d\alpha(t_1, t_2)$ be uniquely determined it is sufficient that the moment problem corresponding to the sequence $\{\mu_{m,0}\}$ be determined (i.e. there exists a unique non-negative measure $d\beta(t)$ such that $\mu_{m,0} = \int_{-\infty}^{\infty} t^m d\beta(t)$).

Proof. Let \mathcal{F} be the r. k. s. corresponding to the r. k. $\mu_{m_1+m_2, n_1-n_2}$. The operators

$$U_s \cong \mu_{m_1+m_2, n_1-n_2-s}, \quad s = 0, \pm 1, \dots,$$

clearly form a group of unitary operators with $U_s^* = U_{-s}$. Let us also consider the operator

$$T \cong \mu_{m_1+m_2+1, n_1-n_2}.$$

By Theorem B of §2 we know that T^* and T^{**} exist and $T^{**} = T$. Therefore T^* is a closed symmetric operator. Further by the same methods as used in §4 we see that the operator given by

$$Jg(m, n) = \bar{g}(m, -n)$$

is a conjugation operator and that J permutes with T . Therefore J permutes with T^* which means that T^* has deficiency index of the form (m, m) .

Let V be the Cayley transform of T^* . The deficiency spaces of T^* are given by $\mathcal{K}^+ = \mathcal{F} \ominus \mathcal{D}_V$ and $\mathcal{K}^- = \mathcal{F} \ominus \mathcal{R}_V$. \mathcal{K}^+ and \mathcal{K}^- consist respectively of elements which satisfy the respective equations

$$Tg = ig \quad \text{and} \quad Th = -ih,$$

that is to say elements of the form

$$g(m, n) = i^m g(0, n) \quad \text{and} \quad h(m, n) = (-i)^m h(0, n).$$

Conversely, it is clear that any elements of this form which belong to \mathcal{F} belong respectively to \mathcal{K}^+ and \mathcal{K}^- .

Let us now show that if $i^m g(0, n) \in \mathcal{K}^+$, then $i^m \bar{g}(0, -n) \in \mathcal{K}^+$ and both of these elements have the same norm. It will then follow by a similar argument that if $(-i)^m h(0, n) \in \mathcal{K}^-$, then $(-i)^m \bar{h}(0, -n) \in \mathcal{K}^-$. First of all let us show that \mathcal{K}^+ reduces any operator U_s . It is clear that T^* and U_s permute on \mathcal{F} . Since T^* is the closure of its restriction to \mathcal{F} it follows immediately that $U_s T^* \subseteq T^* U_s$. From this it follows that $U_s V \subseteq V U_s$. This means that U_s takes \mathcal{D}_V into itself. Further, since it is clear that U_s takes \mathcal{K}^+ into itself, U_s may be written as the direct sum of two operators whose domains are respectively \mathcal{D}_V and \mathcal{K}^+ . Therefore if P is the projection operator whose range is \mathcal{K}^+ , we have $U_s P = P U_s$.

Let $K((m, n), (r, s))$ be the r. k. of \mathcal{K}^+ and $\{\phi_k\}$ a complete orthonormal system in \mathcal{K}^+ . Then

$$K((m, n), (r, s)) = \sum_k \phi_k(m, n) \bar{\phi}_k(r, s) = i^{m-r} \sum_k \phi_k(0, n) \bar{\phi}_k(0, s).$$

On the other hand

$$\begin{aligned} K((m, n), (r, s)) &= P \mu_{m+r, n-s} = P U_{-s} \mu_{m+r, n} \\ &= U_{-s} P \mu_{m+r, n} = U_n U_{-s} P \mu_{m+r, 0} \\ &= U_{n-s} P \mu_{m+r, 0}. \end{aligned}$$

This means

$$K((m, n), (r, s)) = i^{m-r} K'(n-s),$$

where $K'(n-s) = K((0, n), (0, s))$. Therefore K is the product of two positive matrices. If \mathcal{H}_1^+ is the r. k. s. corresponding to i^{m-r} and \mathcal{H}_2^+ the r. k. s. corresponding to K' , then \mathcal{H}^+ is the direct product of \mathcal{H}_1^+ and \mathcal{H}_2^+ (see N. Aronszajn [2]). The elements of \mathcal{H}_2^+ are the restriction of the elements of \mathcal{H}^+ to the set consisting of points of the form $(0, n)$. Therefore, if $i^m g(0, n) \in \mathcal{H}^+$, then $g(0, n) \in \mathcal{H}_2^+$ and $\bar{g}(0, -n) \in \mathcal{H}_2^+$ with $\|g(0, n)\|_2 = \|\bar{g}(0, -n)\|_2$. Consequently,

$$\|i^m g(0, n)\| = \|i^m\|_1 \|g(0, n)\|_2 = \|i^m\|_1 \|\bar{g}(0, -n)\|_2 = \|i^m \bar{g}(0, -n)\|.$$

Define the linear operator V' with domain \mathcal{H}^+ and range \mathcal{H}^- by the formula

$$V' i^m g(0, n) = e^{i\theta} (-i)^m g(0, n),$$

where θ is any real number. It is clear that V' is linear. Further, since

$$\|i^m g(0, n)\| = \|J i^m g(0, -n)\| = \|(-i)^m \bar{g}(0, -n)\| = \|(-i)^m g(0, n)\|,$$

it is clear that V' is an isometric operator and therefore $U = V \oplus V'$ is a unitary operator. The operator U gives rise to a self-adjoint extension of T^* whose Cayley transform is U . Therefore, if the deficiency index is not $(0, 0)$ there exist an infinite number of self-adjoint extensions of T^* of the type we have described. Clearly, if H is a self-adjoint extension of T^* , then $H \subseteq T$.

Suppose that H is a self-adjoint extension of T^* of the type we have described. Every operator U_s permutes with H . In order to show this it is sufficient to show that U_s permutes with U . However, we have already shown that U_s permutes with V and since clearly U_s permutes with V' , the result follows.

If $\{E_{t_1}\}$ is the canonical resolution of the identity associated with H and $\{F_{t_2}\}$ the canonical resolution of the identity associated with U_1 , then by the permutability of U_1 and H , it follows that $E_{t_1, t_2} = E_{t_1} F_{t_2}$ is a projection and the family $\{E_{t_1, t_2}\}$ is a resolution of the identity. Therefore if $f_0 = \mu_{m+0, n-0}$, we have

$$\mu_{m, n} = (H^m U_n f_0, f_0) = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} t_1^m e^{i n t_2} d(E_{t_1, t_2} f_0, f_0).$$

This completes the proof of representation.

To prove the statement about the uniqueness of the measure $d\alpha(t_1, t_2)$ let us suppose that the moment problem associated with the sequence $\{\mu_{m, 0}\}$ is determined. Let \mathcal{H}_1 be the r. k. s. associated with the r. k. $\mu_{m+r, 0}$. The operator

$$T_1 \cong \mu_{m+r+1, 0}$$

must be self-adjoint, for otherwise the moment problem associated with $\{\mu_{m,0}\}$ cannot have a unique solution. Therefore, the function i^m cannot belong to \mathcal{F}_1 . This means the space \mathcal{K}^+ consists only of the zero element. For suppose to the contrary that there exists an element $g(m, n)$ in \mathcal{K}^+ which is not identically zero. Then there exists an n_0 such that $g(0, n_0) \neq 0$. Now the restrictions of the elements of \mathcal{F} to the points (m, n_0) belong to \mathcal{F}_1 . Therefore $g(m, n_0) = i^m g(0, n_0) \in \mathcal{F}_1$ which means that $i^m \in \mathcal{F}_1$. This gives a contradiction. Therefore the operator T must be self-adjoint. By the same method as used in IRI Lemma 1, the self-adjointness of T implies the uniqueness of $d\alpha(t_1, t_2)$.

We take this opportunity to note the following errata in IRI in addition to that published in this journal vol. 74 (1953) p. 536:

- p. 64, line 8. Omit " $s_r^{(k)} d^{(k)} < \pi$."
- p. 65, line -9. For " \mathcal{L}_0 " read " \mathcal{L}_0^\perp ."
- p. 70, line -4. Omit " $y_n d \leq \pi$."
- p. 72, line 4. For " $x \cdot u_k$ " read " $x^{(k)} u_k$."
- pp. 72, 73. For " $T_{x \cdot u_j}$," " $T_{x \cdot u_k}$ " and " $U_{y \cdot u_k}$ " read resp. " $T_{x^{(j)} u_j}$," " $T_{x^{(k)} u_k}$ " and " $U_{y^{(k)} u_k}$ " wherever they appear.

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