

# NORMAL AUTOMORPHISMS AND THEIR FIXED POINTS

BY

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**1. Introduction.** The elements of the centralizer  $T$  of the group of inner automorphisms  $J$  of a group  $G$  (in the group  $A$  of automorphisms of  $G$ ) are called the normal automorphisms of  $G$ . The center  $Z$  of  $G$  is the set of all elements of  $G$  which are fixed by each mapping from  $J$ . Likewise, let  $B$  be the set of fixed points held in common by the mappings from  $T$ .  $G/B$  is abelian, and the elements of  $T$  which induce either the identity or the involution on  $G/B$  form a subgroup  $W$  of  $T$ . We shall investigate the ascending central series of  $W$ . Just as the ascending central series  $\{Z_i\}$  is formed over  $Z=Z_1$ , so an ascending series is formed over  $B$ . Elements of  $G$  lying in members of this  $B$ -series turn out to be fixed points for high powers of normal automorphisms. For automorphisms which induce the identity on  $G/Z_n$ , we show that the common fixed points lie in the centralizer of  $Z_n$  in  $G$ .

The notation is both obvious and conventional.  $G$  denotes a group with automorphism group  $A$  and inner automorphism group  $J$ . The members of the ascending central series of  $G$  are the  $Z_i$ , and the higher commutator subgroups are the  $G^{(i)}$  [3]. If, say, the inner automorphism group of a group  $H$ , different from  $G$ , is to be denoted, we employ the symbols  $J(H)$ , and similarly for other groups or subgroups, such as  $Z_i(H)$ , associated with  $H$ . For a subgroup  $H$  of  $G$ , the centralizer of  $H$  in  $G$  will be denoted by  $C(H; G)$ . If  $x, y \in G$ , then  $(x, y) = x^{-1}y^{-1}xy$ . For normal subgroups  $S$  and  $T$  of a group  $G$ ,  $S \div T$  (following R. Baer) will be the *commutator quotient* of  $S$  by  $T$ , the set of all  $x \in G$  such that  $(x, t) \in S$  for every  $t \in T$ .  $S \div T$  is a normal subgroup of  $G$ . The identity map on a group is indicated by  $\iota$ , and the identity element of a group is to be  $e$ . For a homomorphism  $f$  on  $G$ , the kernel will be written kern  $f$ .  $\oplus$  denotes direct summation of groups. A *periodic group* is one in which each element is of finite order, and an abelian periodic group will be called a *torsion group*. If a periodic group  $G$  has a uniform order on its elements, then  $G$  is said to be *uniform torsion* (*u.t.*), and the least positive uniform order will be called [3] the *exponent* of  $G$ . A group is said to be *torsion-free* if it has no nontrivial elements of finite order. A *complete group* is one in which the  $x^n$  form a set of generators of  $G$  for each positive integer  $n$ . The group of integers is to be  $I$ ; the group of rationals,  $R$ ; the multiplicative group of nonzero rationals,  $R^*$ ; and  $I_n$  is to be the group of integers modulo  $n$ .

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Occasionally, we shall give  $I_n$  its usual representation as the group of integral residue classes, modulo  $n$ , so that  $j_n$  will be the residue class, modulo  $n$ , in which the integer  $j$  lies.

**2. Normal automorphisms.** Let  $G$  be a group, and let  $H$  be a normal subgroup of  $G$ . Let  $\alpha$  be an automorphism of  $G$  such that  $H$  is admissible under both  $\alpha$  and  $\alpha^{-1}$ . That is,  $\alpha(H) \subset H$  and  $\alpha^{-1}(H) \subset H$ . Then  $\alpha$  induces an automorphism  $\alpha'$  on  $G/H$  given by  $\alpha'(xH) = \alpha(x)H$ . In particular, if  $H$  is a characteristic subgroup of  $G$ , then every automorphism  $\alpha$  of  $G$  induces an automorphism  $\alpha'$  on  $G/H$ . If  $\alpha$  induces the identity automorphism  $\iota$  on  $G/Z_1$ ,  $\alpha$  is called a *normal automorphism* (sometimes *center* [3] or *central* [1] automorphism) of  $G$ , and  $\alpha(x) \equiv x \pmod{Z_1}$  for every  $x \in G$ . It is easy to see that  $\alpha \in A$  is normal if, and only if,  $(x, \alpha(y)) = (x, y)$  for every  $x, y \in G$ . Let  $T_1$  be the set of all normal automorphisms of  $G$ . If automorphism composition is interpreted as a multiplication,  $T_1$  becomes a subgroup of the automorphism group  $A$  of  $G$  with  $\iota$  as its identity, and  $T_1$  is normal in  $A$ . It is well known that  $T_1 = C(J; A)$  [3]. An endomorphism  $\gamma$  of  $G$  for which  $\gamma(G) \subset Z_1$  is called a *central endomorphism*. If  $\alpha \in T_1$ ,  $\alpha(x) = x\gamma(x)$  for every  $x \in G$ , where  $\gamma$  is a central endomorphism of  $G$  with the further property (A) that to each  $y \in G$ , there exists a unique  $g = g(y; \gamma) \in G$  with  $\gamma(g) = g^{-1}y$ . We might write  $\alpha = \iota + \gamma$ . If, conversely,  $\gamma$  is a central endomorphism with (A), then the mapping  $\alpha$ , defined by  $\alpha = \iota + \gamma$ , is in  $T_1$ .

Let  $G$  be a group for which  $Z_2 \neq Z_1$ . If  $u \in Z_2$ ,  $u \notin Z_1$ , then the mapping  $\gamma_u$  given by  $\gamma_u(x) = (x, u)$  is readily seen to be a central endomorphism of  $G$ . These  $\gamma_u$  will be called the *Grün endomorphisms* of  $G$ . If  $y \in G$ , then  $\gamma_u(uyu^{-1}) = (uyu^{-1})^{-1}y$  so that  $\gamma_u(x) = x^{-1}y$  has the solution  $x = uyu^{-1}$ . If, conversely,  $x$  is any solution of  $\gamma_u(x) = x^{-1}y$ , then  $u^{-1}xu = y$  so that  $x = uyu^{-1}$ ; and the solution is unique, establishing (A). Hence  $\alpha_u$ , a mapping defined by  $\alpha_u(x) = x\gamma_u(x) = u^{-1}xu = \tau_u(x)$ , is in  $T_1$ . Suppose, conversely, that  $\alpha \in T_1 \cap J$ . Let  $\alpha = \tau_u$ . Then  $u^{-1}xu \equiv x \pmod{Z_1}$  for every  $x \in G$ , so that  $u \in Z_2$  and  $\alpha = \alpha_u$ . We state

**LEMMA 1.**  $T_1 \cap J \cong Z_1(J)$ , and the elements of the former are in one-to-one correspondence with the Grün endomorphisms of  $G$ .

For endomorphisms  $\gamma_\alpha$  and  $\gamma_\beta$  satisfying (A), note that  $\gamma_\alpha(G) \subset \gamma_\beta(G)$  implies  $\gamma_{\alpha\beta}(G) \subset \gamma_\beta(G)$ , so that  $\gamma_{\alpha^n}(G) \subset \gamma_\alpha(G)$  ( $n = 1, 2, 3, \dots$ ).  $\gamma_\iota$  is the trivial endomorphism ( $\gamma_\iota(x) = e$  for every  $x \in G$ ), and  $\gamma_\iota(G) \subset \gamma_\alpha(G)$  for every  $\alpha \in T_1$ .

**LEMMA 2.** For  $\alpha \in T_1$ ,  $\alpha(\gamma_\alpha(G)) = \gamma_\alpha(G) = \alpha(\gamma_{\alpha^{-1}}(G))$ .

**Proof.** For  $x \in G$ ,  $\alpha(\alpha(x)) = \alpha(x)\alpha(\gamma_\alpha(x))$ , so that  $\alpha(\gamma_\alpha(x)) \in \gamma_\alpha(G)$ . Hence  $\alpha(\gamma_\alpha(G)) \subset \gamma_\alpha(G)$ .  $\alpha^{-1}(x) = x\gamma_{\alpha^{-1}}(x)$  implies that  $x = \alpha(x)\alpha(\gamma_{\alpha^{-1}}(x)) = x\gamma_\alpha(x)\alpha(\gamma_{\alpha^{-1}}(x))$ , whence  $\gamma_\alpha(x) = \alpha(\gamma_{\alpha^{-1}}(x^{-1}))$ ; and  $\gamma_\alpha(G) \subset \alpha(\gamma_{\alpha^{-1}}(G))$ . Replacing  $x$  by  $x^{-1}$ , we have  $\gamma_\alpha(x^{-1}) = \alpha(\gamma_{\alpha^{-1}}(x))$ . There exists  $y \in G$  such that  $\alpha(y) = \gamma_\alpha(x^{-1}) = y\gamma_\alpha(y)$ . Since  $\gamma_\alpha(G)$  is a subgroup of  $G$ ,  $y \in \gamma_\alpha(G)$ . Thus,

$\alpha(\gamma_{\alpha^{-1}}(x)) = \alpha(y)$  where  $y \in \gamma_{\alpha}(G)$ , so that  $\alpha(\gamma_{\alpha^{-1}}(G)) \subset \alpha(\gamma_{\alpha}(G))$ .

**3. The common fixed points.** The subgroup  $Z_1$ , the center of  $G$ , is the set of all elements of  $G$  which are fixed by each inner automorphism  $\tau_y$  of  $G$ . For  $T_1 = C(J; A)$ , the set analogous to  $Z_1$  is  $B_1$ , where  $x \in B_1$  if, and only if,  $\alpha(x) = x$  for every  $\alpha \in T_1$ . If  $F(\alpha)$  is the set of the fixed points of  $\alpha \in T_1$ , then  $F(\alpha)$  is a normal subgroup of  $G$ . Since  $B_1 = \bigcap F(\alpha)$ , where the cross-cut is taken over all  $\alpha \in T_1$ ,  $B_1$  is likewise a normal subgroup of  $G$ . Now  $F(\alpha) = \text{kern } \gamma_{\alpha}$ , and  $\gamma_{\alpha}(G)$  is abelian. Thus  $F(\alpha) \supset G'$ , the derivative of  $G$ , for every  $\alpha \in T_1$ , and  $B_1 \supset G'$ . This shows that  $G/B_1$  is abelian and that if  $B_1 = (e)$ , then  $G$  is abelian.

LEMMA 3.  $G' \subset B_1 \subset C(Z_2; G)$ .

**Proof.** If  $x \in B_1$ ,  $\gamma_u(x) = e$  for every Grün endomorphism  $\gamma_u$ ,  $u \in Z_2$ . Consequently  $x$  commutes with every such  $u$ .

COROLLARY. If  $C(Z_2; G) = Z_1$ , then  $G$  is of class 2.

Suppose that  $H$  is a characteristic subgroup of  $G$ , that  $J(H; G)$  is the set of all inner automorphisms  $\tau_v$  of  $G$  where  $v \in H$  (where  $\tau_v(x) = v^{-1}xv$ ), and that  $Q(H; G) = Q(H)$  is the set of all automorphisms of  $G$  such that  $\alpha \in Q(H)$  induces the identity automorphism on  $G/H$ . For instance,  $J(G; G) = J$ , and  $Q(Z_1; G) = T_1$ .  $J(H; G)$  is a normal subgroup of  $Q(H; G)$ . Let  $F = F(Q(H; G))$  be the fixed points common to all mappings in  $Q(H; G)$ , and let  $F^* = F(J(H; G))$  be the fixed points common to all mappings in  $J(H; G)$ .  $F^* \supset F$ . But  $F^* = C(H; G)$ , so that  $F(Q(H; G)) \subset C(H; G)$ . This general result will be used later to establish a variation of Lemma 3.

$G = B_1$  if, and only if,  $G$  has no proper normal automorphisms. By Lemma 3,  $G = B_1$  implies  $G = C(Z_2; G)$  so that every element of  $G$  commutes with every element of  $Z_2$ , and  $Z_2 \subset Z_1$ . Hence the ascending central series of  $G$  breaks off with  $Z_1$  if  $G = B_1$ . Likewise, Lemma 3 has the following obvious

COROLLARY.  $G$  is of class 2 if, and only if,  $B_1 \subset Z_1$ .

In particular,  $G' \subset Z_1$  if, and only if,  $B_1 \subset Z_1$ .

LEMMA 4. (a) If  $T_1$  is finite and if  $Z_1$  is a torsion group, then  $G/B_1$  is a torsion group. (b) If  $Z_1$  is u.t., then  $G/B_1$  is u.t. and  $\exp G/B_1 \mid \exp Z_1$ . (c) If  $Z_1$  is torsion-free, then so is  $G/B_1$ .

**Proof.** (a) For  $x \in G$  and  $\alpha \in T_1$ ,  $\alpha(x) = x\gamma_{\alpha}(x)$ , where  $\gamma_{\alpha}(x) \in Z_1$ . There exists a least positive integer  $n = n(x; \alpha)$  such that  $\gamma_{\alpha}(x^n) = e$ , since  $Z_1$  is a torsion group. Since  $T_1$  is finite, we can form  $n(x)$ , the least common multiple of all such  $n(x; \alpha)$ . For  $\alpha \in T_1$ ,  $\alpha(x^{n(x)}) = x^{n(x)}$  so that  $x^{n(x)} \in B_1$ , and  $G/B_1$  is a torsion group. (b) has a proof which is an obvious modification of the proof of (a). (c) Suppose that  $Z_1$  is torsion-free and that  $x^n \in B_1$ . Then for  $\alpha \in T_1$ ,  $\alpha(x^n) = x^n$ . But  $\alpha(x) = x\gamma_{\alpha}(x)$ , so that  $\gamma_{\alpha}(x^n) = e$ . Since  $\gamma_{\alpha}(x)$  is not a periodic

element,  $\gamma_\alpha(x) = e$  and  $x \in B_1$ . Hence  $G/B_1$  is torsion-free.

**LEMMA 5.** *If  $G/B_1$  is complete, then each  $\gamma_\alpha(G)$  is complete; and if, in addition,  $Z_1$  is torsion-free,  $G/B_1$  and each  $\gamma_\alpha(G)$  are direct sums of copies of  $R$ , the additive group of the rationals.*

**Proof.** For  $z \in \gamma_\alpha(G)$ , there exist  $x \in G$  and  $\alpha \in T_1$  with  $\alpha(x) = xz$ . Since  $G/B_1$  is complete, for each positive integer  $n$  there exists  $y \in G$  with  $x \equiv y^n \pmod{B_1}$ .  $\alpha(y^n x^{-1}) = y^n x^{-1} = y^n \gamma_\alpha(y^n) x^{-1} z^{-1} = y^n x^{-1} \gamma_\alpha(y^n) z^{-1}$ . Hence  $\gamma_\alpha(y^n) = z$ , and  $[\gamma_\alpha(y)]^n = z$ . Since  $\gamma_\alpha(G)$  is abelian, this is enough to show that it is complete. If, in addition,  $Z_1$  is torsion-free, then Lemma 4(c) shows that  $G/B_1$  is torsion-free. Also each  $\gamma_\alpha(G)$  is torsion-free. But torsion-free, complete abelian groups are direct sums of copies of  $R$ .

**4. Automorphisms induced on  $G/B_1$ .** Since  $B_1$  is admissible under each normal automorphism of  $G$ , each such automorphism induces an automorphism on  $G/B_1$ . (See, however, [1] where  $G/G'$  for finite  $G$  is discussed instead.) If  $\alpha \in T_1$  induces the identity on  $G/B_1$ , then  $\alpha(x) = x\gamma_\alpha(x) \equiv x \pmod{B_1}$  so that  $\gamma_\alpha(G) \subset B_1$ . Conversely, if  $\alpha \in T_1$  and if  $\gamma_\alpha(G) \subset B_1$ , then the induced automorphism  $\alpha'$  has the property  $\alpha'(xB_1) = \alpha(x)B_1 = x\gamma_\alpha(x)B_1 = xB_1$  for every  $xB_1 \in G/B_1$ . Thus, a necessary and sufficient condition that  $\alpha \in T_1$  induce  $\iota$  on  $G/B_1$  is that  $\gamma_\alpha(G) \subset B_1$ . Let the set of all such  $\alpha \in T_1$  be denoted by  $V_1$ . By a well known result [3, p. 78] on automorphisms which leave a normal subgroup  $H$  and the factor group  $G/H$  point-wise fixed,  $V_1$  is an abelian group under automorphism composition.  $V_1$  is a normal subgroup of  $T_1$ . For, if  $\alpha \in V_1$ ,  $\beta \in T_1$ , then  $\beta^{-1}\alpha\beta(x) = \beta^{-1}\alpha(x\gamma_\beta(x)) = \beta^{-1}(x\gamma_\beta(x)bc) = xbc$  where  $b, c \in B_1$ , and  $\alpha(x) = xb$ ,  $\alpha\gamma_\beta(x) = \gamma_\beta(x)c$ . This makes  $V_1$  normal in  $T_1$ . Moreover,  $\alpha^{-1}\beta^{-1}\alpha\beta(x) = xc$ , and we have

**LEMMA 6.** *If  $\alpha \in V_1$  and if  $\beta \in T_1$ , then  $\gamma_{(\alpha,\beta)} = \gamma_\alpha\gamma_\beta$ .*

Since  $G/B_1$  is an abelian group, it has the automorphism  $\omega$  given by  $\omega(y) = y^{-1}$  for every  $y \in G/B_1$ .  $\omega^2 = \iota$ , and  $\omega$  is called the *involution automorphism*.

**LEMMA 7.** (a) *If  $\alpha \in T_1$  induces the involution automorphism on  $G/B_1$ , then  $\alpha$  induces the involution automorphism on  $\gamma_\alpha(G)$ .* (b)  *$\alpha \in T_1$  induces  $\omega$  on  $G/B_1$  if, and only if,  $\gamma_\beta\gamma_\alpha(x) = \gamma_\beta(x^{-2})$  for every  $x \in G$  and for every  $\beta \in T_1$ .*

**Proof.** (a) For  $x \in G$ ,  $\alpha(x) = x\gamma_\alpha(x) \equiv x^{-1} \pmod{B_1}$  so that  $x^2\gamma_\alpha(x) \in B_1$  and  $\gamma_\alpha(x^2)\gamma_\alpha(\gamma_\alpha(x)) = e$ . Thus  $\gamma_\alpha(x)\alpha(\gamma_\alpha(x)) = e$ , so that  $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(x^{-1})$ , and  $\alpha$  induces  $\omega$  on  $\gamma_\alpha(G)$ . (b)  $\alpha$  induces  $\omega$  on  $G/B_1$  if, and only if,  $x^2\gamma_\alpha(x) \in B_1$  for every  $x \in G$ . For  $\beta \in T_1$ ,  $\gamma_\beta(x^2\gamma_\alpha(x)) = e$  so that  $\gamma_\beta\gamma_\alpha(x) = \gamma_\beta(x^{-2})$ . Conversely, if  $\gamma_\beta\gamma_\alpha(x) = \gamma_\beta(x^{-2})$  for every  $\beta \in T_1$ , then  $x^2\gamma_\alpha(x) \in B_1$ .

If we let  $W_1$  be the set of all  $\alpha \in T_1$  which induce either  $\iota$  or  $\omega$  on  $G/B_1$ , then  $W_1$  is a group under automorphism composition. Let the set of those elements of  $W_1$  which are not in  $V_1$  be denoted by  $W_1^*$ . Assume, for the pres-

ent, that this set is nonvoid. It is easy to verify that the elements of  $W_1^*$  are carried into elements of  $W_1^*$  by the inner automorphisms of the group  $T_1$ , so that  $W_1$  is a normal subgroup of  $T_1$ . The index  $[W_1: V_1] = 2$ , and  $W_1/V_1 \cong I_2$ ; for, if  $\alpha, \beta \in W_1^*$ , then  $\alpha^{-1}\beta(x) = \alpha^{-1}(x^{-1}b) = \alpha^{-1}(x^{-1})b$ , where  $b \in B_1$ . Since  $\alpha(x) = x^{-1}c$  (where  $c \in B_1$ ),  $\alpha^{-1}(x^{-1}) = xc^{-1}$ , and  $\alpha^{-1}\beta(x) \equiv x \pmod{B_1}$  so that  $\alpha^{-1}\beta \in V_1$ .

**THEOREM 1.** (a) *For a group  $G$ ,  $W_1$  is  $j$ -nilpotent for a given positive integer  $j$ , or  $Z_j(W_1)$  is included properly in  $V_1$ .* (b) *If  $W_1^*$  is nonvoid, then  $\alpha \in Z_j(W_1) \cap V_1$  if, and only if,  $\gamma_\alpha(x^{2^j}) = e$  for every  $x \in G$  and  $\alpha \in V_1$ .*

**Proof.** If  $W_1^*$  is void, then  $V_1 = W_1$  and  $W_1$  is abelian. Let us therefore assume that  $W_1^*$  is nonvoid. First suppose that  $j = 1$ , and consider  $\alpha \in V_1 \cap Z_1(W_1)$ . Choose  $\beta \in W_1^*$ . Then since  $\alpha \in Z_1(W_1)$ ,  $(\alpha, \beta) = \iota$ . Now  $\gamma_{(\alpha, \beta)}(x) = \gamma_\alpha \gamma_\beta(x) = \gamma_\alpha(x^{-2})$ , by Lemmas 6 and 7(b). Since  $\gamma_\iota(x) = e$  for every  $x \in G$ ,  $\gamma_\alpha(x^{-2}) = e$  for every  $x \in G$ . Conversely, if  $\gamma_\alpha(x^2) = e$  for every  $x \in G$ , then  $\gamma_\alpha \gamma_\beta(x) = \gamma_{(\alpha, \beta)}(x) = e$  for every  $x \in G$  and for every  $\beta \in W_1^*$ , by Lemmas 6 and 7(b). But  $\gamma_{(\alpha, \beta)}(x) = e$  for every  $x \in G$  implies that  $(\alpha, \beta) = \iota$  for every  $\beta \in W_1^*$ . Since  $V_1$  is abelian, and since  $\alpha \in V_1$ ,  $\alpha$  is in  $Z_1(W_1)$ . We have verified (b) in the case  $j = 1$ .

Suppose that there exists  $\beta \in Z_1(W_1) \cap W_1^*$ . Since  $[W_1: V_1] = 2$ ,  $\beta \in Z_1(W_1)$  if, and only if,  $W_1^* \subset Z_1(W_1)$ . If  $\alpha \in V_1$ ,  $\beta \in Z_1(W_1) \cap W_1^*$ , then  $(\alpha, \beta) = \iota$  and  $\gamma_{(\alpha, \beta)}(x) = \gamma_\alpha \gamma_\beta(x) = e$  for every  $x \in G$ , by Lemma 6. By Lemma 7(b),  $\beta \in W_1^*$  implies  $\gamma_\alpha \gamma_\beta(x) = \gamma_\alpha(x^{-2}) = e$ . By (b), which has been established for the case  $j = 1$ ,  $\alpha \in Z_1(W_1)$ . Hence  $W_1 = W_1^* \cup V_1 \subset Z_1(W_1)$ . It follows that if  $W_1 \neq Z_1(W_1)$  then  $Z_1(W_1) \subset V_1$ ; and this inclusion must be strict. For, if not,  $Z_1(W_1) = V_1$  and  $W_1/V_1 \cong J(W_1)$ . Since  $W_1/V_1 \cong I_2$ ,  $J(W_1)$  is cyclic, an impossibility [2]. We have now established (a) for the case  $j = 1$ .

Now suppose that the theorem holds for the case  $j - 1$ . If  $\beta \in W_1^*$  and if  $\alpha \in V_1 \cap Z_j(W_1)$ ,  $(\alpha, \beta) \in Z_{j-1}(W_1)$ . If  $x \in G$ ,  $(\alpha, \beta)(x) = x\gamma_\alpha(\gamma_\beta(x)) = x\gamma_\alpha(x^{-2})$ . Noting that  $(\alpha, \beta) \in V_1$  since  $\gamma_\alpha(x^{-2}) \in B_1$ , (b) can be applied for the case  $j - 1$ , and  $\gamma_\alpha[(x^{2^{j-1}})^{-2}] = e$ , whence  $\gamma_\alpha(x^{2^j}) = e$  for every  $x \in G$ . Conversely, suppose that  $\alpha \in V_1$  and that  $\gamma_\alpha(x^{2^j}) = e$  for every  $x \in G$ . Choose  $\beta \in W_1^*$ .  $\gamma_{(\alpha, \beta)}(y) = \gamma_\alpha \gamma_\beta(y) = \gamma_\alpha(y^{-2})$  for every  $y \in G$ . Let  $y = x^{2^{j-1}}$ . Then  $\gamma_\alpha(y^{-2}) = e$  by assumption, and  $\gamma_{(\alpha, \beta)}(x^{2^{j-1}}) = e$  for every  $x \in G$ . Since  $\alpha \in V_1$  implies  $\gamma_\alpha(y^{-2}) \in B_1$ ,  $\gamma_{(\alpha, \beta)}(y) \in B_1$  and  $(\alpha, \beta) \in V_1$ . By (b) for the case  $j - 1$ ,  $(\alpha, \beta) \in Z_{j-1}(W_1)$  for every  $\beta \in W_1^*$ . If  $\beta \in V_1$ , then the fact that  $V_1$  is abelian allows one to conclude that  $(\alpha, \beta) = \iota \in Z_{j-1}(W_1)$ . Hence  $(\alpha, \beta) \in Z_{j-1}(W_1)$  for every  $\beta \in W_1$ , and  $\alpha \in Z_j(W_1)$ . This establishes (b) for the case  $j$ .

Since  $[W_1: V_1] = 2$ , the elements of  $W_1^*$  all have the form  $\beta\alpha$  where  $\alpha \in V_1$ . Suppose now that  $\beta \in Z_j(W_1) \cap W_1^*$  and that  $\alpha$  and  $\delta$  are elements of  $V_1$ .  $\beta\alpha\beta\delta \equiv \beta\alpha\delta\beta \equiv \beta\delta\alpha\beta \equiv \beta\delta\beta\alpha \pmod{Z_{j-1}(W_1)}$ . Likewise,  $\beta\alpha\delta \equiv \delta\beta\alpha \pmod{Z_{j-1}(W_1)}$ . Hence if  $Z_j(W_1) \cap W_1^*$  is nonvoid, then  $W_1^* \subset Z_j(W_1)$ . If  $\alpha \in V_1$ ,  $\beta \in Z_j(W_1) \cap W_1^*$ , then  $(\alpha, \beta) \in Z_{j-1}(W_1)$  and  $\gamma_{(\alpha, \beta)}(x) = \gamma_\alpha \gamma_\beta(x) = \gamma_\alpha(x^{-2})$ . As above,

$\alpha \in V_1$  implies  $(\alpha, \beta) \in V_1$  so that (b) for the case  $j-1$  applies, and  $\gamma_\alpha[x^{(2^{j-1})}]^{-2} = e$ . Thus (b) for the case  $j$ , established above, places  $\alpha \in Z_j(W_1)$ .  $W_1 = W_1^* \cup V_1 \subset Z_j(W_1)$ , and  $W_1 = Z_j(W_1)$ .

If  $W_1 \neq Z_j(W_1)$ , the above shows that  $Z_j(W_1) \subset V_1$ . If the inclusion is not strict, then  $Z_j(W_1) = V_1$  and  $I_2 \cong W_1/V_1 \cong J(W_1/Z_{j-1}(W_1))$ , an impossibility [2]. This completes the proof of the theorem.

**COROLLARY 1.** *Let  $G$  be a group for which  $W_1^*$  is nonvoid. (a) If  $Z_1$  is u.t. with exponent dividing  $2^j$  (where  $j > 1$ ), then  $W_1$  is nilpotent of class  $\leq j$ . (b) If  $Z_1$  is torsion-free and if  $V_1$  is nontrivial, then  $W_1$  is non-nilpotent.*

**Proof.** (a) For  $\alpha, \zeta \in V_1$ ,  $(\alpha, \zeta) = \iota$  since  $V_1$  is abelian. Choose  $\beta \in W_1^*$ . Since  $[W_1: V_1] = 2$ ,  $W_1^*$  is the coset of  $V_1$  in  $W_1$  which contains  $\beta\alpha$ . For  $x \in G$ ,  $\alpha(x) = xb$ ,  $\zeta(x) = xd$ , and  $\beta(x) = x^{-1}c$ , where  $b, d \in Z_1 \cap B_1$  and  $c \in B_1$ .  $(\beta\alpha, \zeta)(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\beta\alpha\zeta(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\beta(xbd) = \alpha^{-1}\beta^{-1}\zeta^{-1}(x^{-1}cbd) = \alpha^{-1}\beta^{-1}(dx^{-1}cbd) = \alpha^{-1}(dxc^{-1}cbd) = dxb^{-1}bd = xd^2$  so that  $(\beta\alpha, \zeta) = \delta_1 \in V_1$ .  $(\beta\alpha, \beta\zeta)(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha\beta\zeta(x) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha\beta(xd) = \alpha^{-1}\beta^{-1}\zeta^{-1}\alpha(x^{-1}cd) = \alpha^{-1}\beta^{-1}\zeta^{-1}(b^{-1}x^{-1}cd) = \alpha^{-1}\beta^{-1}(b^{-1}dxc^{-1}cd) = \alpha^{-1}(b^{-1}dxc^{-1}cd) = b^{-1}dxb^{-1}d = x(b^{-1}d)^2$ , so that  $(\beta\alpha, \beta\zeta) = \delta_2 \in V_1$ . If  $\exp Z_1 \mid 2^j$ , then  $\delta_i(x^{2^{j-1}}) = x^{2^{j-1}}(b^{1-i}d)^{2^j} = x^{2^{j-1}}$ , and  $\gamma_{\delta_i}(x^{2^{j-1}}) = e$  for every  $x \in G$ . By (b) of the theorem,  $\delta_i \in Z_{j-1}(W_1)$ , so that  $W_1^* \subset Z_{j-1}(W_1)$ , and  $W_1$  is nilpotent of class  $\leq j$ . This establishes (a) of the corollary.

(b) Now suppose that  $Z_1$  is torsion-free. By hypothesis, we can find  $\alpha$  and  $\zeta \in V_1$  and  $y \in G$  with  $\alpha(y) \neq \zeta(y)$ , where  $\alpha(y) = yb$  and  $\zeta(y) = yd$ . Construct  $\delta_i$  as in part (a) of the corollary.  $\delta_i(y^{2^{j-1}}) = y^{2^{j-1}}(b^{1-i}d)^{2^j}$ . Since  $b \neq d$ , we can always adjust our notation so that  $d \neq e$ . Since  $Z_1$  is torsion-free,  $b \neq d$  and  $d \neq e$ ,  $(b^{1-i}d)^{2^j} \neq e$  for each positive integer  $j$ , so that, by (b) of the theorem,  $\delta_i \in Z_{j-1}(W_1)$ , and  $W_1^* \not\subset Z_{j-1}(W_1)$  for all such  $j$ , and  $W_1$  is not nilpotent.

**COROLLARY 2.** *Let  $Z_1$  be torsion-free,  $V_1$  be nontrivial,  $W_1^*$  be nonvoid and let  $G$  be complete. Then  $W_1$  has a trivial center.*

**Proof.** By Corollary 1(b),  $W_1$  is not nilpotent. By (a) of the theorem,  $Z_1(W_1)$  is a proper subgroup of  $V_1$ . By (b) of the theorem,  $\alpha \in Z_1(W_1)$  implies  $x^2 \in F(\alpha)$ , the set of all fixed points of  $\alpha$ , for every  $x \in G$ . Since  $G$  is complete,  $G = F(\alpha)$ , and  $\alpha = \iota$ .

It is fairly obvious that  $\alpha$  and  $\beta \in T_1$  induce the same automorphism on  $G/B_1$  if, and only if,  $\alpha \equiv \beta \pmod{V_1}$ ; and an equivalent condition is that  $\gamma_\alpha(x) \equiv \gamma_\beta(x) \pmod{B_1}$  for every  $x \in G$ . It follows that if  $\alpha \equiv \beta \pmod{V_1}$ , then there exists an endomorphism  $\lambda_{\alpha, \beta}$  on  $G$  into  $B_1 \cap Z_1$  such that (1) the kernel of  $\lambda_{\alpha, \beta}$  is just  $F(\alpha^{-1}\beta) = F(\beta^{-1}\alpha)$ ; (2)  $\gamma_\alpha(x) = \gamma_\beta(x)\lambda_{\alpha, \beta}(x)$ ; and (3) for  $g \in G$ ,  $\lambda_{\alpha, \beta}(x) = \beta(x^{-1})g$  has a unique solution  $x = x(g) \in G$ . Conversely, if  $\lambda$  is an endomorphism of  $G$  into  $B_1 \cap Z_1$ , if  $\beta \in T_1$  and if  $\lambda(x) = \beta(x^{-1})g$  has a unique solution  $x = x(g)$  for every  $g \in G$ , then the mapping  $\alpha$  defined by  $\alpha(x) = \beta(x)\lambda(x)$  is a normal automorphism of  $G$  such that  $\alpha \equiv \beta \pmod{V_1}$  and such that  $\lambda = \lambda_{\alpha, \beta}$ . We restate as follows:

LEMMA 8. If  $\beta \in T_1$  and if  $\lambda$  is an endomorphism of  $G$  into  $B_1 \cap Z_1$ , then  $\beta + \lambda \in T_1$  with  $\beta + \lambda \equiv \beta \pmod{V_1}$  if, and only if,  $\iota + \beta^{-1}\lambda \in T_1$ .

Recall that  $\tau_x(y) = x^{-1}yx$ .

LEMMA 9. (a)  $\alpha \in W_1^*$  implies that  $\tau_x \alpha^{-1}(x) = \alpha(x)$  for every  $x \in G$ .  
 (b)  $\alpha \in T_1$  and  $\alpha^2 = \iota$  imply that  $\alpha$  induces  $\omega$  on  $\gamma_\alpha(G)$ . If, in addition,  $\alpha \in Z_1(W_1) \cap V_1$  and if  $W_1^*$  is nonvoid, then  $\gamma_\alpha(G) \subset \ker \gamma_\alpha$ .

**Proof.** (a) is immediate. As for (b),  $\alpha \in T_1$  implies that  $\alpha(x) = x\gamma_\alpha(x)$  and  $\alpha^{-1}(x) = x\alpha^{-1}(\gamma_\alpha(x^{-1})) = x\gamma_{\alpha^{-1}}(x)$ . Hence  $\alpha^{-1}(\gamma_\alpha(x^{-1})) = \gamma_{\alpha^{-1}}(x)$ , or  $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(x^{-1})$ , since  $\alpha^{-1} = \alpha$ , and  $\alpha$  induces  $\omega$  on  $\gamma_\alpha(G)$ . From this,  $\gamma_\alpha(x)\gamma_\alpha^2(x) = \gamma_\alpha(x^{-1})$ , and  $\gamma_\alpha(x^{-2}) = \gamma_\alpha^2(x)$ . By Theorem 1(b),  $\gamma_\alpha(x^{-2}) = e$ , so that  $\gamma_\alpha^2(x) = e$  and  $\gamma_\alpha(G) \subset \ker \gamma_\alpha$ .

5. **The  $B$ -series.**  $G/B_1(G)$  is an abelian group so that all of its automorphisms are normal. Define  $B_2(G)$  as the complete inverse image in  $G$  of  $B_1(G/B_1(G))$  under the natural homomorphism of  $G$  onto  $G/B_1(G)$ . In general, suppose that  $B_j(G)$  is defined. Then  $B_{j+1}/B_j \cong B_1(G/B_j)$ . We let  $B_0(G) = (e)$ . Each  $B_j$  is a normal subgroup of  $G$ , and  $i \leq j$  implies that  $B_i \subset B_j$ , so that the  $B$ -series ascends monotonically in its index. Each  $G/B_j$  is abelian ( $j > 0$ ), and  $B_{j+1}/B_j$  is the set of elements of  $G/B_j$  which are each fixed by all automorphisms of  $G/B_j$  ( $j > 0$ ). If  $B_{j+1} = B_j$ , then for all  $k \geq j$ ,  $B_k = B_j$ .

LEMMA 10. The  $B$ -series breaks off at  $B_1$  if any one of the following holds:

- (a)  $G/B_1$  has no elements of order 2.
- (b)  $Z_1$  is torsion-free, or  $Z_1$  has no elements of order 2 or no  $\gamma_\alpha(G)$ , for  $\alpha \in T_1$ , has elements of order 2.
- (c) To each  $xB_1$  in  $G/B_1$ , there exists an automorphism  $\theta = \theta_x$ , such that  $\theta(xB_1) \neq xB_1$ .
- (d) To each  $x \in G$ , there exists  $\alpha = \alpha_x \in A$  such that  $\alpha$  induces an automorphism on  $G/B_1$ , and  $\alpha(x) \neq x \pmod{B_1}$ .
- (e) The equation  $\xi^2 = \alpha$ , for  $\alpha \in T_1$ , always has a solution in  $T_1$ .

**Proof.** (a) Since  $G/B_1$  is abelian, it has the involution automorphism  $\omega$ . If  $gB_1 \in B_2/B_1$ , then  $\omega(gB_1) = gB_1 = g^{-1}B_1$ , and  $g^2 \in B_1$ . Since  $G/B_1$  has no elements of order 2,  $g \in B_1$  and  $B_2 \subset B_1$ . (b) For  $x^2 \in B_1$  and  $\alpha \in T_1$ ,  $\alpha(x^2) = x^2 = x^2\gamma_\alpha(x^2)$ , and  $\gamma_\alpha(x^2) = e$ . Since  $\gamma_\alpha(G)$  has no elements of order 2,  $\gamma_\alpha(x) = e$  and  $x \in B_1$ . Hence  $G/B_1$  has no elements of order 2, and (a) applies. (c) There is no fixed point common to all automorphisms of  $G/B_1$ , so that  $B_2/B_1$  is trivial, and  $B_2 = B_1$ . (d)  $\alpha$  induces  $\alpha'$ , an automorphism on  $G/B_1$ .  $\alpha'(xB_1) \neq xB_1$  so that (c) can now be applied. (e) If  $g \in B_2$ , then, as we saw in the proof of (a),  $g^2 \in B_1$ . For  $\beta \in T_1$  there exists an induced automorphism  $\beta'$  on  $G/B_1$ . Since  $g \in B_2$ ,  $\beta'(gB_1) = gB_1 = \beta(g)B_1$ . Hence  $\beta(g) \equiv g \pmod{Z_1 \cap B_1}$ .  $\beta^2(g) = \beta(g\gamma_\beta(g)) = \beta(g)\gamma_\beta(g) = g\gamma_\beta(g^2)$ , since  $\gamma_\beta(g) \in B_1 \cap Z_1$ . But  $g^2 \in B_1$  implies that  $\gamma_\beta(g^2) = e$ , so that  $\beta^2(g) = g$ . Since every  $\alpha \in T_1$  is, by hypothesis, a square,  $g \in B_1$ , and  $B_2 \subset B_1$ .

6. **The case  $G = B_2$ .** It is obvious that  $G = B_2$  if, and only if,  $B_1(G/B_1) = G/B_1$ ; that is, if, and only if, the identity is the only normal automorphism of  $G/B_1$ . Since  $G/B_1$  is abelian, we see that  $G = B_2$  if, and only if,  $G/B_1$  has no proper automorphism. But this is equivalent [2; p. 101] to

LEMMA 11.  $G = B_2$  if, and only if  $G/B_1 \cong I_2$ .

Since  $G/B_1 \cong I_2$  in this case, choose  $u \in B_2$ ,  $u \notin B_1$ . Then to each  $x \in G$ ,  $x \notin B_1$ , there exists  $b_x \in B_1$  with  $x = ub_x$ . For  $\alpha \in T_1$ ,  $\alpha(x) = u\gamma_\alpha(u)b_x = x\gamma_\alpha(u)$ . Hence  $\alpha(x) = x\gamma_\alpha(u)$  if  $x \notin B_1$ ,  $= x$  if  $x \in B_1$ . We note that  $\gamma_\alpha(u) \in B_1 \cap Z_1$ , by the proof of Lemma 10(e). Since  $u^2 \in B_1$  (by the proof of Lemma 10(a)),  $\gamma_\alpha(u^2) = e$ . It is clear that if  $\alpha, \beta \in T_1$  then  $\gamma_{\alpha\beta}(u) = \gamma_\alpha(u)\gamma_\beta(u)$ .  $\gamma_\alpha(u) = e$  if, and only if,  $\alpha = \iota$ . Moreover, suppose that  $c \in Z_1 \cap B_1$  and that  $c^2 = e$ . Define  $\alpha$  by  $\alpha(x) = xc$  if  $x \notin B_1$ ,  $= x$  if  $x \in B_1$ .  $\alpha(x) = e$  if, and only if,  $x = e$ . If  $y \in B_1$ ,  $\alpha(y) = y$ ; and if  $y \in G$ ,  $y \notin B_1$ , then  $\alpha(y c^{-1}) = y c^{-1} c = y$ . For  $x, y \in B_1$ ,  $\alpha(xy) = \alpha(x)\alpha(y)$ . If  $x \notin B_1$ ,  $x = ub_x$ .  $\alpha(x) = ub_x c = xc$ . For  $y \in B_1$ ,  $\alpha(xy) = xyc = \alpha(x)\alpha(y)$ . If  $y \notin B_1$ , then  $y = ub_y$  and  $\alpha(y) = yc$ .  $\alpha(xy) = \alpha(u^2 b_x b_y)$ . But  $u^2 \in B_1$ , and  $b_x, b_y \in B_1$ . By the case already established for factors in  $B_1$ ,  $\alpha(xy) = u^2 b_x b_y = xy = xycyc = \alpha(x)\alpha(y)$ , since  $c^2 = e$ . It is thus seen that  $\alpha$  is an automorphism of  $G$  and that  $\alpha \in T_1 \cap V_1$  (since  $\alpha$  induces the identity on  $G/Z_1$  and on  $G/B_1$ ). Let  $K_1$  be the subgroup of  $B_1 \cap Z_1$  generated by the elements of order 2 of that group. We have proved

THEOREM 2. If  $G = B_2$ , then  $T_1$  is an elementary abelian group with exponent 2, and  $T_1 = V_1 \cong K_1$ .

COROLLARY. If  $G = B_2$  and if  $\alpha \in T_1$ ,  $\alpha \neq \iota$ , then  $\gamma_\alpha(G) \cong I_2$ .

**Proof.** By the proof of the theorem,  $\text{kern } \gamma_\alpha = B_1$ . Apply Lemma 11.

## 7. Some properties of the B-series.

LEMMA 12.  $B_{n+1}(G)/B_1(G) \cong B_n(G/B_1(G))$ .

**Proof.** The lemma is valid for  $n=1$ . Suppose that it is true for the case  $j-1$ . Then  $B_1((G/B_1)/(B_j/B_1)) \cong (B_j(G/B_1))/(B_j/B_1)$  since  $B_j/B_1 \cong B_{j-1}(G/B_1)$ , by the induction hypothesis. But  $B_1((G/B_1)/(B_j/B_1)) \cong B_1(G/B_j) \cong B_{j+1}/B_j \cong (B_{j+1}/B_1)/(B_j/B_1)$ . Hence  $B_j(G/B_1) \cong B_{j+1}/B_1$ .

We say that  $G$  is *B-nilpotent of B-class  $n$*  (or  *$n$ -B-nilpotent*) if  $G = B_n$ .

COROLLARY. Suppose that  $G$  is not  $n$ -B-nilpotent. The following are equivalent: (a)  $G$  is  $(n+1)$ -B-nilpotent. (b)  $G/B_1$  is  $n$ -B-nilpotent. (c)  $G/B_n \cong I_2$ .

**Proof.** The equivalence of (a) and (b) follows from the lemma.  $G/B_n \cong I_2$  if, and only if,  $B_1(G/B_n) = G/B_n$ . But  $B_1(G/B_n) \cong B_{n+1}/B_n \cong G/B_n$  if, and only if,  $G = B_{n+1}$ .

LEMMA 13. Let  $G$  be a group which is  $(n+1)$ -B-nilpotent but not  $n$ -B-nilpotent, and suppose that  $Z_k \supset B_n$  and that  $Z_{k-1} \not\supset B_n$ . Then  $G$  is  $k$ -nilpotent.



**Proof.**  $G = B_{n+1}$  implies that  $G/B_n \cong I_2$ , by Lemma 12, Corollary. Since  $G/Z_k \cong (G/B_n)/(Z_k/B_n)$ ,  $G/Z_k$  must be isomorphic to  $I_2$  or  $(e)$ , the only possible homomorphic images of  $G/B_n \cong I_2$ . But  $G/Z_k \cong J(G/Z_{k-1})$ . Since the group of inner automorphisms of a group cannot be a nontrivial cyclic group,  $G/Z_k \cong (e)$ , and  $G = Z_k$ .

**LEMMA 14.** *If  $G$  is  $n$ - $B$ -nilpotent where  $n \geq 2$ , then  $\alpha(\gamma_\alpha(x)) \equiv \gamma_\alpha(x^{-1}) \pmod{B_{n-2}}$ , for every  $\alpha \in T_1$  and for every  $x \in G$ .*

**Proof.** First consider the case  $n = 2$ . If  $G = B_2$ , we see, from the discussion before Theorem 2, that  $\alpha \in T_1$  implies that  $\gamma_\alpha(x) = \gamma_\alpha(u)$  if  $x \notin B_1$ ,  $= e$  if  $x \in B_1$ . Here,  $u$  is a representative of the non-unity coset of  $B_1$  in  $G$ . Since  $\gamma_\alpha(u^2) = e$  and  $\gamma_\alpha(u) \in B_1$ , we obtain  $\gamma_\alpha(x)\alpha(\gamma_\alpha(x)) = \gamma_\alpha(u)\alpha(\gamma_\alpha(u)) = [\gamma_\alpha(u)]^2 = e$  if  $x \notin B_1$ . If  $x \in B_1$ , the calculation still gives  $e$ . Recalling that  $B_0 = (e)$ , we see that the lemma is established for  $n = 2$ .

Suppose that the lemma holds for the case  $n - 1$ . Since  $G = B_n$ ,  $G/B_1$  is  $(n - 1)$ - $B$ -nilpotent, by Lemma 12. For  $\alpha \in T_1$ , consider the induced automorphism  $\alpha'$  on the abelian group  $G/B_1$ . By the induction assumption, if  $\alpha'(xB_1) = (xB_1)(zB_1)$ , then  $zB_1\alpha'(zB_1) \in B_{n-3}(G/B_1)$ . Now  $\alpha(x) = x\gamma_\alpha(x)$ , so that  $\alpha'(xB_1) = x\gamma_\alpha(x)B_1$ , and  $xz \equiv x\gamma_\alpha(x) \pmod{B_1}$ . Hence  $z \equiv \gamma_\alpha(x) \pmod{B_1}$  so that  $zB_1 = \gamma_\alpha(x)B_1$ . A substitution shows that  $\gamma_\alpha(x)B_1\alpha'(\gamma_\alpha(x)B_1) = \gamma_\alpha(x) \cdot \alpha(\gamma_\alpha(x))B_1 \in B_{n-3}(G/B_1)$ . But  $B_{n-3}(G/B_1) \cong B_{n-2}/B_1$ , by Lemma 12. From this we can conclude that  $\gamma_\alpha(x)\alpha(\gamma_\alpha(x)) \in B_{n-2}$  for every  $\alpha \in T_1$  and for every  $x \in G$ . The lemma is established.

**COROLLARY 1.** *If  $G = B_n$ ,  $n \geq 2$ , and if  $\alpha \in T_1$ , then  $\alpha^2(x) \equiv x \pmod{(Z_1 \cap B_{n-2})}$  for every  $x \in G$ .*

**Proof.**  $\alpha(x) = x\gamma_\alpha(x)$  implies that  $\alpha^2(x) = x\gamma_\alpha(x)\alpha(\gamma_\alpha(x))$ . By the lemma,  $\gamma_\alpha(x)\alpha(\gamma_\alpha(x)) \in B_{n-2}$ .

**COROLLARY 2.** *If  $G = B_n$ ,  $n \geq 2$ , and if  $\alpha \in T_1$  induces  $\omega$  on  $\gamma_\alpha(G)$  or on  $G/B_1$ , then  $\alpha^2 = \iota$ .*

**Proof.** If  $\alpha$  induces  $\omega$  on  $\gamma_\alpha(G)$ , then  $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(x^{-1})$ , so that, by the proof of Corollary 1,  $\alpha^2(x) = x$  for every  $x \in G$ . By Lemma 7(a), if  $\alpha$  induces  $\omega$  on  $G/B_1$ , then  $\alpha$  induces  $\omega$  on  $\gamma_\alpha(G)$ .

**COROLLARY 3.** *Let  $M(\alpha)$  be the largest subgroup of  $\gamma_\alpha(G)$  on which  $\alpha$  induces the involution automorphism. If  $G = B_n$ ,  $n \geq 2$ , then  $\gamma_\alpha \text{ kern } \gamma_\alpha^2 = M(\alpha)$ , and  $\gamma_\alpha^2(G)$  is an  $\alpha$ -admissible subgroup of  $\gamma_\alpha(G) \cap B_{n-2}$ .*

**Proof.**  $x \in \text{kern } \gamma_\alpha^2$  if, and only if,  $\gamma_\alpha^2(x) = \gamma_\alpha(x)\alpha(\gamma_\alpha(x)) = e$ ; that is, equivalently,  $\alpha\gamma_\alpha(x) = \gamma_\alpha(x^{-1})$ . But the latter is equivalent to  $\gamma_\alpha(x) \in M(\alpha)$ . By Lemma 2,  $\gamma_\alpha(G)$  is  $\alpha$ -admissible, so that, for given  $x \in G$ ,  $\alpha(\gamma_\alpha(x)) = \gamma_\alpha(y)$  for a suitable  $y = y(x; \alpha)$ . Then  $\gamma_\alpha^2(x) = \gamma_\alpha(x)\gamma_\alpha(y)$ , and  $\alpha(\gamma_\alpha^2(x)) = \alpha(\gamma_\alpha(x)) \cdot \alpha(\gamma_\alpha(y)) = \gamma_\alpha(y)\alpha\gamma_\alpha(y) = \gamma_\alpha^2(y)$ . This shows that  $\gamma_\alpha^2(G)$  is  $\alpha$ -admissible.

LEMMA 15. If  $x \in B_n$ ,  $n \geq 2$ , then  $x^{2^{n-1}} \in B_1$ .

**Proof.** The case  $n=2$  was treated in the proof of Lemma 10(a). Suppose that the lemma is valid for  $n=j$ . If  $x \in B_{j+1}$ ,  $xB_1 \in B_{j+1}/B_1 = B_j(G/B_1)$ . By the induction assumption,  $x^{2^{j-1}}B_1 \in B_1(G/B_1) = B_2/B_1$ , and  $x^{2^{j-1}} \in B_2$ . The case  $n=2$  now shows that  $[x^{2^{j-1}}]^2 = x^{2^j} \in B_1$ .

COROLLARY.  $B_n/B_1$  is u.t. abelian with  $\exp(B_n/B_1) \mid 2^{n-1}$ , so that, for an  $n$ -B-nilpotent group  $G$ ,  $G/B_1$  is u.t. abelian, and an  $n$ -B-nilpotent group with periodic  $B_1$  is itself periodic.

THEOREM 3. If  $G$  is  $n$ -B-nilpotent,  $n \geq 2$ , then  $W_1$  is  $(n-1)$ -nilpotent.

**Proof.** If  $W_1^*$  is void, then  $W_1 = V_1$ , an abelian group. Suppose that  $W_1^*$  is nonvoid.  $x \in B_n$  implies  $x^{2^{n-1}} \in B_1$ , by Lemma 15. If  $\alpha \in V_1$ , then  $\gamma_\alpha(x^{2^{n-1}}) = e$  for every  $x \in G$ , since  $G = B_n$ . By Theorem 1 (b),  $V_1 \subset Z_{n-1}(W_1)$ . By Theorem 1(a),  $W_1$  is  $(n-1)$ -nilpotent.

COROLLARY. If  $G$  with torsion-free  $Z_1$  is  $n$ -B-nilpotent, then  $W_1 = V_1$ , or  $V_1$  is trivial, and  $W_1$  is an elementary abelian group with exponent 2.

**Proof.** If  $W_1^*$  is nonvoid, then Theorem 1, Corollary 1(b), and the present theorem show that  $V_1$  is trivial. Since  $\alpha \in W_1^*$  implies that  $\alpha^2 \in V_1$ ,  $W_1$  is elementary abelian with exponent 2.

LEMMA 16. Each  $B_n$  is  $T_1$ -admissible, and, if  $n \geq 1$ ,  $\gamma_\alpha(B_n) \subset B_{n-1}$  for every  $\alpha \in T_1$ .

**Proof.**  $B_1$  is  $T_1$ -admissible. Suppose that  $B_{n-1}(G)$  is  $T_1$ -admissible for every group  $G$ .  $g \in B_n$  implies that  $gB_{n-1} \in B_1(G/B_{n-1})$ . For  $\alpha \in T_1$ ,  $B_{n-1}$  is both  $\alpha$ - and  $\alpha^{-1}$ -admissible (by the induction assumption), and  $\alpha$  induces an automorphism  $\alpha'$  on the abelian group  $G/B_{n-1}$ . Since  $gB_{n-1} \in B_1(G/B_{n-1})$ ,  $\alpha'(gB_{n-1}) = gB_{n-1} = \alpha(g)B_{n-1}$ , and  $\alpha(g) \equiv g \pmod{B_{n-1}}$ . Hence  $\gamma_\alpha(B_n) \subset B_{n-1}$ . Since  $B_{n-1} \subset B_n$  and  $g \in B_n$ ,  $\alpha(g) \in B_n$  so that  $B_n$  is  $\alpha$ -admissible.

8. **Orbital elements.** An element  $x \in G$  is said to be  $n$ -orbital if  $\alpha^n(x) = x$  for every  $\alpha \in T_1$ . Collecting these  $n$ -orbital elements together in a set  $L_n = L_n(G)$ , we see that  $L_n$  is a subgroup of  $G$ . Since  $T_1 = C(J; A)$ ,  $L_n$  is normal in  $G$ . More generally,  $L_n$  is  $C(T_1; A)$ -admissible. (We shall discuss  $C(T_1; A)$  below.)  $L_1 = B_1$ , and  $m \mid n$  implies  $L_m \subset L_n$ . Thus  $G' \subset B_1 = L_1 \subset L_n$  for every positive integer  $n$ , and  $G/L_n$  is abelian. From the proof of Lemma 10(e), we see that  $x \in B_2$  implies  $\alpha^2(x) = x$  for every  $\alpha \in T_1$ , so that  $B_2 \subset L_2$ .

For positive integers  $s \geq t$ , let  $C(s, t) = s!/t!(s-t)!$ . Consider  $C = C(2^{n-1}, r)$ , where  $n \geq 2$  and  $r \leq n-1$ . If  $r$  is odd,

$$C = 2^{n-1}s \prod_{k=1}^{(r-1)/2} \left( \frac{2^{n-1} - 2k}{2k} \right)$$

where  $s \in R$  is a quotient of odd integers. Let  $k = 2^{c_k}d_k$ , where  $c_k$  is a non-

negative integer, and  $d_k$  is an odd integer. Since  $r$  is odd and  $\leq 2^{n-1}$ ,  $r \leq 2^{n-1} - 1$  and  $k \leq (r-1)/2 \leq 2^{n-2} - 1$ . Thus we have  $c_k \leq n-3$ , and  $(2^{n-1} - 2k)/2k = (2^{n-2-c_k} - d_k)/d_k$ , a quotient of odd integers. We have proved that  $r$  odd implies that  $2^{n-1} \mid C(2^{n-1}, r)$ .  $C(2^{n-1}, r+1) = [(2^{n-1} - r)/(r+1)]C(2^{n-1}, r)$ . For odd  $r \geq 5$ , the exponent of the highest power of 2 dividing into  $r+1$  is  $\leq r-2$ , so that  $2^{n-r+1} \mid C(2^{n-1}, r+1)$ , and  $2^{n-(r+1)} \mid C(2^{n-1}, r+1)$ . If  $r=1$ , then  $r+1=2$ , and  $2^{n-2} = 2^{n-(r+1)} \mid C(2^{n-1}, r+1)$ . If  $r=3$ , then  $r+1=4$ , and  $2^{n-3} = 2^{n-r} \mid C(2^{n-1}, r+1)$ . We summarize in

LEMMA 17. For  $n \geq \max(2, r+1)$ ,  $2^{n-r} \mid C(2^{n-1}, r)$ .

THEOREM 4.  $B_n \subset L_m$ , where  $m = 2^{n-1}$ .

**Proof.** Since the earlier cases have been treated, we assume that  $n \geq 3$ . Suppose that  $g \in B_n$  and that  $\alpha \in T_1$ .  $\alpha^m(g) = \alpha^{m-1}(gb_{n-1})$  where  $b_{n-1} \in B_{n-1}$ , by Lemma 16. Assume, inductively, that  $\alpha^m(g) = \alpha^{m-k} [g \prod_{r=1}^k b_{n-r}^{C(k, r)}]$  where  $b_{n-r} \in B_{n-r}$ , and  $\gamma_\alpha(g) = b_{n-1}$ ,  $\gamma_\alpha(b_t) = b_{t-1}$  ( $t = n-k+1, n-k+2, \dots, n-1$ ). When  $r > n-1$ , we take  $b_{n-r} = e$ . Then

$$\alpha^m(g) = \alpha^{m-(k+1)} \left[ gb_{n-1}^{C(k, 1)+1} \left( \prod_{r=2}^k b_{n-r}^{C(k, r)+C(k, r-1)} \right) b_{n-(k+1)}^{C(k, k)} \right].$$

But  $C(k, 1)+1 = C(k+1, 1)$ ,  $C(k, r)+C(k, r-1) = C(k+1, r)$ , and  $C(k, k) = 1 = C(k+1, k+1)$ . Thus,

$$\alpha^m(g) = \alpha^{m-(k+1)} \left[ g \prod_{r=1}^{k+1} b_{n-r}^{C(k+1, r)} \right],$$

and the induction is complete. Now take  $k=m=2^{n-1}$  and note that  $\alpha^0 = \iota$ . Since  $b$ 's with nonpositive subscript are  $e$ , we can write  $\alpha^m(g) = g \prod_{r=1}^m b_{n-1}^{C(m, r)}$ . By Lemma 15,  $b_{n-r} \in \gamma_\alpha(B_{n-r+1})$  implies that  $b_{n-r}^{2^{n-r}} = e$ . By Lemma 17, however,  $2^{n-r} \mid C(m, r)$ , so that  $\alpha^m(g) = g$ .

COROLLARY. If  $g \in B_n$ ,  $n \geq 2$ , if  $\alpha \in T_1$ , and if  $m = 2^{n-1}$ , then  $\alpha^{m/2}(g) \equiv g \pmod{B_1}$ , and  $\gamma_{\alpha^{m/2}}(g^2) = e$ . In particular, if  $G = B_n$ , then  $\alpha^{m/2} \in V_1$ , and  $T_1/V_1$  is u.t. with exponent dividing  $2^{n-2}$ .

**Proof.** By Lemma 12,  $gB_1 \in B_{n-1}(G/B_1)$ . Let  $\alpha$  induce  $\alpha'$  on  $G/B_1$ . By the theorem,  $\alpha'^{m/2}(gB_1) = gB_1$ , and  $\alpha^{m/2}(g) \equiv g \pmod{B_1}$ ; that is,  $\alpha^{m/2}(g) = gb$  where  $b = \gamma_{\alpha^{m/2}}(g) \in B_1$ . Also by the theorem,  $\alpha^m(g) = g$ . But  $\alpha^m(g) = \alpha^{m/2}(\alpha^{m/2}(g)) = \alpha^{m/2}(gb) = gb^2$ , and  $b^2 = e$ .

LEMMA 18. Let  $n$  be an integer  $\geq 1$ , and let  $G$  be a group for which each automorphism of  $G/B_n$  can be extended to a normal automorphism of  $G$ . Then if  $\gamma_\alpha(g) \in B_n$  for every  $\alpha \in T_1$ ,  $g \in B_{n+1}$ .

**Proof.** By hypothesis,  $\alpha(g) \equiv g \pmod{B_n}$  for every  $\alpha \in T_1$ . Let  $\alpha$  induce  $\alpha'$  on  $G/B_n$ .  $\alpha'(gB_n) = \alpha(g)B_n = gB_n$ . Since the set of induced  $\alpha'$  coincides with  $A(G/B_n) = T_1(G/B_n)$ ,  $gB_n \in B_1(G/B_n) = B_{n+1}/B_n$ , and  $g \in B_{n+1}$ .

**9. The centralizer of  $T_1$ .** Since  $T_1$  is the centralizer of  $J$  in  $A$ ,  $U_1 = C(T_1; A) \supset J$ , where  $U_1$  is a normal subgroup of  $A$ .

**LEMMA 19.** (a)  $B_1(G)$  is  $U_1$ -admissible, and if each automorphism of each  $G/B_i$  ( $i=1, 2, 3, \dots$ ) can be extended to a normal automorphism of  $G$ , then each  $B_n$ ,  $n \geq 2$ , is likewise  $U_1$ -admissible. (b)  $\gamma_\alpha(F(U_1)) \subset F(U_1) \subset Z_1$  for every  $\alpha \in T_1$ . (c) Each  $\theta \in U_1$  induces an automorphism on each  $F(\gamma_\alpha)$ ,  $\alpha \in T_1$ .

**Proof.** (a) For  $\theta \in U_1$  and  $\alpha \in T_1$ ,  $\theta\alpha(x) = \theta(x)\theta\gamma_\alpha(x) = \alpha\theta(x) = \theta(x)\gamma_\alpha\theta(x)$ , so that  $\theta\gamma_\alpha = \gamma_\alpha\theta$  for every  $\alpha \in T_1$ . If  $g \in B_1$ , then  $\gamma_\alpha\theta(g) = \theta\gamma_\alpha(g) = \theta(e) = e$  for  $\alpha \in T_1$ , and  $\theta(g) \in B_1$ . Now suppose that  $B_n$  is  $U_1$ -admissible. For  $g \in B_{n+1}$ ,  $\gamma_\alpha\theta(g) = \theta\gamma_\alpha(g)$ .  $\gamma_\alpha(g) \in B_n$ , by Lemma 16. By the induction assumption,  $\theta\gamma_\alpha(g) \in B_n$ . Applying Lemma 18,  $\theta(g) \in B_{n+1}$ . (b) If  $\theta(g) = g$  for every  $\theta \in U_1$ , then  $\gamma_\alpha\theta(g) = \gamma_\alpha(g) = \theta\gamma_\alpha(g)$ . Hence  $\gamma_\alpha(F(U_1)) \subset F(U_1)$  for every  $\alpha \in T_1$ . If  $\theta(g) = g$  for every  $\theta \in U_1$ , then  $\tau_x(g) = g$  for every  $x \in G$ , since  $J \subset U_1$ . But  $\tau_x(g) = g$  for every  $x \in G$  implies that  $g \in Z_1$ . (c) If  $g \in F(\gamma_\alpha)$ ,  $\gamma_\alpha(g) = g$ , and  $\theta\gamma_\alpha(g) = \gamma_\alpha\theta(g) = \theta(g)$ , so that  $\theta(g) \in F(\gamma_\alpha)$ . Conversely, if  $\theta(g) \in F(\gamma_\alpha)$ , then  $\theta\gamma_\alpha(g) = \gamma_\alpha\theta(g) = \theta(g)$ . Since  $\theta$  is an automorphism,  $\gamma_\alpha(g) = g$ , and  $g \in F(\gamma_\alpha)$ .

**THEOREM 5.** Each element of  $C(T_1; A)$  induces a normal automorphism on  $Z_2$ , and there exists a homomorphism on  $C(T_1; A)$  into  $T_1(Z_2)$  with kernel consisting of all those mappings in  $C(T_1; A)$  which reduce to the identity on  $Z_2$ .

**Proof.**  $\theta \in U_1$  implies that  $\theta$  commutes with every Grün automorphism of  $G$ . If, therefore,  $u \in Z_2$ , then  $\theta(x^{-1}u^{-1}xu) = \theta(x^{-1})\theta(u^{-1})\theta(x)\theta(u) = \theta(x^{-1})u^{-1}\theta(x)u$  for every  $x \in G$ .  $u\theta(u^{-1})$  is, consequently, in the centralizer of every  $\theta(x)$ ,  $x \in G$ . Since  $\theta$  is an automorphism,  $u\theta(u^{-1}) \in Z_1(G) \subset Z_1(Z_2(G))$ , and  $\theta(u) \equiv u \pmod{Z_1(Z_2(G))}$ , so that  $\theta$  restricted to  $Z_2$  is normal thereon.

**COROLLARY.** If  $G$  is of class 2, then  $J \subset C(T_1; A) \subset T_1$ , and  $C(T_1; A) = Z_1(T_1)$ .

**10. The higher normal automorphisms.** If  $\alpha \in A$  has the property  $\alpha(x) \equiv x \pmod{Z_n}$  for every  $x \in G$ , we say that  $\alpha$  is an  $n$ -normal automorphism, and we have described the higher normal automorphisms of  $G$ . Let  $T_n$  be the set of  $n$ -normal automorphisms of  $G$ . Under automorphism composition,  $T_n$  is a normal subgroup of  $G$ , and  $m \leq n$  implies that  $T_m \subset T_n$ .

**THEOREM 6.** (a)  $T_n/T_{n-1}$  is isomorphic to a subgroup of  $T_1(G/Z_{n-1})$ . (b)  $T_n/T_1$  is isomorphic to a subgroup of  $T_n(J)$ .

**Proof.** (a)  $\alpha \in T_n$  induces an automorphism  $\alpha'$  on  $G/Z_{n-1}$ . For every  $x \in G$ ,  $xZ_{n-1} \in G/Z_{n-1}$ , and  $\alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xzZ_{n-1}$  where  $z \in Z_n$ . Then  $zZ_{n-1} \in Z_1(G/Z_{n-1})$ , so that  $\alpha'$  is normal on  $G/Z_{n-1}$ . It is not difficult to see that if  $\alpha, \beta \in T_n$ , then  $(\alpha\beta)' = \alpha'\beta'$ , so that  $(')$  is a homomorphism on  $T_n$  into  $T_1(G/Z_{n-1})$ . Suppose that  $\alpha' = \iota$ . Then  $\alpha'(xZ_{n-1}) = xZ_{n-1}$  for every  $x \in G$ , and

$\alpha(x) \equiv x \pmod{Z_{n-1}}$ . Hence  $\alpha$  induces the identity on  $G/Z_{n-1}$ , and  $\alpha \in T_{n-1}$ . Conversely, if  $\alpha \in T_{n-1}$ ,  $\alpha'(xZ_{n-1}) = \alpha(x)Z_{n-1} = xZ_{n-1}$ , and then  $\alpha' = \iota$  on  $G/Z_{n-1}$ . Therefore,  $\text{kern } (\alpha') = T_{n-1}$ . (b)  $\alpha \in T_{n+1}$  induces an automorphism  $\alpha''$  on  $G/Z_1 \cong J$ , given by  $\alpha''(xZ_1) = \alpha(x)Z_1 = x\gamma(x)Z_1$ , where  $\gamma(x) \in Z_{n+1}$ . Hence  $\alpha''(xZ_1) = xZ_1 \pmod{(Z_{n+1}/Z_1)}$ . Now  $Z_n(J) \cong Z_n(G/Z_1) \cong Z_{n+1}/Z_1$ , as an induction will show. Hence  $\alpha''$  is, effectively, in  $T_n(J)$ .  $\alpha$  induces  $\iota$  if, and only if,  $\alpha(x) \equiv x \pmod{Z_1}$  for every  $x \in G$ , and  $\text{kern } (\alpha'') = T_1$ .

**COROLLARY 1.** *Let  $\alpha'$  be a nontrivial normal automorphism of  $G/Z_n$  which can be extended to a higher normal automorphism  $\alpha$  of  $G$ . Then  $\alpha \notin T_n$ .*

**COROLLARY 2.** *Each  $\alpha \in T_n$  induces a homomorphism of  $G$  and an endomorphism of  $G/Z_{n-1}$  into  $Z_n/Z_{n-1} = Z_1(G/Z_{n-1})$ .*

**Proof.** The endomorphism is  $\gamma_{\alpha'}$ , and the homomorphism is obtained by following the natural mapping  $\phi_{n-1}$  of  $G$  onto  $G/Z_{n-1}$  by  $\gamma_{\alpha'}$ . Moreover,  $\gamma_{\alpha'}\phi_{n-1}(x) = x^{-1}\alpha(x)Z_{n-1}$  for every  $x \in G$ .

Let  $S$  be a set of automorphisms of  $G$  and let  $N(S)$  be the set of all  $g \in G$  such that  $\alpha(g) \equiv g \pmod{Z_1}$  for every  $\alpha \in S$ .

**LEMMA 20.** *If  $K$  is a subgroup of  $A$ , then  $C(K; A) \cap J = J(N(K); G)$ . In particular,  $Z_1(A) \cap J = J(N(A); G)$ .*

**Proof.** If  $\tau_g\alpha = \alpha\tau_g$  for every  $\alpha \in K$ , then  $g^{-1}\alpha(x)g = \alpha(g^{-1})\alpha(x)\alpha(g)$  for every  $x \in G$ , so that  $g\alpha(g^{-1})$  is in the centralizer of every  $\alpha(x)$ . Since  $\alpha$  is an automorphism,  $g\alpha(g^{-1}) \in Z_1$ , and  $\alpha(g) \equiv g \pmod{Z_1}$ , so that  $g \in N(K)$  and  $\tau_g \in J(N(K); G)$ . The proof can be read in reverse to obtain the converse.

**LEMMA 21.** *The following are equivalent: (a)  $J \subset Z_1(A)$ . (b)  $A = T_1$ . Either of these conditions implies that  $G$  is of class 2.*

**Proof.**  $A = T_1$  if, and only if,  $G = N(A)$ . But if the latter holds,  $J(N(A); G) = J$ ; and conversely, if  $J(N(A); G) = J$ ,  $x \in G$  implies the existence of  $y \in N(A)$  with  $\tau_x = \tau_y$ . Then  $x \equiv y \pmod{Z_1}$ , so that, if  $\alpha \in A$ ,  $\alpha(x) \equiv \alpha(y) \equiv y \equiv x \pmod{Z_1}$ . This shows that  $x \in N(A)$  and that  $N(A) = G$ . By Lemma 20,  $J \cap Z_1(A) = J(N(A); G) = J$ , so that  $J \subset Z_1(A)$ , and (a) implies (b). A slight rearrangement of the above argument shows that (b) implies (a). Now if every automorphism of  $G$  is a normal automorphism,  $x^{-1}yx \equiv y \pmod{Z_1}$  for every  $x, y \in G$ . This implies that  $G' \subset Z_1$ , and  $G$  is of class 2.

A similar result is contained in

**THEOREM 7.** *Let  $G$  be a group with the properties (1)  $J \subset T_n$  and (2) each  $\alpha \in A$  induces  $\iota$  on each  $Z_{j+1}/Z_j$  ( $j = 1, 2, \dots, n$ ). Then  $J \subset Z_n(A)$ .*

**Proof.** First, we establish three lemmas:

(R) For a group  $G$ ,  $J \subset T_n$  if, and only if,  $G$  is of class  $n+1$ .

(S) For a group  $G$ ,  $J \cap Z_n(A) \subset J(Z_{n+1}; G)$ .

(T) A group  $G$  with property (2) has the further property that  $J \cap Z_n(A) = J(Z_{n+1}; G)$  (for the  $n$  of property (2)).

To prove (R), use the proof of the last statement of Lemma 21 as a model. As for (S), take  $n=0$ . Then  $J \cap Z_n(A)$  consists of  $\iota$  alone, and the inclusion is trivially valid. Suppose that it holds for  $n=k$ .  $\tau_\theta \in J \cap Z_{k+1}(A)$  implies that  $(\alpha, \tau_\theta) \in Z_k(A)$  for every  $\alpha \in A$ . A brief computation shows that  $(\alpha, \tau_\theta) = \tau_h$ , where  $h = g\alpha^{-1}(g^{-1})$ . By the induction assumption,  $g\alpha^{-1}(g^{-1}) \in Z_{k+1}$ , and this is to be valid for every  $\alpha \in A$ . If we take  $\alpha = \tau_x$ ,  $x \in G$ , then  $gxg^{-1}x^{-1} \in Z_{k+1}$  for every  $x \in G$ , and  $g \in Z_{k+2}$ . But this means that  $J \cap Z_{k+1}(A) \subset J(Z_{k+2}; G)$ .

To prove (T), let  $\{\alpha_i\}$  ( $i=1, 2, \dots, n$ ) be any finite set of elements of  $A$ . For a fixed  $g \in G$ , define  $g_1 = g\alpha_1^{-1}(g^{-1})$ . If  $g_k$  is defined, let  $g_{k+1} = g_k\alpha_{k+1}^{-1}(g_k^{-1})$ . A different finite set of elements of  $A$ , or even the same set in a different order, may very well lead to a different finite sequence  $\{g_i\}$  on  $g$ . Let  $G_i(g) = G_i$  be the set of all  $g_i$  obtained in this fashion for fixed  $g$  and fixed positive integer  $i$ . By Lemma 20,  $\tau_\theta \in Z_1(A)$  if, and only if,  $g \in N(A)$ . But  $g \in N(A)$  if, and only if,  $\alpha(g) \equiv g \pmod{Z_1}$  for every  $\alpha \in A$ . The latter condition is equivalent to  $G_1 \subset Z_1$ . Now suppose that  $\tau_\theta \in Z_k(A)$  if, and only if,  $G_k \subset Z_1$ .  $\tau_\theta \in Z_{k+1}(A)$  if, and only if,  $J(G_1(g); G) \subset Z_k(A)$ . By the induction assumption, this is equivalent to  $G_k(h) \subset Z_1$  for every  $h \in G_1(g)$ . Since  $\bigcup G_k(h) = G_{k+1}(g)$ , where the set union is taken over all  $h \in G_1(g)$ ,  $\tau_\theta \in Z_{k+1}(A)$  if, and only if,  $G_{k+1}(g) \subset Z_1$ .

Now suppose that  $\tau_\theta \in J(Z_{n+1}; G)$ . Then  $g \in Z_{n+1}$  and  $G_1(g) \subset Z_n$ , since each  $\alpha \in A$  induces the identity on  $Z_{n+1}/Z_n$ . Assume, inductively, that  $G_k(g) \subset Z_{n-k+1}$ . Since each member of  $A$  induces the identity on  $Z_{n-k+1}/Z_{n-k}$ ,  $G_{k+1}(g) \subset Z_{n-k}$ . In particular,  $G_n(g) \subset Z_1$ . By the above,  $J(Z_{n+1}; G) \subset Z_n(A)$ . Along with (S), this is enough to establish (T).

To prove the theorem, note that  $J \subset T_n$  implies, by (R), that  $G$  is of class  $n+1$ . Therefore, in (T), replace  $Z_{n+1}$  by  $G$ . Since  $J(G; G) = J$ , the theorem is proved.

For a subgroup  $K$  of  $A$ , it is clear that  $F(K) \subset N(K)$ , that  $Z_1 \subset N(K)$ , and that  $N(K)$  is  $K$ -admissible. We prove a preliminary result on  $Q(H; G)$  for a characteristic subgroup  $H$  of  $G$ .

**LEMMA 22.** *Let  $H$  be a characteristic subgroup of  $G$ . (a)  $Q(H; G) \cap J = J(H \div G; G)$ . (b)  $\alpha$  and  $\beta$  induce the same automorphism on  $G/H$  if, and only if,  $\alpha \equiv \beta \pmod{Q(H; G)}$ . In particular,  $\tau_x \equiv \tau_y$  if, and only if,  $x \equiv y \pmod{H \div G}$ .*

**Proof.** (a)  $\tau_\theta \in Q(H; G)$  if, and only if,  $\tau_\theta(x) \equiv x \pmod{H}$  for every  $x \in G$ . But this latter condition is equivalent to  $(g, x^{-1}) \in H$  for every  $x \in G$ , and this is true if, and only if,  $g \in H \div G$ . (b) is obvious.

**LEMMA 23.** (a)  $T_n \div J = T_{n+1}$ . (b)  $(T_n, J) \subset J(Z_n; G)$ . (c)  $T_n \cap J = J(Z_{n+1}; G)$ .

**Proof.** (a) and (b) can be established by routine arguments. To verify (c),

replace  $H$  by  $Z_n$  in Lemma 22(a), and note that  $Q(Z_n; G) = T_n$  and that  $Z_n \div G = Z_{n+1}$ .

**COROLLARY.** (a) For a subgroup  $K$  of  $A$  and for a positive integer  $n$ ,  $N(K) = Z_n$  if, and only if,  $C(K; A) \cap J = T_{n-1} \cap J$ . (b)  $N(K) = Z_1$  if, and only if,  $C(K; A) \cap J$  is trivial. (c) If  $G$  is  $n$ -nilpotent, and if  $K$  is a subgroup of  $A$ , then  $K \subset T_1$  if, and only if,  $C(K; A) \cap J = T_{n-1} \cap J$ .

**Proof.** (a) If  $N(K) = Z_n(G)$ , then  $J(Z_n; G) = J(N(K); G) = C(K; A) \cap J$ , by Lemma 20. Since  $J(Z_n; G) = T_{n-1} \cap J$  (by (c) of the lemma), half the statement is established. Conversely, suppose that  $C(K; A) \cap J = T_{n-1} \cap J = J(Z_n; G)$ . One can readily check the equivalence of the following statements: (1)  $x \in Z_n$ . (2)  $\tau_x \alpha = \alpha \tau_x$  for every  $\alpha \in K$ . (3)  $\tau_x \alpha \alpha^{-1}(y) = \alpha \tau_x \alpha^{-1}(y)$  for every  $y \in G$ ,  $\alpha \in K$ . (4)  $x^{-1}yx = \alpha(x^{-1})y\alpha(x)$  for every  $y \in G$  and every  $\alpha \in K$ . (5)  $\alpha(x)x^{-1} \in Z_1$  for every  $\alpha \in K$ . (6)  $x \in N(K)$ . (b) follows from (a) by taking  $n = 1$ . (c)  $K \subset T_1$  if, and only if,  $N(K) = G$ . Since  $G = Z_n$ , (a) is applicable.

**THEOREM 8.**  $G^{(n)} \subset F(T_n) \subset C(Z_n; G)$ .

**Proof.** If  $\alpha \in T_n$ , then  $\alpha(x^{-1}y^{-1}xy) = t^{-1}x^{-1}u^{-1}y^{-1}xtyu$  where  $t, u \in Z_n$ . Hence  $\alpha(x^{-1}y^{-1}xy) \equiv x^{-1}y^{-1}xy \pmod{Z_{n-1}}$ , and  $\alpha$  induces the identity on  $G'/(Z_{n-1} \cap G')$ . Suppose, inductively, that  $\alpha \in T_n$  induces the identity on  $G^{(k)}/(Z_{n-k} \cap G^{(k)})$ . A set of generators of  $G^{(k+1)}$  is all  $(x, y)$  where  $x, y \in G^{(k)}$ .  $\alpha(x^{-1}y^{-1}xy) = t^{-1}x^{-1}u^{-1}y^{-1}xtyu$ , where  $t, u \in Z_{n-k}$ . Hence  $\alpha(x^{-1}y^{-1}xy) \equiv x^{-1}y^{-1}xy \pmod{Z_{n-k-1}}$ ; and our induction shows that  $\alpha \in T_n$  induces the identity on each  $G^{(k)}/(Z_{n-k} \cap G^{(k)})$ . Now take  $k = n$  so that  $Z_{n-k} = (e)$ . That is, each  $\alpha \in T_n$  induces the identity on  $G^{(n)}$ , whence  $G^{(n)} \subset F(T_n)$ . By the discussion after Lemma 3,  $F(T_n) \subset C(Z_n; G)$ .

**COROLLARY 1.**  $G^{(n)} \subset F(J(Z_{n+1}; G))$ .

**Proof.**  $T_n \supset J(Z_{n+1}; G)$ , by Lemma 23(c).

**COROLLARY 2.** If  $F(T_n) = (e)$  or if  $F(J(Z_{n+1}; G)) = (e)$  for some positive integer  $n$ , then  $G$  is solvable [3].

**COROLLARY 3.**  $J(G^{(n)}; G) \subset C(T_{n+1}; A)$ .

**Proof.**  $\alpha \in T_{n+1}$ , and  $g \in G^{(n)}$  imply that  $\alpha(g) \equiv g \pmod{Z_1}$ , by the proof of the theorem. Hence  $J(G^{(n)}; G) \subset J(N(T_{n+1}); G) = C(T_{n+1}; A) \cap J$ , by Lemma 20.

If we let  $U_n = C(T_n; A)$ , then, by Lemma 20,  $J(N(T_n); G) \subset U_n$ . As in Lemma 19(a),  $F(T_n)$  is  $U_n$ -admissible.

**11. Examples.** (A) For positive integers  $n > 2$ , let  $D_n$  denote the  $n$ th dihedral group, the group of isometries of a regular  $n$ -gon.  $D_n$  is the semidirect product of  $I_n$  and of  $I_2$  with the multiplication rules  $(x_n, 0_2)(y_n, z_2) = (x_n + y_n, z_2)$  and  $(x_n, 1_2)(y_n, z_2) = (x_n - y_n, 1_2 + z_2)$ . For  $n > 2$ , there is a non-trivial element in the center if, and only if,  $n$  is even; and in this case, the

center consists of two elements,  $(0_n, 0_2)$  and  $(h_n, 0_2)$ , where  $h$  is an integer such that  $2h=n$ . Since  $D_n$  is a group with two generators, there are three non-trivial possibilities for central endomorphisms. The verification of the following results is easy:  $T_1(D_{4k})$  is isomorphic to the Klein four group,  $I_2 \oplus I_2$ . Let us denote the four group by  $\mathfrak{B}$ .  $B_1(D_{4k})$  consists of all  $(x_{4k}, 0_2)$  where  $x$  is even, so that  $B_1(D_{4k}) \cong I_{2k}$ . Likewise,  $B_1(D_{4k}) = D'_{4k}$ , and, in fact,  $D_{4k}/B_1(D_{4k}) \cong \mathfrak{B}$ . It follows that the  $B$ -series breaks off at  $B_1(D_{4k})$ .  $T_1(D_{4k+2}) \cong I_2$ .  $B_1(D_{4k+2})$  consists of all  $(x_{4k+2}, 0_2)$ , so that  $B_1(D_{4k+2}) \cong I_n$ .  $D_{4k+2}/B_1(D_{4k+2}) \cong I_2$  so that, by Lemma 11,  $D_{4k+2}$  is 2- $B$ -nilpotent. If  $n=4$ , then  $D_4/Z_1(D_4)$  is isomorphic to  $\mathfrak{B}$  whence  $D_4$  is of class 2. Then  $T_2(D_4) = A(D_4)$ , and it can be readily verified that  $A(D_4) \cong D_4$  and that  $F(T_2(D_4)) = B_1(D_4) = Z_1(D_4)$ .

(B) Let  $G$  be a group of type  $(2^\infty)$ .  $G$  is isomorphic to the additive group, modulo 1, of the rationals  $k/2^n$ , where  $k$  is an odd integer or 0. Since  $G$  is abelian,  $T_1(G) = A(G)$ .  $G$  has a nontrivial automorphism  $\alpha(k/2^n) = 1 - (k/2^n)$  corresponding to the conjugation automorphism on the representation of  $G$  on the unit circle. The only fixed points are  $1=0$  and  $1/2$ . Conversely, if  $\beta$  is any automorphism of  $G$ ,  $2\beta(1/2) = \beta(1) = 1=0$ , so that  $\beta(1/2) = 1/2$  or  $0$ . Thus  $B_1 \cong I_2$ . Since  $G/B_1 \cong G$ ,  $B_2$  consists of  $0, 1/4, 1/2$ , and  $3/4$ , and  $B_2 \cong I_4$ . In general,  $B_n \cong I_{2^n}$ .  $G = \bigcup B_n$  where the union is taken over all positive integral values of  $n$ .

(C) Let  $G$  be the multiplicative group of all nonsingular 2 by 2 matrices over the field of rationals,  $R$ . It is well known that  $Z_1$  consists of all

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad u \neq 0,$$

and that  $Z_2 = Z_1$ . By Lemma 1, we have an example of a group for which  $T_1 \cap J$  is trivial. Let  $\mu$  be an endomorphism of the multiplicative group of nonzero rationals  $R^*$  where  $x\mu(x^2a) = 1$  has a unique solution  $x = x(a; \mu)$  for every  $a \in R^*$ . Let  $d_1$  and  $d_2$  be integers with the restriction  $|d_i| = 1$ . Define a mapping  $\alpha = \alpha(\mu; d_1, d_2)$  on  $G$  by

$$\alpha \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r\mu(r) & 0 \\ 0 & \mu(r) \end{pmatrix} \quad \text{for every } r \in R^*,$$

$$\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & d_1 \\ d_1 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_2 & d_2 \\ 0 & d_2 \end{pmatrix}.$$

Then it is possible to prove that  $\alpha$  is a normal automorphism of  $G$ , and each normal automorphism of  $G$  is such an  $\alpha(\mu; d_1, d_2)$ . A matrix  $M \in G$  is in  $B_1$  if, and only if, it can be factored (without regard to the order of the factors) into a product of an even number of factors

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$



an even number of factors

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and a set of factors

$$\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix} \quad (i = 1, 2, \dots, n),$$

where  $a_1 a_2 \cdots a_n = 1$ .  $V_1$  turns out to be the set of all  $\alpha(\mu; d_1, d_2)$  with  $\mu(x) = \pm 1$  for every  $x \in R^*$ .  $W_1^*$  consists of all  $\alpha$  with  $\mu(x) = \pm(1/x)$ , and  $[T_1: V_1]$  is equal to the number of normal subgroups of index 2 in  $G$  which contain

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$W_1$  for this group is abelian. Now  $W_1^*$  is nonvoid,  $V_1$  is nontrivial and  $Z_1$  has the periodic element

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, this example shows that we cannot drop the hypothesis of aperiodicity for  $Z_1$  in Theorem 1, Corollary 1(b).  $G/B_1 \cong \mathfrak{B} \oplus R^*$  whence  $B_2 = B_1$ . Let  $\mu$  be an endomorphism of  $R^*$  such that, to each positive prime  $p$ , there exists a positive prime  $q$  with  $q|\mu(p)| = 1 = q|\mu(q)|$ . Then  $\alpha(\mu; d_1, d_2) \in T_1$ , so that, for this group,  $T_1$  is far from trivial and  $T_1 \neq W_1$ . Negatively, one can show, for instance, that if  $\mu$  is an endomorphism of  $R^*$  for which  $|\mu(p)|$  is always a product of  $k$  positive primes (or a product of the reciprocals of  $k+1$  positive primes) for every positive prime  $p$ , then the corresponding  $\alpha$  is not an automorphism.

(D) Let  $G$  be a group with generators  $a, b$ , and  $c$ , where  $a^2 = e$ ,  $ab = ba$ ,  $ac = ca$ , and  $bc = cba$ . Then every element of  $G$  can be written uniquely as a product  $a^i b^j c^k$  where  $i$  is 0 or 1, and  $j$  and  $k$  range over the integers.  $Z_1 \cong I_2$  and  $G/Z_1 \cong R \oplus R$  so that  $G$  is nilpotent of class 2. One can verify that  $T_1 \cong \mathfrak{B}$ . An element is in  $B_1$  if, and only if,  $j$  and  $k$  are both even. Under any automorphism  $\alpha$  each center element,  $a^i$ , is fixed. There is an automorphism  $\beta$  which changes the sign of  $j$  in each term. Its set of fixed points is precisely all elements with  $j=0$ . There is an automorphism  $\delta$  which changes the sign of  $k$  in each term, and the corresponding fixed points are all elements with  $k=0$ . The cross-cut of these two sets of fixed points is  $Z_1$ , so that  $Z_1 = F(A) = F(T_2)$ , and this latter set is included in  $F(T_1) = B_1$  properly. (In the example of  $D_4$  above,  $F(T_1) = F(T_2)$  for the class 2 group  $D_4$ .)

Presumably, by extending the group of this example or by considering  $n$  by  $n$  triangular matrices with a diagonal of unities, one could exhibit groups with significant  $T_n$ , for  $n > 2$ .

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