

ON BIEBERBACH-EILENBERG FUNCTIONS. II

BY

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1. Let us denote by C the class of functions $f(z)$ regular for $|z| < 1$ which have power series expansion about $z=0$ beginning

$$f(z) = a_1z + a_2z^2 + \dots$$

and which satisfy the condition

$$(1) \quad f(z_1)f(z_2) \neq 1, \quad |z_1| < 1, \quad |z_2| < 1.$$

Let $C(\lambda)$ denote the subclass of C consisting of functions $f(z)$ for which $|a_1| = \lambda$, $0 < \lambda \leq 1$. If $f \in C(\lambda)$ it is subordinate to a function $g(z) \in C$ univalent in $|z| < 1$ [9] for which then $|g'(0)| \geq \lambda$. Thus the image of $|z| < 1$ by $w = g(z)$ covers the circle $|w| < \lambda/4$ and hence by condition (1), for $|z| < 1$, $|g(z)| < 4/\lambda$. Consequently, for $|z| < 1$, $|f(z)| < 4/\lambda$ so that the functions in $C(\lambda)$ are uniformly bounded. Rogosinski [9] raised the question of the best possible bound $P(\lambda)$ for the functions in $C(\lambda)$. He gave certain upper estimates for $P(\lambda)$ but not the precise value. The object of the present paper is to give the complete solution of this problem by a method used earlier to solve certain other questions for the same class of functions [5].

2. We begin by solving first the related problem of the closest boundary point to the origin of the image of $|z| < 1$ by a univalent function in $C(\lambda)$. In the ζ -plane ($\zeta = \xi + i\eta$) we regard the domain $\Delta(t)$ defined by

$$\begin{aligned} 0 < \eta < \pi, & \quad \xi < 0, \\ 0 < \eta < \pi - t, & \quad \xi = 0, \\ -t < \eta < \pi - t, & \quad \xi > 0, \end{aligned}$$

where $0 \leq t < \pi$. Let us denote the following boundary points of $\Delta(t)$ in the manner indicated: $0, B$; $-it, C$; $i(\pi - t), E$; $\pi i, F$. Further let the boundary point of $\Delta(t)$ at infinity between F and B in the natural cyclic order be denoted by A , the corresponding point between C and E by D . We map $\Delta(t)$ conformally onto the left-hand half-plane $\Re w < 0$ in such a way that A goes into $w=0$, D into $w=\infty$. Let us denote by P the point $\zeta = (\pi - t)i/2$. Rotation of $\Delta(t)$ through 180° about P corresponds in the w -plane to a linear transformation of $\Re w < 0$ onto itself interchanging 0 and ∞ . This transformation has the form $w^* = a/w$, a real and positive. The fixed point $-a^{1/2}$ (positive root) of the latter transformation is the image of P . We adjust the original mapping from the ζ -plane to the w -plane so that this becomes the point -1 and the linear transformation becomes $w^* = 1/w$.

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Let the images of B and C be i/l and i/m , $l \geq m > 0$. Then the images of E and F are $-il$, $-im$. If we extend ζ as a (non-single-valued) function of w to the whole w -plane by reflection in various segments of the imaginary axis we see at once that $d\zeta^2$ is a quadratic differential on the w -sphere with double poles at 0 , ∞ , simple poles at i/m , $-im$ and simple zeros at i/l , $-il$ (except in the case $t=0$ when the simple poles and zeros coincide in pairs and cancel). Indeed we can write

$$d\zeta^2 = K \frac{(w + il)(w - i/l)}{w^2(w + im)(w - i/m)} dw^2$$

with K a suitable positive constant. In case $t=0$ this assumes a simple limiting form. The curves on which $d\zeta^2 > 0$ will be called trajectories, those on which $d\zeta^2 < 0$ will be called orthogonal trajectories.

Let us regard now the mapping from the domain $|z| < 1$, $\Re z < 0$ in the z -plane onto the domain $0 < \eta < \pi$, $\xi < 0$ in the ζ -plane by the function $\zeta = \log(z/i)$ (with a suitable determination). Combining this with the above mapping from the ζ -plane into the w -plane and extending it by reflection across the segments joining i and $-i$ in the z -plane and i/l and $-im$ in the w -plane we obtain a function regular and univalent for $|z| < 1$. We denote this function by $f(z; t)$ and the image of $|z| < 1$ under it by $D(t)$. The latter domain is bounded by orthogonal trajectories joining i/l and $-il$ and a rectilinear slit along an orthogonal trajectory from $-im$ to $-il$. The latter slit degenerates to a point when $t=0$. Further $f(z; t) \in C$ since the transformation $w^* = 1/w$ carries $D(t)$ into its exterior; also $f'(0; t) > 0$ and $f(z; 0) \equiv z$.

The quantities m and l depend on the parameter t and when appropriate will be denoted by $m(t)$ and $l(t)$.

3. THEOREM 1. *Let $g(w)$ be regular and univalent in $D(t)$ with $g(0)=0$, $|g'(0)|=1$ and such that $g(w_1)g(w_2) \neq 1$, $w_1, w_2 \in D(t)$. Let μ denote the modulus of the boundary point of the image of $D(t)$ under $w'=g(w)$ closest to $w'=0$. Then $\mu \geq m(t)$ and equality can be obtained only for $g(w) = \pm \phi(w)$, provided $t > 0$, where $\phi(0)=0$ and ϕ is a conformal mapping of $D(t)$ upon itself. When $t=0$, equality is attained only for $g(w) = e^{i\theta}\phi(w)$, θ real, ϕ as before.*

The proof of this result depends on the consideration of the following module problem. Let L be a Jordan curve in the w -plane enclosing $w=0$ and having reflectional symmetry in the imaginary w -axis. Let L^* be the image of L under the transformation $w^* = 1/w$. We suppose that L^* is exterior to L and denote by $\mathfrak{D}$ the doubly-connected domain bounded by L and L^* . Let G be a point on the positive imaginary axis in $\mathfrak{D}$ and G^* its image under the transformation $w^* = 1/w$. Let C_1 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L from G , G^* , and L^* . Let C_2 denote the class of rectifiable Jordan curves lying in $\mathfrak{D}$ and separating L^* from L , G and G^* .

Let ρ be a real-valued non-negative function of integrable square over $\mathfrak{D}$ and such that, for $c \in C_i$ ($i=1, 2$), $\int_c \rho |dw|$ exists and that $\int_c \rho |dw| \geq 1$. Then let the greatest lower bound of $\iint_{\mathfrak{D}} \rho^2 du dv$ ($w=u+iv$) for all such functions ρ be denoted by $M(L, G)$. This actually is a minimum attained for a particular function ρ . This can be proved by reducing the problem to a hexagon problem by a method similar to that of [2] or by a general construction method, but this result is not needed here.

Let us return now to the quadratic differential $d\zeta^2$ which we denote by $Q(w)dw^2$. We have seen that $D(t)$ is bounded by the union of certain orthogonal trajectories. The orthogonal trajectories interior to $D(t)$ are Jordan curves with reflectional symmetry in the imaginary w -axis. The orthogonal trajectory of this set which meets the positive imaginary axis at the point ir ($r>0$) will be denoted by $H(r)$. As r tends to zero $H(r)$ tends to circular form [10; 6]. Let $H(r)$ for r sufficiently small play the role of L in the preceding module problem and let the point $w=i/m$ play the role of G . Let the doubly-connected domain bounded by $H(r)$ and the boundary of $D(t)$ be denoted by $E(r)$ and its image under the transformation $w^*=1/w$ be denoted by $E^*(r)$. These two domains have equal module (for the class of curves separating the boundaries) which we denote by $M(r)$. We then verify readily that for a suitable constant ν the metric $\nu|Q(w)|^{1/2}|dw|$ provides the extremal metric in the module problem defining $M(L, G)$ and this independently of the value of r . Further, in this case, $M(L, G) = 2M(r)$.

With again the choice $L=H(r)$ but for a point H with affix ih , $h>1/m$, we have $M(L, G) \geq M(L, H) + d$ where d (>0) is independent of r but depends on h . This can be seen in various ways, perhaps most easily by observing that it is possible to modify the function $|Q(w)|^{1/2}$ by setting it equal to zero in a sufficiently small neighborhood of G to obtain a function admissible in the competition for the greatest lower bound $M(L, H)$, independent of the choice of r .

Let now $E'(r)$ be the image of $E(r)$ under the function g of Theorem 1. We shall suppose it to lie again in the w -plane. Let K_1 be the bounded continuum complementary to $E'(r)$ bounded by the image L' of L . Let K_2 be the other continuum complementary to $E'(r)$. We obtain from $E'(r)$ a circularly symmetrized domain $\tilde{E}(r)$ in the following manner. Let the intersections of $|w|=R$ with K_1 and K_2 have respectively angular Lebesgue measure $l_1(R)$ and $l_2(R)$. Let $\tilde{K}_1$ be the set defined by $\pi/2 - l_1(R)/2 \leq \Phi \leq \pi/2 + l_1(R)/2$ for those values of R for which K_1 meets $|w|=R$ where R, Φ are polar coordinates in the w -plane. Let $\tilde{K}_2$ be the set defined by $-\pi/2 - l_2(R)/2 \leq \Phi \leq -\pi/2 + l_2(R)/2$ for those values of R for which K_2 meets $|w|=R$. The complement of $\tilde{K}_1 \cup \tilde{K}_2$ is a doubly-connected domain which we denote by $\tilde{E}(r)$.

Let $E'(r)$, $\tilde{E}(r)$ have modules $M'(r)$, $\tilde{M}(r)$. Clearly $M'(r) = M(r)$ while by a standard symmetrization argument $M'(r) \leq \tilde{M}(r)$ [8]. Under the assumption of Theorem 1 the point of K_2 closest to $w=0$ has modulus μ , thus the

point $-i\mu$ is a point of $\tilde{K}_2$ and a boundary point of $\tilde{E}(r)$. We verify at once that $\tilde{E}(r)$ does not overlap with its image $\tilde{E}^*(r)$ under the transformation $w^*=1/w$ (as a consequence of the condition $g(w_1)g(w_2) \neq 1$, $w_1, w_2 \in D(t)$). Let us now denote the point i/μ by H and let $\mathfrak{E}(r)$ be the intersection of $\tilde{E}(r)$ with the exterior of L . Let this domain, which is doubly-connected at least for r small enough, have module $\mathfrak{M}(r)$ (for the class of curves separating its boundaries as usual). Since $H(r)$ tends to circular form as r tends to zero we have that $\tilde{M}(r) - \mathfrak{M}(r)$ approaches zero as r tends to zero. We see also that $\mathfrak{E}(r)$ separates L from H , H^* and L^* , and $\mathfrak{E}^*(r)$, its image by the transformation $w^*=1/w$, separates L^* from L , H and H^* .

Suppose now that $\mu < m(t)$. Then, by an earlier remark, $M(L, G) \geq M(L, H) + d$ where $d (> 0)$ is independent of r . On the other hand, by a standard argument [4], $M(L, H) \geq 2\mathfrak{M}(r)$. Thus, combining this with the statements $M(L, G) = 2M(r)$, $\tilde{M}(r) - \mathfrak{M}(r) = o(1)$, we have

$$2M(r) \geq 2\tilde{M}(r) + d + o(1),$$

a contradiction to the result $\tilde{M}(r) \geq M(r)$. Thus $\mu \leq m(t)$.

Suppose next that $\mu = m(t)$. If $\tilde{K}_2$ did not coincide with the complement of $D(t)$ it would follow by a standard form of argument [3] that we would have $M(r) \geq \tilde{M}(r) + \delta + o(1)$ for $\delta (> 0)$ independent of r for r sufficiently small. This would be in contradiction to the inequality $\tilde{M}(r) \geq M(r)$. Further, unless $\tilde{K}_2$ is obtained from K_2 by a rigid rotation about $w=0$, we shall have $\tilde{M}(r) \geq M(r) + p$ where p is a positive constant independent of r , for r sufficiently small [7, Theorem 3]. In this case we would have $M(L, G) \geq 2\mathfrak{M}(r)$ and we would get

$$2M(r) \geq 2\tilde{M}(r) + o(1)$$

contrary to the inequality $\tilde{M}(r) \geq M(r) + p$.

Thus equality is possible at most when $g(w)$ has the form $e^{i\theta}\phi(w)$, θ real, with $\phi(w)$ a conformal mapping of $D(t)$ onto itself and $\phi(0)=0$. The function $e^{i\theta}\phi(w)$ can satisfy the conditions of Theorem 1 only if neither 1 or -1 is interior to the image of $D(t)$ by this function. Let us assume first that $t > 0$. Then the only boundary points of $D(t)$ on $|w|=1$ are the points $w = \pm 1$. This can be verified by observing that $|w|=1$ has the direction of an orthogonal trajectory only at the points $w = \pm i$ and that this situation will occur between any two boundary points on $|w|=1$. That $d\zeta^2 < 0$ on $|w|=1$ only at $w = \pm i$ is the consequence of a simple direct numerical calculation. Thus the open arc $|w|=1$, $\Im w > 0$ is interior to $D(t)$, the open arc $|w|=1$, $\Im w < 0$ is exterior to $D(t)$ and the only values of $e^{i\theta}$ for which $e^{i\theta}\phi(w)$ satisfies the conditions on $g(w)$ in Theorem 1 are ± 1 . When $t=0$ all values $e^{i\theta}$, θ real, are clearly admissible. This completes the proof of Theorem 1.

4. The uniqueness part of Theorem 1 implies that no two values $f'(0; t)$ are equal. By an argument involving the theory of normal families we see that

$f'(0; t)$ tends to zero as t approaches π . Also for $t=0$, $f'(0; 0)=1$. Thus as t takes the values $0 \leq t < \pi$, $f'(0; t)$ takes the values $1 \geq f'(0; t) > 0$. In particular for $1 \geq \lambda > 0$ there is a unique such function with $f'(0; t) = \lambda$. We denote this function by $F(z; \lambda)$ and the modulus of the closest boundary point to the origin of the image of $|z| < 1$ under this function by $\mu(\lambda)$. Clearly $F(z; \lambda) \in C(\lambda)$. We then have the following result.

THEOREM 2. *If $f(z)$ is a univalent function in $C(\lambda)$, then for the modulus μ of the boundary point of the image of $|z| < 1$ under f closest to the origin we have $\mu \geq \mu(\lambda)$ and equality can be attained only for $f(z) = \pm F(e^{i\psi}z; \lambda)$, ψ real, when $\lambda < 1$ and for $f(z) = e^{i\theta}z$, θ real, when $\lambda = 1$.*

This is an immediate consequence of Theorem 1.

5. We now turn to the determination of $P(\lambda)$, the best possible uniform bound for functions in $C(\lambda)$. We observe first that we obtain the same result if we restrict ourselves to univalent functions in $C(\lambda)$. Indeed $f(z) \in C(\lambda)$ is subordinate to a univalent function $g(z) \in C$ [9]. The function $g(z)$ will not in general be in $C(\lambda)$ but $|g'(0)| \geq \lambda$. Thus, producing a slit from the boundary of the image under g of $|z| < 1$ toward the origin and letting $h(z)$ be a function mapping $|z| < 1$ on the slit domain with $h(0) = 0$, for a suitable length of slit we shall have $|h'(0)| = \lambda$. Further the least upper bound of $|h(z)|$ for $|z| < 1$ will be the same as the least upper bound of $|g(z)|$ for $|z| < 1$ and this is at least as large as the same quantity for $f(z)$.

Now, if for $f(z)$ a univalent function in C we have

$$\text{l.u.b.}_{|z| < 1} |f(z)| = M,$$

for the modulus μ of the boundary point of the image of $|z| < 1$ under f closest to the origin we have $\mu \leq 1/M$ so $M \leq 1/\mu$. Thus $P(\lambda) \leq (\mu(\lambda))^{-1}$. We shall now show that the precise value is $P(\lambda) = (\mu(\lambda))^{-1}$.

First we observe by an argument similar to that of the first paragraph of this section that for $\lambda' \geq \lambda$, $P(\lambda') \leq P(\lambda)$.

Let t be the value corresponding to λ as in §4. We may assume $t > 0$, $\lambda < 1$ since in the special case excluded the result is evident. Let us denote by $D(t, \epsilon)$ the domain obtained from $D(t)$ by shortening the slit from $-il$ to $-im$ to a slit from $-il$ to $-i(m+\epsilon)$, $0 < \epsilon < l-m$. Let $f(z; t, \epsilon)$ be the function mapping $|z| < 1$ onto $D(t, \epsilon)$ such that $f(0; t, \epsilon) = 0$, $f'(0; t, \epsilon) > 0$. Clearly $f'(0; t, \epsilon) > \lambda$. Let $U(\sigma)$ denote the open set of points whose distance from the segment joining i/l to $i/(m+\epsilon)$ is less than σ . Let $U^*(\sigma)$ be the image of $U(\sigma)$ by the transformation $w^* = 1/w$. Let

$$D(t, \epsilon, \sigma) = (D(t, \epsilon) \cup U(\sigma)) - \bar{U}^*(\sigma).$$

For σ sufficiently small this is a simply-connected domain containing the point $w=0$. Let $f(z; t, \epsilon, \sigma)$ be the function with $f(0; t, \epsilon, \sigma) = 0$, $f'(0; t, \epsilon, \sigma) > 0$

mapping $|z| < 1$ onto $D(t, \epsilon, \sigma)$. By the general theory of kernels of domains [1], $f(z; t, \epsilon, \sigma)$ converges to $f(z; t, \epsilon)$ as σ tends to zero. Thus $f'(0; t, \epsilon, \sigma) > \lambda$ for σ sufficiently small. However

$$\text{l.u.b.}_{|z|<1} |f(z; t, \epsilon, \sigma)| = (m + \epsilon)^{-1} + \sigma.$$

Thus by the monotone property given above

$$P(\lambda) \geq (m + \epsilon)^{-1} + \sigma$$

for ϵ, σ positive, sufficiently small. Hence

$$P(\lambda) \geq m^{-1} = (\mu(\lambda))^{-1}.$$

We summarize these statements in our principal result.

THEOREM 3. *If $P(\lambda)$ is the best possible uniform bound for functions in $C(\lambda)$, $P(\lambda) = (\mu(\lambda))^{-1}$. There is no function $f(z)$ in $C(\lambda)$ having $P(\lambda)$ as least upper bound of $|f(z)|$, $|z| < 1$, except for $\lambda = 1$.*

The final statement of Theorem 3 follows from the uniqueness part of Theorem 2, since for such a function $f(z)$ the modulus μ of the boundary point of the image of $|z| < 1$ under f closest to the origin would satisfy $\mu \leq \mu(\lambda)$, thus $\mu = \mu(\lambda)$ and $f(z)$ would be $e^{i\theta} F(e^{i\psi} z; \lambda)$, θ, ψ real. However $|F(e^{i\psi} z; \lambda)|$, for $|z| < 1$, has least upper bound strictly less than $(\mu(\lambda))^{-1}$ except when $\lambda = 1$.

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