ASYMPTOTIC EXPANSIONS FOR THE WHITTAKER FUNCTIONS OF LARGE COMPLEX ORDER m

BY

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1. Introduction. The Whittaker functions, customarily denoted $M_{k,m}(x)$ and $W_{k,m}(x)$, are solutions of the well-known differential equation

$$(1.1) \qquad \frac{d^2W}{dx^2} + \left\{ -\frac{1}{4} + \frac{k}{x} + \frac{1/4 - m^2}{x^2} \right\} W = 0.$$

As this equation is important to the discussion of numerous physical problems, there is reason to find forms which represent the Whittaker functions asymptotically as to the variable x or one of the parameters k and m. In this paper we find expressions which represent these functions asymptotically as to m. It is supposed that x, k, and m are complex and that the parameter k is bounded. The arguments of x and m are restricted to lie in the ranges

$$-\pi < \arg(x) \leqq \pi$$

and

$$-3\pi/2 < \arg(m) \leqq 3\pi/2,$$

respectively. The leading terms of the asymptotic developments are explicitly obtained. These involve only simple rational and exponential functions of x. The algorithm in [6] can be used to find more terms, if desired.

To the author's knowledge but one other investigation of the structures of the Whittaker functions for numerically large values of m and bounded values of k has been made, namely an analysis by Fisher [4]. He has studied their structures for numerically large m by means of difference equations under the assumption that $|x/2m|$ be bounded from one. This restriction has not been invoked here. The asymptotic forms found in this paper agree with those he has obtained in the case that $|x/2m| < 1$. For $|x/2m| > 1$, however, his expressions are of a different nature as they are also asymptotic in k.

Erdélyi has investigated the case of x, k, and m all real and large in absolute value by the saddle point method under the limitation that the ratios of x to k and x to m be fixed [3]. Recently Chang, Chu, and O'Brien have

Presented to the Society, November 28, 1953; received by the editors November 23, 1953.

(1) This paper is based on the author's thesis, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Wisconsin. The author is indebted to Professor R. E. Langer, under whose direction the thesis was written, for numerous suggestions and criticisms, and to the Office of Naval Research, which partially supported the research.

found an asymptotic expansion for the function $W_{k,m}(x)$ when x, k, and m are all numerically large, x is a positive real, and the ratios of x to k and x to m are nearly fixed [1; 2]. There have been numerous researches on the structures of the Whittaker functions for numerically large k and for numerically large k and x. A partial summary of these is given by Chang, Chu, and O'Brien at the end of [1]. A most complete discussion not listed there is due to Ziebur [9].

To obtain our results, we use the theory of asymptotic solutions of ordinary linear differential equations developed by Langer [5; 6]. Our discussion consists of two parts. Part I, which constitutes the principal portion, deals with the Whittaker functions. The Whittaker equation (1.1) is transformed there into the form

$$\frac{d^2u}{dz^2} + \{m^2q_0(z) + mq_1(z)\}u = 0. \tag{1.2}$$

In order that Langer's theory may be applied to yield the asymptotic expansions of the solutions of this equation for complex values of z in an unbounded region, it must be somewhat extended. This is done in Part II, which could be said to be in the nature of an appendix wherein assertions made in Part I as to the asymptotic forms of the solutions of equation (1.2) are proved.

Part I. Asymptotic forms for the Whittaker functions

2. **Preliminaries.** Two pairs of linearly independent solutions of the Whittaker equation (1.1) have been intensively studied in mathematical literature. The solutions of one pair are commonly denoted $M_{k,m}(x)$ and $M_{k,-m}(x)$. When $2m$ is not an integer, they are expressible in terms of hypergeometric series [8, p. 337]. The solutions of the other pair are usually designated $W_{k,m}(x)$ and $W_{-k,m}(-x)$ and are defined for all values of m [8, p. 339].

In the analysis which follows we find forms representing the solutions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ asymptotically as to m for the values of x and m specified by the relations

$$\begin{aligned} &\text{(a)}\ 0 \leqq \arg(x) \leqq \pi, \\ &\text{(b)}\ -\pi/2 < \arg(m) \leqq \pi/2, \end{aligned} \tag{2.1}$$

respectively, k being a bounded complex parameter(2). Since k is not otherwise restricted, from these forms representations for the function $W_{k,m}(e^{-\pi i}x)$ can be obtained. Once this is accomplished, it is clear that the asymptotic expansion of the function $W_{k,m}(x)$ for arguments of x in the range

$$-\pi < \arg(x) \leqq \pi$$

will be at hand. Moreover, since

(2) When $\pi/2 - \arg(m) < \epsilon$, we consider values of x in the region for which $\pi + \epsilon > \arg(x) > \epsilon$; when $\pi/2 + \arg(m) < \epsilon$, we consider values of x in the region for which $\pi - \epsilon > \arg(x) > -\epsilon$. The analysis in these cases differs little from that given in Part I; consequently, it is omitted.

$$W_{k,e^{\pm\pi i}m}(x) = W_{k,m}(x) \qquad [7, \text{ p. } 116],$$

asymptotic forms for the function $W_{k,m}(x)$ for arguments of m satisfying the condition

$$-3\pi/2 < \arg(m) \leqq 3\pi/2$$

can be found from those obtained under the restriction (2.1b).

Henceforward, then, it is assumed that x and m satisfy the conditions (2.1). There are known relations of linear dependence, see for example [7, p. 117], which permit the derivation of the asymptotic expansions of the solutions $M_{k,m}(x)$ and $M_{k,-m}(x)$ from those for the solutions $W_{k,m}(x)$ and $W_{-k,m}(-x)$.

3. **Transformation of the Whittaker equation.** If we choose

$$\text{(3.1)} \qquad \begin{aligned} &\text{(a)}\ z = \log(x/2im), \\ &\text{(b)}\ u = e^{-z/2}W/2i \end{aligned}$$

as new independent and dependent variables in the Whittaker equation, the differential equation

$$\text{(3.2)} \qquad \frac{d^2u}{dz^2} + \{m^2(e^{2z} - 1) + m(2ike^z)\}u = 0$$

is obtained. The relation (3.1a) maps the region (2.1a) of the x-plane conformally onto the horizontal strip of the z-plane for which

$$\text{(3.3)} \qquad -\pi/2 - \arg(m) \leqq I(z) \leqq \pi/2 - \arg(m).$$

Since the variable of immediate interest is x and since it is a single-valued function of z, we may confine our attention to the values of z specified by the relation

$$-\pi < \arg(z) \leqq \pi.$$

In doing this we cut the x-plane from the point $x = 2im$ to the origin.

The coefficients of m and m^2 in equation (3.2) are seen to be single-valued analytic functions of z in the cut strip (3.3). For the values of z under consideration the coefficient $(e^{2z} - 1)$ has a simple zero at the origin and is elsewhere bounded from zero [see note to the condition (2.1a)]. The origin is therefore a first order turning point for equation (3.2). The point $x = 2im$ is carried into the origin by the transformation (3.1a). Thus it is seen that $x = 2im$ is a significant point for the study of the Whittaker equation.

4. **Solutions in a region containing the point** $x = 2im$. We first consider equation (3.2) in the sub-region of the strip (3.3) for which

$$\text{(4.1)} \qquad \mathrm{Re}(z) < \beta,$$

where β is an arbitrary positive real number. (As seen on the x-plane this is the region for which $|x/2m| < e^\beta$.) In this sub-region equation (3.2) has a turning point of the first order and is of the form (1.2) with $q_0(z)=e^{2z}-1$ and $q_1(z)=2ike^z$.

The theory of asymptotic solutions of equations of the type (1.2) is given in Part II. It involves several functions of $q_0(z)$ and $q_1(z)$. Among these we have evaluated the following in terms of the coefficients of equation (3.2):

$$\text{(4.2)}\quad\begin{aligned}
&\text{(a)} && \phi(z) = (e^{2z}-1)^{1/2} = \left|e^{2z}-1\right|^{1/2}\exp\left\{(i/2)\arg(e^{2z}-1)\right\},\\
&\text{(b)} && \Phi(z) = \phi(z) + (i/2)\log\left[\frac{i-\phi(z)}{i+\phi(z)}\right],\\
&\text{(c)} && \xi(z,m) = m\Phi(z),\\
&\text{(d)} && \mu_0(z) = \cos\left\{ik\log\left[e^z+\phi(z)\right]\right\},\\
&\text{(e)} && \mu_1(z) = [\phi(z)]^{-1}\sin\left\{ik\log\left[e^z+\phi(z)\right]\right\}.
\end{aligned}$$

The function Φ defines a mapping of the region (3.3) onto a portion of a Riemann surface which will be denoted R_Φ. It can be shown by calculation that the function Φ is bounded from zero in the strip (3.3) except at the origin [5, p. 467]. Thus the origin and the point at infinity are the branch points of the mapping from z to Φ. This mapping, which has been discussed by Langer [5, p. 466], is described in the adjoining figures. The regions on the z-strip (3.3) and on R_Φ which are bounded by the curves OA and CD, OB and CB, OA and FG, and OE and FE correspond. The complete surface R_Φ is formed by joining the two parts of Fig. 2 along the line OA.

A surface R_ξ is obtained from R_Φ by a rotation counterclockwise about the origin through the angle $\arg(m)$ and by an expansion outward from the origin by the factor $|m|$.

We also note that when the real part of z is negative and its absolute value large,

$$\text{(4.3)}\quad\begin{aligned}
&\text{(a)} && \phi = \pm\, i + O(e^{2z}),\\
&\text{(b)} && \Phi \doteq \pm\, i[z+\log e/2] + O(e^{2z}),\\
&\text{(c)} && \Psi'/\Psi = O(z^{-1}),\\
&\text{(d)} && \mu_0 = \cos\left[k\pi/2 + o(z)\right],\\
&\text{(e)} && \mu_1 = \frac{\sin\left[\mp k\pi/2 + o(z)\right]}{\pm\, i + O(e^{2z})},\\
&\text{(f)} && \mu_j^{(n)} = O(e^z), \qquad j = 1, 2;\ n = 1, 2, 3, \cdots,\\
&\text{(g)} && D_0(z,m) = O(1),\\
&\text{(h)} && K(z,m) = O(z^{-2}),
\end{aligned}$$

where the functions $\Psi(z)$, $D_0(z, m)$, and $K(z, m)$ are defined in §12 of Part II. In these formulas the upper sign is to be used when $I(z)\geqq 0$; the lower sign is to be used when $I(z)<0$.

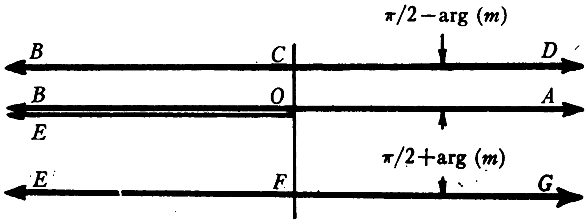

FIG. 1. The z-strip (3.3)

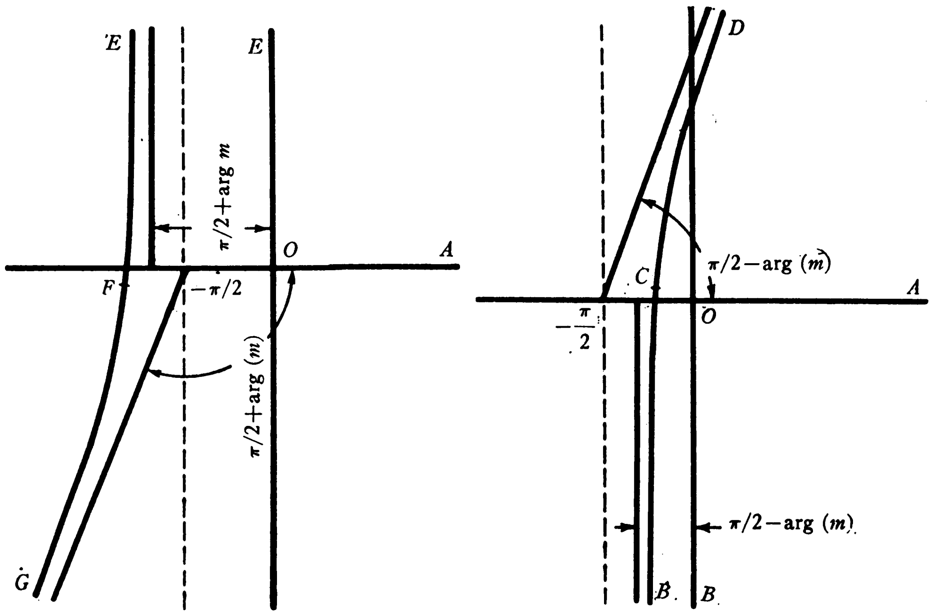

FIG. 2. R_Φ

From these and the previous facts obtained concerning the coefficients of equation (3.2) it is seen that the hypotheses in §§11, 12, and 14 of Part II[8] are fulfilled by this equation in the region (4.1). It follows that in the region (4.1) the differential equation (3.2) admits of pairs of solutions $u_{h,1}(z, m)$ and $u_{h,2}(z, m)$ which have the asymptotic forms

(4.4)
$$\text{(a)}\quad u_{h,1}(z, m) = m^{-1/6}\phi^{-1/2}e^{i\xi}(\mu_0 + i\mu_1\,\phi)\left\{1 + \frac{E(z, m)}{m} + \frac{E(\xi)}{\xi}\right\},$$
$$\text{(b)}\quad u_{h,2}(z, m) = m^{-1/6}\phi^{-1/2}e^{-i\xi}(\mu_0 - i\mu_1\,\phi)\left\{1 + \frac{E(z, m)}{m} + \frac{E(\xi)}{\xi}\right\},$$

([8]) The hypotheses of §§12 and 14 are italicized for convenience in reference.

when $(h-1)\pi+\epsilon \leqq \arg(\xi) \leqq (h+1)\pi-\epsilon$, $h=0, \pm 1, \pm 2$. The letter E is used as a generic symbol to denote a bounded function of the argument or arguments indicated.

5. **Solutions when $|x/2m|$ is large.** In order to identify the Whittaker functions in terms of the solutions $u_{h,1}(z, m)$ and $u_{h,2}(z, m)$ of §4 and also to find their structures when $|x/2m|$ is large, we next consider the differential equation (3.2) in the portion of the region (3.3) for which

$$\text{(5.1)} \qquad \operatorname{Re}(z) \geqq \alpha,$$

where α is any positive real number less than β. (As seen on the x-plane this is the region for which $|x/2m| \geqq e^{\alpha}$.) Equation (3.2) has no turning point in the domain (5.1); therefore, it is an equation of the type considered in §16 of Part II. The theory presented there involves functions $\psi(z, m)$, $\theta(z, m)$, and $\zeta(z, m)$, which for equation (3.2) are found to be

$$\text{(5.2)} \qquad \begin{aligned} &\text{(a)}\ \psi(z, m) = \phi(z) + \frac{ike^{z}}{m\phi(z)}, \\ &\text{(b)}\ \theta(z, m) = \frac{\psi''}{2\psi} - \frac{3\psi'^2}{4\psi^2} + \frac{k^2e^{2z}}{\phi^2}, \\ &\text{(c)}\ \zeta(z, m) = m\phi + \frac{im}{2}\log\left[\frac{i-\phi}{i+\phi}\right] + ik\log\left[e^{z}+\phi\right]. \end{aligned}$$

We now discuss some of their properties. A simple calculation shows that the function $\psi(z, m)$ is bounded from zero in the region (5.1) for $|m|$ sufficiently large. When the real part of z is positive and large,

$$\psi(z, m) = e^{z} + O(1),$$

so that for such values of z, $\theta(z, m) = O(1)$. The relation between z and ζ determines a mapping of the domain (5.1) onto a region which we designate T_ζ. From the formulas (4.2b), (4.2c), and (5.2c) it is seen that

$$\text{(5.3a)} \qquad \zeta = \xi + ik\log\left[e^{z} + (e^{2z}-1)^{1/2}\right];$$

consequently, when the real part of z is positive and large,

$$\text{(5.3b)} \qquad \zeta = me^{z} - m\pi/2 + ik(z + \log 2) + o(z).$$

Employing these facts, the nature of the region T_ζ is easily determined. T_ζ is illustrated in the adjoining Fig. 3. It is apparent that the points of T_ζ can all be connected to the image on T_ζ of the "point" $x = -\infty$ (approached along the ray $\arg(x) = \pi$) by curves upon which the imaginary part of ζ is monotonically increasing. The image on T_ζ of the "point" $x = +\infty$ (approached along the ray $\arg(x) = 0$) has a similar property.

It is now clear that when the differential equation (3.2) is considered in the region (5.1), it fulfills the hypotheses set forth in §16 of Part II. If the lower limits of integration which appear in the relations (16.6a) and (16.6b) are chosen as the images on the region (5.1) of the "points" $x=-\infty$ and $x=+\infty$, respectively, a particular pair of solutions, $U_1(z, m)$ and $U_2(z, m)$, of equation (3.2) is determined. For values of z in the region (5.1) these solutions have the asymptotic forms

$$(5.4)\quad \begin{aligned} &\text{(a)}\ U_1(z, m) = e^{i\zeta}\psi^{-1/2}\left\{1 + \frac{E(z, m)}{m}\right\},\\ &\text{(b)}\ U_2(z, m) = e^{-i\zeta}\psi^{-1/2}\left\{1 + \frac{E(z, m)}{m}\right\}. \end{aligned}$$

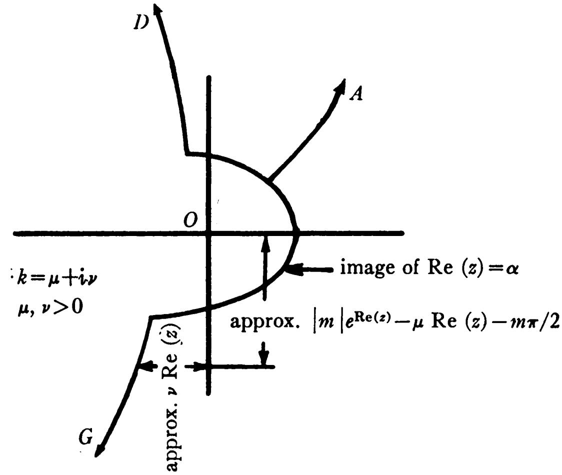

FIG. 3. T_ζ

6. **Identification of $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ when $|x/2m| \geqq e^{\alpha}$.** Among any three solutions of equation (3.2) there exists a relation of linear dependence. Thus identities subsist which relate the solutions $(e^{-z/2}/2i)W_{k,m}(x)$ and $(e^{-z/2}/2i)W_{-k,m}(e^{-\pi i}x)$ to the pair of solutions $U_1(z, m)$ and $U_2(z, m)$. These may be written as

$$(6.1)\quad \begin{aligned} &\text{(a)}\ \frac{e^{-z/2}}{2i} W_{k,m}(x) = C_{1,1}U_1(z, m) + C_{1,2}U_2(z, m),\\ &\text{(b)}\ \frac{e^{-z/2}}{2i} W_{-k,m}(e^{-\pi i}x) = C_{2,1}U_1(z, m) + C_{2,2}U_2(z, m). \end{aligned}$$

The coefficients $C_{h,j}$ are constant as to z but may depend upon the parameters k and m.

We first consider the identity (6.1a). The function $W_{k,m}(x)$ is known to have the property:

(6.2) $$\lim_{|x|\to\infty} e^{x/2}x^{-k}W_{k,m}(x) = 1, \qquad |\arg(x)| < \pi \text{ [8, p. 343]}.$$

The approximation

(6.3) $$e^{i\zeta} = x^{-k}e^{x/2}i^{k-m}m^{k}\{1 + o(z)\}$$

follows from the relation (5.3b) by the transformation (3.1a). Let x be chosen satisfying the condition (2.1a) with $\operatorname{Re}(x) > 0$ and such that the corresponding z is in the region (5.1). For such an x, employing the approximation (6.3), the identity (6.1a) with $U_1(z, m)$ and $U_2(z, m)$ replaced by their asymptotic expansions (5.4) takes the form

$$\begin{aligned} W_{k,m}(x) = {} & 2iC_{1,1}e^{x/2}x^{-k}i^{k-m}m^{k}\left\{1 + \frac{E(z, m)}{m}\right\}\{1 + o(z)\} \\ & + 2iC_{1,2}e^{-x/2}x^{k}i^{-k+m}m^{-k}\left\{1 + \frac{E(z, m)}{m}\right\}\{1 + o(z)\}. \end{aligned}$$

Upon multiplication of both sides of this expression by the quantity $e^{x/2}x^{-k}$, it becomes

$$\begin{aligned} e^{x/2}x^{-k}W_{k,m}(x) = {} & 2iC_{1,1}e^{x}x^{-2k}i^{k-m}m^{k}\left\{1 + \frac{E(z, m)}{m}\right\}\{1 + o(z)\} \\ & + 2iC_{1,2}i^{-k+m}m^{-k}\left\{1 + \frac{E(z, m)}{m}\right\}\{1 + o(z)\}. \end{aligned}$$

After taking limits as $\operatorname{Re}(x)$ becomes positively infinite, it is found with the aid of the limit (6.2) that

$$C_{1,1} = 0; \quad \text{and hence,} \quad C_{1,2} = (1/2)i^{k-m-1}m^{k}\{1 + E(m)/m\}.$$

The conclusion that for values of z in the region (5.1)

(6.4) $$W_{k,m}(x) = i^{k-m}m^{k}e^{z/2}U_2(z, m)\left\{1 + \frac{E(m)}{m}\right\}$$

is thus obtained.

If x and k are replaced by $e^{-\pi i}x$ and $-k$, respectively, the limit (6.2) becomes

$$\lim_{|x|\to\infty} e^{-x/2}x^{k}W_{-k,m}(e^{-\pi i}x) = e^{k\pi i}, \qquad |\arg(e^{-\pi i}x)| < \pi.$$

Now let x be chosen satisfying the condition (2.1a) with $\operatorname{Re}(x) < 0$ and such that the corresponding z is in (5.1). Then the identity (6.1b), when multiplied by the quantity $e^{-x/2}x^{k}$, may be cast into the form

$$e^{-x/2}x^k W_{-k,m}(e^{-\pi i}x) = 2iC_{2,1}m^k i^{k-m}\left\{1+\frac{E(z,m)}{m}\right\}\{1+o(z)\}$$
$$+ 2iC_{2,2}m^{-k}i^{m-k}e^{-x}x^{2k}\left\{1+\frac{E(z,m)}{m}\right\}\{1+o(z)\}.$$

Taking limits as Re (x) becomes negatively infinite we find, using the limit (6.5), that

$$C_{2,1} = (1/2)i^{k+m-1}m^{-k}\left\{1+\frac{E(m)}{m}\right\}, \quad \text{and} \quad C_{2,2} = 0.$$

Substitution of these values for the coefficients in the relation (6.1b) establishes the result that for z in the domain (5.1)

$$(6.6) \qquad W_{-k,m}(e^{-\pi i}x) = i^{m+k}m^{-k}e^{z/2}U_1(z, m)\left\{1+\frac{E(m)}{m}\right\}.$$

7. **Identification of $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ when $|x/2m| \leqq e^\beta$.** As the functions described by the representations (4.4) and (5.4) solve the same differential equation, namely equation (3.2), there exist dependence relations between them. It is through these relations that we shall identify the functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ in terms of the solutions $u_{h,j}(z, m)$ of §4. Since the constants α and β associated with the regions (4.1) and (5.1) were chosen so that $\alpha<\beta$, these regions overlap. For values of z in their intersection

$$-\pi+\epsilon < \arg(\xi) < \pi-\epsilon.$$

We are thus led to consider the identities

$$(7.1) \qquad U_j(z, m) = B_{j,1}u_{0,1}(z, m) + B_{j,2}u_{0,2}(z, m), \qquad j = 1, 2.$$

The coefficients, which are independent of z, are given by the formulas[4]

$$(7.2) \qquad B_{j,1} = \frac{W(u_{0,2}, U_j)}{W(u_{0,2}, u_{0,1})}, \qquad B_{j,2} = \frac{W(u_{0,1}, U_j)}{W(u_{0,1}, u_{0,2})}.$$

From the identifications (4.2) it can be shown that

$$\mu_0 \pm i\mu_1\phi = (e^z+\phi)^{\mp k}.$$

From this and the relation (5.3a) the equalities

$$e^{\pm i\xi}(\mu_0 \pm i\mu_1\phi) = e^{\pm i\zeta}$$

are obtained. Thus the representations (4.4) with $h=0$ can be written in the form

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(4) The symbol $W(u, v)$ denotes the Wronskian of the two functions u and v.

$$(7.3) \qquad u_{0,i}(z, m) = m^{-1/6}\phi^{-1/2}e^{\pm i\zeta}\left\{1 + \frac{E(z, m)}{m} + \frac{E(\xi)}{\xi}\right\}, \qquad i = 1, 2.$$

Further, it is seen from the relations (4.2a) and (5.2a) that when z is in the region (5.1)

$$\psi(z, m) = \phi(z) + \frac{E(z, m)}{m} \cdot$$

We now calculate the coefficients (7.2). From the formulas (5.4) and (7.3) it is found that

$$\begin{aligned} W(u_{0,1}, u_{0,2}) &= -2im^{2/3}\{1 + E(m)/m\}, \\ W(u_{0,1}, U_1) &= -\ im^{5/6}e^{2i\zeta}\{E(m)/m + E(\xi)/\xi\}, \\ W(u_{0,1}, U_2) &= -\ 2im^{5/6}\{1 + E(m)/m\}, \\ W(u_{0,2}, U_1) &= 2im^{5/6}\{1 + E(m)/m\}, \\ W(u_{0,2}, U_2) &= -\ im^{5/6}e^{-2i\zeta}\{E(m)/m + E(\xi)/\xi\}. \end{aligned}$$

Since $W(u_{0,2}, U_2)$ is constant as to z, in computing its value we may choose any point z_1 which is in both the regions (4.1) and (5.1). In particular we choose z_1 as such a point for which Re $(i\zeta)$ is maximum. Then,

$$W(u_{0,2}, U_2) \sim e^{-2i\zeta_1}E(m)m^{-1/6}, \qquad \zeta_1 = \zeta(z_1, m);$$

and hence,

$$B_{2,1} \sim e^{-2i\zeta_1}E(m)m^{-5/6}, \qquad B_{2,2} \sim m^{1/6}.$$

Therefore the identity (7.1) with $j=2$ becomes

$$(7.4) \qquad U_2(z, m) \sim e^{-2i\zeta_1}E(m)m^{-5/6}u_{0,1}(z, m) + m^{1/6}u_{0,2}(z, m).$$

We may rewrite this relation in the form

$$U_2(z, m) \sim \frac{e^{i(\zeta-2\zeta_1)}}{m}\phi^{-1/2} + e^{-i\zeta}\phi^{-1/2}.$$

Because we have chosen z_1 so that Re $(i\zeta)$ is a maximum at ζ_1, we see that the second term on the right-hand side of the last representation is the dominant one even though the quantity $e^{i\zeta}$ may be large.

In evaluating $W(u_{0,1}, U_1)$ we may also choose z to be any point which is in both the regions (4.1) and (5.1). In this instance we choose z to be z_2, a point for which Re $(i\zeta)$ is a minimum. Then

$$W(u_{0,1}, U_1) \sim -\ ie^{2i\zeta_2}E(m)m^{-1/6},$$

$$\zeta_2 = \zeta(z_2, m);$$

and hence

$$B_{1,1} \sim m^{1/6},$$
$$B_{1,2} \sim e^{2i\zeta_2}E(m)m^{-5/6}.$$

Consequently, with $j=1$ the identity (7.1) becomes

$$U_1(z, m) \sim m^{1/6}\, u_{0,1}(z, m) + e^{2i\zeta_2}E(m)m^{-5/6}u_{0,2}(z, m). \tag{7.5}$$

We note that because we have picked z_2 so that Re $(i\zeta)$ is a minimum at ζ_2, the first term on the right-hand side of the above representation is the dominant one even though $e^{-i\zeta}$ may be large.

The formulas (6.4), (6.6), (7.4), and (7.5) lead to these results:

(7.6)

$$\text{(a)} \quad W_{k,m}(x) \sim i^{k-m}m^k e^{z/2}\left\{e^{-2i\zeta_1}m^{-5/6}E(m)u_{0,1}(z, m) + m^{1/6}u_{0,2}(z, m)\right\},$$

$$\text{(b)} \quad W_{-k,m}(e^{-\pi i}x) \sim i^{k+m}m^{-k}e^{z/2}\left\{m^{1/6}u_{0,1}(z, m) + e^{2i\zeta_2}E(m)m^{-5/6}u_{0,2}(z, m)\right\}.$$

8. **The asymptotic forms.** With the identification of the functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ in terms of the solutions $u_{0,j}(z, m)$ completed, we may use the relations (14.6) and (15.7) of Part II to compute the asymptotic expansions of these functions throughout the image on R_ξ of the region (4.1). By way of summary the results of these calculations are included in the second of the succeeding tables. In these calculations only the dominant terms of the coefficients of $e^{i\xi}$ and $e^{-i\xi}$ have been kept.

Having obtained the representations for the Whittaker functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$, the identity

(8.1)

$$M_{k,m}(x) = \frac{\Gamma(2m+1)}{\Gamma(m+1/2-k)}e^{-k\pi i}W_{-k,m}(e^{-\pi i}x) + \frac{\Gamma(2m+1)}{\Gamma(m+1/2+k)}e^{(m-k+1/2)\pi i}W_{k,m}(x) \qquad [7, p. 117$$

can be employed to find similar expressions for the functions $M_{k,m}(x)$ and $M_{k,-m}(x)$. The formula

$$\Gamma(\lambda + a) \sim \lambda^{\lambda+a-1/2}e^{-\lambda}(2\pi)^{1/2}, \quad |\arg(\lambda + a)|, \quad |\arg(\lambda)| \leqq \pi - \delta \tag{8.2}$$

[8, p. 279]

allows reduction of the gamma functions present in the coefficients of the identity (8.1) to more suitable form. The asymptotic forms of the functions $M_{k,m}(x)$ and $M_{k,-m}(x)$ are given in the third and fourth tables. Two sets of forms are shown for the function $M_{k,-m}(x)$ because of the limitation imposed on arg (m) by the relation (8.2).

In these tables the abbreviations

$$
(8.3)\qquad
\begin{aligned}
&\text{(a)}\quad \omega_1 = e^{(-m+k)\pi i/2}m^{k},\\
&\text{(b)}\quad \omega_2 = e^{(m+k)\pi i/2}m^{-k},\\
&\text{(c)}\quad \omega_3 = e^{(m-k+1)\pi i/2}2^{2m+1/2}m^{m+1/2}e^{-m},\\
&\text{(d)}\quad \omega_4 = e^{(2-5m-3k)\pi i/2}2^{-2m+1/2}m^{-m+1/2}e^{m},\\
&\text{(e)}\quad \omega_5 = e^{(1-m+k)\pi i/2}2^{-2m+1/2}m^{-m+1/2}e^{m}
\end{aligned}
$$

have been used.

The geometric determination of the images of the regions $\Xi^{(j)}$ upon the region (4.1) and upon the x-plane can be found in Langer's work [5, p. 372]. This information can, therefore, be omitted here.

TABLE 1. Forms when $|x/2m| \geqq e^{\alpha}$

$W_{k,m}(x)$	$\omega_1\psi^{-1/2}e^{z/2-i\zeta}$
$W_{-k,m}(e^{-\pi i}x)$	$\omega_2\psi^{-1/2}e^{z/2+i\zeta}$
$M_{k,m}(x)$	$\omega_3\psi^{-1/2}\{e^{i(\zeta-\pi/2)}+e^{-i\zeta}\}e^{z/2}$
$M_{k,e^{\pi i}m}(x)$ $(-\pi/2<\arg(m)<0)$	$\omega_4\psi^{-1/2}e^{z/2-i\zeta}+\omega_5\psi^{-1/2}e^{z/2+i\zeta}$
$M_{k,e^{-\pi i}m}(x)$ $(0<\arg(m)\leqq\pi/2)$	$\omega_5\psi^{-1/2}e^{z/2-i\zeta-\pi i/2}+\omega_4e^{4m\pi i}\psi^{-1/2}e^{z/2+i\zeta+\pi i/2}$

$$z=\log(x/2im),\qquad \psi=(e^{2z}-1)^{1/2}+ikm^{-1}e^{z}(e^{2z}-1)^{-1/2},$$

$$\zeta=m(e^{2z}-1)^{1/2}+(i/2)m\log\left[\frac{i-(e^{2z}-1)^{1/2}}{i+(e^{2z}-1)^{1/2}}\right]+ik\log\left[e^{z}+(e^{2z}-1)^{1/2}\right]$$

TABLE 2. Forms for $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ when ξ is in $\Xi^{(j)}$

j	$W_{k,m}(x)$	$W_{-k,m}(e^{-\pi i}x)$
2	$\omega_1\phi^{-1/2}\{e^{i(\xi+\pi/2)}[e^{z}+\phi]^{-k}\}e^{z/2}$	$\omega_2\phi^{-1/2}\{e^{i\xi}[e^{z}+\phi]^{-k}+e^{i(\pi/2-\xi)}[e^{z}+\phi]^{k}\}e^{z/2}$
1	$\omega_1\phi^{-1/2}\{e^{i(\xi+\pi/2)}[e^{z}+\phi]^{-k}+e^{-i\xi}[e^{z}+\phi]^{k}\}e^{z/2}$	$\omega_2\phi^{-1/2}\{e^{i\xi}[e^{z}+\phi]^{-k}\}e^{z/2}$
0	$\omega_1\phi^{-1/2}\{e^{-i\xi}[e^{z}+\phi]^{k}\}e^{z/2}$	$\omega_2\phi^{-1/2}\{e^{i\xi}[e^{z}+\phi]^{-k}\}e^{z/2}$
-1	$\omega_1\phi^{-1/2}\{e^{-i\xi}[e^{z}+\phi]^{k}\}e^{z/2}$	$\omega_2\phi^{-1/2}\{e^{i\xi}[e^{z}+\phi]^{-k}+e^{-i(\xi+\pi/2)}[e^{z}+\phi]^{k}\}e^{z/2}$
-2	$\omega_1\phi^{-1/2}\{e^{i(\xi-\pi/2)}[e^{z}+\phi]^{-k}+e^{-i\xi}[e^{z}+\phi]^{k}\}e^{z/2}$	$\omega_2\phi^{-1/2}\{e^{-i(\xi+\pi/2)}[e^{z}+\phi]^{k}\}e^{z/2}$

$$\phi=(e^{2z}-1)^{1/2},\qquad \xi=m\phi+\frac{im}{2}\log\left[\frac{i-\phi}{i+\phi}\right]$$

$$\Xi^{(j)}:\ (j-1)\pi+\epsilon\leqq\arg(\xi)\leqq(j+1)\pi-\epsilon$$

TABLE 3. Forms for $M_{k,m}(x)$ when ξ is in $\Xi^{(j)}$

j	
2	$\omega_3\phi^{-1/2}\{e^{-i\xi}[e^z+\phi]^k\}e^{z/2}$
1	$\omega_3\phi^{-1/2}\{e^{-i\xi}[e^z+\phi]^k\}e^{z/2}$
0	$\omega_3\phi^{-1/2}\{e^{i(\xi-\pi/2)}[e^z+\phi]^{-k}+e^{-i\xi}[e^z+\phi]^k\}e^{z/2}$
-1	$\omega_3\phi^{-1/2}\{e^{i(\xi-\pi/2)}[e^z+\phi]^{-k}\}e^{z/2}$
-2	$\omega_3\phi^{-1/2}\{e^{i(\xi-\pi/2)}[e^z+\phi]^{-k}\}e^{z/2}$

TABLE 4a. Forms for $M_{k,e^{\pi i}m}(x)$ when $-\pi/2<\arg(m)<0$ and ξ is in $\Xi^{(j)}$

j	
2	$\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}+e^{-i(\xi-\pi/2)}[e^z+\phi]^k\}e^{z/2}$
1	$\omega_4\phi^{-1/2}\{e^{-i\xi}[e^z+\phi]^k\}e^{z/2}+\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}\}e^{z/2}$
0	$\omega_4\phi^{-1/2}\{e^{-i\xi}[e^z+\phi]^k\}e^{z/2}+\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}\}e^{z/2}$
-1	$\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}+e^{-i(\xi+\pi/2)}[e^z+\phi]^k\}e^{z/2}$
-2	$\omega_4\phi^{-1/2}\{e^{i(\xi-\pi/2)}[e^z+\phi]^{-k}\}e^{z/2}+\omega_5\phi^{-1/2}\{e^{-i(\xi+\pi/2)}[e^z+\phi]^k\}e^{z/2}$

TABLE 4b. Forms for $M_{k,e^{-\pi i}m}(x)$ when $0<\arg(m)\leqq\pi/2$ and ξ is in $\Xi^{(j)}$

j	
2	$\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}\}e^{z/2}+e^{4m\pi i}\omega_4\phi^{-1/2}\{-e^{-i\xi}[e^z+\phi]^k\}e^{z/2}$
1	$\omega_5\phi^{-1/2}\{e^{i\xi}[e^z+\phi]^{-k}+e^{-i(\xi+\pi/2)}[e^z+\phi]^{+k}\}e^{z/2}$
0	$\omega_5\phi^{-1/2}\{e^{-i(\xi+\pi/2)}[e^z+\phi]^{+k}\}+e^{4m\pi i}\omega_4\phi^{-1/2}\{e^{i(\xi+\pi/2)}[e^z+\phi]^{-k}\}e^{z/2}$
-1	$\omega_5\phi^{-1/2}\{e^{-i(\xi+\pi/2)}[e^z+\phi]^{+k}\}e^{z/2}+e^{4m\pi i}\omega_4\phi^{-1/2}\{e^{i(\xi+\pi/2)}[e^z+\phi]^{-k}\}e^{z/2}$
-2	$\omega_5\phi^{-1/2}\{-e^{i\xi}[e^z+\phi]^{-k}+e^{-i(\xi+\pi/2)}[e^z+\phi]^k\}e^{z/2}$

9. Representations in terms of x. Since

$$\phi(x) = \{ | x/2m |^2 \exp \{2i[\arg (x) - \arg (m) - \pi/2]\} - 1\}^{1/2},$$

arg (ϕ) is greater or less than zero according as arg (x) is greater or less than $\pi/2+\arg (m)$. In light of the convention (4.2a) that the quantity $[(x/2im)^2 - 1]^{1/2}$ be interpreted as

$$| (x/2m)^2 - 1 |^{1/2} \exp \{(i/2) \arg [(x/2im)^2 - 1]\},$$

it is convenient in succeeding formulas to write

$$\phi(x) = i\{(x^2e^{-2\pi i}/4m^2) + 1\}^{1/2} \qquad \text{when } \arg (\phi) > 0,$$
$$\phi(x) = - i\{(x^2/4m^2) + 1\}^{1/2} \qquad \text{when } \arg (\phi) < 0.$$

Reduction of the forms appearing in Tables 3 and 4 yields these results for values of x such that $|x/2m| <1$:

$$\begin{aligned} M_{k,\pm m}(\pm 2mx) \sim {}& 2^{\pm 2m+1/2}(\pm m)^{\pm m+1/2} \exp \{ \pm m[(x^2e^{-2\pi i} + 1)^{1/2} - 1]\} \\ & \cdot [(x^2e^{-2\pi i} + 1)^{1/2} - 1]^{\pm m}[(x^2e^{-2\pi i} + 1)^{1/2} - x]^k \frac{x^{\mp m+1/2}}{(x^2e^{-2\pi i} + 1)^{1/4}}, \end{aligned} \tag{9.1a}$$

when $\arg (x)>\pi/2+\arg (m)$; and

$$\begin{aligned} M_{k,\pm m}(\pm 2mx) \sim {}& 2^{\pm 2m+1/2}(\pm m)^{\pm m+1/2}x^{\mp m+1/2} \exp \{ \pm m[(x^2 +1)^{1/2} - 1]\} \\ & \cdot [(x^2 + 1)^{1/2} - 1]^{\pm m}[(x^2 + 1)^{1/2} - x]^k(x^2 + 1)^{-1/4}, \end{aligned} \tag{9.1b}$$

when $\arg (x)<\pi/2+\arg (m)$. These representations include only dominant terms. Similar formulas can be obtained for the functions $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$.

In the case that $|x| >2|m|$ the reduction of the forms in the tables leads to less simple descriptions of the functions $M_{k,\pm m}(x)$ than those above. If m is restricted to positive real values, however, $M_{k,m}(x)$ is described by the forms (9.1) when $|x/2m| >1$; but it is found that

$$\begin{aligned} M_{k,-m}(- 2mx) \sim {}& \frac{\sin (1/2 + m - k)}{\sin (2m\pi)} 2^{-2m+1/2}(-m)^{-m+1/2}x^{-m+1/2} \\ & \cdot \exp \{m[(x^2e^{-2\pi i} + 1)^{1/2} + 1]\} [(x^2e^{-2\pi i} + 1)^{1/2} - 1]^m \\ & \cdot [(x^2e^{-2\pi i} + 1)^{1/2} + x]^k(x^2e^{-2\pi i} + 1)^{-1/4}, \end{aligned} \tag{9.2a}$$

when $\arg (x)>\pi/2$; and

$$\begin{aligned} M_{k,-m}(- 2mx) \sim {}& \frac{\sin (1/2 + m + k)}{\sin (2m\pi)} 2^{-2m+1/2}(-m)^{-m+1/2}x^{-m+1/2} \\ & \cdot \exp \{m[(x^2 + 1)^{1/2} + 1]\} [(x^2 + 1)^{1/2} - 1]^m \\ & \cdot [(x^2 + 1)^{1/2} + x]^k(x^2 + 1)^{-1/4}, \end{aligned} \tag{9.2b}$$

when $\arg(x) < \pi/2$. The representations (9.1) are in agreement with those derived by Fisher [4, p. 533] subject to the condition that $|x/2m|$ be less than one, provided that the correct determination of the argument of the quantity $(x^2/4m^2+1)^{1/2}$ which appears in his formulas is made.

A reduction of the forms in Table 1 yields the results

$$
(9.3)\quad
\begin{aligned}
&\text{(a)} \quad W_{k,m}(2imx) \sim m^k[x^2-1]^{-1/4}x^{1/2+m}[x+(x^2-1)^{1/2}]^{+k} \\
&\qquad\qquad \cdot [(i+(x^2-1)^{1/2}]^{-m} \\
&\qquad\qquad \cdot \exp\{i[k\pi/2 - m(x^2-1)^{1/2}]\}, \\
&\text{(b)} \quad W_{-k,m}(2e^{-\pi i/2}mx) \sim m^{-k}[x^2-1]^{-1/4}x^{1/2-m}[x+(x^2-1)^{1/2}]^{-k} \\
&\qquad\qquad \cdot [i+(x^2-1)^{1/2}]^{m} \\
&\qquad\qquad \cdot \exp\{i[k\pi/2 + m(x^2-1)^{1/2}]\},
\end{aligned}
$$

for $|x/2m| > 1$. It should be noted that the algorithm Langer has developed in [6] can be employed to refine our approximations to include terms of order m^{-n}, n a positive integer.

10. **Formulas for $W_{k,m}(x)$ and $W_{-k,m}(e^{-\pi i}x)$ when $|\xi|$ is small.** If the modulus of ξ is not large, the asymptotic expansions given in the tables of §8 do not apply. In this instance approximation by means of the relation (15.9) of Part II is appropriate. The results which follow are obtained from this relation and the formulas (7.6).

$$
(10.1)\quad
\begin{aligned}
&\text{(a)} \quad W_{k,m}(x) = i^{k-m}m^{k+1/6}[1+O(m^{-1})](x/2im)^{1/2} \\
&\qquad\qquad \cdot [\gamma_{1,2}^{(0)}y_1 + \gamma_{2,2}^{(0)}y_2 + O(m^{-2/3})], \\
&\text{(b)} \quad W_{-k,m}(e^{-\pi i}x) = i^{m+k}m^{-k+1/6}[1+O(m^{-1})](x/2im)^{1/2} \\
&\qquad\qquad \cdot [\gamma_{1,1}^{(0)}y_1 + \gamma_{2,1}^{(0)}y_2 + O(m^{-2/3})].
\end{aligned}
$$

It is assumed that the modulus of m is sufficiently large. The coefficients $\gamma_{h,j}^{(0)}$ are evaluated by Langer in [5, p. 460]. The functions $v_j(z, m)$ and $y_j(z, m)$, which are defined by the formulas (15.1) and (15.2), involve Bessel functions of orders $\pm 1/3$ and their derivatives; and since the modulus of ξ is small, the ordinary series for the Bessel functions can be used in the relations (10.1). For use in these formulas the approximations

$$
\mu_0(z) \sim 1, \qquad \mu_1(z) \sim ik, \qquad D_0(z, m) \sim 1,
$$
$$
\xi \sim m[(x/2im)^2 - 1]^{3/2}/3
$$

are also permissible.

Part II. The asymptotic solutions of a certain type of linear differential equation

11. **Introduction.** The following discussion concerns differential equations

of the type

$$\frac{d^2u(z,\lambda)}{dz^2} + \{\lambda^2 q_0(z) + \lambda q_1(z)\}u(z,\lambda) = 0, \tag{11.1}$$

where the symbol λ denotes a complex parameter large in absolute value. Analytic forms are found which represent solutions of this equation asymptotically as to λ in a closed simply-connected region, S_z, of the complex z-plane (with the point at infinity deleted). The coefficients $q_0(z)$ and $q_1(z)$ are considered to be single-valued analytic functions in S_z.

Investigations of equations of this type have been made previously but, to the author's knowledge, never in an unbounded region in which $q_0(z)$ has a single simple zero or in an unbounded region in which $q_0(z)$ has no zeros. The analysis presented here relies heavily on two of these investigations by Langer. The first is a discussion for real values of the independent variable on a bounded interval of an equation of which equation (11.1) is a special case [6]. The second treats equation (11.1) in the case that $q_1(z)$ is identically zero [5]. In these two papers asymptotic expansions for the solutions of the differential equation under consideration are found in terms of known ones for the Bessel functions. Such is the case here; in fact, both the formal structure and the proofs of this analysis closely parallel work in [5] and [6]. The ensuing discussion is, therefore, succinct.

Equation (11.1), henceforward to be referred to as the "given equation," is first considered in the circumstance that $q_0(z)$ has a single simple zero in S_z. It is then discussed for $[q_0(z)]^{-1}$ bounded there.

12. **Preliminaries.** It is assumed in §§12–15 that $q_0(z)$ *has one simple zero in* S_z. Its reciprocal is therefore bounded except in the neighborhood of the zero of $q_0(z)$. Without loss of generality, this zero may be supposed to occur at the origin; and by the transfer of a constant to λ^2 the argument of $z^{-1}q_0(z)$ may be adjusted to zero there. It is also assumed that $\int_0^z q_0^{1/2}(t)dt$ *is bounded from zero except at the origin.*

Let us now recall from [6] the definitions of the following functions

(12.1)

(a) $\phi(z) = q_0^{1/2}(z) = |q_0(z)|^{1/2} \exp\{(i/2)\arg[q_0(z)]\}$,

(b) $\Phi(z) = \int_0^z \phi(t)dt$,

(c) $\Psi(z) = \Phi(z)^{1/6}\phi(z)^{-1/2}, \qquad \Psi(0) = [3q_0'(0)/2]^{1/6}$,

(d) $\xi(z,\lambda) = \lambda\Phi(z)$,

(e) $\mu_0(z) = \cos\left\{\int_0^z \frac{q_1(t)dt}{2\phi(t)}\right\}$,

$$\text{(f)} \qquad \mu_1(z) = \frac{1}{\phi(z)} \sin \left\{ \int_0^z \frac{q_1(t)dt}{2\phi(t)} \right\},$$

$$\text{(g)} \quad D_0(z, \lambda) = 1 + \frac{\mu_0\mu_1' - \mu_0'\mu_1}{\lambda} - \frac{\Psi''}{\Psi}\frac{\mu_1^2}{\lambda^2},$$

$$\text{(12.1)} \quad \text{(h)} \quad g_1(z, \lambda) = \mu_0'' + \frac{\Psi''}{\Psi}\mu_0 + \left\{ 2\,\frac{\Psi''}{\Psi}\mu_1' + \left(\frac{\Psi''}{\Psi}\right)'\mu_1 \right\}\lambda^{-1},$$

$$\text{(i)} \qquad g_2(z) = \mu_1'' + \frac{\mu_1\Psi''}{\Psi},$$

$$\text{(j)} \quad K(z, \lambda) = \frac{-1}{D_0}\left[g_1(\mu_0 + \mu_1'/\lambda) + g_2\left(q_0\mu_1 - \frac{\mu_0'}{\lambda} - \frac{\mu_1\Psi''}{\lambda^2\Psi} \right)\right] + \frac{D_0''}{2D_0} - \frac{3}{4}\left[\frac{D_0'}{D_0}\right]^2.$$

The hypotheses upon the coefficient $q_0(z)$ make it clear that $\Psi(z)$ has neither zeros nor poles in S_z.

The region S_z is to be considered as being covered by a two-sheeted Riemann surface appropriate to a single-valued representation of the function $\xi(x, \lambda)$. This surface will also be designated S_z, and henceforward the nomenclature S_z will refer to this surface. The relations (12.1b) and (12.1d) define a mapping of S_z onto a region S_ξ which is a portion of a three-sheeted Riemann surface with branch points at zero and infinity. It is convenient to the discussion of the given equation to subdivide S_ξ into quadrants of the form (only a finite number of these will be distinct)

$$\text{(12.2)} \quad \Xi_{k,1}\colon (k - 1/2)\pi \leqq \arg(\xi) \leqq k\pi, \qquad \Xi_{k,2}\colon k\pi \leqq \arg(\xi) \leqq (k + 1/2)\pi,$$
$$k = 0, \pm 1, \pm 2, \cdots.$$

Langer has found it possible to choose pairs of linearly independent solutions of the differential equation

$$\text{(12.3)} \qquad \frac{d^2v}{dz^2} + \left\{\lambda^2 q_0(z) - \Psi''/\Psi\right\} v = 0$$

with particularly simple asymptotic espansions with respect to ξ in these quadrants $\Xi_{k,l}$. These solutions are designated here $v_{k,1}(z, \lambda)$ and $v_{k,2}(z, \lambda)$. They and their first derivatives possess the asymptotic expansions

$$\text{(12.4)} \quad \begin{aligned} &\text{(a)} \quad v_{k,j}(z, \lambda) \sim \Psi(z)\xi^{-1/6}e^{\pm i\xi}[1 + E(\xi)/\xi], \\ &\text{(b)} \quad v_{k,j}'(z, \lambda) \sim \Psi'\xi^{-1/6}e^{\pm i\xi}[1 + E(\xi)/\xi] \pm \frac{i\lambda^{2/3}\xi^{1/6}e^{\pm i\xi}}{\Psi}[1 + E(\xi)/\xi] \end{aligned}$$

for ξ in either $\Xi_{k,1}$ or $\Xi_{k,2}$. The notation is adopted that the upper sign is to be used when $j=1$, the lower sign when $j=2$.

13. **The "related equation" when $q_0(z)$ has a zero.** The functions

$$y_{k,j}(z, \lambda) = \frac{\mu_0(z)}{D_0^{1/2}(z, \lambda)} v_{k,j}(z, \lambda) + \frac{\mu_1(z)}{D_0^{1/2}(z, \lambda)} \frac{v'_{k,j}(z, \lambda)}{\lambda}, \tag{13.1}$$

$$j = 1, 2;\ k = 0, \pm 1, \cdots,$$

are solutions of the differential equation

$$\frac{d^2y}{dz^2} + \{\lambda^2 q_0(z) + \lambda q_1(z) + K(z, \lambda)\} y = 0 \qquad [6, \text{p. } 468]. \tag{13.2}$$

Because of its resemblance to the given equation it will be called the "related equation." That

$$|v_{k,j}(z, \lambda)| < M \quad \text{and} \quad |v'_{k,j}(z, \lambda)| < M|\lambda|^{2/3}, \quad \text{when} \quad |\xi| \leqq N,$$

is known[5]. These relations establish the inequality

$$|y_{k,j}(z, \lambda)| < M \qquad \text{for } |\xi| \leqq N. \tag{13.3}$$

Differentiation of the formula (13.1) yields the equality

$$y'_{k,j}(z, \lambda) = \left\{\lambda\left(\frac{\mu_0}{D_0^{1/2}}\right)' + \frac{\mu_0}{D_0^{1/2}}\left(-\lambda^2\phi^2 + \frac{\Psi''}{\Psi}\right)\right\} \frac{v_{k,j}}{\lambda}$$
$$+ \left\{\lambda\left(\frac{\mu_0}{D_0^{1/2}}\right) + \left(\frac{\mu_1}{D_0^{1/2}}\right)'\right\} \frac{v'_{k,j}}{\lambda}.$$

The images upon S_z of points in S_ξ are determined by the inverse of the relation $\xi=\lambda\int_0^z\phi(t)dt$. If the modulus of λ is taken to be large, the points of S_z corresponding to those points of S_ξ for which $|\xi| \leqq N$ will lie in a neighborhood of the origin. In fact, at these points

$$|\lambda z^{3/2} O(1)| \leqq N \quad \text{so that} \quad z = O(\lambda^{-2/3}).$$

This implies $\lambda^2\phi^2 = O(\lambda^{4/3})$; and thus $|y'_{k,j}(z, \lambda)| < M|\lambda|^{2/3}$ when $|\xi| \leqq N$. We also note that

$$W(y_{k,1}, y_{k,2}) = W(v_{k,1}, v_{k,2}) = -2i\lambda^{2/3}.$$

14. **Solutions of the given equation for ξ in $\Xi_{k,l}$.** It is now shown that the asymptotic forms for the solutions $y_{k,j}(z, \lambda)$ also asymptotically represent solutions of the given equation[6]. This equation may be written in the form

[5] Here and henceforward the letters M and N are used as generic symbols for positive constants.

[6] This analysis can obviously be extended to include the algorithm for the solution of the given equation which Langer presents in [6].

$$\frac{d^2u}{dz^2} + \{\lambda^2 q_0(z) + \lambda q_1(z) + K(z, \lambda)\} u = K(z, \lambda)u.$$

This is to be considered to be a nonhomogeneous differential equation having the related equation as its homogeneous associate. Since $W(y_{k,1}, y_{k,2}) = -2i\lambda^{2/3}$, there exist solutions of this equation such that

$$(14.1)\quad u_{k,j}(z, \lambda) = y_{k,j}(z, \lambda) + \frac{1}{2i\lambda^{2/3}} \int_*^z \{ y_{k,1}(z, \lambda) y_{k,2}(t, \lambda) - y_{k,1}(t, \lambda) y_{k,2}(z, \lambda) \} K(t, \lambda) u_{k,j}(t, \lambda) dt.$$

The choice of the lower limit of integration is determined below. If the definitions

$$(14.2)\quad \begin{aligned} &(a)\ \ Y_j(z, \lambda) = y_{k,j}(z, \lambda) \frac{e^{\mp i\xi}}{\Psi(z)}, \\ &(b)\ \ U_j(z, \lambda) = u_{k,j}(z, \lambda) \frac{e^{\mp i\xi}}{\Psi(z)}, \\ &(c)\ \ \xi_1 = \lambda \int_0^t \phi(\tau) d\tau, \\ &(d)\ \ K_j(z, t, \lambda) = \pm \frac{K(t, \lambda)\Psi^2(t)}{2i} [Y_j(z, \lambda) Y_{3-j}(t, \lambda) - Y_j(t, \lambda) Y_{3-j}(z, \lambda) \exp \{ \mp 2i(\xi - \xi_1) \}] \end{aligned}$$

are made as in [5], then the equation (14.1) may be written in the form

$$(14.3)\quad U_j(z, \lambda) = Y_j(z, \lambda) + \lambda^{-2/3} \int_*^z K_j(z, t, \lambda) U_j(t, \lambda) dt.$$

We now explain some notation. The symbol Γ is to designate a set of ordinary curves in S_z upon whose images, Γ^*, on S_ξ the imaginary part of ξ is monotone (nonincreasing or nondecreasing). The symbol Ω is to designate a region of S_z whose image, Ω^*, on S_ξ has the property that each of its points can be joined to the origin and to a point ξ_m at which $|I(\xi)|$ is maximum or infinite by Γ^*-curves. We call Γ_1^* that one of the two Γ^*-curves from the point ξ to ξ_m and from ξ to the origin upon which $I(\xi)$ is algebraically a minimum at the point ξ, and Γ_2^* that one upon which $I(\xi)$ is algebraically a maximum at the point ξ.

It is assumed that *every point of S_z can be included in some Ω region for $|\lambda|$ sufficiently large.* It is also hypothesized that *within S_z the functions*

$$(\mu_j/D_0^{1/2}), \qquad (\mu_j/D_0^{1/2})', \qquad \Psi'/\Psi, \quad \textit{and} \quad \phi$$

are bounded in absolute value and that

$$\int_{\Gamma_j}\left|\frac{K(t,\lambda)}{\phi(t)}\right|\,|dt|$$

is uniformly bounded as to z with the integration being taken over arcs Γ_j on which $|z| \geqq N$. From the formulas (12.4), (13.1), and (13.3) together with the just mentioned hypothesis as to ϕ, Ψ, μ_j, and D_0 it is found that

$$|Y_j| < M, \quad \text{when} \quad |\xi| \leqq N,$$

and

$$|\xi^{1/6}Y_j| < M, \quad \text{when} \quad |\xi| > N, \qquad \text{for any } z \text{ in } S_z.$$

Let us fix our attention upon a specific quadrant $\Xi_{k,l}$. With Ω any appropriate region whose image, Ω^*, on $\Xi_{k,l}$ fulfills our requirements, the integration in equation (14.3) is to be taken over a curve Γ_j in Ω. This determines the lower limit of integration * to be either the origin or the image on Ω of the point ξ_m. Now Langer's reasoning [5, pp. 454–458] may be adapted in its entirety and carried forward to establish the conclusion that(7)

$$\xi^{s/6}U_j(z,\lambda) = \xi^{s/6}Y_j(z,\lambda) + E(z,\lambda)/\lambda$$

for z in the region Ω. In view of the definitions (14.2) this last result may be rewritten in the form

$$u_{k,j}(z,\lambda) = y_{k,j}(z,\lambda) + \Psi\xi^{-s/6}e^{\pm i\xi}\frac{E(z,\lambda)}{\lambda}. \tag{14.4}$$

A similar discussion leads to the conclusion that

$$u'_{k,j}(z,\lambda) = y'_{k,j}(z,\lambda) + \Psi\xi^{-s/6}e^{\pm i\xi}E(z,\lambda) \tag{14.5}$$

for z in Ω. The asymptotic expansions for the solutions $u_{k,j}(z,\lambda)$, when ξ is in the region Ω^* on $\Xi_{k,l}$, can be obtained from the relations (14.4), (13.1), and (12.4). When only the dominant terms are included, these reduce to the form

$$u_{k,j}(z,\lambda) \sim \lambda^{-1/6}\phi^{-1/2}e^{\pm i\xi}(\mu_0 \pm i\mu_1\phi)\left\{1 + \frac{E(z,\lambda)}{\lambda} + \frac{E(\xi)}{\xi}\right\}, \quad \xi \text{ in } \Xi_{k,l}. \tag{14.6}$$

15. **The forms of $u_{k,j}(z,\lambda)$ when ξ is not in $\Xi_{k,l}$.** The relations (14.4) describe a pair of solutions of the given equation for values of z confined to any specific region Ω on S_z. The forms of these solutions are now deduced for all admitted values of z and λ. To do this a new pair of solutions, $u_1(z,\lambda)$ and $u_2(z,\lambda)$, of the given equation is introduced; and the coefficients in the dependence relations which subsist between these solutions and the $u_{k,j}(z,\lambda)$ are determined.

The formulas

(7) When $|\xi| \leqq N$, $s=0$; when $|\xi| > N$, $s=1$.

$$v_j(z, \lambda) = \Psi(z)\xi^{1/3}J_{\mp 1/3}(\xi), \qquad j = 1, 2, \tag{15.1}$$

define a pair of solutions of the differential equation (12.3). There correspond to these the functions

$$y_j(z, \lambda) = \frac{\mu_0(z)}{D_0^{1/2}(z, \lambda)} v_j(z, \lambda) + \frac{\mu_1(z)}{D_0^{1/2}(z, \lambda)} \frac{v_j'(z, \lambda)}{\lambda}, \qquad j = 1, 2, \tag{15.2}$$

which are solutions of the related equation. That the Bessel functions have the property

$$\lim_{x \to 0} x^\nu J_{-\nu}(x) = 2^\nu/\Gamma(1 - \nu)$$

is well known. From this it follows that

$$\begin{aligned} y_1(0, \lambda) &= O(1), & y_1'(0, \lambda) &= O(1), \\ y_2(0, \lambda) &= O(\lambda^{-1/3}), & y_2'(0, \lambda) &= O(\lambda^{2/3}). \end{aligned} \tag{15.3}$$

The solutions $v_j(z, \lambda)$ of equation (12.3) are linearly independent, as are also the solutions $v_{h,j}(z, \lambda)$ so that there exist relations of the form

$$\begin{aligned} &\text{(a)} \quad v_j(z, \lambda) = \overset{(h)}{c}_{j,1} v_{h,1}(z, \lambda) + \overset{(h)}{c}_{j,2} v_{h,2}(z, \lambda), \\ &\text{(b)} \quad v_{h,j}(z, \lambda) = \overset{(h)}{\gamma}_{1,j} v_1(z, \lambda) + \overset{(h)}{\gamma}_{2,j} v_2(z, \lambda), \qquad j = 1, 2. \end{aligned} \tag{15.4}$$

The values of the coefficients $\gamma_{k,j}$ and $c_{j,k}$ are known [5, p. 460]. Replacing the $v_j(z, \lambda)$ in the relation (15.2) by the right-hand side of the identity (15.4a), the identity

$$y_j(z, \lambda) = \overset{(h)}{c}_{j,1} y_{h,1}(z, \lambda) + \overset{(h)}{c}_{j,2} y_{h,2}(z, \lambda) \tag{15.5a}$$

is obtained. The corresponding relation

$$y_{h,j}(z, \lambda) = \overset{(h)}{\gamma}_{1,j} y_1(z, \lambda) + \overset{(h)}{\gamma}_{2,j} y_2(z, \lambda) \tag{15.5b}$$

is similarly found.

Formulas for the solutions $u_{h,j}(z, \lambda)$ of the given equation when ξ is not restricted to the quadrant $\Xi_{h,l}$ can now be deduced. A pair of solutions, $u_1(z, \lambda)$ and $u_2(z, \lambda)$, of the given equation associated with the solutions $y_j(z, \lambda)$ of the related equation is determined by the conditions

$$\begin{aligned} u_j(0, \lambda) &= y_j(0, \lambda), \\ u_j'(0, \lambda) &= y_j'(0, \lambda), \qquad j = 1, 2. \end{aligned} \tag{15.6}$$

Between these solutions and the $u_{h,j}(z, \lambda)$ there exist identities in the forms

$$\begin{aligned} &\text{(a)} \quad u_{h,j}(z, \lambda) = \overset{(h)}{\alpha}_{1,j} u_1(z, \lambda) + \overset{(h)}{\alpha}_{2,j}(z, \lambda), \\ &\text{(b)} \quad u_j(z, \lambda) = \overset{(h)}{a}_{j,1} u_{h,1}(z, \lambda) + \overset{(h)}{a}_{j,2} u_{h,2}(z, \lambda), \qquad j = 1, 2, \end{aligned} \tag{15.7}$$

in which the coefficients are as yet undetermined. The Wronskians which determine them are constants and are easily evaluated at the origin through use of the formulas (14.4), (14.5), (15.5), the conditions (15.6), and the evaluations (15.3). Thus it is found that

$$(15.8)\qquad \begin{aligned} &\text{(a)}\ \alpha_{k,j}^{(h)} = \gamma_{k,j}^{(h)} + O(\lambda^{-2/3}),\\ &\text{(b)}\ a_{j,k}^{(h)} = c_{j,k}^{(h)} + O(\lambda^{-2/3}), \qquad j, k = 1, 2. \end{aligned}$$

Now that these coefficients are known, the asymptotic forms of the solutions $u_j(z, \lambda)$ and $u_{h,j}(z, \lambda)$ for ξ in any quadrant $\Xi_{m,n}$ are obtainable through the representations (14.6). The relation (15.7) also reveals that

$$(15.9)\qquad u_{h,j}(z, \lambda) = \gamma_{1,j}^{(h)} y_1(z, \lambda) + \gamma_{2,j}^{(h)} y_2(z, \lambda) \quad \text{when} \quad |\xi| < N.$$

When, for a fixed h, ξ passes from $\Xi_{h,1}$ or $\Xi_{h,2}$ into an adjacent quadrant, each of the formulas deducible through the relations (15.7) changes to the extent of a replacement of one of its coefficients. The one which changes, however, always belongs to that term which is asymptotically sub-dominant. The affected term does not become large until arg (ξ) changes by an amount $\pi/2$. Thus, the asymptotic expansions for the solutions $u_j(z, \lambda)$ and $u_{h,j}(z, \lambda)$, which can be found from the identities (15.8) for ξ in any quadrant $\Xi_{m,n}$, represent these solutions for all sufficiently large values of $|\xi|$ in the larger region $\Xi^{(m)}$ defined by

$$\Xi^{(m)}: (m-1)\pi + \epsilon \leqq \arg(\xi) \leqq (m+1)\pi - \epsilon, \qquad m = 0, \pm 1, \pm 2, \cdots,$$
$$\epsilon > 0 \text{ and sufficiently small.}$$

16. **Solutions when $q_0(z)$ has no zeros in S_z.** We now turn our attention to the case of $[q_0(z)]^{-1}$ bounded in S_z. With $[q_0(z)]^{1/2}$ as the root $|q_0^{1/2}| \exp\{(i/2) \arg (q_0)\}$ of $q_0(z)$, the relation

$$(16.1)\qquad \psi(z, \lambda) = q_0^{1/2}(z) + \frac{q_1(z)}{2\lambda q_0^{1/2}}$$

defines a function $\psi(z, \lambda)$ whose reciprocal is bounded in S_z for $|\lambda|$ sufficiently large if we assume $q_1(z)/[q_0(z)]^{1/2}$ to be bounded there.

Let the function $\zeta(z, \lambda)$ be chosen as any function satisfying the condition

$$(16.2)\qquad \frac{d\zeta}{dz} = \lambda\psi(z, \lambda).$$

This function defines a mapping of S_z onto a region T_ζ which is assumed to have the properties:

(a) In T_ζ there is a point ζ_{m+} at which $I(\zeta)$ is a maximum and a point ζ_{m-} at which $I(\zeta)$ is a minimum. (These points may be at infinity.)

(b) The points of T_ζ may all be connected to ζ_{m+} by curves upon which $I(\zeta)$ is monotonically increasing and to ζ_{m-} by curves upon which $I(\zeta)$ is monotonically decreasing. The images of ζ_{m+} and ζ_{m-} on S_z are denoted z_{m+} and z_{m-} respectively.

The differential equation

$$\frac{d^2Y}{dz^2} + \{\lambda^2 q_0(z) + \lambda q_1(z) + \theta(z, \lambda)\}Y = 0, \tag{16.3}$$

in which

$$\theta(z, \lambda) = q_1^2/4q_0 + \psi''/2\psi - 3\psi'^2/4\psi^2, \tag{16.4}$$

has a pair of linearly independent solutions

$$Y_j(z, \lambda) = e^{\pm i\zeta}\psi^{-1/2}(z, \lambda), \qquad j = 1, 2. \tag{16.5}$$

Equation (16.3) takes the part of the related equation in this discussion. It is easily shown that

$$W(Y_1, Y_2) = -2i\lambda.$$

The given equation can be rewritten in the form

$$\frac{d^2u}{dz^2} + \{\lambda^2 q_0(z) + \lambda q_1(z) + \theta(z, \lambda)\}u = \theta(z, \lambda)u.$$

Considering this to be a nonhomogeneous differential equation which has equation (16.3) as its homogeneous associate, it is seen to have a pair of solutions $U_1(z, \lambda)$ and $U_2(z, \lambda)$ which satisfy the relations

$$\text{(a)}\quad U_1(z, \lambda) = Y_1(z, \lambda) + \frac{1}{2i\lambda}\int_{z_{m+}}^{z} \{Y_1(z, \lambda)Y_2(t, \lambda) - Y_2(z, \lambda)Y_1(t, \lambda)\} \cdot U_1(t, \lambda)\theta(t, \lambda)dt,$$

(16.6)

$$\text{(b)}\quad U_2(z, \lambda) = Y_2(z, \lambda) + \frac{1}{2i\lambda}\int_{z_{m-}}^{z} \{Y_1(z, \lambda)Y_2(t, \lambda) - Y_2(z, \lambda)Y_1(t, \lambda)\} \cdot U_2(t, \lambda)\theta(t, \lambda)dt.$$

If $\int_{z_0}^{z}|\theta(t, \lambda)/\psi(t, \lambda)|\,|dt|$ (z_0 may be any fixed point of S_z) is assumed to be uniformly bounded as to z in S_z, then an analysis similar to that of §14 shows that the solutions $U_1(z, \lambda)$ and $U_2(z, \lambda)$ have the forms([8])

$$U_j(z, \lambda) = Y_j(z, \lambda)\left\{1 + \frac{E(z, \lambda)}{\lambda}\right\}, \qquad j = 1, 2, \tag{16.7}$$

for z in S_z.

([8]) The remark given as footnote six also applies here.

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