EIGENFUNCTION EXPANSIONS ASSOCIATED WITH SINGULAR DIFFERENTIAL OPERATORS[1]

BY

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1. Introduction. In this paper we shall prove some expansion theorems connected with pairs of so-called adjoint differential operators reducible to the form

$$\Omega = \frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x) \tag{1.1}$$

and

$$\Omega^* = \frac{d}{dx}\left[\frac{d}{dx} - b(x)\right] + c(x), \tag{1.2}$$

on an interval $-\infty \leqq r_1 \leqq x \leqq r_2 \leqq \infty$; the coefficient $b(x)$ is assumed continuous, but not necessarily bounded, in (r_1, r_2), and $c(x)$ is assumed continuous in $[r_1, r_2]$. We shall impose other conditions of a global character on $b(x)$. (See §2.)

Motivated by applications to diffusion theory, we shall consider (1.1) in the Banach space $C[r_1, r_2]$ of functions continuous on $[r_1, r_2]$ and (1.2) in the space $L(r_1, r_2)$ of functions integrable on (r_1, r_2). More specifically, Ω will be considered as an operator from $C[r_1, r_2]$ to itself and Ω^* as an operator from $L(r_1, r_2)$ to itself. For a more detailed discussion of the domains of Ω and Ω^*, see the end of §2 and §3.

The method we use here, which is entirely real, may be outlined as follows: we start with (1.1) under boundary conditions of the type (3.3)–(3.5). Under our restrictions on $b(x)$, we find that the eigenfunctions of Ω under the boundary conditions contain a set orthonormal and complete in a certain weighted L_2 space (§4). To bridge the gap between the Hilbert space and the Banach spaces we use the theory of semi-groups. Feller [1] has shown that to each set of boundary conditions of the type we consider, there corresponds a semi-group $\{T_t\}$ which is strongly continuous at $t=0$ in the space X given by our Definition 5.1 (see our Lemma 5.1). We are able to obtain an expansion for T_tf in terms of the eigenfunctions of Ω in X (Lemmas 5.2 and 5.3). This expansion gives the so-called Dirichlet representation of the semi-group (see Hille [2, p. 346 and also formulas (21.3.4)–(21.3.6)]). From the strong

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continuity of $\{T_t\}$ at $t=0$, we immediately obtain an expansion theorem for functions $f \in X$ (Theorems 5.1–5.3). The results are then extended to include an expansion theorem for bounded measurable functions (Theorem 5.4). All of these are expansions in the sense of generalized Abel summability. In §6 we deduce expansion theorems for (1.2) from our previous results.

If d^2/dx^2 in (1.1) and the term d/dx in brackets in (1.2) have a coefficient $a(x)$ which is positive and differentiable in (r_1, r_2), we can reduce $a(x)$ to 1 by the canonical change of variable, $\eta(x) = \int_{x_0}^{x} a(s)^{-1/2} ds$. The classification of boundaries which we state in §2, I–IV for $a(x) \equiv 1$ can be found for the general $a(x)$ in [1, §23].

We remark here that if we make the change of independent variable $\xi(x) = \int_{x_0}^{x} e^{B(s)} ds$ with $B(s) = \int_{x_0}^{s} b(t) dt$ in (1.1), then the operator is transformed into

$$\frac{d}{d\xi}\left(e^{2B(x)} \frac{d}{d\xi}\right) + c(x). \tag{1.3}$$

In this form it is a special case of the operators considered in [3; 4; 6; 8]. However, we are imposing rather strong restrictions on $b(x)$ and $c(x)$, and the few results we do use about the operators in the L_2 space do not follow directly from their very general theorems.

§2 is devoted to the behavior of solutions of

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = \lambda u \tag{1.4}$$

with $\lambda > \text{l.u.b. } c(x)$ for $x \in [r_1, r_2]$ under various conditions on $b(x)$. Some properties of the Green functions are derived in §3 and used in §4 to obtain information about the spectrum of Ω. In §4 we also state properties of the solutions of (1.4) which hold for all real λ and are not covered by §2. Some of these are known, but since the proofs are simple, it seems desirable to include them for completeness. §§5 and 6 are devoted exclusively to the expansion theorems.

2. **Classification of boundaries.** In [1] Feller has given a classification of boundaries for operators of the types (1.1) and (1.2) with $c(x) \equiv 0$. For the time being we shall be interested in (1.1) alone. With only minor alterations in his proofs we can show that the four categories he has considered have the same properties in the case of the more general $c(x) \in C[r_1, r_2]$. Since the changes required are so slight, we omit the proofs in I–IV below.

NOTATION. *We let $L_p[g, a, b]$ denote the space of measurable functions f for which $\int_a^b |f(x)|^p g(x) dx < \infty$.*

For simplicity we assume that $0 \in (r_1, r_2)$ and define $B(x) = \int_0^x b(s) ds$. We give the classification for the boundary r_2. The only changes required for r_1 are to interchange the subscripts 1 and 2 and the words nonincreasing and nondecreasing in the statements below. In what follows c^+ denotes the positive part of c; both c and c^+ are elements of $C[r_1, r_2]$.

I. *The boundary r_2 is called* regular *if*

$$e^{B(x)} \in L(0, r_2) \quad \text{and} \quad e^{-B(x)} \in L(0, r_2). \tag{2.1}$$

If $\lambda > \|c^+\|$ then (1.4) has two nonincreasing solutions $u_1(x)$ and $u_2(x)$ with $u_1(x) \to 0$, $e^{B(x)}u_1'(x) \to -1$ as $x \to r_2$ and $u_2(x) \to 1$, $e^{B(x)}u_2'(x) \to 0$ as $x \to r_2$. Each solution is bounded in $[0, r_2]$, but is non-negative and nonincreasing if and only if $u = p_1u_1 + p_2u_2$ with $p_j \geqq 0$.

II. *r_2 is an* exit *boundary if it is not regular and*

$$e^{-B(x)} \int_0^x e^{B(s)} ds \in L(0, r_2)(^2). \tag{2.2}$$

If $\lambda > \|c^+\|$ then each solution $u(x)$ of (1.4) is bounded in $[0, r_2]$; there exists a nonincreasing solution $u_2(x)$ such that $u_2(x) \to 0$, $e^{B(x)}u_2'(x) \to -\alpha \leqq 0$ as $x \to r_2$ and $u_2 \in L(e^B, 0, r_2)$. No independent solution tends to 0 as $x \to r_2$ or is in $L(e^B, 0, r_2)$.

III. *r_2 is an* entrance *boundary if it is not regular and*

$$e^{B(x)} \int_0^x e^{-B(s)} ds \in L(0, r_2)(^2). \tag{2.3}$$

If $\lambda > \|c^+\|$ then (1.4) has a nonincreasing solution $u_2(x)$ with $u_2(x) \to 1$, $e^{B(x)}u_2'(x) \to 0$ as $x \to r_2$. No independent solution is bounded in $[0, r_2]$, but every solution $u(x)$ satisfies $u \in L(e^B, 0, r_2)$: furthermore $\lim_{x \to r_2} u'(x)e^{B(x)}$ exists and is finite.

IV. *r_2 is a* natural boundary *in all other cases.*

If $\lambda > \|c^+\|$, then (1.4) has a nonincreasing solution u_2 with $u_2(x) \to 0$, $e^{B(x)}u_2'(x) \to 0$ as $x \to r_2$ and $u_2 \in L(e^B, r_1, r_2)$; no independent solution is in $L(e^B, 0, r_2)$.

If (1.4) has two independent solutions in $L_2(e^B, 0, r_j)$ for some value of λ, then we say, following Weyl [8], that (1.1) is of the *limit circle* type at r_j; otherwise (1.1) is said to be of the *limit point* type at r_j. It is easily shown, using the method of proof given in [4, p. 61], that if two independent solutions are in $L_2(e^B, 0, r_j)$ for one value of λ, the same is true for all values of λ. The following theorem gives the connection between this classification and that of Feller in I–IV.

THEOREM 2.1. *If r_j is a regular boundary, then* (1.1) *is of limit circle type at r_j. If r_j is either an exit or a natural boundary, then* (1.1) *is of limit point type at r_j. If r_j is an entrance boundary, then* (1.1) *is of the limit circle type at r_j if and only if*

(2) Condition (2.2) is equivalent to $e^{B(x)}\int_x^{r_2} e^{-B(s)}ds \in L(0, r_2)$ and condition (2.3) is equivalent to $e^{-B(x)}\int_x^{r_2} e^{B(s)}ds \in L(0, r_2)$.

(2.4) $$e^{B(x)}\left[\int_0^x e^{-B(s)}ds\right]^2 \in L(0, r_j).$$

Proof. For simplicity, we consider the boundary r_2. We need check these statements for one value of λ only. Assume that $\lambda > \|c^+\|$. That a regular boundary is of limit circle type follows directly from I above.

If r_2 is an exit boundary, we note that (2.2) implies that $e^{-B(x)} \in L(0, r_2)$. Therefore, since r_2 is not regular, $e^B \notin L(0, r_2)$. A positive nondecreasing solution u_1 of (1.4) for $\lambda > \|c^+\|$ is obtained by prescribing $u_1(0) > 0$ and $u_1'(0) > 0$, because no solution of (1.4) for $\lambda > \|c^+\|$ can have a positive maximum. Clearly $u_1 \notin L_2(e^B, 0, r_2)$. If r_2 is a natural boundary, construct u_1 in the same way. By IV above, $u_1 \notin L(e^B, 0, r_2)$ and hence $u_1 \notin L_2(e^B, 0, r_2)$.

If r_2 is an entrance boundary, then $e^B \in L(0, r_2)$ and $e^{-B} \notin L(0, r_2)$. If u_2 is defined as in III, then it is easily verified that an independent solution u_1 is given by

(2.5) $$u_1(x) = u_2(x)\int_0^x \frac{e^{-B(s)}}{u_2^2(s)}\,ds.$$

Now $0 < m \leqq u_2(x) \leqq M < \infty$ for $x \in [0, r_2]$ so that

(2.6) $$\frac{m}{M^2}\int_0^x e^{-B(s)}ds \leqq u_1(x) \leqq \frac{M}{m^2}\int_0^x e^{-B(s)}ds.$$

Thus $u_1 \in L_2(e^B, 0, r_2)$ if and only if (2.4) holds.

In this paper we shall deal only with regular, exit, and entrance boundaries. A study of natural boundaries is in progress, but will be deferred to a later paper.

Finally, we make a few remarks concerning the domains of Ω and Ω^*. We wish to consider Ω as an operator from $C[r_1, r_2]$ to itself. The domain of Ω consists of those elements $F \in C[r_1, r_2]$ for which $\Omega F(x)$ is continuous at each point $x \in (r_1, r_2)$ and for which $\lim_{x \to r_j} \Omega F(x) = l_j$ exists and is finite. We define $\Omega F(r_j) = l_j$, so that $\Omega F \in C[r_1, r_2]$. If r_j is an entrance boundary, we can say a bit more about the domain. Suppose, for example, that r_2 is an entrance boundary and that $\Omega F = f \in C[r_1, r_2]$. It is easily verified that

(2.7) $$F(x) = F(0) + e^{B(x)}F'(x)\int_0^x e^{-B(s)}ds + \int_0^x e^{-B(s)}ds\int_s^x e^{B(t)}\{c(t)F(t) - f(t)\}\,dt.$$

From (2.3)([2]) the limit as $x \to r_2$ of the third term on the right exists. On the other hand $e^{-B} \notin L(0, r_2)$; otherwise, r_2 would be a regular boundary since (2.3) implies that $e^B \in L(0, r_2)$. Therefore, in order that $F \in C[r_1, r_2]$ it is necessary that $\lim_{x\to r_2}\{F'(x)e^{B(x)}\}=0$. We have thus shown that if r_j is an entrance boundary and $F\in$domain Ω, then $\lim_{x\to r_j}\{F'(x)e^{B(x)}\}=0$.

On the other hand, we wish to consider Ω^* as an operator from $L(r_1, r_2)$ to itself. The domain of Ω^* consists of those elements $G\in L(r_1, r_2)$ for which $G'-b(x)G(x)$ is absolutely continuous on $[r_1, r_2]$. For such G we clearly have $\Omega^*G\in L(r_1, r_2)$. If r_j is an exit boundary, we can give more information about the domain of Ω^*. Suppose that r_2 is an exit boundary, and denote $G'(x)-b(x)G(x)$ by $\Phi(x)$. It is easily seen that

$$\int_0^x G(x)dx = e^{-B(x)}G(x)\cdot\int_0^x e^{B(s)}ds - \int_0^x e^{B(s)}ds\int_s^x \Phi(t)e^{-B(t)}dt. \tag{2.8}$$

If $G\in$ domain Ω^*, then $\Phi\in C[r_1, r_2]$ and by (2.2)(2) the second integral on the right approaches a limit as $x\to r_2$. Since $e^B\notin L(0, r_2)$, we must have $\lim_{x\to r_2}\{G(x)e^{-B(x)}\}=0$ in order that G be in $L(r_1, r_2)$. In other words, if r_j is an exit boundary and $G\in$ domain Ω^*, then $\lim_{x\to r_j}\{G(x)e^{-B(x)}\}=0$.

3. **The Green functions.** The properties of the Green functions will play an important role in our treatment of the eigenfunction expansions. We define, following Feller [1],

DEFINITION 3.1. *Suppose that* $\lambda\geqq\|c^+\|$ (see beginning of §2) *and that* $u_1(x)$ *is a non-negative, nondecreasing solution of* (1.4), *while* $u_2(x)$ *is a non-negative, nonincreasing solution of* (1.4). *We assume that the* u_i *are normed so that* $u_2(0)u_1'(0)-u_1(0)u_2'(0)=1$. *Then the function,*

$$G(x, y; \lambda) = \begin{cases} u_1(x)u_2(y)e^{B(y)}, & x\leqq y, \\ u_2(x)u_1(y)e^{B(y)}, & x\geqq y, \end{cases} \tag{3.1}$$

is called a regular Green function for (1.1).

We shall treat (1.4) only under certain boundary conditions which will be given below. Imposing such a condition means that we are dealing with a contraction of Ω. For example, if we wish to solve (1.4) under the boundary condition $y(r_1)=y(r_2)=0$, then we are actually solving

$$\lambda y - \bar{\Omega} y = 0, \tag{3.2}$$

where $\bar{\Omega}$ is the contraction of Ω to the set of functions $F\in$ domain Ω for which $F(r_1)=F(r_2)=0$.

The boundary conditions we consider in this paper will depend on the nature of the coefficient $b(x)$ at the boundaries. If r_j, for $j=1$ and/or 2 is a *regular* boundary, we shall impose a so-called "elastic barrier" condition,

$$q_j y(r_j) + (-1)^j p_j \lim_{x\to r_j}\{e^{B(x)}y'(x)\} = 0, \tag{3.3}$$

where $0\leqq p_j/q_j\leqq\infty$. In case r_j is an *exit* boundary, we shall impose the so-called "absorbing barrier" condition,

$$y(r_j) = 0. \tag{3.4}$$

If r_j is an entrance boundary, we need impose no boundary condition at r_j, as will be seen in the subsequent developments. We point out here that if $y \in C[r_1, r_2]$ satisfies (3.2), then by our remarks at the end of §2, it follows that if r_j is an entrance boundary, the condition

$$\lim_{x \to r_j} \left\{ e^{B(x)} y'(x) \right\} = 0 \tag{3.5}$$

is automatically satisfied. This is the "reflecting barrier" condition.

As mentioned above, it is convenient to introduce an operator $\bar{\Omega}$, which takes into account the boundary conditions.

DEFINITION 3.2. *We define $\bar{\Omega}$ to be the contraction of the operator Ω to the set of functions $F \in$ domain Ω (defined at the end of §2) for which*

(i) *condition* (3.3) *is satisfied with $F = y$ for a given pair of real numbers p_j, q_j for which $0 \leqq p_j/q_j \leqq \infty$, whenever r_j is a* regular *boundary*;

(ii) *condition* (3.4) *is satisfied with $F = y$ if r_j is an* exit *boundary.*

If r_1 and r_2 are both entrance boundaries, then we define $\bar{\Omega} = \Omega$.

From now on we shall use the notation $\bar{\Omega}$ with the understanding that it should be interpreted according to the nature of the boundaries; for example, if r_1 is an exit boundary and r_2 is an entrance boundary, then $\bar{\Omega}$ is the contraction of Ω to the set of functions $F \in$ domain Ω for which $F(r_1) = 0$.

The following theorem gives additional information about the regular Green functions. *From now on we assume p_j and q_j to be fixed.*

THEOREM 3.1. *If $G(x, y; \lambda)$ is defined as in Definition 3.1 and if neither boundary is regular, then for each $f \in C[r_1, r_2]$ and $\lambda > \|c^+\|$,*

$$F(x) = \int_{r_1}^{r_2} G(x, y; \lambda) f(y) dy \tag{3.6}$$

is in domain $\bar{\Omega}$ and is the unique solution in $C[r_1, r_2]$ of

$$\lambda F - \bar{\Omega} F = f. \tag{3.7}$$

Moreover,

$$(\lambda - \|c^+\|)\|F\| \leqq \|f\|. \tag{3.8}$$

If r_j is a regular boundary for $j = 1$ and/or 2 then among the regular Green functions there is one and only one for which the above conclusion is true.

Proof. A proof of this theorem can be found in [1, §§13, 14 and 15], for $c \equiv 0$. The modifications required for this case are minor ones.

DEFINITION 3.3. *Given p_j, q_j, and $\bar{\Omega}$ defined as in Definition 3.2, let $G(x, y; \lambda)$ be the regular Green function described in Theorem 3.1. We define a new kernel,*

$$K(x, y; \lambda) = G(x, y; \lambda) e^{B(y)} \tag{3.9}$$

associated with $\bar{\Omega}$.

THEOREM 3.2. *If neither boundary is natural, then for each* $\lambda^* > \|c^+\|$,

$$(3.10) \qquad I = \int_{r_1}^{r_2}\int_{r_1}^{r_2} K^2(x, y; \lambda^*) e^{B(x)} e^{B(y)} dx dy < \infty.$$

Proof. Defining u_1 and u_2 as in Definition 3.1, we have

$$(3.11) \qquad \begin{aligned} I = &\int_{r_1}^{r_2} \left\{ e^{B(x)} u_1^2(x) \int_x^{r_2} u_2^2(s) e^{B(s)} ds \right\} dx \\ &+ \int_{r_1}^{r_2} \left\{ e^{B(x)} u_2^2(x) \int_{r_1}^{x} u_1^2(s) e^{B(s)} ds \right\} dx. \end{aligned}$$

Since u_1 and u_2 are monotonic we have

$$(3.12) \qquad \int_{r_1}^{r_2} K^2(x, y; \lambda^*) e^{B(y)} dy \leqq u_1(x) u_2(x) \int_{r_1}^{r_2} K(x, y; \lambda^*) e^{B(y)} dy.$$

Therefore by (3.8),

$$(3.13) \qquad \int_{r_1}^{r_2} K^2(x, y; \lambda^*) dy \leqq \frac{u_1(x) u_2(x)}{\lambda^+ - \|c^+\|}.$$

By I–III of §2,

$$(3.14) \qquad \int_{r_1}^{r_2} u_1(x) u_2(x) e^{B(x)} dx < \infty,$$

which completes the proof.

For later convenience we make the definition:

DEFINITION 3.4. *Given* $\bar{\Omega}$ *defined as in Definition* 3.2, *we define* $\bar{\Omega}^*$ *to be the operator* Ω^* *in* (1.2) *contracted to the subset of functions* G *in* $L(r_1, r_2)$ *for which* $G \in$ *domain* Ω^* *and*

$$(3.15) \qquad q_j \lim_{x \to r_j} G(x) e^{-B(x)} + (-1)^j p_j \lim_{x \to r_j} [G'(x) - b(x) G(x)] = 0$$

if r_j *is a regular boundary;*

$$(3.16) \qquad \lim_{x \to r_j} [G'(x) - b(x) G(x)] = 0$$

if r_j *is an entrance boundary. If both* r_1 *and* r_2 *are exit boundaries then* $\bar{\Omega}^* = \Omega^*$.

4. **The eigenfunctions of** $\bar{\Omega}$. In this section we wish to show that the eigenfunctions in $L_2(e^B, r_1, r_2)$ of the integral operator with kernel $K(x, y; \lambda^*)$ with $\lambda^* > \|c^+\|$ (see beginning of §2) are the same as those of $\bar{\Omega}$ in $C[r_1, r_2]$. In Lemmas 4.1–4.3, we state a few properties of solutions of (4.1), which hold for all real λ.

LEMMA 4.1. *If* r_j *is an exit boundary, then in* $[0, r_j]$ *for each real* λ,

(a) *every solution of*

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = \lambda y \tag{4.1}$$

has at most a finite number of zeros;

(b) *every solution is bounded;*

(c) *there exists a solution* $y \in C[0, r_j]$ *for which* $y(r_j) = 0$ *and* $y \in L(e^B, 0, r_j)$. *No independent solution has either of these properties.*

Proof. (a) It suffices to prove (a) for $\lambda < -\|c\|$, for (4.1) can be written $d(e^By')/dx = \{\lambda - c(x)\}e^By$, and the assertion for $\lambda \geqq -\|c\|$ would then follow by the Sturm comparison theorem. Suppose that there exists a solution $y(x)$ of (4.1) with $\lambda < -\|c\|$ having infinitely many zeros in $[0, r_2]$, for example. Let $x_0 > 0$ be a point at which $y(x)$ has a relative maximum. From $\lambda - c(x) < 0$ we see that $y(x)$ cannot have a negative maximum; hence $y(x_0) > 0$. Let x_1 be the next point after x_0 for which $y(x_1) = 0$. Since $y(x)$ can have no positive minimum, it must be nonincreasing (x_0, x_1), and we have

$$\begin{aligned} 0 < y(x_0) &= \int_{x_0}^{x_1} e^{-B(s)}ds \int_{x_0}^{s} [c(t) - \lambda]y(t)e^{B(t)}dt \\ &\leqq y(x_0)\int_{x_0}^{x_1} e^{-B(s)}ds \int_{x_0}^{s} [c(t) - \lambda]e^{B(t)}dt. \end{aligned} \tag{4.2}$$

Therefore, letting $\beta = \max\ [c(t) - \lambda] < \infty$ for $0 \leqq t \leqq r_2$,

$$\frac{1}{\beta} \leqq \int_{x_0}^{r_1} e^{-B(s)}ds \int_{x_0}^{s} e^{B(t)}dt. \tag{4.3}$$

For an oscillatory solution x_0 can be chosen as near to r_2 as we please; this, however, contradicts (2.2). From now on we assume $j = 2$ for simplicity.

(b) From §2, we may suppose $\lambda \leqq \|c^+\|$. It follows from part (a) that if $y(x)$ is a solution of (4.1), then there exists a point ξ such that $y(x)$ is of constant sign in $[\xi, r_2]$, say $y(x) > 0$. Let $u(x)$ be a solution of (4.1) with $\lambda > \|c^+\|$ such that $u(\xi) = y(\xi)$ and $u'(\xi) = y'(\xi)$. Then by a well-known argument, $u(x) > y(x)$ in $[\xi, r_2]$. Since $u(x)$ is bounded in $[\xi, r_2]$ (cf. II, §2) the result follows.

(c) First we show that there exists a solution $w(x)$ such that $w(x) \to 1$ as $x \to r_2$ by the method of successive approximations. It is sufficient to construct such a solution in an interval $[\eta, r_2]$ in which

$$\int_{x}^{r_2} e^{-B(s)}ds \int_{\eta}^{s} e^{B(t)} \mid c(t) - \lambda \mid dt < \epsilon < 1. \tag{4.4}$$

We define

$$(4.5)\qquad w_0(x) \equiv 1,\ w_{n+1}(x) = 1 + \int_x^{r_2} e^{-B(s)}ds \int_\eta^s e^{B(t)}[c(t) - \lambda]w_n(t)dt.$$

It is easily verified that $|w_n(x)-w_{n-1}(x)|<\epsilon^n$ for $x\in[\eta,\ r_2]$ and since $0<\epsilon<1$, w_n converges to a limit function w. Since $|w_n(x)|<1/(1-\epsilon)$ in $[\eta, r_2]$, we conclude by dominated convergence in (4.5) that $w(x)$ is a solution of (4.1) with $\lim_{x\to r_2} w(x)=1$. Clearly $w\notin L_2(e^B,\ 0,\ r_2)$ since $e^B\notin L[0,\ r_2]$.

We now construct a solution $y(x)$ with $y(r_2)=0$. Choose η so that $M>w(x)>m>0$ in $[\eta,\ r_2]$. Define

$$(4.6)\qquad y(x) = w(x)\int_x^{r_2} \frac{e^{-B(s)}ds}{[w(s)]^2}, \qquad \eta \leqq x \leqq r_2.$$

Then $y(x)$ is a solution of (4.1) with $y(r_2)=0$, and

$$(4.7)\qquad 0 < \int_\eta^{r_2} e^{B(x)}y(x)dx \leqq \frac{M}{m^2}\int_\eta^{r_2} e^{B(s)}ds \int_s^{r_2} e^{-B(t)}dt < \infty.$$

Therefore $y\in L(e^B,\ 0,\ r_2)$, and since y is bounded we have $y\in L_2(e^B,\ 0,\ r_2)$.

LEMMA 4.2. *If r_j is an entrance boundary, then in $[0,\ r_j]$ for each real λ*

(d) *every solution of* (4.1) *has at most a finite number of zeros*;

(e) *there exists a bounded solution y_j with $\lim_{x\to r_j} e^{B(x)}y_j'(x)=0$. No independent solution is bounded in $[0,\ r_j]$, but $\lim_{x\to r_j} e^{B(x)}y'(x)$ exists for each independent solution y and is finite and different from zero.*

Proof. (d) Let us suppose that r_2, for example, is an entrance boundary and that there exists a solution with infinitely many zeros in $(0,\ r_2)$. As pointed out in the proof of part (a), it is sufficient to prove that this is impossible for $\lambda<-\|c\|$. From (2.7) with $F=y$ and $f=\lambda y$ we have, after a change of the order of integration in the last integral,

$$(4.8)\qquad \begin{aligned} y(x) &= y(0) + e^{B(x)}y'(x)\int_0^x e^{-B(s)}ds \\ &\quad + \int_0^x e^{B(t)}\{c(t)y(t) - \lambda y(t)\}dt\int_0^t e^{-B(s)}ds. \end{aligned}$$

Now let x_k be a point at which $y(x)$ has a relative minimum and x_{k+1} the first point after x_k where $y(x)$ has a relative maximum. Since $\lambda-c(x)<0$, $y'(x_k)=y'(x_{k+1})=0$, $y''(x_k)>0$, $y''(x_{k+1})<0$, we conclude that $\mu_k=y(x_k)<0$ and $\mu_{k+1}=y(x_{k+1})>0$. From (4.8) we have, letting $\beta=\max\ \{c(t)-\lambda\}$ for $r_1\leqq t\leqq r_2$,

$$(4.9)\qquad \begin{aligned} 0 < \mu_{k+1} - \mu_k &= \int_{x_k}^{x_{k+1}} e^{B(t)}\{c(t) - \lambda\}y(t)dt\int_0^t e^{-B(s)}ds \\ &\leqq \beta\{\mu_{k+1} - \mu_k\}\int_{x_k}^{x_{k+1}} e^{B(t)}dt\int_0^t e^{-B(s)}ds, \end{aligned}$$

or

$$\frac{1}{\beta} \leq \int_{x_k}^{x_k+1} e^{B(t)}dt \int_0^t e^{-B(s)}ds. \tag{4.10}$$

For an oscillatory solution x_k can be chosen arbitrarily close to r_2 but this contradicts (2.3).

(e) We can again construct a bounded solution by the method of successive approximations. Suppose r_2 is an entrance boundary. We define

$$w_0(x) \equiv 1,\ w_n(x) = 1 + \int_x^{r_2} e^{-B(t)}dt \int_t^{r_2} w_{n-1}(s)\{c(s) - \lambda\} e^{B(s)}ds. \tag{4.11}$$

It is sufficient to construct a bounded solution in some neighborhood $[\eta, r_2]$. We shall find it convenient to choose η so that for $x \in [\eta, r_2]$,

$$\int_x^{r_2} e^{-B(t)}dt \int_t^{r_2} | c(s) - \lambda | e^{B(s)}ds < \epsilon < 1. \tag{4.12}$$

We then have $|w_1(x) - w_0(x)| < \epsilon$ and by induction $|w_{n+1}(x) - w_n(x)| < \epsilon^{n+1}$ for $x \in [\eta, r_2]$. Hence $w_n(x)$ converges to a limit function $w(x)$. Since $|w_n(x)| < 1/(1-\epsilon)$, for $x \in [\eta, r_2]$, we conclude by dominated convergence in (4.11) that w is a solution of (4.1) with $\lim_{x\to r_2} w(x) = 1$. Our first statement in part (e) is then proved setting $w = y_2$.

If $y(x)$ is a bounded solution of (4.1), then from (4.8) it follows that if r_2 is an entrance boundary $\lim_{x\to r_2} y'(x)e^{B(x)} = 0$. Otherwise $y(x)$ would be unbounded in $[0, r_2]$ by the definition of an entrance boundary and our assumption that $c(x) \in C[r_1, r_2]$. If there were a solution $y(x)$ independent of $y_2(x)$ and bounded in $[0, r_2]$, then

$$\lim_{x\to r_*} \{ y_2(x)y'(x)e^{B(x)} - y(x)y_2'(x)e^{B(x)} \} = 0. \tag{4.13}$$

This, however, is impossible since the product of $e^{B(x)}$ and the Wronskian of y and y_2 is a nonzero constant. Therefore no solution independent of y_2 is bounded in $[0, r_2]$.

We can now prove our final statement that $\lim_{x\to r_2} y'(x)e^{B(x)} = c \neq 0$ for an unbounded solution y. Choose x_0 large enough that $y_2(x) > 0$ in $[x_0, r_2]$. Then a solution $y(x)$ independent of $y_2(x)$ is given by

$$y(x) = y_2(x) \int_{x_0}^{x} e^{-B(s)} y_2^{-2}(s)ds \tag{4.14}$$

for $x \in [x_0, r_2]$. For this solution,

$$y'(x)e^{B(x)} = y_2'(x)e^{B(x)} \int_x^{x} e^{-B(x)} y_2^{-2}(s)ds + [y_2(x)]^{-1}. \tag{4.15}$$

Now $y_2(x)\to 1$ as $x\to r_2$. It follows from (4.8) with $y=y_2$ that $y_2'(x)e^{B(x)} = o\{(\int_{x_0}^{x}e^{-B})^{-1}\}$. Hence $\lim_{x\to r_2} y'(x)e^{B(x)}=1$.

This completes the proof of Lemma 4.2.

LEMMA 4.3. *If λ is real and r_j is a regular boundary, then in $[0, r_j]$ each solution of* (4.1) *is bounded and has at most a finite number of zeros.*

Proof. The proof of this lemma is the same as that of Lemma 4.1 parts (a) and (b).

These lemmas make it possible to prove the main result of this section.

THEOREM 4.1. *Suppose $\lambda^*>\|c^+\|$ and $K(x, y; \lambda^*)$ is defined according to Definition* 3.3; *then $\phi_n(x)$ is an eigenfunction in $C[r_1, r_2]$ of $\bar{\Omega}$ corresponding to the eigenvalue λ_n if and only if $\phi_n\in L_2(e^B, r_1, r_2)$ and*

$$\phi_n(x) = (\lambda^* - \lambda_n)\int_{r_1}^{r_2} K(x, y; \lambda^*)\phi_n(y)e^{B(y)}dy. \tag{4.16}$$

Proof. We first note that I–III, §2 imply that $\bar{\Omega}$ has no eigenvalues larger than $\|c^+\|$. The necessity of the two conditions then follows from Lemmas 4.1 and 4.2 and the definition of the Green function.

Suppose now that $\phi_n\in L_2(e^B, r_1, r_2)$ and (4.9) holds. By the definition of $K(x, y; \lambda^*)$, we have

$$\begin{aligned}\phi_n(x) = (\lambda^* - \lambda_n)\Big[u_2(x)&\int_{r_1}^{x} u_1(s)\phi_n(s)e^{B(s)}ds \\ &+ u_1(x)\int_{x}^{r_2} u_2(s)\phi_n(s)e^{B(s)}ds\Big].\end{aligned} \tag{4.17}$$

The integrals in (4.10) exist for each $x\in(r_1, r_2)$ by Schwarz's inequality and the properties of u_1 and u_2 stated in I–III, §2. Therefore $\phi_n(x)$ is continuous in (r_1, r_2), and

$$\begin{aligned}\phi_n'(x) = (\lambda^* - \lambda_n)\Big[u_2'(x)&\int_{r_1}^{x} u_1(s)\phi_n(s)e^{B(s)}ds \\ &+ u_1'(x)\int_{x}^{r_2} u_2(s)\phi_n(s)e^{B(s)}ds\Big].\end{aligned} \tag{4.18}$$

Let us consider the boundary r_2 for simplicity. If r_2 is entrance limit circle, then both u_1 and $u_2\in L_2(e^B, 0, r_2)$; furthermore, $\lim_{x\to r_2} e^{B(x)}u_2'(x)=0$ and $\lim_{x\to r_2} e^{B(x)}u_1'(x)$ exists. It follows then from (4.11) that $e^{B(x)}\phi_n'(x)\to 0$ as $x\to r_2$ and $\phi_n\in C[0, r_2]$.

If r_2 is of the entrance limit point or the exit type at r_2, then the very fact that $\phi_n(x)$ is a solution in $(0, r_2)$ of 4.1 for $\lambda=\lambda_n$ with $\phi_n\in L_2(e^B, r_1, r_2)$ implies by Lemmas 4.1 (c) and 4.2 (e) that $\phi_n(x)$ satisfies the appropriate boundary conditions at r_2.

If r_2 is regular, then from (4.10) it follows that $\phi_n \in C[0, r_2]$; the result then follows from Theorem 3.1 and the two preceding paragraphs.

We can now make several useful observations about the eigenfunctions and eigenvalues of $\bar{\Omega}$. First of all, it is easily verified that if $\int_{r_1}^{r_2} K(x, y; \lambda^*) \cdot f(y)e^{B(y)}dy \equiv 0$ for some $f \in L_2(e^B, r_1, r_2)$ then $f(x) = 0$ almost everywhere. From this and Theorem 3.2 we conclude that the values of λ_n for which (4.9) holds form a denumerably infinite discrete set such that

$$\sum_{n=0}^{\infty} \lambda_n^{-2} < \infty. \tag{4.19}$$

Finally, since $\|c^+\|$ is an upper bound for the eigenvalues of $\bar{\Omega}$, there must be an N such that

$$\lambda_n < 0, \qquad n \geqq N. \tag{4.20}$$

The eigenfunctions ϕ_n of (4.9), which are also eigenfunctions of $\bar{\Omega}$, contain an orthonormal system with weight e^B, which is complete in $L_2(e^B, r_1, r_2)$.

That Theorem 4.1 is not true under general boundary conditions may be illustrated as follows: if both boundaries are exit boundaries and $c(x) \equiv 0$, then 1 is an eigenfunction in $C[r_1, r_2]$ of Ω under the boundary condition $\lim_{x \to r_j} e^{B(x)} u'(x) = 0$, corresponding to the eigenvalue $\lambda = 0$; however, $1 \notin L_2(e^B, r_1, r_2)$.

5. Expansions in eigenfunctions of $\bar{\Omega}$.

LEMMA 5.1. *The operator $\bar{\Omega}$ of Definition 3.2 is the infinitesimal generator of a semi-group $\{T_t\}$ from $C[r_1, r_2]$ to itself, strongly continuous at $t=0$, except when $p_j = 0$ in (3.2) or when r_j is an exit boundary; in this case the statement is true if $C[r_1, r_2]$ is replaced by the subspace of functions vanishing at r_j. The semi-group has the further properties:*

$$\|T_t f\| \leqq \|f\| e^{kt} \tag{5.1}$$

where $k = \|c^+\|$ (defined at the beginning of §2), and

$$f \geqq 0 \to T_t f \geqq 0. \tag{5.2}$$

The resolvent of $\{T_t\}$ is the transformation $f \to F$ given by (3.6). The infinitesimal generator of the adjoint semi-group $\{T_t^*\}$ from $L(r_1, r_2)$ to itself is the operator $\bar{\Omega}^*$ defined in Definition 3.4.

Proof. All the statements except the last follow directly from the Hille-Yosida Theorem and Theorem 3.1. The proof of the last statement is the same as that given in [1] for the case $c \equiv 0$.

DEFINITION 5.1. *With the operator $\bar{\Omega}$ we associate a Banach space X given by: (1) $C[r_1, r_2]$ if $p_j \neq 0$ in (3.3) or if r_j is not an exit boundary; (2) the subspace* *of* $C[r_1, r_2]$ *of functions vanishing at* r_j *if* $p_j=0$ *in* (3.3), *or if* r_j *is an exit boundary.*

LEMMA 5.2. *If neither boundary is natural, then for* $f\in X$ *we have*

$$T_t f(x) = \int_{r_1}^{r_2} f(y)\Gamma(x, y; t)e^{B(y)}dy, \qquad t > 0, \tag{5.3}$$

where

$$\Gamma(x, y; t) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)e^{\lambda_n t}. \tag{5.4}$$

For each $t>0$, *the series in* (5.4) *converges uniformly in the region* $r_1 \leqq {}^x_y \leqq r_2$ *if neither boundary is of entrance type; otherwise, the series converges uniformly in the region* $r_1' \leqq {}^x_y \leqq r_2'$ *where* $r_j' \neq r_j$ *if* r_j *is an entrance boundary. The representation* (5.3) *holds for* $x\in[r_1, r_2]$ *in the former case and for* $x\in[r_1', r_2']$ *in the latter.*

Proof. We first prove (5.3) for a function $g(x)$ which vanishes in the neighborhood of r_j if r_j is an exit boundary. Suppose that $g(x)\neq 0$ for x in some interval $[s_1, s_2]$. Our first step will be to investigate the convergence of the series,

$$\sum_{n=0}^{\infty} (g, \phi_n)\phi_n(x)e^{\lambda_n t} \tag{5.5}$$

where

$$(g, \phi_n) = \int_{r_1}^{r_2} g(x)\phi_n(x)e^{B(x)}dx. \tag{5.6}$$

Since g vanishes outside $[s_1, s_2]$, we know that $g\in L_2(e^B, r_1, r_2)$, and therefore that $(g, \phi_n)\to 0$ as $n\to\infty$. Furthermore, it follows from (4.16) and (3.13) that

$$|\phi_n(x)| \leqq M|\lambda_n|, \qquad n > n_0, \tag{5.7}$$

for x in any closed interval not containing an entrance boundary. Therefore, the series (5.5) for each $t>0$ converges uniformly in $[r_1', r_2']$, since (4.19) and (4.20) hold.

We can show in addition that (5.5) converges to $T_t g(x)$ in $[r_1', r_2']$, by showing that it converges in the mean of $L_2(e^B, r_1, r_2)$ to $T_t g(x)$. By Lemma 5.1, the infinitesimal generator of the adjoint semigroup $\{T_t^*\}$ is $\bar{\Omega}^*$. Moreover, it is easily verified that if $\lambda_n\phi_n - \bar{\Omega}\phi_n = 0$, then $\lambda_n(\phi_n e^B) - \bar{\Omega}^*(\phi_n e^B) = 0$. By Lemmas 4.1–4.3, $\phi_n e^B \in L(r_1, r_2)$. Therefore,

$$\int_{r_1}^{r_2} (T_t g)\phi_n e^B = \int_{r_1}^{r_2} g T_t^*(e^B\phi_n) = e^{\lambda_n t}\int_{r_1}^{r_2} g\phi_n e^B. \tag{5.8}$$

This shows that $T_t g \in L_2(e^B, r_1, r_2)$ and (5.5) therefore converges in the mean of $L_2(e^B, r_1, r_2)$ to $T_t g$ by the last paragraph of §4.

All we must show now is that integration and summation may be interchanged in (5.5). Letting $S_N(x, y)$ be the Nth partial sum of (5.4), we have for fixed $x \in [r_1', r_2']$,

$$\int_{r_1}^{r_2} S_N^2(x, y) e^{B(y)} dy \leqq \sum_{n=0}^{\infty} \{\phi_n(x)\}^2 e^{2\lambda_n t} < \infty. \tag{5.9}$$

Since $g \in L_2(e^B, s_1, s_2)$ and g vanishes outside $[s_1, s_2]$, we have

$$\begin{aligned} T_t g(x) &= \lim_{N\to\infty} \int_{r_1}^{r_2} g(y) S_N(x, y) e^{B(y)} dy \\ &= \int_{r_1}^{r_2} g(y) \lim_{N\to\infty} S_N(x, y) e^{B(y)} dy = \int_{r_1}^{r_2} g(y) \Gamma(x, y; t) e^{B(y)} dy, \end{aligned} \tag{5.10}$$

by a well-known theorem (cf. [7, p. 139]).

We have now proved the theorem for functions which vanish in the neighborhood of an exit boundary. To prove the representation in the general case, it will be enough to prove it for $f \geqq 0$ in X. We can form a sequence of functions $g_\nu(x) \in X$ vanishing in the neighborhood of r_j if r_j is an exit boundary, and satisfying

$$\lim_{\nu\to\infty} \|g_\nu - f\| = 0 \tag{5.11}$$

and

$$0 \leqq g_\nu(x) \leqq g_{\nu+1}(x) \leqq f(x). \tag{5.12}$$

From (5.2) and (5.10), it follows that

$$\Gamma(x, y; t) \geqq 0.$$

Therefore, since T_t is a continuous transformation, (5.3) follows from (5.10) and monotonic convergence.

THEOREM 5.1. *If neither boundary is natural, then each $f \in X$ can be expressed as*

$$f(x) = \lim_{t\to 0} \int_{r_1}^{r_2} f(y) \left[\sum_{n=0}^{\infty} e^{\lambda_n t} \phi_n(x) \phi_n(y) \right] e^{B(y)} dy, \tag{5.13}$$

the convergence as $t \to 0$ being uniform for $x \in [r_1, r_2]$ if neither boundary is an entrance boundary; the convergence as $t \to 0$ is uniform for $x \in [r_1', r_2']$ with $r_j' \neq r_j$ if r_j is an entrance boundary.

Proof. This is an immediate consequence of Lemma 5.2, (5.1), and the strong continuity of $\{T_t\}$ at $t=0$ for $f\in X$.

LEMMA 5.3. *Suppose that each boundary is either a regular or an entrance boundary. Let* $[r_1', r_2']$ *denote a closed interval contained in* $[r_1, r_2]$ *with* $r_j' \neq r_j$ *if* r_j *is an entrance boundary. Then for each* $f\in X$ *we have*

$$T_tf(x) = \sum_{n=0}^{\infty} \phi_n(x)(f, \phi_n)e^{\lambda_n t}, \tag{5.14}$$

where (f, ϕ_n) *is defined as in* (5.6). *For each* $t>0$ *the series in* (5.14) *converges uniformly in* $[r_1', r_2']$.

Proof. By Lemma 5.2, all we need show is that for $x\in[r_1', r_2']$

$$\begin{aligned}\int_{r_1}^{r_2} f(y)\left[\lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(x)\phi_n(y)e^{\lambda_n t}\right]e^{B(y)}dy \\ = \lim_{N\to\infty}\sum_{n=0}^{N} e^{\lambda_n t}\phi_n(x)\int_{r_1}^{r_2} f(y)\phi_n(y)e^{B(y)}dy.\end{aligned} \tag{5.15}$$

Under our hypotheses, we have

$$f \in L_2(e^B, r_1, r_2).$$

Furthermore, (5.9) holds for fixed $x\in[r_1', r_2']$, so that by the theorem we have quoted before in [7, p. 139], we conclude that (5.15) holds. This completes the proof.

THEOREM 5.2. *If each boundary is either entrance or regular, then each* $f\in X$ *has the expansion*

$$f(x) = \lim_{t\to 0} \sum_{n=0}^{\infty}{}^{*} \phi_n(x)(f, \phi_n)e^{\lambda_n t}, \tag{5.16}$$

where $\sum_{n=0}^{*\infty}$ *coincides with* $\sum_{n=0}^{\infty}$ *unless* $x=r_j$ *with* r_j *an entrance boundary, in which case*

$$\sum_{n=0}^{\infty}{}^{*} \phi_n(r_j)(f, \phi_n)e^{\lambda_n t} = \lim_{x\to r_j}\sum_{n=0}^{\infty}\phi_n(x)(f, \phi_n)e^{\lambda_n t}. \tag{5.17}$$

The limit as $t\to 0$ *in* (5.16) *is uniform for* $x\in[r_1, r_2]$. *For each* $t>0$, *the convergence of the series is described in Lemma* 5.3.

Proof. This theorem follows immediately from Lemma 5.3, combined with the strong continuity of $\{T_t\}$ at $t=0$.

Under certain additional hypotheses in the cases of entrance and exit boundaries, the expansion theorems can be somewhat strengthened. In the next theorem we give a set of sufficient conditions for improving the results

THEOREM 5.3. *Suppose that each boundary is either an exit boundary satisfying*

$$e^{-B(x)}\left[\int_0^x e^{B(s)}ds\right]^2 \in L(0, r_i), \tag{5.18}$$

an entrance limit circle, or a regular boundary; then (5.14) *holds for each* $x \in [r_1, r_2]$, *the series converging uniformly for* $x \in [r_1, r_2]$. *Consequently, each* $f \in X$ *has the expansion*

$$f(x) = \lim_{t \to 0} \sum_{n=0}^{\infty} \phi_n(x)(f, \phi_n)e^{\lambda_n t} \tag{5.19}$$

the limit as $t \to 0$ *being uniform for* $x \in [r_1, r_2]$.

Proof. Suppose first that r_2 is entrance limit circle. Under this condition we can show that (5.7) holds in $[0, r_2]$. Since $\phi_n(x)$ is an eigenfunction of $\overline{\Omega}$, we have

$$\lim_{x \to r_2} \phi_n'(x)e^{B(x)} = 0.$$

Therefore,

$$-\phi_n'(x)e^{B(x)} = \int_x^{r_2} [\lambda_n - c(s)]\phi_n(s)e^{B(s)}ds. \tag{5.20}$$

Integrating once more and using (4.16) and (3.13), we have

$$\begin{aligned} |\phi_n(x)| &= \left|\phi_n(0) - \int_0^x [\lambda_n - c(s)]\phi_n(s)e^{B(s)}ds \int_0^s e^{-B(t)}dt\right| \\ &\leqq \frac{|\lambda_n - \lambda^*|}{|\lambda^* - k|} u_1(0)u_2(0) + (|\lambda_n| + \|c\|) \int_0^{r_2} e^{B(s)}ds \\ &\qquad \cdot \left[\int_0^s e^{-B(t)}dt\right]^2. \end{aligned} \tag{5.21}$$

Since the integral on the extreme right converges by Theorem 2.1, we have (5.7) holding in $[0, r_2]$. From (4.16) and (3.13) and §2, I and II, we know this to hold also if r_2 is either an exit or a regular boundary. Summing up, we have shown that *under the hypotheses of this theorem* (5.7) *holds for* $x \in [r_1, r_2]$ *with M independent of x and n.*

We shall now prove a similar result for the generalized Fourier coefficients; namely, that *under the hypotheses of this theorem,*

$$\int_{r_1}^{r_2} |f(x)\phi_n(x)| e^{B(x)}dx \leqq C\|f\|\lambda_n^2, \qquad n > N_0, \tag{5.22}$$

for $f \in X$, with C independent of x and n.

If neither boundary is an exit boundary, then there is no difficulty, since the left side of (5.22) is then bounded by $\|f\| [\int_{r_1}^{r_2} e^{B(x)} dx]^{1/2}$. Now suppose that r_2 is an exit boundary; then

$$(5.23)\quad \phi_n(x) = -\phi_n'(0) \int_x^{r_2} e^{-B(s)} ds - \int_x^{r_2} [\lambda_n - c(s)] e^{-B(s)} ds \int_0^s e^{B(t)} \phi_n(t) dt.$$

Therefore, from (5.23) and a change in the order of integration, we have

$$(5.24)\quad \begin{aligned} \int_0^{r_2} |\phi_n(x)| e^{B(x)} dx \leqq |\phi_n'(0)| \int_0^{r_2} e^{B(x)} \left[\int_x^{r_2} e^{-B(s)} ds \right] dx \\ + M |\lambda_n| \cdot (|\lambda_n| + \|c\|) \int_0^{r_2} e^{-B(s)} ds \left[\int_0^s e^{B(x)} dx \right]^2, \end{aligned}$$

where M is the constant we found in the first part of the proof. It follows from (4.18) that $|\phi_n'(0)| \leqq C_1 |\lambda_n|$ for n sufficiently large. Since the two integrals in (5.24) converge, we have proved (5.22).

From what we have just proved, combined with (4.19) and (4.20), we conclude that for each $t > 0$, the series on the right-hand side of (5.19) converges uniformly for $x \in [r_1, r_2]$ and defines a bounded transformation from X to X. This transformation agrees with T_t on the set of functions $f \in X$ which are also in $L_2(e^B, r_1, r_2)$; since this set is dense in X, it follows that T_t must be given by the series on the right of (5.19). The conclusion of the theorem then follows from the strong continuity of $\{T_t\}$ at $t = 0$.

If $p_j = 0$ in (3.3) or if r_j is an exit boundary for $j = 1$ and/or 2, then X is a proper subspace of $C[r_1, r_2]$. The following theorem extends Theorem 5.1 to a wide class of functions outside the space X.

THEOREM 5.4. *If $f(x)$ is a bounded measurable function on $[r_1, r_2]$, and neither boundary is natural, then* (5.13) *holds at every point of continuity $x \in (r_1, r_2)$. The convergence as $t \to 0$ is bounded uniformly in x.*

Proof. We have already observed that $\Gamma(x, y; t) \geqq 0$. We shall now show that

$$(5.25)\quad 0 \leqq \int_{r_1}^{r_2} \Gamma(x, y; t) e^{B(y)} dy \leqq 1.$$

This needs proof, of course, only if X is a proper subspace of $C[r_1, r_2]$. We can approximate the function $f(x) \equiv 1$ pointwise by a sequence $g_\nu(x) \in X$ satisfying (5.12); this immediately gives (5.25) by monotonic convergence.

It is now easy to show that

$$(5.26)\quad \lim_{t \to 0} \int_{r_1}^{r_2} \Gamma(x_0, y; t) e^{B(y)} dy = 1$$

for $x_0 \in (r_1, r_2)$. Let $h \in X$ with $h(x) \leqq 1$ and $h(x_0)=1$ for some $x_0 \in (r_1, r_2)$; from (5.25) we have

$$(5.27) \qquad \int_{r_1}^{r_2} h(y)\Gamma(x_0, y; t)e^{B(y)}dy \leqq \int_{r_1}^{r_2} \Gamma(x_0, y; t)e^{B(y)}dy \leqq 1,$$

and since the limit as $t \to 0$ of the left side is 1 by Theorem 5.1, we conclude that (5.26) holds.

We have now proved the theorem for functions in the subspace $\overline{C}$ of $C[r_1, r_2]$, defined by

$$(5.28) \qquad \overline{C}[r_1, r_2] = \{f \mid f \in C[r_1, r_2] \text{ and } f(r_1) = f(r_2)\}.$$

To obtain the result for a general bounded measurable function, we can choose two functions g_1 and g_2 in $\overline{C}[r_1, r_2]$ satisfying

$$(5.29) \qquad g_1(x) \leqq f(x) \leqq g_2(x),$$

and

$$(5.30) \qquad g_j(x_0) = f(x_0), \qquad j = 1, 2,$$

for a given $x_0 \in (r_1, r_2)$ which is a point of continuity of $f(x)$. Since from (5.25)

$$(5.31) \qquad \begin{aligned} \int_{r_1}^{r_2} g_1(y)\Gamma(x_0, y; t)e^{B(y)}dy &\leqq \int_{r_1}^{r_2} f(y)\Gamma(x_0, y; t)e^{B(y)}dy \\ &\leqq \int_{r_1}^{r_2} g_2(y)\Gamma(x_0, y; t)e^{B(y)}dy, \end{aligned}$$

the result follows from the preceding part of the proof. That the convergence is bounded follows from (5.25).

6. **Expansions in eigenfunctions of $\overline{\Omega}^*$.** These expansion theorems follow almost directly from those of §5.

Suppose we are given a set of boundary conditions of type (3.15)–(3.16). We let Y denote the space of totally finite signed measures on the interval $[r_1, r_2]$, unless $p_j=0$ in (3.15) or r_j is an exit boundary; in these cases let Y denote the space of measures which give the point r_j zero measure.

As we have observed before [preceding (5.8)], if ϕ_n is an eigenfunction of $\overline{\Omega}$ corresponding to the eigenvalue λ_n, then

$$(6.1) \qquad \psi_n = e^B \phi_n$$

is an eigenfunction of $\overline{\Omega}^*$. Therefore the eigenfunctions of $\overline{\Omega}^*$ contain an orthonormal set $\{\psi_n\}$ with weight function e^{-B}, complete in $L_2(e^{-B}, r_1, r_2)$.

THEOREM 6.1. *For each $\mu \in Y$, we have*

$$(6.2) \qquad \lim_{t \to 0} \int_{r_1}^{r_2} f(y)dy \int_{r_1}^{r_2} \left[\sum_{n=0}^{\infty} \psi_n(x)\psi_n(y)e^{\lambda_n t} \right] e^{-B(x)} d_x\mu = \int_{r_1}^{r_2} f(y) d_y\mu$$

for every $f \in X$.

Proof. From Lemma 5.2, we conclude that the convergence as $t \to 0$ in (5.13) is bounded in x. Therefore, (6.2) follows from an application of Fubini's theorem, (6.1), and bounded convergence.

THEOREM 6.2. *Given $\mu \in Y$, let $\mu(x)$ denote $\mu[0, x]$; if x_1 and x_2 are points of continuity of $\mu(x)$, then*

$$\lim_{t\to 0} \int_{x_1}^{x_2} dy \int_{r_1}^{r_2} \left[\sum_{n=0}^{\infty} \psi_n(y)\psi_n(x)e^{\lambda_n t} \right] e^{-B(x)} d_x\mu = \mu(x_2) - \mu(x_1). \tag{6.3}$$

Proof. Let $f(x)$ be the characteristic function of $[x_1, x_2]$. The result then follows from Theorem 5.4, Fubini's theorem, and bounded convergence.

It is easily verified that the adjoint semi-group to $\{T_t\}$ is given by

$$\begin{aligned} T^*\mu(S) &= \int_S dy \int_{r_1}^{r_2} \Gamma(x, y; t) e^{B(y)} d_x\mu \\ &= \int_S dy \int_{r_1}^{r_2} \left[\sum_{n=0}^{\infty} \psi_n(y)\psi_n(x)e^{\lambda_n t} \right] e^{-B(x)} d_x\mu. \end{aligned} \tag{6.4}$$

The range of T_t^* lies in the space Y' of absolutely continuous measures (see [1; Theorem 13.3]); the semi-group contracted to this space forms a semi-group which is strongly continuous at the origin. Since Y' is equivalent to the space $L[r_1, r_2]$, we can state the following theorems:

THEOREM 6.3. *If neither boundary is natural and $g \in L(r_1, r_2)$, then,*

$$g(y) = \underset{t\to 0}{\text{l.i.m.}} \int_{r_1}^{r_2} g(x) \left[\sum_{n=0}^{\infty} \psi_n(x)\psi_n(y)e^{\lambda_n t} \right] e^{-B(x)} dx. \tag{6.5}$$

THEOREM 6.4. *Under the hypotheses of Theorem 5.3, if $g \in L(r_1, r_2)$ then*

$$g = \underset{t\to 0}{\text{l.i.m.}} \sum_{n=0}^{\infty} \psi_n(g, \psi_n)e^{\lambda_n t}, \tag{6.6}$$

where

$$(g, \psi_n) = \int_{r_1}^{r_2} g(x)\psi_n(x)e^{-B(x)} dx = \int_{r_1}^{r_2} g(x)\phi_n(x) dx. \tag{6.7}$$

The series in (6.6) converges in the mean of $L(r_1, r_2)$ for each $t > 0$.

Proof. The proof of this theorem follows easily from the information about ϕ_n which was developed in the proof of Theorem 5.3.

Added in proof. Some of our results overlap those contained in [9], which appeared in print after the present paper was submitted.

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