ON LINEAR, SECOND ORDER DIFFERENTIAL EQUATIONS IN THE UNIT CIRCLE

BY

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1. **Introduction.** In the differential equation

$$W'' + p(z)W' + q(z)W = 0 \qquad (' = d/dz), \tag{1}$$

let z be a real or complex variable, $q(z)$ a continuous and $p(z)$ a continuously differentiable function on the domain under consideration. The function

$$I(z) = q(z) - (1/4)p(z) - (1/2)p'(z) \tag{2}$$

is called the *invariant* of (1). If $W_1(z)$, $W_2(z)$ are linearly independent solutions of (1) and if $u = W_1/W_2$, then

$$2I = \{u, z\}, \tag{3}$$

where $\{u, z\}$ is the Schwarzian parameter

$$\{u, z\} = (u''/u')' - (1/2)(u''/u')^2. \tag{4}$$

The change of dependent variables $W \to w = W \exp((1/2)\int^z p dz)$ transforms (1) into the normal form

$$w'' + I(z)w = 0. \tag{5}$$

Hence, in considering zeros of solutions of (1), it can be assumed that (1) has the form (5). The term "solution" will always mean a non-trivial ($\not\equiv 0$) solution.

This note will be concerned principally with solutions $w = w(z)$ of the differential equation (5) under the assumption that $I(z)$ is analytic on the circle $|z| < 1$. (Unless the contrary is stated below, it will be supposed that $I(z)$ satisfies this assumption.)

The following terminology will be used: If no solution of a differential equation has two zeros (on a given z-set), then the differential equation will be said to be *disconjugate* (on that set) [11]. Similarly, if no solution has an infinite set of zeros, the differential equation will be called *non-oscillatory*. (In contrast to the situation on the real field, where Sturm's separation theorem is valid, it is possible that a solution of (5) can have a finite number of zeros on $|z| < 1$ and that another solution has an infinite number of zeros there.)

2. **Reduction to a real independent variable.** The results on the zeros of solutions of (5) in the case that z is a complex variable will be deduced from cases where $I(z)$ is a complex-valued function of a real variable (for example, $z = x + iy$ for fixed y). The transfer of these results from the real case will be possible because of the following "comparison" theorem (cf., e.g., [9, p. 319]):

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(*) Let $I(z)$ be a continuous, complex-valued function of a real variable z on some interval. If

$$v'' + \Re(I(z))v = 0 \tag{6}$$

is disconjugate on the given interval, then (5) is disconjugate on that interval.

In this assertion, (6) can of course be replaced by any Sturm majorant, for example, by

$$v'' + |I(z)|v = 0. \tag{7}$$

The comparison theorem (*) is a particular case of the following trivial fact: If A_1 and A_2 are self-adjoint (bounded or unbounded) operators in Hilbert space and $A_1 \geqq \text{const.} > 0$, then $\lambda = 0$ is not in the point spectrum of $A_1 + iA_2$. Theorem (*) follows by choosing $A_1 f$ to be the differential operator $f'' + \Re I(z)f$ associated with boundary conditions $f = 0$ at the end points of the interval and $A_2 f$ to be $\Im(I(z))f$.

The transformation rule

$$\{u, Z\} = \{u, z\}(dz/dZ)^2 + \{z, Z\} \tag{8}$$

for the Schwarzian derivative under the change of (real or complex) independent variables $z \to Z$ supplies, by (3), the transformation rule for the invariant of (1) or (5). In particular, (8) reduces to

$$\{u, Z\} = \{u, z\}(dz/dZ)^2 \quad \text{if} \quad Z = (\alpha z + \beta)/(\gamma z + \delta), \tag{9}$$

$\alpha\delta - \beta\gamma \neq 0$, since $w = \alpha z + \beta$, $\gamma z + \delta$ are linearly independent solutions of the equation $w'' = 0$ which has the invariant $I(z) = 0$.

Assertion (*) immediately implies two results of Nehari [7], which state that if $u = u(z)$ is an analytic function on the unit circle $|z| < 1$, then $u(z)$ is *schlicht* on $|z| < 1$ whenever $I(z)$, defined by (3), satisfies either of the inequalities

$$|I(z)| \leqq \pi^2/4, \tag{10_1}$$

$$|I(z)| \leqq 1/(1 - |z|^2)^2 \tag{10_2}$$

[7, p. 549 and p. 545]. In fact, $u(z)$ is *schlicht* on $|z| < 1$ whenever (5) is disconjugate there. That either (10_1) or (10_2) implies that (5) is disconjugate on $|z| < 1$ can be deduced from (*) as follows:

Ad (10_1). Suppose that some solution $w = w(z)$ of (5) has two zeros in $|z| < 1$, then the remark concerning (9) (in fact, the case $Z = e^{i\phi}z$) shows that there is no loss of generality in supposing that these zeros are on the same horizontal segment $z = x + iy$, with $y = \text{const.}$ and $x^2 < 1 - y^2$. Since the length of the x-interval joining the two zeros is less than 2, the inequality (10_1) shows that no solution of (7) (hence, by (*), no solution of (5)) can have two zeros on such an interval.

Ad (10_2). As verified by direct computation by Nehari [7, p. 547], the inequality (10_2) for the invariant of (5) is unchanged by conformal mappings $z \to Z$ of the unit circle $|z| < 1$ onto $|Z| < 1$. In fact, if

$$ds = |dz|/(1 - |z|^2) \tag{11}$$

denotes the non-euclidean arc length, which is invariant under the mapping $z \to Z$, then (9) can be written as

$$(1 - |Z|^2)^2 |\{u, Z\}| \, ds = (1 - |z|^2)^2 |\{u, z\}| \, ds. \tag{12}$$

Thus the invariance of (10_2) follows from (3).

If the assertion concerning (10_2) is false, then some solution $w = w(z)$ of (5) has at least two zeros on $|z| < 1$. In view of the invariance of (10_2), it can be supposed that these zeros are real. Either one of the following two equivalent arguments shows, by (*), that this leads to a contradiction. First, for real z, (7) has the Sturm majorant

$$d^2v/dx^2 + v/(1 - x^2)^2 = 0, \qquad -1 < x < 1,$$

which is disconjugate since it possesses the solution $v = (1 - x^2)^{1/2}$ having no zeros on $-1 < x < 1$; cf. [6]. Second, the change of variables

$$s = (1/2) \log (1 + z)/(1 - z)$$

satisfying (11) for real $z = x$ transforms the invariant $|I(z)|$ of (7), according to (3) and (8), into the non-positive function $|I(z)|(1 - |z|^2)^2 - 1$ for $z = z(s)$, $-\infty < s < \infty$. Hence, in the case (10_2), (5) is disconjugate on $|z| < 1$.

The constants $\pi^2/4$ and 1 are the best possible in (10_1) and (10_2), respectively; [7, p. 549] and [6, p. 552].

3. **A criterion for disconjugateness.** The condition (10_2) can be replaced by a somewhat different criterion:

(i) If C is a circular arc in $|z| < 1$ orthogonal to the boundary $|z| = 1$, then the inequality

$$\int_C (1 - |z|^2) |I(z) dz| \leqq 2 \tag{13}$$

implies that (5) is disconjugate on C.

If $z = x$ in (7) is a real variable on some interval $a < x < b$ and $I(z)$ is continuous on this interval, then, according to [4], (7) is disconjugate on this interval if

$$\int_a^b (b - x)(x - a) |I(x)| \, dx \leqq b - a \tag{14}$$

(cf. [8] for a generalization to complex variables). Hence, if C is the real interval $-1 < x < 1$, (13) and (*) imply that no solution of (5) has two zeros on the real axis. If C is any circular arc orthogonal to $|z|=1$, there exists a transformation $z \rightarrow Z$ of the type in (9) of the circle $|z|<1$ onto $|Z|<1$ such that the image of C is the real segment $-1<X<1$, where $Z=X+iY$. It follows from (9) that no solution of (5) has two zeros on C if

$$\int_{-1}^{1}(1-|Z|^2)|I(z)(dz/dZ)^2 dZ| \leqq 2, \qquad Z = X. \tag{15}$$

Note that (11) and (12) imply that

$$(1-|Z|^2)|\{u, Z\}dZ| = (1-|z|^2)|\{u, z\}dz|. \tag{16}$$

Consequently, the integrals in (13) and (15) are equal, and so (15) follows from (13). Thus (13) assures that no solution of (5) has two zeros on C.

4. **Disconjugateness and $\mu(1)$.** If C is a line segment contained in $|z|<1$, a sufficient criterion for (7) (hence for (5)) to be disconjugate on C is

$$\int_C |I(z)dz| \leqq 4/L, \quad \text{where } L \text{ is the length of } C \tag{17}$$

(Liapounoff; cf., e.g., [4]). If C is a chord of $|z|=1$, this can be improved to

$$\int_C (1-|z|^2)|I(z)dz| \leqq L \tag{18}$$

by the criterion (14); see [4].

An inequality of Fejér and Riesz [3] states that

$$\int_C |I(z)dz| \leqq (1/2)\mu(1), \tag{19}$$

if C is the real line segment $-1<x<1$,

$$\mu(1) = \lim_{r\to 1}\mu(r), \tag{20}$$

and

$$\mu(r) = \int_{|z|=r}|I(z)dz|. \tag{21}$$

According to a remark of Nehari [8, p. 695], (19) is valid if C is any circular arc in $|z|<1$ orthogonal to $|z|=1$. Hence, the inequality

$$\int_C (1-|z|^2)|I(z)dz| \leqq \int_C |I(z)dz|$$

and (i) give the following:

(ii) The differential equation (5) is disconjugate on $|z|<1$ whenever

$$\mu(1) \leqq 4. \tag{22}$$

The weakened form $\mu(1) \leqq 2$ of this condition follows from Nehari's inequality (21) in [8] and his remark following it. The above use of the inequality (19) is similar to the procedure of Nehari.

The constant 4 in (22) is the ratio of the constants 2 in (13) and 1/2 in (19). Although the inequalities (13) and (19) cannot be improved, it remains undecided whether or not (22) is the "best" possible.

Nehari's inequality leading to the weakened form $\mu(1) \leqq 2$ of (22) has the following consequence: The inequality

$$\mu(r) \leqq 4/L \qquad (L > 0) \tag{23}$$

implies that no solution of (5) has two zeros $z = z_1, z_2$ in the circle $|z| < r$ satisfying $|z_1 - z_2| \leqq L$.

5. **Solutions satisfying** $w(0) = 0$. The inequality (23) can be considerably sharpened for r near 1 in dealing with a particular solution of (5).

(iii) If $w = w(z)$ is a solution of (5) satisfying $w(0) = 0$, then $w(z)$ has no zero different from $z = 0$ in $|z| < 1$ if

$$\mu(r) \leqq 1/2r(1 - r) \quad \text{for} \quad 1/2 \leqq r < 1. \tag{24}$$

In order to prove this, grant, for a moment, the fact that no solution of (7) (hence no solution of (5)) can have two zeros on a radius $z = te^{i\phi}$, where $0 \leqq t < 1$, if

$$\int_0^r t^2 \left| I(te^{i\phi}) \right| dt \leqq r/4(1 - r) \quad \text{for} \quad 0 < r < 1. \tag{25}$$

The inequality of Fejér and Riesz implies the second of the inequalities

$$\int_0^r t^2 \left| I(te^{i\phi}) \right| dt \leqq r^2 \int_{-r}^r \left| I(te^{i\phi}) dt \right| \leqq (1/2) r^2 \mu(r)$$

and so (24) implies (25). Thus, in order to prove (iii), it is sufficient to prove the statement concerning (25).

Let $q_1(s)$, $q_2(s)$ be real-valued, continuous functions on $0 < s < \infty$ such that

$$q_1(s) \geqq 0 \quad \text{and} \quad \int^\infty q_1(s) ds < \infty.$$

If the first of the differential equations

$$d^2v/ds^2 + q_k(s)v = 0 \qquad (k = 1, 2) \tag{26}$$

is disconjugate on $0 < s < \infty$ and if

$$\int_s^\infty \left| q_2(s) \right| ds \leqq \int_s^\infty q_1(s) ds \qquad (0 < s < \infty), \tag{27}$$

then (26_2) is disconjugate on $0<s<\infty$ (cf. [5, p. 245] and [11]). The choice $q_1(s)=1/4s^2$ (Kneser) gives the sufficient condition

$$\int_s^\infty |q_2(s)|\,ds \leqq 1/4s \qquad (0<s<\infty) \tag{28}$$

for (26_2) to be disconjugate on $0<s<\infty$.

If $q(t)$ is continuous in the differential equation

$$d^2v/dt^2 + q(t)v = 0 \qquad (0<t<1), \tag{29}$$

the change of independent variables $s=(1-t)/t$ (which maps $0<t<1$ onto $\infty>s>0$) alters the invariant $q(t)$ of (29) to $q_2(s)=q(t)(dt/ds)^2$, by (9). Since (28) is transformed into

$$\int_0^t t^2|q(t)|\,dt \leqq t/4(1-t) \qquad (0<t<1), \tag{30}$$

the statement concerning (25) follows.

By using functions other than $q_1(s)=1/4s^2$, for example,

$$q_1(s) = (1/4s^2)(1 + 1/\log^2 s),$$

it is possible to refine (24) somewhat. It is also possible to refine (iii) in the following direction: Let $0\leqq\alpha<1$. There exists a constant $K=K$ (independent of α and $I(z)$) such that if

$$\mu(r) \leqq K(1-\alpha)^2/(1-r) \quad \text{for} \quad 1/2 \leqq r < 1,$$

then no solution of (5) which has a zero in the circle $|z|\leqq\alpha$ has another zero on $|z|<1$.

6. **The solutions in the case $\mu(1)<\infty$.** If the condition (22) for (5) to be disconjugate on $|z|<1$ is weakened to

$$\mu(1) < \infty, \tag{31}$$

then, according to Nehari [8], (5) is non-oscillatory on $|z|<1$. Actually, (31) implies a great deal more about the solutions $w(z)$ of (5) than the fact that $w(z)$ has only a finite number of zeros on $|z|<1$.

Let (31) hold. Then there exists a function $\psi(\theta)$ of bounded variation on $|\theta|\leqq\pi$ such that

$$\psi(\theta) = \lim_{r\to 1}\int_0^\theta |I(re^{i\phi})|\,d\phi \tag{32}$$

holds at every continuity point of $\psi(\theta)$.

The properties of the solutions $w(z)$ of (5) under the assumption (31) can be described as follows:

$w(z)$ is uniformly continuous on $|z|<1$; in fact, the derivative $w'(z)$ is bounded on $|z|<1$. In addition, the radial limits $w'(e^{i\theta})=\lim w'(re^{i\theta})$, as $r\to 1$, exist for all θ. The function $w'(z)$ on $|z|\leqq 1$ (defined to be $w'(e^{i\theta})$ at $z=e^{i\theta}$) is continuous on $|z|\leqq 1$ except possibly at the points $e^{i\theta}$ where θ is a discontinuity point of $\psi(\theta)$. The point $z=e^{i\theta}$ is a continuity point of $w'(z)$ if $w(e^{i\theta})=0$. Finally, there exists one and only one solution $w=w(z)$ of (5) for which w and w' have preassigned radial limits $w(e^{i\theta})$, $w'(e^{i\theta})$ at a given point $z=e^{i\theta}$ of $|z|=1$.

In order to verify these properties, note that if, on a fixed radius, one has

$$\int_0^1 (1-t)\,|\,I(te^{i\theta})\,|\,dt < \infty, \tag{33}$$

then the radial limits $w(e^{i\theta})$, $w'(e^{i\theta})$ belonging to a solution $w(z)$ of (5) exist ([1]; cf. [2, pp. 368–370] and [10, pp. 261–268]). Furthermore, there exists one and only one solution having preassigned radial limits $w(e^{i\theta})$, $w'(e^{i\theta})$ for a fixed θ; cf. [10]. Clearly, (19) and (31) imply (33) for every θ.

Every solution $w=w(z)$ of (5) satisfies

$$w(z) = w(0) + zw'(0) - \int_0^z (z-\zeta)I(\zeta)w(\zeta)d\zeta. \tag{34}$$

Hence

$$|\,w(re^{i\theta})\,| \leqq A + r\int_0^r |\,I(te^{i\theta})w(te^{i\theta})\,|\,dt \qquad (A = |\,w(0)\,| + |\,w'(0)\,|).$$

Consequently, a standard inequality gives

$$|\,w(re^{i\theta})\,| \leqq A \exp r\int_0^r |\,I(te^{i\theta})\,|\,dt,$$

and so, by the inequality of Fejér and Riesz,

$$|\,w(re^{i\theta})\,| \leqq A \exp (1/2)r\mu(r) \leqq A \exp (1/2)\mu(1).$$

This proves that w is bounded on $|z|<1$. By (34),

$$w'(z) = w'(0) - \int_0^z I(\zeta)w(\zeta)d\zeta. \tag{35}$$

Consequently,

$$|\,w'(re^{i\theta})\,| \leqq |\,w'(0)\,| + \text{Const.}\int_0^r |\,I(te^{i\theta})\,|\,dt \leqq |\,w'(0)\,| + \text{Const. } \mu(1),$$

so that $w'(z)$ is bounded on $|z|<1$.

The relation (35) shows that $w'(z)$ is continuous at $z=e^{i\theta}$ if

$$\int_{\theta}^{\theta+h} \left| I(re^{i\phi})w(re^{i\phi}) \right| d\phi \to 0, \quad \text{as} \quad (h, r) \to (0, 1).$$

In view of the continuity of $w(z)$, this is the case when θ is a continuity point of $\psi(\theta)$ or when $w(e^{i\theta}) = 0$. This completes the proof of the properties of $w(z)$ enumerated above.

It is clear that these properties imply that $w(z)$ has a finite number of zeros on $|z| \leqq 1$. For otherwise there is a point $z = e^{i\theta}$ which is a cluster point of zeros of $w(z)$. Then $w(e^{i\theta}) = 0$ and so $w'(z)$ is continuous at $z = e^{i\theta}$. Consequently $w'(e^{i\theta}) = 0$. But the only solution $w(z)$ belonging to the (radial) limits $w(e^{i\theta}) = 0$, $w'(e^{i\theta}) = 0$ is the trivial solution $w(z) \equiv 0$.

7. **An upper estimate for the number of zeros in $|z| < r$.** The inequality (23) furnishes an upper estimate for the number $N(r) = N(r; w(z))$ of zeros of a solution $w(z)$ in the circle $|z| < r$ (<1):

(iv) Let there exist on $0 < r < 1$ a positive, continuously differentiable, non-decreasing function $\lambda = \lambda(r)$ satisfying

$$\mu(r) \leqq \lambda(r) \tag{36}$$

and

$$d\lambda/dr = O(\lambda^2(r)), \qquad \text{as } r \to 1. \tag{37}$$

Then

$$N(r) = O\left(\int_0^r \lambda^2(r)dr\right), \qquad \text{as } r \to 1. \tag{38}$$

(iv) shows that if $z_1, z_2, \cdots$ are the zeros of a solution $w(z)$ of (5), then

$$\sum (1 - |z_k|) = \int (1 - r)dN(r) < \infty$$

is implied by

$$\int^1 (1 - r)\lambda^2(r)dr < \infty. \tag{39}$$

Simple examples seem to indicate that (38) can be improved to the corresponding relation in which $\lambda^2(r)$ is replaced by $\lambda(r)$. But this possibility will remain undecided.

In order to prove (iv), note that, according to the inequalities (23) and (36), the distance L between any pair of zeros of a solution $w(z)$ on $|z| \leqq r$ satisfies $L \geqq 4/\lambda(r)$. In the proof of (38), it can therefore be supposed that $\lambda(r) \to \infty$, as $r \to 1$.

If r is sufficiently near 1, the ring $r - 2/\lambda(r) \leqq |z| \leqq r$ can be divided into $2(2\pi r)/(2/\lambda(r)) = 2\pi r\lambda(r)$ sets (curvilinear quadrilaterals) such that the distance between any pair of points in any of the sets is less than $4/\lambda(r)$. Consequently, the ring contains at most $2\pi r\lambda(r)$ zeros of a solution $w=w(z)$. The width of the ring is $\Delta r=2/\lambda(r)$. Hence, the number of zeros is at most $2\pi r\lambda(r) \leqq \text{Const. } \lambda^2(r)\Delta r$. On the other hand, the monotony of λ implies

$$\int^r \lambda^2(r)dr \geqq \sum \lambda^2(r-\Delta r)\Delta r,$$

if the interval $(0, r)$ is divided into a finite number of pieces. Hence, it is clear that (38) follows if it is verified that

$$\lambda(r) = O(\lambda(r - 2/\lambda(r))), \qquad \text{as } r \to 1.$$

But this is a consequence of (37), which implies that

$$\log\,[\lambda(r)/\lambda(r - 2/\lambda(r))] = \int \lambda^{-1}d\lambda = O\left(\int \lambda dr\right) = O(1),$$

as $r \to 1$, where the limits of integration are $r - 2/\lambda(r)$ and r.

References

1. M. Bôcher, *On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic*, Trans. Amer. Math. Soc. vol. 1 (1900) pp. 40–53.

2. O. Dunkel, *Regular singular points of a system of homogeneous linear differential equations of the first order*, Proceedings of the American Academy of Arts and Sciences vol. 38 (1902–03) pp 341–370.

3. L. Fejér and F. Riesz, *Über einige funktionentheoretische Ungleichungen*, Math. Zeit. vol. 11 (1921) pp. 305–314.

4. P. Hartman and A. Wintner, *On an oscillation criterion of Liapounoff*, Amer. J. Math. vol. 73 (1951) pp. 885–890.

5. E. Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 234–252.

6. ———, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 552–553.

7. Z. Nehari, *The Schwarzian derivative and schlicht functions*, ibid. pp. 545–551.

8. ———, *On the zeros of solutions of second order linear differential equations*, Amer. J. Math. vol. 76 (1954) pp. 689–697.

9. C. T. Taam, *Oscillation theorems*, ibid. vol. 74 (1952) pp. 317–324.

10. A. Wintner, *Asymptotic integrations of the adiabatic oscillator*, ibid. vol. 69 (1947) pp. 251–272.

11. ———, *On the non-existence of conjugate points*, ibid. vol. 73 (1951) pp. 368–380.

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