

THE SYMMETRIC DERIVATIVE ON THE $(k-1)$ - DIMENSIONAL HYPERSPHERE

BY

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1. Introduction. Let x be a point on the unit $(k-1)$ -dimensional hypersphere Ω in Euclidean k -space, $k \geq 3$, and let μ be a completely additive set function of bounded variation defined on the Borel sets of Ω . Let $D(x, h)$ represent the spherical cap on Ω obtained by intersecting Ω with a sphere whose center is x and radius is $2 \sin h/2$, and let $|D(x, h)|$ be the $(k-1)$ -dimensional volume of $D(x, h)$. Then μ will be said to have a symmetric derivative at x , designated by $\mu_s(x)$, if $|D(x, h)|^{-1} \mu[D(x, h)]$ tends to $\mu_s(x)$ as h tends to zero.

Let $S[d\mu] = \sum_{n=0}^{\infty} Y_n(x)$ be the Stieltjes series of surface harmonics defined by μ . We shall show in this paper that if $\mu_s(x_0)$ exists and is finite and μ satisfies the global condition $|\mu|[D(x'_0, \epsilon)] = 0$ for some $\epsilon > 0$, where x'_0 is the point diametrically opposite to x_0 and $|\mu|$ is the total variation of μ , then S is summable (C, δ) , $\delta > (k-2)/2 + 1$, to $\mu_s(x_0)$. This result generalizes the well-known result for Fourier-Stieltjes series where $\delta > 1$, see [8, p. 55]. In case the global condition is not satisfied, we obtain that $S[d\mu]$ is summable (C, η) to $\mu_s(x_0)$ where $\eta > k-2$ for $k \geq 4$ and $\eta > 3/2$ for $k = 3$.

In the special case when μ is absolutely continuous and $Y_n(x_0) = O(n^{-1})$, we shall show that a necessary and sufficient condition that $S[d\mu]$ converges at x_0 to the finite value β is that $\mu_s(x_0)$ exists and equals β . This fact generalized a result previously obtained by Hardy and Littlewood [5, p. 229] for Fourier series.

2. Definitions and notation. λ will always designate the value $(k-2)/2$, and P_n^λ will designate the Gegenbauer (ultraspherical) polynomials defined by the equation,

$$(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} r^n P_n^\lambda(\cos \theta).$$

With the help of these functions, we can associate to every additive set function defined on the Borel sets of Ω and of bounded variation there, a sequence of surface harmonics by means of the equation

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$$Y_n(x) = \frac{\Gamma(\lambda)(n + \lambda)}{2\pi^{\lambda+1}} \int_{\Omega} P_n^{\lambda}(\cos \gamma) d\mu(y)$$

where γ is the angle between x and y , that is if $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$,

$$\cos \gamma = (x, y) = x_1 y_1 + \dots + x_k y_k.$$

As shown in [3, Chap. 11], $r^n Y_n(x)$ gives rise to an homogeneous harmonic polynomial of degree n in Euclidean k -space. We define $S[d\mu] = \sum_{n=0}^{\infty} Y_n(x)$ and call this the Stieltjes-series of surface harmonics associated to μ .

By the terminology $\sum_{n=0}^{\infty} Y_n(x)$ is (C, α) summable to a given value, we mean the usual Cesaro summability defined for example in [8, Chap. 3].

We note that $D(x_0, h)$ defined in the introduction is the set

$$\{x, (x, x_0) \geq \cos h\}.$$

$[\lambda]$ will mean the integral part of λ and $|\mu|(E)$ will stand for the total variation of μ in E .

3. Statement of main results. We shall prove the following theorem for Stieltjes-series of surface harmonics:

THEOREM 1. *Let μ be a completely additive set function defined on the Borel sets of Ω and of bounded variation on Ω , and let $S[d\mu] = \sum_{n=0}^{\infty} Y_n(x)$. Suppose $\mu_s(x_0)$ exists and is finite. Then $S[d\mu]$ is (C, η) , $\eta > \max(3/2, k-2)$, summable to $\mu_s(x_0)$. If, furthermore, μ satisfies the condition that $|\mu|[D(x'_0, \epsilon)] = 0$ for some $\epsilon > 0$, where x'_0 is diametrically opposite to x , then $S[d\mu]$ is (C, δ) summable to $\mu_s(x_0)$, $\delta > \lambda + 1$.*

Concerning the convergence of series of surface harmonics, we shall prove the following theorem:

THEOREM 2. *Let f be an integrable function on Ω and define $\mu(E)$, for E a Borel set on Ω , by $\mu(E) = \int_E f(x) d\Omega(x)$ where $d\Omega(x)$ is the $(k-1)$ -dimensional volume element on Ω . Let $S[f] = \sum_{n=0}^{\infty} Y_n(x)$, and suppose that $Y_n(x_0) = O(n^{-1})$. Then a necessary and sufficient condition that $S[f]$ converges at x_0 to β is that $\mu_s(x_0) = \beta$.*

REMARK 1. By [1, Theorem 2], it is easily seen that the condition $|\mu|[D(x'_0, \epsilon)] = 0$ for some $\epsilon > 0$ in Theorem 1 can be replaced by the condition that in $D(x'_0, \epsilon)$, μ is absolutely continuous with $\mu(E) = \int_E f(y) d\Omega(y)$ and that

$$\int_{D(x'_0, \epsilon)} \frac{|f(y)|}{[1 - (x'_0, y)]^{\lambda/2}} d\Omega(y) < \infty.$$

4. Fundamental lemmas. Before proceeding with the proof of these theorems we shall prove some lemmas. By $S_n^{\alpha, \lambda}(\cos \theta)$, we shall designate the sum

$$S_n^{\alpha, \lambda}(\cos \theta) = \sum_{j=0}^n (j + \lambda) P_j^{\lambda}(\cos \theta) A_{n-j}^{\alpha}$$

where $\sum_{n=0}^{\infty} A_n x^n = (1-x)^{-(\alpha+1)}$ with $\alpha > 1$. By $[g(\cos \theta)]'$, we shall mean $dg(\cos \theta)/d\theta$.

LEMMA 1. $|[S_n^{\alpha, \lambda}(\cos \theta)]'| \leq K(\epsilon) n^{\lambda+1} \theta^{-(\alpha+\lambda+1)}$ for $n^{-1} \leq \theta \leq \pi - \epsilon$ and $[\lambda] + 2 > \alpha > [\lambda] + 1$, where $K(\epsilon)$ is a constant depending on ϵ but not on n .

We first observe that

$$(1) \quad \sum_{n=0}^{\infty} (n + \lambda) [P_n^{\lambda}(\cos \theta)]' r^n = - \frac{2\lambda(\lambda + 1)r(1 - r^2) \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+2}}.$$

Let us suppose that $\lambda > 1$. Then since

$$(2) \quad \frac{1}{(1 - r)^{\alpha+1}} \frac{(1 - r^2)r \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+2}} \\ = \frac{1}{(1 - r)^{\alpha}} \frac{(1 - r^2)r \sin \theta}{(1 - 2r \cos \theta + r^2)^{\lambda+1}} - \frac{1}{(1 - r)} \frac{1}{(1 - 2r \cos \theta + r^2)}$$

we obtain from (1) that

$$(3) \quad [S_n^{\alpha, \lambda}(\cos \theta)]' = K_1 \sum_{j=0}^n [S_j^{\alpha-1, \lambda-1}(\cos \theta)]' T_{n-j}(\cos \theta)$$

where $T_n(\cos \theta) = \sum_{j=0}^n P_j^1(\cos \theta)$ and K_1 is a constant. But

$$P_j^1(\cos \theta) = \sin(j+1)\theta / \sin \theta;$$

therefore,

$$(4) \quad T_n(\cos \theta) = \frac{\cos \theta/2 - \cos(2n+3)\theta/2}{2 \sin \theta \sin \theta/2}.$$

So if we assume that $[S_n^{\alpha-1, \lambda-1}(\cos \theta)]'$ satisfies the conclusion of the lemma, we see from (3) and (4) that

$$|[S_n^{\alpha, \lambda}(\cos \theta)]'| \leq K(\epsilon) \theta^{-(\alpha+\lambda-1)-2} \sum_{j=0}^n j^{\lambda}.$$

Therefore in order to prove the lemma we need prove it only in the special cases $\lambda = 1/2$ and $\lambda = 1$.

To do this we introduce

$$S_n^{\alpha, 0}(\cos \theta) = A_n^{\alpha}/2 + \sum_{j=1}^n (\cos j\theta) A_{n-j}^{\alpha}$$

and observe by [8, p. 56] that

$$(5) \quad |[S_n^{\alpha,0}(\cos \theta)]'| \leq K(\epsilon)n\theta^{-(\alpha+1)} \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon, 1 < \alpha < 2.$$

For the case $\lambda=1/2$, we rewrite (2) in the form

$$\begin{aligned} & \frac{1}{(1-r)^{\alpha+1}} \frac{(1-r^2)r \sin \theta}{(1-2r \cos \theta + r^2)^{5/2}} \\ &= \frac{1}{(1-r)^{\alpha+1}} \frac{(1-r^2)r \sin \theta}{(1-2r \cos \theta + r^2)^2} \frac{1}{(1-2r \cos \theta + r^2)^{1/2}} \end{aligned}$$

and obtain that

$$(6) \quad [S_n^{\alpha,1/2}(\cos \theta)]' = K_2 \sum_{i=0}^n [S_i^{\alpha,0}(\cos \theta)]' P_{n-i}^{1/2}(\cos \theta)$$

where K_2 is a constant. From [7, p. 160], $|P_n^{1/2}(\cos \theta)| \leq (n \sin \theta)^{-1/2}$. So we conclude from (5) and (6) that

$$|[S_n^{\alpha,1/2}(\cos \theta)]'| \leq K(\epsilon)n^{3/2}\theta^{-(\alpha+3/2)} \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon \text{ and } 1 < \alpha < 2$$

which proves the lemma in the special case $\lambda=1/2$.

For the case $\lambda=1$, let us assume that $2 < \alpha < 3$. Then by (2), we obtain that

$$[S_n^{\alpha,1}(\cos \theta)]' = K_3 \sum_{j=0}^n [S_j^{\alpha-1,0}(\cos \theta)]' T_{n-j}(\cos \theta)$$

and (4) and (5) give us that

$$|[S_n^{\alpha,1}(\cos \theta)]'| \leq K(\epsilon)n^2\theta^{-(\alpha+2)} \quad \text{for } n^{-1} \leq \theta \leq \pi - \epsilon,$$

and the proof to the lemma is complete.

LEMMA 2. $|[S_n^{\alpha,\lambda}(\cos \theta)]'| \leq Kn^{\alpha+2\lambda+2}$ for $0 \leq \theta \leq n^{-1}$ and $\alpha \geq 0$, where K is a constant independent of n .

To prove this lemma we write

$$\begin{aligned} & \frac{1}{(1-r)^{\alpha-1}} \frac{(1-r^2)r \sin \theta}{(1-2r \cos \theta + r^2)^{\lambda+2}} \\ &= \frac{(1-r^2)r \sin \theta}{(1-r)^{\alpha+1}(1-2r \cos \theta + r^2)^{\lambda+3/2}} \frac{1}{(1-2r \cos \theta + r^2)^{1/2}} \end{aligned}$$

and obtain that

$$(7) \quad [S_n^{\alpha,\lambda}(\cos \theta)]' = K_1 \sum_{i=0}^n [S_i^{\alpha,\lambda-1/2}(\cos \theta)]' P_{n-i}^{1/2}(\cos \theta),$$

$S_n^{\alpha,0}(\cos \theta)$ being defined as in Lemma 1.

If we assume that the conclusion to the lemma holds for $S_n^{\alpha,\lambda-1/2}(\cos \theta)$, we can use the fact that $|P_n^{1/2}(\cos \theta)| \leq 1$ and obtain from (7) that

$$|[S_n^{\alpha,\lambda}(\cos \theta)]'| \leq K_2 \sum_{j=0}^n j^{\alpha+2\lambda+1} \leq Kn^{\alpha+2\lambda+2}.$$

So to prove the lemma, we need only show that the conclusion of the lemma holds for $S_n^{\alpha,0}(\cos \theta)$. But

$$|[S_n^{\alpha,0}(\cos \theta)]'| = \left| \sum_{j=0}^n j \sin j\theta A_{n-j}^{\alpha} \right| \leq Kn^{\alpha+2},$$

and the proof is complete.

We now state a lemma of Kogbetliantz [6, p. 139].

LEMMA 3. For $-1 < \alpha \leq k-1$ and for $0 \leq \theta \leq \pi$

$$|S_n^{\alpha,\lambda}(\cos \theta)| \leq K(n+1)^{k-2}(\sin \theta/2)^{-(\alpha+1)}.$$

We next define the expression $B_n^{\lambda}(h)$ by

$$(8) \quad B_n^{\lambda}(h) = \int_0^h P_n^{\lambda}(\cos \theta) (\sin \theta)^{2\lambda} d\theta / P_n^{\lambda}(1) \int_0^h (\sin \theta)^{2\lambda} d\theta$$

and prove the following lemmas:

LEMMA 4. $|B_n^{\lambda}(h)| \leq K(hn)^{-\lambda}$ for $0 \leq h \leq \pi/2$ where K is a constant independent of n .

LEMMA 5. $|B_{n+1}^{\lambda}(h) - B_n^{\lambda}(h)| \leq Kh^2n$ for $0 \leq h \leq \pi$ where K is a positive constant independent of n .

To prove Lemma 4 we have by [7, p. 80, pp. 166 and 167] that

$$(9) \quad \begin{aligned} |P_n^{\lambda}(\cos \theta)| &\leq K_1 \theta^{-\lambda} n^{\lambda-1} && \text{for } 0 \leq \theta \leq \pi/2, \\ P_n^{\lambda}(1) &= \binom{n+2\lambda-1}{n}, \\ |P_n^{\lambda}(\cos \theta)| &\leq P_n^{\lambda}(1). \end{aligned}$$

We conclude that the left side of the inequality in Lemma 4 is majorized by a constant multiple of

$$n^{\lambda-1} \int_0^h \theta^{-\lambda} \theta^{2\lambda} d\theta / n^{2\lambda-1} \int_0^h \theta^{2\lambda} d\theta,$$

which gives the right side of the inequality, and the lemma is proved.

To prove Lemma 5, we let $g(n)$ equal the square of the normalizing coefficient of $P_n^\lambda(r)$, that is [3, p. 174]

$$(10) \quad g(n) = \int_{-1}^1 [P_n^\lambda(r)]^2 (1-r^2)^{\lambda-1/2} dr = \frac{\pi^{1/2} \Gamma(\lambda + 1/2)}{(n + \lambda) \Gamma(\lambda)} P_n^\lambda(1)$$

and obtain from the Christoffel-Darboux formula [3, p. 159] that

$$(11) \quad \frac{P_n^\lambda(\cos \theta) P_{n+1}^\lambda(1) - P_{n+1}^\lambda(\cos \theta) P_n^\lambda(1)}{2 \sin^2 \theta/2} = 2g(n) \frac{n + \lambda}{n + 1} \sum_{j=0}^n [g(j)]^{-1} P_j^\lambda(\cos \theta) P_j^\lambda(1).$$

From (8) and (9), we see that the left side of the inequality in Lemma 5 is majorized by a constant multiple of

$$\max_{0 \leq \theta \leq h} \frac{|P_{n+1}^\lambda(1) P_n^\lambda(\cos \theta) - P_n^\lambda(1) P_{n+1}^\lambda(\cos \theta)|}{n^{4\lambda-2}}.$$

But by (9), (10), and (11) this expression in turn is majorized by a constant multiple of

$$\frac{h^2}{n^{4\lambda-2}} n^{2\lambda-2} \sum_{j=0}^n j j^{2\lambda-1} = h^2 O(n),$$

which is the right side of the inequality in Lemma 5, and the proof of the lemma is complete.

5. Proof of Theorem 1. Let us first suppose that $\mu(E)=0$ for E a Borel set contained in $D(x'_0, \epsilon)$ where ϵ is a positive number between 0 and $\pi/2$. Then with $\delta > \lambda + 1$ but less than $[\lambda] + 2$, we have

$$(12) \quad S_n^\delta(x) = \sum_{j=0}^n Y_j(x) A_{n-j}^\delta = \frac{\Gamma(\lambda)}{2\pi^{\lambda+1}} \int_{\Omega - D(x'_0, \epsilon)} S_n^{\delta, \lambda}(\cos \gamma) d\mu(y)$$

with $\cos \gamma = (x, y)$ and $S_n^{\delta, \lambda}(\cos \gamma)$ defined in the beginning of §4.

Next, we introduce the continuous function $f(x)$ which has the following properties:

- (i) $f(x) = 0$ for x in $D(x'_0, \epsilon)$,
- (ii) $f(x_0) = \mu_*(x_0)$,
- (iii) $\int_\Omega f(x) d\Omega(x) = \mu[D(x_0, \pi - \epsilon)]$,

and set $\mu_1(E) = \int_E f(x) d\Omega(x)$ for E a Borel set on Ω . By [1, Theorem 2], $R_n^\delta(x_0)/A_n^\delta \rightarrow \mu_*(x_0)$ for $\delta > \lambda$ when R_n is defined by the last expression in (12) with μ replaced by μ_1 . Consequently to prove the second part of this theorem it is sufficient to show that

$$(13) \quad \frac{2\pi^{\lambda+1}}{\Gamma(\lambda)} [R_n^\delta(x_0) - S_n^\delta(x_0)] = \int_{\Omega-D(x_0', \epsilon)} S_n^{\delta, \lambda}[(x_0, y)] d\mu_2(y) = o(n^\delta)$$

with $[\lambda] + 2 > \delta > \lambda + 1$ and $\mu_2 = \mu_1 - \mu$.

To do this, we define a one-dimensional completely additive set function σ of bounded variation on $[0, \pi]$ in the following manner:

To every Borel set Z on $[0, \pi]$ associate the set Z^{k-1} on Ω where $Z^{k-1} = \{x, x \in \Omega \text{ and } (x, x_0) = \cos \theta \text{ where } \theta \in Z\}$. Then σ is defined by $\sigma(Z) = \mu_2(Z^{k-1})$. So in particular if $Z = [0, h]$, $\sigma(Z) = \mu_2[D(x_0, h)]$.

With this definition we obtain that

$$(14) \quad \begin{aligned} \int_{\Omega-D(x_0', \epsilon)} S_n^{\delta, \lambda}[(x_0, y)] d\mu_2(y) &= \int_0^{\pi-\epsilon} S_n^{\delta, \lambda}(\cos \theta) d\sigma(\theta) \\ &= - \int_0^{\pi-\epsilon} \mu_2[D(x_0, \theta)] [S_n^{\delta, \lambda}(\cos \theta)]' d\theta \end{aligned}$$

since both $\mu_2[D(x_0, \pi - \epsilon)]$ and $\mu_2[D(x_0, 0)]$ are zero.

Now by construction $\mu_2[D(x_0, \theta)] = o[|D(x_0, \theta)|] = o(\theta^{2\lambda+1})$ as $\theta \rightarrow 0$. Therefore by Lemma 2,

$$(15) \quad \begin{aligned} \int_0^{\pi-\epsilon} \mu_2[D(x_0, \theta)] [S_n^{\delta, \lambda}(\cos \theta)]' d\theta &= o\left(\int_0^{\pi-\epsilon} \theta^{2\lambda+1} n^{\delta+2\lambda+2} d\theta\right) \\ &= o(n^\delta), \end{aligned}$$

and by Lemma 1,

$$(16) \quad \begin{aligned} \left| \int_{\pi^{-1}}^h \mu_2[D(x_0, \theta)] [S_n^{\delta, \lambda}(\cos \theta)]' d\theta \right| \\ \leq \psi_1(h) n^{\lambda+1} \int_{\pi^{-1}}^h \theta^{2\lambda+1} \theta^{-(\delta+\lambda+1)} d\theta \leq \psi_2(h) n^\delta, \end{aligned}$$

where $\psi_i(h) (i=1, 2)$ tends to zero with h .

From Lemma 1, we also obtain that

$$(17) \quad \int_h^{\pi-\epsilon} \mu_2[D(x_0, \theta)] [S_n^{\delta, \lambda}(\cos \theta)]' d\theta = O(n^{\lambda+1-\delta} n^\delta) = o(n^\delta).$$

We therefore conclude from (13), (14), (15), (16), and (17) that

$$\limsup_{n \rightarrow \infty} |R_n^\delta(x_0) - S_n^\delta(x_0)| n^{-\delta} = 0.$$

Since $R_n^\delta(x_0)/A_n^\delta \rightarrow \mu_s(x_0)$, the second part of Theorem 1 is proved.

To prove the first part of the theorem, we suppose that $k-2 < \eta < k-1$ for $k=4$ and $3/2 < \eta < 2$ for $k=3$ and set $\mu = \mu_3 + \mu_4$, where for Borel sets E ,

$\mu_3(E) = 0$ for $E \subset \Omega - D(x_0, 3\pi/4)$ and $\mu_4(E) = 0$ for $E \subset D(x_0, 3\pi/4)$. Then

$$\frac{S_n^\eta(x_0)}{A_n^\eta} = \frac{\Gamma(\lambda)}{2A_n^\eta \pi^{\lambda+1}} \int_{\Omega} S_n^{\eta, \lambda}(\cos \gamma) [d\mu_3(y) + d\mu_4(y)] = I_n^\eta + II_n^\eta.$$

By the part of the theorem proved above, $I_n^\eta \rightarrow \mu_s(x_0)$ for $\eta > \lambda + 1$. By Lemma 3 with $k-2 < \eta < k-1$,

$$|II_n^\eta| \leq \frac{K(n+1)^{k-2} \int_{\Omega - D(x_0, 3\pi/4)} |d\mu_4(y)|}{\left(\sin \frac{3\pi}{8}\right)^{\eta+1} n^\eta}.$$

We conclude that II_n^η tends to zero as $n \rightarrow \infty$, and the proof of Theorem 1 is complete.

REMARK 2. To show that η cannot be taken equal to $k-2$ in the above proof, choose μ_4 to be the mass distribution with mass 1 at x_0' and zero elsewhere. Then

$$II_n^{k-2} = \frac{\Gamma(\lambda)}{2A_n^{k-2} \pi^{\lambda+1}} \sum_{j=0}^n (j+\lambda)(-1)^j P_j^\lambda(1) A_{n-j}^{k-2}.$$

By [8, p. 43] for II_n^{k-2} to tend to zero, $(n+\lambda)P_n(1)$ would have to be $o(n^{k-2})$. But $(n+\lambda)P_n^\lambda(1)n^{-(k-2)} \rightarrow 1/(k-3)!$ as $n \rightarrow \infty$. So in Theorem 1, η must be chosen greater than $k-2$.

6. Proof of Theorem 2. The sufficiency condition of Theorem 2 follows immediately from Theorem 1 and the usual Tauberian theorem for Cesàro summability [4, p. 121].

To prove the necessity, we set $F_h(x) = |D(x, h)|^{-1} \int_{D(x, h)} f(z) d\Omega(z)$ and obtain that

$$\begin{aligned} & \int_{\Omega} P_n^\lambda[(x, y)] F_h(y) d\Omega(y) \\ (18) \quad &= |D(x, h)|^{-1} \int_{\Omega} f(z) \left[\int_{\Omega} P_n^\lambda[(x, y)] X_{D(y, h)}(z) d\Omega(y) \right] d\Omega(z) \\ &= |D(x, h)|^{-1} \int_{\Omega} f(z) \left[\int_{D(z, h)} P_n^\lambda[(x, y)] d\Omega(y) \right] d\Omega(z) \end{aligned}$$

where $X_E(x)$ stands for the characteristic function of the set E .

Then by [3, p. 243],

$$(19) \quad P_n^\lambda[(x, y)] = P_n^\lambda(1) |\Omega| [h(n)]^{-1} \sum_{j=1}^{h(n)} S_n^j(x) S_n^j(y)$$

where $h(n)$ is the maximum number of linearly independent surface harmon-

ics of degree n on Ω , and $S_n^j(x)$, $j=1, \dots, h(n)$, are a set of linear independent, orthonormal surface harmonics on Ω .

On the other hand, by [3, p. 240] with $z=(1, 0, \dots, 0)$ and x given by spherical coordinates in terms of z , $S_n^j(x)$ can be chosen as a constant multiple of

$$(20) \quad Y(m_q; \theta_q, \pm \phi) = e^{\pm i m_{2\lambda} \phi} \prod_{q=0}^{2\lambda-1} (\sin \theta_{q+1})^{m_{q+1}} P_{m_q - m_{q+1}}^{m_{q+1} + \lambda - 1/2 q}(\cos \theta_{q+1})$$

where $n = m_0 \geq m_1 \geq \dots \geq m_{2\lambda} \geq 0$. [For the spherical coordinate notation see [3, p. 233] with $p=2\lambda$.] Let us call $S_n^1(x)$ the function obtained in (20) when the sequence $(n, 0, \dots, 0)$ is used. Then $S_n^1(x)$ is the function $P_n^\lambda[(x, z)]$, normalized.

We shall now show that,

$$(21) \quad \int_{D(s, h)} S_n^j(y) d\Omega(y) = 0, \quad \text{for } j \neq 1.$$

For there must then exist some $q \neq 0$ such that $m_q \neq 0$. Let q_1 be the last such q . Recalling that

$$d\Omega(y) = (\sin \theta_1)^{2\lambda} (\sin \theta_2)^{2\lambda-1} \dots (\sin \theta_{2\lambda}) d\theta_1 \dots d\theta_{2\lambda} d\phi$$

where $0 \leq \theta_q \leq \pi$ ($q=1, \dots, 2\lambda$), and $0 \leq \phi \leq 2\pi$, we see immediately that (21) holds in case $q_1 = 2\lambda$. Let us suppose then that $q_1 \neq 2\lambda$. Then

$$(\sin \theta_{q_1+1})^{m_{q_1+1}} P_{m_{q_1} - m_{q_1+1}}^{m_{q_1+1} + \lambda - 1/2 q_1}(\cos \theta_{q_1+1}) = P_{m_{q_1}}^{\lambda - 1/2 q_1}(\cos \theta_{q_1+1}).$$

But by [2, p. 177]

$$\int_0^\pi P_n^{\lambda - 1/2 q}(\cos \theta) (\sin \theta)^{2\lambda - q} d\theta = 0 \quad \text{for } n \neq 0$$

and consequently (21) holds.

We thus obtain from (18), (19), (20), and (21) that

$$(22) \quad \begin{aligned} & \int_{\Omega} P_n^\lambda[(x, y)] F_h(y) d\Omega(y) \\ &= \zeta(n) \int_{D(w, h)} P_n^\lambda[(y, w)] d\Omega(y) \int_{\Omega} f(z) P_n^\lambda[(x, z)] d\Omega(z) \end{aligned}$$

where

$$\begin{aligned} \zeta(n) &= \frac{P_n^\lambda(1) |\Omega|}{h(n) |D(w, h)|} \left\{ \int_{\Omega} (P_n^\lambda[(x, w)])^2 d\Omega(x) \right\}^{-1} \\ &= \frac{1}{D(w, h) P_n^\lambda(1)} \quad \text{by [3, p. 236, formula 29].} \end{aligned}$$

Consequently with $S[f] = \sum_{n=0}^{\infty} Y_n(x)$ and $B_n^\lambda(h)$ as in Lemmas 4 and 5, we conclude from (22) that

$$S[F_h] = \sum_{n=0}^{\infty} Y_n(x) B_n^\lambda(h)$$

and, furthermore, from the continuity of $F_h(x)$, from the fact that $Y_n(x) = O(n^{-1})$, and from Lemma 4, that

$$(23) \quad F_h(x_0) = \sum_{n=0}^{q[h^{-1}]} Y_n(x_0) B_n^\lambda(h) + \sum_{n=q[h^{-1}]+1}^{\infty} Y_n(x_0) B_n^\lambda(h)$$

where q is a fixed, large positive integer.

Since there clearly is no loss of generality in assuming that

$$R_n = \sum_{j=0}^n Y_j(x_0) \rightarrow 0,$$

we shall make this assumption and shall show this implies that $F_h(x_0) \rightarrow 0$.

By Lemma 4 the second sum on the right in (23) is majorized by a constant multiple of

$$(24) \quad h^{-\lambda} \sum_{n=q[h^{-1}]}^{\infty} n^{-1(+\lambda)} = O(q^{-\lambda}).$$

Using Abel summation by parts on the first sum on the right side in (23), we obtain that this sum is equal to

$$(25) \quad \sum_{n=0}^{q[h^{-1}]-1} R_n [B_n^\lambda(h) - B_{n+1}^\lambda(h)] + R_{q[h^{-1}]} B_{q[h^{-1}]}^\lambda(h).$$

By Lemma 4 the second term in (25) tends to zero as $h \rightarrow 0$. By Lemma 5 the first term in (25) is majorized by

$$h^2 \sum_{n=0}^{q[h^{-1}]} o(n)$$

which tends to zero with h . We consequently conclude from (23) and (24) that

$$\limsup_{h \rightarrow 0} |F_h(x_0)| = O(q^{-\lambda}).$$

But this fact implies that $\lim_{h \rightarrow 0} F_h(x_0) = 0$, and the proof of Theorem 2 is complete.

In closing we point out that with no change in the proof of Theorem 2 the condition that μ be absolutely continuous can be replaced with one requiring only that $\mu [O(x, h)]$ be continuous at x_0 for h small.

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Page 329, lines 20–21. For “those with superscript “ Q ” partial recursive, uniformly in Q ;” read “those with superscript “ Q ” partial recursive uniformly in Q ;”.