

# THE MAXIMUM OF SUMS OF STABLE RANDOM VARIABLES

BY

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1. **Introduction.** Let  $X_1, X_2, \dots$  be identically distributed independent random variables and set  $S_n = X_1 + \dots + X_n$ . In this paper we obtain the limiting distribution of

$$\max_{1 \leq k \leq n} S_k$$

for random variables  $X_i$  which have a stable distribution.

In case the  $X_i$  are normally distributed the limiting distribution of  $\max S_k$  has been known for some time and turns out to be the truncated normal distribution; the limiting distribution of  $\max |S_k|$  is also known and is a theta function (cf. [1]). The same results obtain when the  $X_i$  are not necessarily identically distributed but merely such that the central limit theorem applies to them (cf. [2]). In these cases the limiting distributions are the same as the distributions of the corresponding functionals for the Wiener stochastic process which can be, in turn, formulated as boundary value problems for the simple diffusion equation, whose solutions are classical.

But in the case the  $X_i$  belong to the domain of attraction of a stable law other than the normal, the problem cannot be reduced to a diffusion equation, and none of the standard methods seems to work. In one particular case a solution has been given by Kac and Pollard [3]. They found the limiting distribution of  $\max |S_k|$  when the  $X_i$  are identically distributed Cauchy variables. However their method failed to yield anything for the one-sided maximum,  $\max S_k$ .

In this paper we find the limiting distribution of  $\max S_k$  when the variables  $X_i$  have a symmetric stable distribution of index  $\gamma$ ,  $0 < \gamma \leq 2$ . Corresponding results could no doubt be obtained under the condition the  $X_i$  merely belong to the domain of attraction of a stable law, using the present method, but we restrict our treatment to the present case for simplicity. The problem of determining the limiting distribution of  $\max |S_k|$  for general stable variables is still apparently open.

2. **An initial reduction.** The results of this paper are based upon, and made possible by, the following basic result of Spitzer [4]. Let  $X_1, X_2, \dots$  be

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independent, identically distributed random variables and set for  $\xi \geq 0$ :

$$(2.1) \quad \begin{aligned} \phi_n(\xi) &= E(\exp(-\xi \max(0, S_1, \dots, S_n))) & (n = 1, 2, \dots), \\ \phi_0(\xi) &\equiv 1, \end{aligned}$$

$$(2.2) \quad \psi_n(\xi) = E(\exp(-\xi \max(0, S_n))) \quad (n = 1, 2, \dots),$$

then we have [4].

THEOREM 1.

$$\sum_{n=0}^{\infty} \phi_n(\xi) t^n = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \psi_n(\xi) t^n\right).$$

Before proceeding we make the following remarks.

(i) The limiting distribution of  $n^{-1/\gamma} \max(S_1, \dots, S_n)$  exists; that is,  $\Pr\{n^{-1/\gamma} \max(S_1, S_2, \dots, S_n) < x\} \rightarrow F(x)$  where  $F(x)$  is a nondegenerate distribution function, see e.g. Kac and Pollard [3]. As a matter of fact if  $X(t)$  is a symmetric stable process of index  $\gamma$  we have

$$F(x) = \Pr\left\{\sup_{0 \leq t \leq 1} X(t) < x\right\}$$

and also

$$\Pr\left\{\sup_{0 \leq t \leq y} X(t) < x\right\} = F(xy^{-1/\gamma}).$$

In addition if we let  $T(z)$  be the first passage time for the boundary  $z$

$$T(z) = \sup_t \{t \mid X(\tau) < z, 0 < \tau \leq t\}$$

we have

$$\Pr\{T(z) < t\} = 1 - F(zt^{-1/\gamma})$$

with this same function  $F(x)$ .

(ii) The limiting distributions of  $n^{-1/\gamma} \max(S_1, S_2, \dots, S_n)$  and  $n^{-1/\gamma} \max(0, S_1, \dots, S_n)$  are the same, for

$$\Pr\{S_k \leq 0, k = 1, \dots, n\} = 2^{-2n} \binom{2n}{n} \rightarrow 0.$$

Now let  $\phi_n(\xi)$  and  $\psi_n(\xi)$  be as in (2.1) and (2.2) respectively, and suppose the  $X_i$  have a symmetric stable distribution of index  $\gamma$

$$E(e^{itX_i}) = e^{-|t|^\gamma}, \quad 0 < \gamma \leq 2.$$

Let  $f_n(x)$  be the density for  $S_n = X_1 + \dots + X_n$ . Then, the following formal operations being easily justified,

$$\begin{aligned}
f_n(x) &= \frac{1}{\pi} \int_0^\infty \cos xy e^{-n|y|^\gamma} dy, \\
\psi_n(\xi) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty e^{-n|y|^\gamma} \int_0^\infty e^{-\xi x} \cos xy dx dy \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp(-n\xi^\gamma y^\gamma) \frac{dy}{1+y^2}, \\
\sum_1^\infty \frac{1}{n} \psi_n(\xi) t^n &= -\frac{1}{2} \log(1-t) - \frac{1}{\pi} \int_0^\infty \frac{\log(1-t \exp(-\xi^\gamma y^\gamma))}{1+y^2} dy, \\
\log(1-t) + \sum_1^\infty \frac{1}{n} \psi_n(\xi) t^n &= -\frac{1}{\pi} \int_0^\infty \log\left(\frac{1-t \exp(-\xi^\gamma y^\gamma)}{1-t}\right) \frac{dy}{1+y^2},
\end{aligned}$$

for  $|t| < 1$ . We have then, when  $t \rightarrow 1$ ,

$$\begin{aligned}
\log(1-t) + \sum_1^\infty \frac{1}{n} \psi_n(\xi) t^n &\rightarrow -\frac{1}{\pi} \int_0^\infty \log\left(\lim_{t \rightarrow 1} \frac{1-t \exp(-(1-t)\xi^\gamma y^\gamma)}{1-t}\right) \frac{dy}{1+y^2} \\
&= -\frac{1}{\pi} \int_0^\infty \frac{\log(1+\xi^\gamma y^\gamma)}{1+y^2} dy.
\end{aligned}$$

Upon using Theorem 1, we have thus proved

**THEOREM 2.**

$$\lim_{t \rightarrow 1} (1-t) \sum_0^\infty \phi_n((1-t)^{1/\gamma} \xi) t^n = g(\xi),$$

where

$$(2.3) \quad g(\xi) = \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(1+\xi^\gamma y^\gamma)}{1+y^2} dy\right).$$

**3. An integral equation.** Let  $F(x)$  be the limiting distribution and  $\phi(\xi)$  be the Laplace transform of  $F(x)$ :

$$F(x) = \lim_{n \rightarrow \infty} \Pr \{n^{-1/\gamma} \max(S_1, S_2, \dots, S_n) < x\},$$

$$\phi(\xi) = \int_0^\infty e^{-\xi x} dF(x) = \lim_{n \rightarrow \infty} \phi_n(\xi/n^{1/\gamma}).$$

We define a random variable  $N$ , independent of the  $X_j$ , whose distribution is given by

$$\Pr \{N = j\} = (1 - t)^j, \quad j = 0, 1, \dots; 0 < t < 1.$$

Then the result of Theorem 2 can be expressed as

$$\lim_{t \rightarrow 1} E(\phi_N((1 - t)^{1/\gamma} \xi)) = g(\xi).$$

Now

$$\lim_{t \rightarrow 1} \Pr \{(1 - t)N < x\} = 1 - e^{-x},$$

and hence

$$\begin{aligned} \lim_{t \rightarrow 1} E(\phi_N((1 - t)^{1/\gamma} \xi)) &= \lim_{t \rightarrow 1} E(\phi_N([(1 - t)N]^{1/\gamma} \xi N^{-1/\gamma})) \\ &= \int_0^\infty e^{-x} \phi(\xi x^{1/\gamma}) dx \end{aligned}$$

since  $\lim_{t \rightarrow 1} E(\phi_N(\xi N^{-1/\gamma})) \rightarrow \phi(\xi)$  uniformly in  $\xi \geq 0$ .

We have thus proved

**THEOREM 3.** *The Laplace transform  $\phi(\xi)$  of the limiting distribution  $F(x)$  of  $n^{-1/\gamma} \max(S_1, S_2, \dots, S_n)$  satisfies the integral equation*

$$(3.1) \quad \int_0^\infty e^{-x} \phi(\xi x^{1/\gamma}) dx = g(\xi),$$

where  $g(\xi)$  is given by (2.3).

The integral equation (3.1) has a unique solution, and in fact is easily solved by Mellin transforms. Let  $\mathfrak{F}(s)$  be the Mellin transform of the limiting distribution function  $F(x)$  (whose Laplace transform is  $\phi(\xi)$ ), and let  $g(s)$  be the Mellin transform of  $g(\xi)$ , given by (2.3);

$$\begin{aligned} \mathfrak{F}(s) &= \int_0^\infty x^{s-1} dF(x), \\ g(s) &= \int_0^\infty \xi^{s-1} g(\xi) d\xi. \end{aligned}$$

On taking the transform of both sides of (3.1) we obtain

$$(3.2) \quad \mathfrak{F}(1 - s) \Gamma(s) \Gamma(1 - s\gamma^{-1}) = g(s),$$

$$(3.3) \quad \mathfrak{F}(s) = \frac{g(1 - s)}{\Gamma(1 - s) \Gamma(1 - \gamma^{-1} + s\gamma^{-1})}$$

for  $|\Re s|$  sufficiently small.

From (3.3) it is not difficult to deduce that  $F(x)$  is absolutely continuous.

Designating the corresponding density by  $f(x)$  we have on formally inverting (3.3).

THEOREM 4. *The limiting density  $f(x)$  for  $n^{-1/\gamma} \max (S_1, S_2, \dots, S_n)$  is*

$$(3.4) \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\Gamma(1-s)\Gamma(1-\gamma^{-1}+s\gamma^{-1})} \int_0^\infty z^{-s} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(1+z^\gamma y^\gamma)}{1+y^2} dy dz\right) dy dz ds.$$

It seems difficult to simplify (3.4) further for general values of  $0 < \gamma \leq 2$ , but the cases  $\gamma=1$ ,  $\gamma=2$  can be reduced.

(i)  $\gamma=2$ ; the  $X_i$  are normally distributed, mean 0, variance 2. In this case it is simple to deduce that  $g(\xi)$ , given by (2.3) is  $g(\xi) = (\xi+1)^{-1}$  and  $g(s) = \Gamma(s)\Gamma(1-s)$ . Hence (3.3) gives

$$\mathfrak{F}(s) = \frac{\Gamma(s)}{\Gamma(2^{-1} + 2^{-1}s)} = \pi^{-1/2} 2^{s-1} \Gamma\left(\frac{s}{2}\right) = \pi^{-1/2} \int_0^\infty x^{s-1} e^{-x^2/4} dx$$

and hence

$$f(x) = \pi^{-1/2} e^{-x^2/4}, \quad x \geq 0,$$

or the truncated normal.

(ii)  $\gamma=1$ ; the  $X_i$  have a Cauchy distribution with common density  $\pi^{-1}(1+x^2)^{-1}$ . Here (3.2) yields

$$\mathfrak{F}(1-s) = \frac{g(s)}{\Gamma(s)\Gamma(1-s)} = \frac{\sin \pi s}{\pi} g(s)$$

or

$$(4.1) \quad \int_0^\infty x^{s-1} \frac{1}{x} f\left(\frac{1}{x}\right) dx = -\frac{1}{2\pi i} \int_0^\infty [(ze^{\pi i})^{s-1} - (ze^{-\pi i})^{s-1}] g(z) dz$$

where

$$(4.2) \quad g(z) = \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(1+zy)}{1+y^2} dy\right).$$

Regarding  $z$  as a complex variable, the function  $g(z)$  of (4.2) is analytic in the entire plane cut along the negative real axis. For  $-1/2 < \Re s < 1/2$  a standard transformation of the integral on the right hand side of (4.1) yields the result

$$(4.3) \quad \frac{1}{x} f\left(\frac{1}{x}\right) = -\frac{g(xe^{\pi i}) - g(xe^{-\pi i})}{2\pi i}, \quad x > 0,$$

with  $g(z)$  given by (4.2). This can also be deduced by observing that for  $\gamma=1$  the left hand side of the integral equation (3.1) can be written as the Stieltjes transform of  $(1/x)f(1/x)$  whose inversion is known to be (4.3); see Widder [5].

It is simple to deduce that for  $x>0$

$$g(xe^{\pm \pi i}) = \exp \left( -\frac{1}{\pi} \int_0^x \frac{\log w}{1+w^2} dw \mp i \tan^{-1} x \right) (1+x^2)^{-1/4}$$

and hence

$$\begin{aligned} \frac{1}{x} f\left(\frac{1}{x}\right) &= \frac{1}{\pi} \frac{x}{(1+x^2)^{3/4}} \exp \left( -\frac{1}{\pi} \int_0^x \frac{\log w}{1+w^2} dw \right), \\ f(x) &= \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \exp \left( -\frac{1}{\pi} \int_0^x \frac{\log w}{1+w^2} dw \right), \quad x > 0. \end{aligned}$$

It seems worthy of note that for  $x \rightarrow \infty$   $f(x) \sim \pi^{-1}x^{-2}$ , exactly the behavior of the parent Cauchy variables. Also for  $x \rightarrow 0$ ,  $f(x) \sim \pi^{-1}x^{-1/2}$ .

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