RECURSIVE AND RECURSIVELY ENUMERABLE ORDERS[1]

BY

H. G. RICE

I. Basic concepts

1. Recursive enumerability and recursiveness. We consider infinite sequences of non-negative integers, free from repetitions. A familiar equivalence relation between such sequences is that based on sets: two sequences are equivalent when they enumerate the same set. The equivalence classes under this relation, with the necessary operations introduced, form a system isomorphic to the algebra of all infinite sets of non-negative integers, with the set of all such integers, ϵ, as a distinguished element.

In this algebra we can introduce the concepts of recursive enumerability and recursiveness as follows. If an equivalence class α contains a sequence produced by a recursive function, we say that α corresponds to a recursively enumerable (r.e.) set (or, informally, that α *is* an r.e. set). In each equivalence class there is a sequence which is in order of size; we call this the *principal sequence* of its class. If the principal sequence of α is produced by a recursive function, we say that α is a recursive set [7, p. 291].

This development is interesting chiefly because it can be generalized. For any equivalence relation between sequences, we call recursively enumerable any element of the equivalence class algebra which contains a sequence produced by a recursive function. If a nontrivial principal sequence can be defined for each element, we call recursive those elements whose principal sequences are produced by recursive functions. (A trivial principal sequence would be one which was never produced by a recursive function, or which was produced by a recursive function whenever any sequence of its equivalence class was.)

In this paper we study the system arising from a particular equivalence relation based on order. The set equivalence relation ignores differences in order between sequences; the order equivalence relation will ignore differences in sets. However, the introduction of recursion theory produces strong interconnections between the two systems, which we will use constantly in attempting to gain information about them both.

2. Orders.

Definition. Two sequences $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are equivalent if the same permutation of their terms arranges each in order of size.

Presented to the Society, December 28, 1951; received by the editors March 10, 1955 and, in revised form, March 17, 1956.

(1) Part of the work in this paper was done with the support of the National Science Foundation, Grant No. NSF G1011.

Formally, suppose that $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are the sequences of values produced by the functions $f(x)$ and $g(x)$ respectively. Then the sequences are equivalent if and only if there exists a function $p(x)$, producing an enumeration of ϵ without repetitions (i.e., a permutation of the non-negative integers), such that $f(p(x))$ and $g(p(x))$ produce sequences in order of size(2).

For a given $f(x)$, $p(x)$ is evidently unique.

Equivalence classes under this relation will be called *orders*, and denoted by α^*, β^*, etc. In each order there is one sequence enumerating ϵ, which we designate as the *principal sequence* of its order. (In the order containing the sequence produced by $f(x)$, the principal sequence will be the inverse permutation to that given by $p(x)$.) We now define an operation on orders.

DEFINITION. Let $p(x)$ and $q(x)$ produce the principal sequences of orders α^* and β^* respectively. Then $\gamma^* = \alpha^*\beta^*$ is the order whose principal sequence is produced by $p(q(x))$.

We note that if $f(x)$ produces any sequence of α^*, then $f(q(x))$ produces a sequence of γ^*. In particular, if $f(x)$ is monotone increasing, then $f(q(x))$ produces a sequence of β^*.

The equivalence class algebra here is the group of permutations of the non-negative integers. The operation is the iteration of permutations. The identity is the order ϵ^*, all of whose sequences are in order of size. The group, of course, is neither commutative nor denumerable. The inverse of an order α^* will be written $(\alpha^*)^{-1}$.

We now introduce recursive enumerability and recursiveness.

DEFINITION. If at least one of the sequences of an order α^* is produced by a recursive function, we say that α^* is an *r.e. order*. If the principal sequence of α^* is produced by a recursive function, we say that α^* is a *recursive order*.

We denote the class of all sets by V, of all r.e. sets by F, and of all recursive sets by E. The corresponding concepts for orders we denote by V^*, F^* and E^*.

3. **Repetitions.** In excluding sequences which contain repetitions, we follow the precedent of the theory of sets. In determining the set enumerated by a sequence, only the first occurrence of a given number is significant; all later occurrences are ignored. This practice can be formalized as follows.

Let $f(x)$ be a function with infinite range. We define the "sequence of first occurrences" associated with the sequence of values produced by $f(x)$ as the sequence produced by $g(x)$, where $g(0) = f(0)$ and

$$g(x+1) = f\left(\mu y\left[\prod_{z=0}^{x} |f(y) - g(z)| \neq 0\right]\right).$$

Two sequences will be equivalent if they have the same sequence of first oc-

(2) An equivalent definition may be heuristically helpful: the sequences produced by $f(x)$ and $g(x)$ are equivalent when, for all x and y, $f(x) < f(y)$ if and only if $g(x) < g(y)$. See §10.

currences. In this way a sequence free from repetitions can be considered as the principal sequence of an equivalence class of sequences, and its set or order properties extended to all sequences of its class. (This is an example of a trivial principal sequence, as we defined it in §1.)

4. The classification of sequences.

THEOREM 1. *If two sequences are contained in the same set equivalence class* α *and the same order equivalence class* α^*, *then they are identical.*

Proof. If the sequences $f(0), f(1), f(2), \cdots$ and $g(0), g(1), g(2), \cdots$ are both in α^*, then there exists a function $p(x)$ giving such a permutation that $f(p(x))$ and $g(p(x))$ both produce sequences in order of size. If $f(0)$, $f(1)$, $f(2), \cdots$ and $g(0), g(1), g(2), \cdots$ are both in α, then $f(p(x)) \equiv g(p(x))$, since both functions give enumerations of the same set in order of size. If we denote $\mu y[p(y)=x]$ by $p^{-1}(x)$, then $f(p(p^{-1}(x))) \equiv g(p(p^{-1}(x)))$; i.e., $f(x)=g(x)$.

We see, then, that the intersection of a set and an order is a unique sequence. These two equivalence relations suffice for the complete classification of infinite sequences free from repetitions. If the intersection of a set α and an order α^* is a sequence produced by a recursive function, we say that *α has a recursive enumeration in α^*.*

For a complete classification of sequences produced by everywhere-defined functions, repetition properties of course have to be taken into account. This is undoubtedly worthy of study, but beyond the scope of the present paper.

II. RECURSIVE ORDERS

5. The groups E^* and Q^*.

THEOREM 2. *E^* is a subgroup of V^*.*

Proof. If $p(x)$ and $q(x)$ are recursive functions producing the principal sequences of orders α^* and β^*, then $p(q(x))$ and $p^{-1}(x)=\mu y[p(y)=x]$ are recursive functions producing the principal sequences of $\alpha^*\beta^*$ and $(\alpha^*)^{-1}$.

The nondenumerability of V^* and the denumerability of E^* establish E^* as a proper subgroup of V^*. E^* is not invariant in V^*, as Theorem 3 will show, but in Theorem 4 we determine how much of E^* is invariant.

THEOREM 3. *If $(\alpha^*)^{-1}E^*\alpha^* \subset E^*$, then $\alpha^* \in E^*$.*

Proof. Let $q(x)$ produce the principal sequence of $(\alpha^*)^{-1}$. If $p(x)$ produces the permutation

$$0 \to 1, \qquad 2x+1 \to 2x+3, \qquad 2x+2 \to 2x,$$

then $p(x)$ is a recursive function and by hypothesis $r(x)=q(p(q^{-1}(x)))$ is again a recursive function. If $q(0)=a$, then $q(1)=q(p(0))=r(q(0))=r(a)$. In general, $q(2x+3)=r(q(2x+1))$, and $q(2x+2)=r^{-1}(q(2x))$.

DEFINITION. $\alpha^* \in Q^*$ when all transforms of α^* under inner automorphisms of V^* are in E^*.

It is easily shown by group-theoretical arguments that Q^* is a subgroup of E^*, and invariant in V^*.

THEOREM 4. *The principal sequences of orders of Q^* are just those permutations moving only finitely many numbers.*

Proof. Any permutation moving only finitely many numbers can be produced by a recursive function. For such a permutation can be written explicitly as a product of a finite number of cycles of finite length. Let $p(x)$ give such a permutation, and let us transform it by an arbitrary permutation given by $q(x)$: $q^{-1}(p(q(x)))$. A number a is moved by this permutation only if $q(a)$ is moved by $p(x)$. But this can happen for only finitely many a's. So any transform of $p(x)$ again moves only finitely many numbers. If $p(x)$ produces the principal sequence of α^*, then $\alpha^* \in Q^*$.

To show the converse, we show that any permutation moving infinitely many numbers has nondenumerably many different transforms(3), whereas E^* is denumerable. We first observe that if $p(x)$ moves infinitely many numbers, there exist disjoint infinite sets α and β such that $p(\alpha) = \beta$. For if $p(x)$ contains an infinite cycle and a is a number in that cycle, the set α generated by applying $p(x)$ an odd number of times to a is disjoint from and mapped onto the set β generated by applying $p(x)$ an even number of times to a. If $p(x)$ contains an infinite number of cycles, the set α of smallest numbers of these cycles is mapped onto a set β disjoint from α.

Now let γ be an arbitrary infinite set disjoint from β. There exists a permutation $q(x)$ mapping γ onto α, and β onto β. The inverse of $q(x)$ then also maps β onto β. The transform of $p(x)$ by $q(x)$ maps γ onto β, and hence corresponding to distinct sets γ_0 and γ_1 must be distinct transforms of $p(x)$. But there are of course nondenumerably many such sets γ.

The cosets α^*E^* and $E^*\alpha^*$ will thus not coincide for $\alpha^* \in V^* - E^*$, although they overlap on at least $\alpha^*Q^* = Q^*\alpha^*$. We will find the left cosets α^*E^* playing a significant role in later developments (§§7 and 9), but the right cosets remain somewhat mysterious.

All of the concepts in this paper can of course be relativized by replacing "recursive" by "recursive in functions $\psi_1(x), \psi_2(x), \cdots, \psi_k(x)$" [5, p. 275 and p. 326]. In particular, the hierarchies of functions and sets studied by Kleene, Mostowski and Davis [5, Chapter XI], will be paralleled, through a generalization of Theorem 2, by hierarchies of groups similar to E^*. In this paper, however, we shall not concern ourselves with these generalizations, hoping rather to develop on the lowest level of the theory ideas and results of sufficient interest to be perhaps worth generalizing.

---

(3) The author is indebted to the referee for this fact, the use of which gives a proof much shorter than the original one.

III. Recursively enumerable orders

6. Recursive enumerations.

THEOREM 5. *If α has a recursive enumeration in α^*, then the functions $h(x)$, producing the principal sequence of α, and $p(x)$, producing the principal sequence of α^*, are recursive in each other.*

Proof. Let $f(x)$ produce a recursive enumeration of α in α^*. Then since $f(x)=h(p(x))$, $h(x)=f(p^{-1}(x))$ and $p(x)=\mu y[h(y)=f(x)]$.

Thus all recursive enumerations of sets of E must be in orders of E^*, and of sets of $F-E$ in orders of F^*-E^*. Since $F-E$ is not empty, F^*-E^* is not empty.

Theorem 6 will show that each set of F has recursive enumerations in denumerably many orders of F^*. To see that each order α^* of F^* contains recursive enumerations of denumerably many sets of F, we observe that if $f(x)$ produces a recursive enumeration of α_0 in α^*, then $f(x)+n$, for example, produces a recursive enumeration in α^* of a set $\alpha_n \neq \alpha_0$.

7. Left cosets of E^*.

THEOREM 6. *A set α with a recursive enumeration in α^* has a recursive enumeration in β^* if and only if α^* and β^* are in the same left coset of E^*.*

Proof. Let $f(x)$ produce a recursive enumeration of α in α^*. Suppose $\beta^*=\alpha^*\gamma^*$ for some γ^* in E^*. Then if $p(x)$ is the recursive function producing the principal sequence of γ^*, $f(p(x))$ produces a recursive enumeration of α in β^*.

Suppose $g(x)$ produces a recursive enumeration of α in β^*. Then $\mu y[f(y)=g(x)]$ produces the principal sequence of an order γ^* of E^* such that $\beta^*=\alpha^*\gamma^*$.

COROLLARY A. *If α^* and β^* are in the same left coset of E^* and $\alpha^* \in F^*-E^*$, then $\beta^* \in F^*-E^*$.*

COROLLARY B. *Every infinite recursive set has recursive enumerations in all orders of E^*.*

THEOREM 7. *F^*-E^* is the union of denumerably many left cosets of E^*.*

Proof. Theorem 6, Corollary A establishes that F^*-E^* is a union of left cosets. This union cannot be more than denumerable, since F^*-E^* is denumerable and the cosets are disjoint from each other. To show that the union is infinite, we construct a denumerable sequence of r.e. sets, no two of which have recursive enumerations in the same order.

Let $\alpha_0 \in F-E$, with $f(x)$ a recursive function enumerating α_0 in α^*. Consider the set α_n obtained from α_0 by dropping from α_0 its n smallest elements. $\alpha_n \in F-E$, since it is obtained by removing a finite number of elements from α_0. Now suppose that there exists a recursive function $g(x)$ which enumerates α_n in α^*. We deduce from this that α_n must be recursive.

Let $t(x)=g(\mu y[f(y)=x])$. $t(x)$ is partial recursive, defined on α_0, and $t(f(x))=g(x)$. If $q(x)$ produces the principal sequence of $(\alpha^*)^{-1}$, then $h(x)=f(q(x))$ produces the principal sequence of α_0, and $t(h(x))=g(q(x))$ produces the principal sequence of α_n; moreover $t(h(x))=h(x+n)$. Hence, if $t^m(a)$ means $t(x)$ applied m times to a,

$$\begin{aligned}\alpha_n = t(\alpha_0) = &\{t(h(0)), t^2(h(0)), t^3(h(0)), \cdots\}\\ &\cup \{t(h(1)), t^2(h(1)), t^3(h(1)), \cdots\}\\ &\cup \cdots \cup \{t(h(n-1)), t^2(h(n-1)), t^3(h(n-1)), \cdots\}\end{aligned}$$

where $t^m(h(a))$, as a function of m, recursively enumerates a set in order of size. So α_n, as a finite union of recursive sets, is recursive.

Hence α_0 and α_n do not have recursive enumerations in the same order. In fact, the above proof holds for any pair from the denumerable sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$.

Partial recursive functions which share with $t(x)$ the property of preserving order on a set α will receive further consideration in Chapter V, where the above example will be generalized(4).

IV. Recursively bounded and recursively comparable orders

For the further study of r.e. orders, we isolate two of their properties, giving rise to the two classes discussed in this chapter. Since these classes are of some interest in their own right, we state our theorems as generally as possible, leaving to corollaries the applications to F^*.

8. Recursively bounded orders.

Definition. An order α^*, with principal sequence produced by $p(x)$, is *recursively bounded* when there exists a recursive function $f(x)$ such that $p(x)\leqq f(x)$ for all x. We denote the class of recursively bounded orders by B^*.

Theorem 8. *$F^*\subset B^*$.*

Proof. Every sequence of an order α^* is an upper bound for the principal sequence, and if $\alpha^*\in F^*$, there will be such a sequence produced by a recursive function.

Theorem 9. *B^* is a semigroup.*

Proof. Let α^* and β^* be orders of B^*, with $p_1(x)$ and $p_2(x)$ producing their principal sequences and recursive functions $f_1(x)$ and $f_2(x)$ bounding $p_1(x)$ and $p_2(x)$ respectively. If $\alpha^*\beta^*=\gamma^*$, then $q(x)=p_1(p_2(x))$ produces the principal sequence of γ^*.

Now $p_1(p_2(x))\leqq f_1(p_2(x))$, and $p_2(x)\leqq f_2(x)$. Set $g(x)=\max\,[f_1(0), f_1(1), \cdots, f_1(f_2(x))]$. $g(x)$ is a recursive function bounding $q(x)$, and $\gamma^*\in B^*$.

(4) The proof used here, and the generalization, are adapted from a suggestion of the referee.

COROLLARY. *B^* is a union of both right and left cosets of E^*.*

In general, if S is a semigroup imbedded in a group G, S induces a relation in G: aRb if and only if $as=b$ for some s in S. This relation is transitive, reflexive if the identity of G is in S, but not anti-symmetric if S includes a proper subgroup of G. (Similarly for $sa=b$.) This is the case with B^* and V^*.

DEFINITION. $\alpha^* R_B \beta^*$ when $\alpha^*\xi^*=\beta^*$ for some ξ^* in B^*.

We note that if $(\alpha^*)^{-1}\in B^*$ and $\beta^*\in B^*$, then $\alpha^* R_B \alpha^*$.

9. Recursively comparable orders.

DEFINITION. An order α^*, with principal sequence produced by $p(x)$, is *recursively comparable* when there exists a recursive function $g(x, y)$ (the *comparing function*) which equals 1 when $p(x)<p(y)$ and 0 otherwise. We denote the class of recursively comparable orders by C^*.

THEOREM 10. *$F^*\subset C^*$.*

Proof. Let $f(x)$ be a recursive function producing a sequence of an order α^*. Then $g(x, y)=1\dot{-}(1\dot{-}(f(y)\dot{-}f(x)))$ is the recursive comparing function for α^*.

THEOREM 11. *If $\alpha^*\in C^*$ and $\alpha^* R_B \beta^*$, and if $p(x)$ and $q(x)$ produce the principal sequences of α^* and β^* respectively, then $p(x)$ is recursive in $q(x)$.*

Proof. Suppose $\alpha^*\rho^*=\beta^*$, $\rho^*\in B^*$, and $g(x, y)$ and $h(x)$ are the recursive comparing and bounding functions for α^* and ρ^* respectively.

Informally, we compute $p^{-1}(0)$ by finding $z_0=q^{-1}(0)$. Then $p^{-1}(0)$ is the argument producing the smallest number in the set $\{p(0), p(1), \cdots, p(h(z_0))\}$, which the comparing function $g(x, y)$ enables us to find. We then compute $p^{-1}(1)$ by a similar process, excluding the number $p^{-1}(0)$ already found. To compute $p(a)$, we continue the process until $a=p^{-1}(b)$ for some b.

Formally, with $s(x)=q^{-1}(x)$,

$$r(0) = \mu z\left[\sum_{u=0}^{h(s(0))} g(u, z) = 0\right],$$

$$r(x+1) = \mu z\left[\sum_{u=0}^{h(s(x+1))} g(u, z)\cdot \prod_{v=0}^{x} |r(v)-u| + \sum_{w=0}^{x}(1 \dot{-} |r(w)-z|) = 0\right],$$

$$p(x) = r^{-1}(x).$$

COROLLARY A. *If $\alpha^*\in C^*-E^*$, then $(\alpha^*)^{-1}\notin B^*$.*

For otherwise $\alpha^* R_B \epsilon^*$ and $\alpha^*\in E^*$.

COROLLARY B. *If $\alpha^*\in F^*-E^*$, $(\alpha^*)^{-1}\in F^*$.*

COROLLARY C. *B^* is not a group.*

COROLLARY D. *No element of F^*-E^* is of finite (group-theoretic) order.*

For if $(\alpha^*)^{-1} = (\alpha^*)^n$, then $(\alpha^*)^{-1} \in B^*$, by Theorem 9.

THEOREM 12. *C^* is a union of left cosets of E^*, but the only right coset of E^* which is included in C^* is E^* itself.*

Proof. Let $\alpha^* \in C^* - E^*$, with principal sequence produced by $p(x)$ and recursive comparing function $g(x, y)$. If $\beta^* \in E^*$, with principal sequence produced by $q(x)$, then $g(q(x), q(y))$ is the recursive comparing function for $\alpha^*\beta^*$, proving the first assertion of the theorem.

To prove the second, we choose $q(x)$ to be the recursive function giving the permutation $0 \rightarrow 1$, $2x+1 \rightarrow 2x+3$, $2x+2 \rightarrow 2x$. From $\beta^*\alpha^* \in C^*$ we deduce the contradiction $\alpha^* \in E^*$. For suppose $h(x, y)$ is the recursive comparing function for $\beta^*\alpha^*$. Denote $p^{-1}(i)$ by a_i, and suppose we know a_0, a_1 and $a_2 \cdot a_4$ is unique in having the properties: $p(a_1) < p(a_4)$, $q(p(a_4)) < q(p(a_1))$ and $a_4 \neq a_2$. A systematic search through the values of $h(x, y)$ and $g(x, y)$ will then find a_4. Similarly, a_3 is unique in having the properties: $p(a_3) < p(a_4)$, $q(p(a_4)) < q(p(a_3))$ and $a_3 \neq a_1$. In general,

$$a_{2i+4} = \mu y[h(y, a_{2i+1}) \cdot g(a_{2i+1}, y) \cdot (1 \dot{-} (1 \dot{-} | y - a_{2i+2} |)) = 1],$$
$$a_{2i+3} = \mu y[h(a_{2i+4}, y) \cdot g(y, a_{2i+4}) \cdot (1 \dot{-} (1 \dot{-} | y - a_{2i+1} |)) = 1].$$

COROLLARY. *F^* and C^* are not semigroups.*

10. **Further theorems on B^* and C^*.** The theorems of this section will be in justification of the introduction of B^* and C^*.

THEOREM 13. *$B^* \not\subset C^*$.*

Proof. Let ϕ_n be the transposition $2n \leftrightarrow 2n+1$. If $p(x)$ gives a permutation consisting of one occurrence each of some of the transpositions ϕ_n, then $p(x)$ is determined by the set α of all n such that ϕ_n occurs in the permutation. Take α as a set of $V - E$. Then if $p(x)$ produces the principal sequence of α^*,

(1) $p(2x) \dot{-} 2x$ is the characteristic function of α. So $\alpha^* \notin E^*$.
(2) $p(x) \leqq x+1$ for all x. So $\alpha^* \in B^*$.
(3) $(\alpha^*)^{-1} = \alpha^*$. So $(\alpha^*)^{-1} \in B^*$, and by Theorem 10, Corollary A, $\alpha^* \notin C^*$.

COROLLARY. *B^* is not denumerable.*

For α may be chosen in nondenumerably many ways.

There are also nondenumerably many elements of B^* whose inverses are not in B^*. If S is a semigroup, and G the group of all elements of S which have inverses in S, then if $a, b \in S$ and $ab \in G$, we have $a, b \in G$. For if $ab = c \in G$, $a^{-1} = bc^{-1} \in S$. So if α^* is as above, and $\beta^* \in F^* - E^*$, $\alpha^*\beta^*$ does not have its inverse in B^*.

We now prepare to establish the converse of Theorem 13. A comparing function $g(x, y)$ uniquely determines an order α^*, for the function $p(y)$ equal to the number of x's for which $g(x, y) = 1$ produces the principal sequence of the order for which $g(x, y)$ is the comparing function. Hence C^* is denumerable.

The four conditions necessary and sufficient that a function $g(x, y)$ with range $\{0, 1\}$ be a comparing function for some order are:

(a) $g(x, x)=0$;

(b) if $x \neq y$, then $g(x, y)=1$ if and only if $g(y, x)=0$ (antisymmetry);

(c) if $g(x, y)=g(y, z)=1$, then $g(x, z)=1$ (transitivity);

(d) for no y are there infinitely many x's such that $g(x, y)=1$.

The necessity is easily seen. To verify the sufficiency we need to show that for every n, there exists a unique y such that for exactly n x's, $g(x, y)=1$. It is convenient to think of the values of $g(x, y)$ arranged in a table, with the values of x across the top and the values of y down the left-hand side. Then to show that no two rows, say the ith and jth, can contain the same number of 1's, we observe that by condition (b), there is a 1 at either the ith row and jth column, or at the jth row and ith column, say the latter. Then by condition (c), every column which intersects the ith row in a 1 must also intersect the jth row in a 1. I.e., the jth row has a 1 for every 1 in the ith row. and at least one more, at the ith column.

To show that for every n there is a row with just n 1's, we note first that condition (d) guarantees that for infinitely many n's such a row will exist. Suppose it exists and is the ith for m. Then if the j_0th, j_1th, $\cdots$, j_{m-1}th are the m columns intersecting the ith row in 1's, the j_0th, j_1th, $\cdots$, j_{m-1}th rows contain less than m 1's. Since no two have the same number of 1's, there must be a one-to-one correspondence between the m numbers $0, 1, \cdots, m-1$ and these m rows, giving a row with n 1's for every $n<m$.

Suppose now that we attempt to make such a table, choosing the values of $g(0, 0)$, $g(1, 0)$, $g(1, 1)$, $g(2, 0)$, $g(2, 1)$, $g(2, 2)$, $\cdots$ in that order. Condition (b) then provides the values of $g(0, 1)$, $g(0, 2)$, $g(1, 2)$, $\cdots$. Condition (a) requires that $g(0, 0)$, $g(1, 1)$, $g(2, 2)$, $\cdots$ be 0, and condition (c) will sometimes require a certain value to be 0 or 1, but at other times we will have a free choice. If we have some effective procedure for making the free choices when they occur, a procedure which obeys condition (d), then we will produce the table for a recursive comparing function, and determine an order of C^*. The proof of Theorem 14 involves just such a procedure.

Lemma. *If $p(x)$ produces the principal sequence of an order $\alpha^*$of C^*-E^*, then there exists no general recursive function $f(x)k$ with infinite range, such that $p(f(x))$ is recursive.*

Proof. An alternative statement is that $p(x)$ is not effectively computable on any infinite r.e. subset of its domain. E.g., the set of all x's for which $p(x)=x$ must be finite or immune.

For the proof, we observe that if we know the value a of $p(x_a)$, the comparing function enables us to find the a numbers y for which $p(y)<p(x_a)$,

and the largest of these is an upper bound for $p^{-1}(w)$, for all $w<p(x_a)$. If we suppose the lemma false, then $h(w)=f(\mu x[w<p(f(x))])$ is general recursive, and

$$\mu y\left[\sum_{z=0}^{y} g(z, h(w)) = p(h(w))\right]$$

is a recursive upper bound for $p^{-1}(w)$, contradicting, by Theorem 11, Corollary A, the hypothesis that $\alpha^*\in C^*-E^*$.

THEOREM 14. $C^*\not\subset B^*$.

Proof. Let α^* be an order of C^*-E^*. Let $h(x, y)$ be the recursive comparing function for α^*. We construct a table for the comparing function $g(x, y)$ of an order β^* by the following procedure.

Suppose, at the intersection of the ith row and jth column $(i<j)$ we have a free choice, not dictated by condition (c). Then we look at the first $j+1$ values of $h(x, i)$, $x=0, 1, \cdots, j$. If there are no 1's among them, we set $g(j, i)=0$. If there are some 1's, suppose k is the largest $y\leqq j$ such that $h(y, i)=1$. If there are already k or more 1's in the ith row of our new table, we set $g(j, i)=0$. If not, $g(j, i)=1$.

This procedure for deciding our free choices is certainly effective. It satisfies condition (d), since no row of the table of $h(x, y)$ contains infinitely many 1's. $g(x, y)$ is then the recursive comparing function for an order β^* of C^*.

Formally,

$$g(x, x) = 0;$$

$$g(x, y) = 1 \dot{-} g(y, x), \qquad \text{if } x < y;$$

$$g(x+1, 0) = 1 \dot{-} \left(1 \dot{-} \left(((x+1) \dot{-} \mu z_{z\leqq x+1}[h((x+1) \dot{-} z, 0) = 1]\right) \dot{-} \sum_{z=0}^{x} g(z, 0)\right)\Big), \qquad \text{if } y = 0 \leqq x;$$

$$g(x+1, y+1) = \left(1 \dot{-} \left(1 \dot{-} \prod_{z=0}^{y}(g(x+1, z) + g(z, y+1))\right)\right) \cdot\left(1 \dot{-} \left(\prod_{z=0}^{y} (g(z, x+1) + g(y+1, z)\right)\right) \cdot\left(1 \dot{-} \left(((x+1) \dot{-} \mu z_{z\leqq x+1}[h((x+1) \dot{-} z, y+1) = 1]) \dot{-} \sum_{z=0}^{x} g(z, y+1)\right)\right)\Big), \qquad \text{if } 0 < y < x,$$

with the convention that the bounded μ-function has its upper bound as value if no number within the bounds has the required property.

We now show that $\beta^* \notin B^*$. First, note that certain rows of the table of values of $g(x, y)$, namely, those which represent a larger number in the principal sequence of β^* than any row higher in the table, are assured of getting at least as many 1's as the directions from the table of $h(x, y)$ call for. We may call them *free* rows, since condition (c) will not place any restriction on the number of 1's they can receive. We can recursively enumerate in order of size the locations of the free rows of the table of $g(x, y)$; the ith row is free if and only if it has no 0's to the left of the main diagonal, so

$$f(0) = 0,\ f(x+1) = \mu y\left[(1 \dot{-} ((f(x) + 1) \dot{-} y)\cdot \prod_{z=0}^{y \dot{-} 1} g(z, y) = 1\right]$$

produces the desired enumeration. Now if the ith row is free, the number of 1's in it is not less than the largest y for which $h(y, i) = 1$.

Suppose $\beta^* \in B^*$, and let $b(x)$ be a recursive upper bound for β^*. Then if $p(x)$ produces the principal sequence of α^*, we can effectively compute the values of $p(f(x))$, contradicting the lemma. For

$$p(f(x)) = \sum_{z=0}^{b(x)} h(z, x).$$

THEOREM 15. $F^* = B^* \cap C^*$.

Proof. We have $F^* \subset B^* \cap C^*$, by Theorems 8 and 10. Suppose $h(x)$ and $g(x, y)$ are recursive bounding and comparing functions respectively for an order α^* of $B^* \cap C^*$, whose principal sequence is produced by $p(x)$. We construct a recursive function $f(x)$ which produces a sequence of α^*.

$$f(0) = 2^{h(0)},$$

$$f(1) = \left[\frac{f(0)}{2}\right]\cdot(1 \dot{-} g(0, 1)) + (f(0) + 2^{h(1)})\cdot g(0, 1).$$

For the general case we construct the auxiliary functions $u(x+1)$, $v(x+1)$ and $w(x+1)$.

$$u(x+1) = \prod_{z=0}^{x} g(z, x+1) + 2\prod_{z=0}^{x} g(x+1, z).$$

$u(x+1) = 1$ if $p(x+1)$ is the largest, and 2 if it is the smallest, of the set $\{p(0), p(1), \cdots, p(x+1)\}$. Otherwise, $u(x+1) = 0$.

$$v(x+1) = \mu y\left[(g(y, x+1) + \sum_{z=0}^{x} g(z, y)\cdot g(x+1, z)) \cdot \left| u(x+1) - 1 \right| = 0\right].$$

$v(x+1)$ is the least y such that $p(x+1)<p(y)$, and for no $z<x+1$ is $p(x+1)<p(z)<p(y)$, unless this y exceeds $x+1$. Then $v(x+1)=0$.

$$w(x+1) = \mu y\Big[(g(x+1, y) + \sum_{z=0}^{x} g(y, z)\cdot g(z, x+1)) \cdot |u(x+1) - 2| = 0\Big].$$

$w(x+1)$ is the counterpart, with inequalities reversed, of $v(x+1)$. The factor $|u(x+1)-2|$ gives $w(x+1)=0$ in the case $p(x+1)=0$. Now

$$\begin{aligned} f(x+1) = {} & (1 \dot{-} |u(x+1) - 2|)\Big(f\Big(\mu y\Big[\sum_{z=0}^{x} g(y, z) = 0\Big]\Big) + 2^{h(x+1)}\Big) \\ & + (1 \dot{-} |u(x+1) - 2|)\cdot\Big[f\Big(\mu y\Big[\sum_{z=0}^{x} g(z, y) = 0\Big]\Big)\Big/2\Big] \\ & + (1 \dot{-} u(x+1))\cdot[(f(v(x+1)) - f(w(x+1)))/2]. \end{aligned}$$

The key to this construction is the leaving of enough room between the values of $f(x)$ as they are assigned for as many other values as may later need to be inserted. This the upper bound enables us to do.

The effect of Theorem 15 is to interpret a property of orders (considered as classes of sequences) in terms of properties directly applicable to permutations and the functions producing them. The analogous process for sets may also be fruitful; an example appears in §16. However, the natural operations on sets are not so conveniently expressed in terms of operations on monotone functions, so principal sequences will probably play a less important role in the theory of sets than in the theory of orders.

V. Order-equivalence

11. Recursive order-preserving transformations.

Definition. A partial recursive function $t(x)$ gives a *recursive order-preserving transformation on α* when:

(1) α is included in the domain of $t(x)$;

(2) if $x, y\in\alpha$ and $x<y$, then $t(x)<t(y)$.

Hence $t(x)$ is one-to-one between α and $t(\alpha)$.

We call $t(\alpha)$ an *order-transform* of α. Clearly the relation "β is an order-transform of α" is reflexive and transitive; when it is also symmetric we say that α and β are *order-equivalent.* In §16 we present an example of a case where symmetry does not hold, so that α need not always be order-equivalent to $t(\alpha)$.

Theorem 16. *If β is an order-transform of α, then β is Turing reducible* [7, pp. 311–312] *to α.*

Proof. Let $h_1(x)$ and $h_2(x)$ produce the principal sequences, and $c_1(x)$ and $c_2(x)$ be the characteristic functions, of α and $\beta = t(\alpha)$ respectively. Then

$$c_2(x) = 1 \dot{-} \prod_{z=0}^{x} | h_2(z) - x |,$$

$$h_2(x) = t(h_1(x)),$$

$$h_1(x) = \mu y \left[\sum_{z=0}^{y} c_1(z) = x + 1 \right].$$

The second step of this proof brings out the fact that if $t(x)$ sends α into β, preserving order, then the values of $t(x)$ on α are uniquely determined. If $s(x)$ and $t(x)$ preserve order on α and $s(\alpha) = t(\alpha)$, then $s(x) = t(x)$ for all x in α.

The generalization mentioned in §7 of the example used in the proof of Theorem 7 can now be stated: if there exists a recursive order-preserving transformation $t(x)$ on a set α such that $t(\alpha) \subset \alpha$ and $\alpha - t(\alpha)$ contains just $n \neq 0$ elements, then $\alpha \in E$. Suppose a is the largest element of $\alpha - t(\alpha)$, and α_0 is the set of all elements of α greater than a. Then setting $\alpha_n = t(\alpha_0)$, the argument of Theorem 7 applies. Since $\alpha - \alpha_n$ is finite, the recursiveness of α_n gives the result.

12. Order-equivalence between r.e. sets.

THEOREM 17. *If α is an infinite r.e. set, then β is an order-transform of α if and only if β is r.e., and α and β have recursive enumerations in the same order.*

Proof. If $f(x)$ produces a recursive enumeration of α in α^*, and $t(x)$ gives a recursive order-preserving transformation on α, then $t(f(x))$ produces a recursive enumeration of $t(\alpha)$ in α^*. Conversely, if $g(x)$ produces a recursive enumeration of β in α^*, then $t(x) = g(\mu y[f(y) = x])$ gives a recursive order-preserving transformation on α, and $t(\alpha) = \beta$.

COROLLARY A. *An r.e. set is order-equivalent to all its order-transforms.*

COROLLARY B. *F and E are closed under recursive order-preserving transformations.*

The symmetry extends to subsets of α and $t(\alpha)$; if $t(x)$ gives a recursive order-preserving transformation on an r.e. set α, and $\gamma \subset \alpha$, then the order-equivalence between α and $t(\alpha)$ gives an order-equivalence between γ and $t(\gamma)$. However, order-equivalence can hold in other cases too. For an example of two order-equivalent sets which have no order-equivalent r.e. supersets, we let $t(x)$ be the recursive function giving the permutation $2x \rightarrow 2x+1$, $2x+1 \rightarrow 2x$. Let α be any set not in F which contains exactly one number from each pair $2x$, $2x+1$. Then $t(x)$ preserves order on α and on $t(\alpha)$ as well, and $t(t(\alpha)) = \alpha$. Now suppose there exists a partial recursive function $s(x)$ which preserves order on an r.e. superset γ of α, and such that $s(\alpha) = t(\alpha)$; i.e., $s(x) = t(x)$ for all x in α.

Since γ is r.e. and α is not, $\alpha \neq \gamma$. If $2x \in \gamma - \alpha$, $s(2x) \neq t(2x)$, since $2x+1 \in \alpha$ and $s(2x+1) = t(2x+1) = 2x$, so that $s(2x) < 2x$ while $t(2x) = 2x+1$. If $2x+1 \in \gamma - \alpha$, then $2x \in \alpha$ and $s(2x) = t(2x) = 2x+1$, so that $s(2x+1) > 2x+1$ while $t(2x+1) = 2x$. Hence α is the set of those elements x of γ for which $s(x) = t(x)$. But this set is clearly r.e., contrary to hypothesis.

Theorem 16 established the Turing reducibility of β to α when β and α are order-equivalent. Let us examine this reduction when α is r.e., with $f(x)$ producing a recursive enumeration without repetitions of α. We can find, in terms of x, a bound for y in the third equation (i.e., a bound on the number of questions to be asked about membership in α). If there are at least x numbers w for which $f(w) < f(y)$, then $f(y)+1$ is such a bound. In spite of this, the reduction of $t(\alpha)$ to α is not reduction by truth tables [7, pp. 299–301]. For in a reduction by truth tables the process must be effective, although perhaps incorrect, even when wrong answers are given to questions about membership in α. In this case, clearly $h_2(x)$ in the second equation will have no value if, by using wrong values of $c_1(x)$, we give $h_1(x)$ a value not in the domain of $t(x)$. So in general the very effectiveness of the process, not merely its correctness, depends on having the correct values for $c_1(x)$. (We give, after Theorem 18, an example in which $t(x)$ cannot be extended to a general recursive function.)

On the other hand, the sets α_n of Theorem 7 are clearly reducible to each other, although not order-equivalent. So we do not have here the most general Turing reducibility, but rather a special kind not considered by Post.

13. **Strong order-equivalence.** If $t(x)$ gives a recursive order-preserving transformation on a recursive superset (e.g., ϵ) of α, we say that α and $t(\alpha)$ are *strongly order-equivalent*. If α and β are strongly order-equivalent, then α and β will have much in common. For example, creative sets are strongly order-equivalent only to creative sets. Let α be creative and θ recursive, with $\alpha \subset \theta$. If $t(x)$ gives a recursive order-preserving transformation on θ, then suppose ξ is an r.e. subset of $(t(\alpha))'$. Let $s(x)$ give the inverse recursive order-preserving transformation.from $t(\theta)$ to θ. $s(\xi \cap t(\theta)) \cup \theta'$ is an r.e. subset of α', and we can obtain an element a of $\alpha' - s(\xi \cap t(\theta)) \cup \theta'$. Then $a \in \theta$, and $t(a) \in t(\theta) \cap ((t(\alpha))' \cup \xi')$.

THEOREM 18. *If α is not finite or immune, and β is any infinite set, then α includes a subset strongly order-equivalent to β.*

Proof. α has an infinite r.e. subset, and hence an infinite recursive subset ξ [7, p. 291]. If $h(x)$ produces the principal sequence of ξ, then $h(\beta)$ is a subset of α, strongly order-equivalent to β.

COROLLARY. *Every such α has a creative subset.*

On the other hand, if α is immune, and includes a subset which is an order-transform of β, then β is also immune. For by Theorem 17, Corollary B, any r.e. subset of β goes into an r.e. subset of α.

We now show the existence of two order-equivalent r.e. sets which are not strongly order-equivalent. Denote by $\Phi(n, x)$ the nth partial recursive function of one variable in Kleene's enumeration [4, pp. 50–58], modified so that to every n corresponds a function. Let α be the set of all n such that $\Phi(n, n)$ is defined. It is easily seen that α is r.e. If $t(x)$ is the partial recursive function

$$2x + (1 \dot{-} |2x - \Phi(x, x)|),$$

then $t(x)$ gives a recursive order-preserving transformation on α.

Now if there exists a partial recursive function $s(x)$, giving a recursive order-preserving transformation on a recursive superset θ of α, then by defining $f(x) = s(x)$ for $x \in \theta$, $f(x) = 0$ for $x \in \theta'$, we have a general recursive function $f(x) = \Phi(r, x)$ for some r, such that $\Phi(r, x) = t(x)$ for all x in α. But this is impossible, for $r \in \alpha$. If $\Phi(r, r) = 2r$, $t(r) = 2r+1$ and if $\Phi(r, r) \neq 2r$, $t(r) = 2r$. Hence α and $t(\alpha)$ are not strongly order-equivalent.

We remark that the impossibility of extending $t(x)$ to a general recursive function is not, in general, essential to escape strong order-equivalence between r.e. sets. The function $s(x) = [x/2]$ is general recursive, preserves order on $t(\alpha)$, and $s(t(\alpha)) = \alpha$. However, there does not exist a recursive superset θ of $t(\alpha)$ on which $s(x)$ preserves order, or what is the same thing here, is univalent(5). For then $s(\theta)$ would be a recursive superset of α, and there would be an inverse recursive order-preserving transformation from $s(\theta)$ to θ, sending α into $t(\alpha)$.

The fact that the elements of α' are not uniquely transformed by $t(x)$ suggests the possibility (somewhat limited by Theorem 19 below) that sets whose complements have quite different properties may be order-equivalent. (In particular, the complements need not be order-equivalent.) This would be in sharp contrast to the classification into simple sets, creative sets, etc., in which the complements play an essential role.

14. Hypersimple sets. In this section we establish some restrictions on the kinds of sets which can be order-equivalent to a hypersimple set.

Following Dekker [2, p. 497], we call an infinite sequence of finite sets $\phi_0, \phi_1, \phi_2, \cdots$ a *discrete array* when no ϕ_i is empty, $\phi_i \cap \phi_j = o$ for $i \neq j$, and there exist general recursive functions $a(i, x)$ and $b(i)$ such that ϕ_i is the range of $a(i, x)$ for fixed i, and contains $b(i)+1$ elements. We may suppose that the sequence $a(i, 0), a(i, 1), \cdots, a(i, b(i))$ is in order of size and free from repetitions. Then an infinite set is *hyperimmune* if and only if its complement includes at least one of the sets ϕ_i from every discrete array $\phi_0, \phi_1, \phi_2, \cdots$. An r.e. set whose complement is hyperimmune is *hypersimple.*

Let α be a hypersimple set, and β a set order-equivalent to α, with $f(x)$

(5) This is to say that every recursive superset of $t(\alpha)$ contains both members of infinitely many pairs $\{2x, 2x+1\}$.

and $g(x)$ producing recursive enumerations of α and β, respectively, in the same order α^*.

LEMMA. *There are only finitely many x_i's for which there exists a y such that $f(x_i)<f(y)$ (hence $g(x_i)<g(y)$) and $g(y)-g(x_i)<f(y)-f(x_i)$.*

Proof. Suppose there exist infinitely many such x_i's. Define[6]

$$h(0) = \mu y[((f(K(y)) + 1) \dot{-} f(L(y))) + (((g(L(y)) \dot{-} g(K(y))) + 1) \dot{-} (f(L(y)) \dot{-} f(K(y)))) = 0],$$

$$h(x+1) = \mu y[((f(K(y)) + 1) \dot{-} f(L(y))) + (((g(L(y)) \dot{-} g(K(y))) + 1) \dot{-} (f(L(y)) \dot{-} f(K(y)))) + (f(L(h(x))) \dot{-} f(K(y))) = 0].$$

$h(x)$ is such that $f(K(h(x)))<f(L(h(x)))$,

$$g(L(h(x))) - g(K(h(x))) < f(L(h(x))) - f(K(h(x)))$$

and $f(L(h(x)))\leqq f(K(h(x+1)))$. Set

$$\phi_i = \{f(K(h(i))) + 1, f(K(h(i))) + 2, \cdots, f(L(h(i))) - 1\}.$$

It is not hard to show formally that $\phi_0, \phi_1, \phi_2, \cdots$ is a discrete array. However, each ϕ_i has $(f(L(h(i)))-f(K(h(i))))-1$ elements, of which at most $(g(L(h(i)))-g(K(h(i))))-1$ can be in α. So α does not include any of the ϕ_i's, contradicting its hypersimplicity.

DEFINITION. A *gap* in a set is a pair of elements a, b such that $a+1<b$ and for no c, $a<c<b$, is c an element of the set. The *width* of the gap is $(b-a)-1$.

We wish to interpret the lemma in terms of gaps. Let $q(x)$ produce the principal sequence of $(\alpha^*)^{-1}$. Then $f(q(x))$ and $g(q(x))$ produce (not recursively) the principal sequences of α and β respectively. The lemma implies that there exist only finitely many gaps $f(q(i)), f(q(i+1))$ in α which do not correspond to gaps $g(q(i)), g(q(i+1))$ in β, and that where gaps in α do correspond to gaps in β, the gap in α will be wider than the gap in β only finitely many times.

THEOREM 19. *If α is hypersimple and β is order-equivalent to α, then $\beta'=\gamma_0\cup\gamma_1$, where γ_0 and γ_1 are disjoint, γ_0 is r.e. (perhaps empty) and γ_1 is hyperimmune.*

Proof. We consider two cases.

CASE 1. All but finitely many of the gaps in β correspond to gaps in α. From the lemma, we know that the gaps in α can be wider than the gaps in β only finitely many times. Suppose that the gaps in β are wider than the gaps in α infinitely many times. Define

(6) $K(x)$ and $L(x)$ are the Cantor enumerating functions whose sequences of values are 0, 0, 1, 0, 1, 2, 0, $\cdots$ and 0, 1, 0, 2, 1, 0, 3, $\cdots$ respectively. We also mention $J(x, y)$, for which $J(K(x), L(x))=x$. See Julia Robinson, Proc. Amer. Math. Soc. vol. 1 (1950) p. 704.

$$h(0) = \mu y[((f(K(y)) + 1) \dot{-} f(L(y))) + (((f(L(y)) \dot{-} f(K(y))) + 1) \dot{-} (g(L(y)) \dot{-} g(K(y)))) = 0],$$

$$h(x+1) = \mu y[((f(K(y)) + 1) \dot{-} f(L(y))) + (((f(L(y)) \dot{-} f(K(y))) + 1) \dot{-} (g(L(y)) \dot{-} g(K(y)))) + (f(L(h(x))) \dot{-} f(K(y))) = 0].$$

$h(x)$ is such that $f(K(h(x))) < f(L(h(x)))$,

$$f(L(h(x))) - f(K(h(x))) < g(L(h(x))) - g(K(h(x)))$$

and $f(L(h(x))) \leqq f(K(h(x+1)))$. Set

$$\phi_i = \{f(K(h(i))) + 1, f(K(h(i))) + 2, \cdots, f(L(h(i))) - 1\}.$$

It is not hard to show formally that $\phi_0, \phi_1, \phi_2, \cdots$ is a discrete array. However, β has a gap somewhere between $g(K(h(i)))$ and $g(L(h(i)))$, since there are at most $(f(L(h(i))) - f(K(h(i)))) - 1$ elements of β greater than $g(K(h(i)))$ and less than $g(L(h(i)))$. So α has a gap somewhere between $f(K(h(i)))$ and $f(L(h(i)))$ (with finitely many exceptions), and ϕ_i contains an element of α' for all but finitely many i's, contradicting the hypersimplicity of α. (With respect to the finitely many exceptions, see [2, Theorem 1.4]).

We now have that, with finitely many exceptions, every gap in β corresponds to a gap in α and conversely, and these corresponding gaps have the same width. So there exists an $\bar{x}$ such that either for all $\bar{x} < x$, $g(x) = f(x) + n$ for a fixed n, or else for all $\bar{x} < x$, $f(x) = g(x) + n$. From this it is easy to see that β is hypersimple. To the elements of the sets $\phi_0, \phi_1, \phi_2, \cdots$ of any discrete array we can apply a translation by n and obtain a new array which with finitely many exceptions, is discrete and bears the same relationship to β as $\phi_0, \phi_1, \phi_2, \cdots$ bears to α.

On the other hand, suppose that β is hypersimple and order-equivalent to α. Then the lemma works both ways to give us immediately the situation in the previous paragraph, and the elements of β are obtained, with finitely many exceptions, by translating the elements of α by n.

In this case γ_0 is empty.

CASE 2. β has infinitely many gaps which do not correspond to gaps in α. Then the function

$$h(0) = \mu y[\,|\,(f(K(y)) + 1) - f(L(y))\,| + ((g(K(y)) + 2) \dot{-} g(L(y))) = 0],$$

$$h(x+1) = \mu y[\,|\,(f(K(y)) + 1) - f(L(y))\,| + ((g(K(y)) + 2) \dot{-} g(L(y))) + ((h(x) + 1) \dot{-} y) = 0]$$

has as range the set of numbers i such that $f(K(i)) + 1 = f(L(i))$, but $g(K(i)) + 1 < g(L(i))$, so that $g(K(i)), g(L(i))$ is a gap in β. The union of the finite sets

$$\{g(K(h(x))) + 1, g(K(h(x))) + 2, \cdots, g(L(h(x))) - 1\}$$

is an infinite r.e. subset γ_0 of β'.

All the elements of $\gamma_1=\beta'-\gamma_0$ occur in gaps in β which correspond to gaps in α. Now suppose $\phi_0, \phi_1, \phi_2, \cdots$ is a discrete array such that γ_1 has at least one element in common with each ϕ_i. If $a(i, x)$ and $b(i)$ are the functions described in the definition of a discrete array, define $l(x)$ and $u(x)$:

$$l(0) = \mu y[g(K(y)) \dot{-} a(L(y), 0) = 0],$$
$$l(x+1) = \mu y[(g(K(y)) \dot{-} a(L(y), 0)) + ((g(u(x)) + 1) \dot{-} g(K(y))) = 0],$$
$$u(x) = \mu y[a(L(l(x)), b(L(l(x)))) \dot{-} g(y) = 0].$$

$g(K(l(x)))$ is not greater than any element of $\phi_{L(l(x))}$, $g(u(x))$ is not less than any element of $\phi_{L(l(x))}$, and $g(u(x))<g(K(l(x+1)))$.

Somewhere between $g(K(l(x)))$ and $g(u(x))$ is a gap in β, which corresponds to a gap in α somewhere between $f(K(l(x)))$ and $f(u(x))$. Then the finite sets

$$\{f(K(l(x))) + 1, f(K(l(x))) + 2, \cdots, f(u(x)) - 1\}$$

form a discrete array, no set of which is included in α, contradicting the hypersimplicity of α. Hence γ_1 is hyperimmune.

The possibility of Case 2 is seen by considering the transformation given by $t(x)=2x$. In such a case, α' and $(t(\alpha))'$ cannot be order-equivalent.

As special cases of Theorem 19 we see that a hypersimple set cannot be order-equivalent to a creative set nor to a set which is simple but not hypersimple.

Theorem 19 contains the beginnings of a classification of r.e. orders (and of left cosets of E^*) by the kinds of sets which can have recursive enumerations in them.

VI. Other applications

This chapter contains a variety of topics suggested by or making use of the ideas so far developed.

15. **Recursive order types**(7). In §10 we gave four conditions on a function $g(x, y)$, with range $\{0, 1\}$, which are necessary and sufficient that it be the comparing function of a permutation of the set of non-negative integers ϵ. The first three are needed for $g(x, y)$ to determine a linear ordering; the fourth, (d), ensures that this ordering is of order type ω. If we drop condition (d), then $g(x, y)$ may, in general, determine an ordering of ϵ having some other order type.

Let us say that if $g(x, y)$ is a recursive function satisfying conditions (a)–(c), $g(x, y)$ determines a *recursively comparable ordering of* ϵ. An order type represented by a recursively comparable ordering of ϵ we call a *recursive order type.*

THEOREM 20. *The recursive order types which are well-ordered are just the infinite recursive ordinal numbers.*

(7) See also Clifford Spector, J. Symbolic Logic vol. 20 (1955) p. 152.

Proof. Markwald [6, p. 139] defines a recursive ordinal number α as one which can be represented by a general recursive function $f(x, y, z)$ such that the relation

$$x \leq_f y \leftrightarrow (\exists z)(f(x, y, z) = 0)$$

gives a well-ordering of the set

$$\delta = \{x \mid (\exists y)(\exists z)(f(x, y, z) \cdot f(y, x, z) = 0)\}$$

with ordinal number α. He shows the equivalence of this definition to that of Church and Kleene [1, pp. 11–21].

Now if $g(x, y)$ is a recursive comparing function for a well-ordered recursive order type α, then α is a recursive ordinal number with $f(x, y, z) = g(y, x) + 0 \cdot z$ and $\delta = \epsilon$.

To show the converse, we note, after Markwald, that the set δ is r.e. Let $h(x)$ be a general recursive function enumerating δ without repetitions. Then for all x and y there exists a z such that

$$f(h(x), h(y), z) \cdot f(h(y), h(x), z) = 0$$

so that $f(h(x), h(y), z)$ gives a well-ordering of ϵ with ordinal number α. The recursive comparing function is

$$g(x, y) = (1 \dot{-} f(h(x), h(y), \mu z[f(h(x), h(y), z) \cdot f(h(y), h(x), z) = 0])) \cdot (1 \dot{-} (1 \dot{-} |x - y|)).$$

16. Recursively bounded sets(8).

DEFINITION. An infinite set α, with principal sequence produced by $h(x)$, is *recursively bounded* when there exists a recursive function $f(x)$ such that $h(x) \leq f(x)$ for all x. We denote the class of recursively bounded sets by B.

THEOREM 21. *$V - B$ is the class of hyperimmune sets.*

Proof. If $\alpha \in B$, with principal sequence produced by $h(x)$ and recursive bounding function $f(x)$, we construct a discrete array $\phi_0, \phi_1, \phi_2, \cdots$ (§14) such that no ϕ_i is a subset of α'. Define

$$g(0) = f(0), g(x + 1) = f(g(x) + 1).$$

Then

$$\phi_0 = \{0, 1, \cdots, g(0)\},$$
$$\phi_{i+1} = \{g(i) + 1, g(i) + 2, \cdots, g(i + 1)\},$$

with

(8) *Added in proof.* See U. T. Medvedev, Dokaldy Akademii Nauk SSSR. vol. 102 (1955) pp. 211–214.

$$a(0, x) = g(0) \dot{-} (g(0) \dot{-} x),$$
$$a(i+1, x) = g(i+1) \dot{-} ((g(i+1) \dot{-} g(i)) \dot{-} (x+1)),$$

and

$$b(0) = g(0), b(i+1) = g(i+1) \dot{-} g(i).$$

Clearly α contains at least one element of ϕ_0. In the set

$$\phi_0 \cup \phi_1 = \{0, 1, \cdots, f(f(0)+1)\}$$

α has at least $f(0)+1$ elements, not all of which can be in ϕ_0. So α contains at least one element of ϕ_1. In general, α has at least $g(k)+1$ elements in common with $\bigcup_{i=0}^{k+1} \phi_i$, at most $g(k)$ of which can be in $\bigcup_{i=0}^{k} \phi_i$. So no ϕ_{k+1} is included in α'.

On the other hand, suppose $\phi_0, \phi_1, \phi_2, \cdots$ is a discrete array, no set of which is included in α'. Then $a(x, b(x))$, giving the largest element of ϕ_x, is a recursive bounding function for the principal sequence of α.

We now show how every set of $V-B$ is derived in a simple way from the principal sequence of an order of V^*-B^*.

Definition. If $p(x)$ produces the principal sequence of α^*, we call $p(y)$ a *maximal value* of $p(x)$ when $p(x) < p(y)$ for all $x < y$.

Lemma. *Every infinite set α is the set of maximal values for some $p(x)$.*

Proof. Let $h(x)$ produce the principal sequence of α, and $g(x)$ the principal sequence of α'. Then if $p(0) = h(0)$, $p(h(x)+1) = h(x+1)$, $p(g(x)+1) = g(x)$, it is clear that $p(x)$ produces a permutation with α as its set of maximal values. For if $p(x) \in \alpha$, $x \leqq p(x)$, while if $p(x) \in \alpha'$, $p(x) < x$.

Theorem 22. *Let α be the set of maximal values of the function $p(x)$ producing the principal sequence of α^*. Then $\alpha^* \in B^*$ if and only if $\alpha \in B$.*

Proof. If $f(x)$ is a recursive function bounding the principal sequence of α, then $f(x)$ bounds the principal sequence of α^*.

Let $f(x)$ be a recursive function such that $p(x) \leqq f(x)$, and let $h(x)$ produce the principal sequence of α. Then from the definition of maximal value, $h(0) = p(0) \leqq f(0)$. Now $\mu y[h(0) < p(y)] \leqq h(0)+1$ since there are only $h(0)$ non-negative integers less than $h(0)$. So $p(x)$ must take on the value $h(1)$ for some $x \leqq h(0)+1$. Then $h(1) \leqq \max [f(0), f(1), \cdots, f(f(0)+1)]$. In general, $h(x) \leqq g(x)$, where

$$g(0) = f(0), g(x+1) = \max [f(0), f(1), \cdots, f(g(x)+1)].$$

Corollary. *Every set of $V-B$ (every hyperimmune set) is the set of maximal values of the principal sequence of an order of V^*-B^*, and conversely.*

From our present standpoint let us consider the method given by Dekker [3] for constructing hypersimple sets. He found that if $f(x)$ recursively enumerates a set of $F-E$, then the set β of all x's for which there exists a y

such that $x<y$ and $f(y)<f(x)$ is hypersimple. That is, β is r.e. and β' is hyperimmune.

Suppose $f(x)$ produces its sequence of values in an order α^*. Then Dekker's hypothesis gives $\alpha^* \in F^* - E^*$. It is not hard to see that β' is the set of maximal values of the principal sequence of $(\alpha^*)^{-1}$, which by Theorem 11, Corollary B, is not in B^*. Hence β' is hyperimmune. By Theorem 10, $\alpha^* \in C^*$, and this is enough to give the recursive enumerability of β, since β is the set of rows, in the table of the recursive comparing function of α^*, which have at least one 1 to the right of the main diagonal.

Finally we observe that the original function $f(x)$ is unnecessary in this argument, and we need only $\alpha^* \in C^* - E^*$ to say that the set of all x's which are not maximal values of $(\alpha^*)^{-1}$ is hypersimple. Since $F^* - E^* \subset C^* - E^*$, this is an extension of Dekker's result.

We can now provide the example promised in §11, of an order-transform of a set α which is not order-equivalent to α. Let $t(x)$ be a general recursive function which takes on every value infinitely often. If $f(x)$ produces the principal sequence of a set β of $V-B$ and $h(x)$ the principal sequence of any infinite set γ, we form $g(x)$:

$$g(0) = \mu y[(f(0) \dot{-} y) + |t(y) - h(0)| = 0],$$
$$g(x+1) = \mu y[(f(x+1) \dot{-} y) + (g(x) \dot{-} y) + |t(y) - h(x+1)| = 0].$$

$g(x)$ produces the principal sequence of a set α which, since $f(x) \leqq g(x)$, is in $V-B$. Moreover, since $t(g(x)) = h(x)$, $t(x)$ gives a recursive order-preserving transformation on α and $t(\alpha) = \gamma$. So all infinite sets of non-negative integers can be generated from the sets of $V-B$ by recursive order-preserving transformations. But if γ is recursive, for example, α and γ cannot be order-equivalent.

17. **Automorphisms of** V. Since an automorphism of the Boolean algebra V must transform the class of unit sets one-to-one onto itself, and is uniquely determined by this transformation, we can associate each such automorphism with a function $p(x)$ producing a permutation; the automorphism is $\alpha \rightarrow p(\alpha)$. Hence V^* is the group of automorphisms of V. This suggests problems in two directions. One may search for classes of sets closed under a certain class of automorphisms, or one may attack the apparently more difficult question of finding the class of all automorphisms under which a certain class of sets is closed. Along the first line, Dekker [2, p. 496]([9]) has observed some classes closed under automorphisms in E^*. We establish as a corollary to Theorem 23 a result of this kind for automorphisms in B^*.

THEOREM 23. *If the sequence produced by $f(x)$ is recursively bounded, then $f(x)$ enumerates a set of B in an order of B^*.*

([9]) We suggest replacing Dekker's term "recursively isomorphic" by "conjugate under a recursive automorphism," or "recursively conjugate."

Proof. We have observed (Theorem 8) that $f(x)$ bounds the principal sequence of the order in which it produces its values, so that a recursive bound for $f(x)$ serves equally for the principal sequence.

Suppose $h(x)$ produces the principal sequence of the set α enumerated by $f(x)$. If $g(x)$ is a recursive function such that $f(x) \leqq g(x)$, we have $h(x) \leqq \max [g(0), g(1), \cdots, g(x)]$, since $h(0) \leqq f(0)$ and for any x, not all the $x+1$ values $f(y)$ for $y \leqq x$ can be less than $h(x)$.

COROLLARY. *B is closed under automorphisms in B^*. For if $\alpha \in B$, $\alpha^* \in B^*$, $h(x)$ and $p(x)$ producing the principal sequences of α and α^* respectively, then $p(h(x))$ produces a bounded enumeration of $p(\alpha)$, by the argument of Theorem* 9.

However, this result is not a contribution along the second line mentioned above, for we can give an example of an automorphism not in B^* under which B is closed. For this we use the permutation constructed in the Lemma to Theorem 22. If $h_1(x)$ and $h_2(x)$ produce the principal sequences of α and α' respectively, then $p(0) = h_1(0)$, $p(h_1(x)+1) = h_1(x+1)$, $p(h_2(x)+1) = h_2(x)$.

We first show that for any number $a+1$, there exists at most one $y \leqq a+1$ for which $a+1 < p(y)$. The only candidates for such a $p(y)$ are the maximal values of $p(x)$ produced by arguments $z \leqq a+1$, since if $p(z)$ is not maximal, $p(z) = z-1 < a+1$. Suppose b is the largest number not exceeding $a+1$ for which $p(b)$ is maximal. If $b=0$, $p(0)$ is the only maximal value produced by an argument not exceeding $a+1$. If $0 < b \leqq a+1$, then the maximal value preceding $p(b)$ is $b-1 < a+1$, so for all $x \leqq a+1$, $x \neq b$, $p(x) < a+1$.

Now suppose $g_1(x)$ and $g_2(x)$ produce the principal sequences of a set β and its transform $p(\beta)$ respectively. There are $n+2$ elements of β not exceeding $g_1(n+1)$, and since for at most one $m \leqq n+1$, $g_1(n+1) < p(g_1(m))$, there are at least $n+1$ elements of $p(\beta)$ not exceeding $g_1(n+1)$. I.e., $g_2(n) \leqq g_1(n+1)$. Hence if a recursive function $f(x)$ bounds $g_1(x)$, $f(x+1)$ bounds $g_2(x)$. This holds whether or not the set α of maximal values of $p(x)$ is in B.

18. **Computable groups.** As a condition for the "effectiveness" of an infinite group $\mathfrak{G}$, isomorphism with a subgroup of E^* suggests itself. This condition appears to be extremely weak, as evidenced by the fact that E (under symmetric difference), and hence every one of its subgroups, is isomorphic to a subgroup of E^*. For to the recursive set α, consisting of the numbers a_i, we can let correspond the recursive order whose principal sequence is the permutation consisting of the transpositions $(2a_i, 2a_i+1)$. Symmetric difference of sets corresponds to iteration of the corresponding permutations.

A probably stronger condition is computability.

DEFINITION. If there exists such a one-to-one correspondence between the elements of an infinite group $\mathfrak{G}$ and the non-negative integers that a recursive function $g(x, y)$ exists, equal to the number corresponding to the product of the elements corresponding to x and y, then $\mathfrak{G}$ is *computable.*

THEOREM 24. *If G is computable, then G is isomorphic to a subgroup of E^*.*

Proof. By the group properties of G, $g(x, i)$ for fixed i produces the principal sequence of an order of E^*. This is, of course, simply the Cayley representation of a group as a group of permutations(10), and the set of orders obtained in this way forms a subgroup G^* of E^*, isomorphic to $\mathfrak{G}$.

Although we have no counterexample to offer, it seems highly improbable that the converse of this theorem is true. In the discussion below, we develop some sufficient conditions for computability, and see that Q^* and Q are computable.

Suppose there exists a one-to-one correspondence between the elements of a group G and sets S of finite "words" (sequences of symbols) on the alphabet $0, 1, 2, \cdots$ (or a recursive subset thereof), such that if w and v are words corresponding to elements a and b respectively, then wv (the symbols of w followed by the symbols of v) corresponds to the product ab.

EXAMPLE 1. A group $\mathfrak{G}$ with n generators $a_1, a_2, \cdots, a_n$. Let $a_i \to 2i-2$, $a_i^{-1} \to 2i-1$. Inversely, two integers j and k correspond to the same generator (or inverse of a generator) when $j \equiv k \pmod{2n}$.

EXAMPLE 2. Q under symmetric difference.

$$\alpha = \{a_1, a_2, \cdots, a_k\} \to a_1a_2 \cdots a_k = w \in S_\alpha.$$

$w = a_1a_2 \cdots a_k \in S_\alpha$, where α consists of those integers occurring an odd number of times in w.

EXAMPLE 3. Q^*. Let α^* be the order whose principal sequence moves only the finite cycle $(a_1a_2 \cdots a_k)$, with $a_1 \leqq a_i$. If $a_1 \neq 0$,

$$(a_1a_2 \cdots a_k) = (0a_1)(0a_2) \cdots (0a_k)(0a_1) \to a_1a_2 \cdots a_ka_1 = w \in S_{\alpha^*};$$

if $a_1 = 0$, $(0a_2 \cdots a_k) = (0a_2) \cdots (0a_k) \to a_2 \cdots a_k = w \in S_{\alpha^*}$.

$w = a_1a_2 \cdots a_k \in S_{\alpha^*}$, where the principal sequence of α^* is the permutation given by the product of the transpositions $(0a_1)(0a_2) \cdots (0a_k)$.

In a well-known manner we can set up an effective one-to-one correspondence between words and non-negative integers, using the functions $J(x, y)$, $K(x)$ and $L(x)$(6). If $w = a_1a_2 \cdots a_k \to n$, denote it by w_n. The numbering gives a recursive function $g_1(x, y)$ such that $w_xw_y = w_{g(x,y)}$.

Now suppose there exists a recursive function $f(x, y)$ which equals 1 when w_x and w_y are in the same correspondence class S, and 0 otherwise. Such a function can be constructed for Examples 2 and 3, since this amounts merely to the formalization of obviously effective procedures. In Example 1 the situation is more complicated, and may depend on the structure of the group.

The recursive function $f(x, y)$ enables us to select a single x (the smallest) from each $\{x \mid w_x \in S\}$. If $h(x)$ is defined by

$$h(0) = 0,\ h(x+1) = \mu y\left[((h(x) + 1) \dot{-} y) + \sum_{z=0}^{y-1} f(z, y) = 0\right],$$

(10) See, e.g., Birkhoff and MacLane, *Survey of modern algebra*, rev. ed., New York, 1953, p. 132, Theorem 8.

then let x correspond to the set S containing $w_{h(x)}$ and corresponding to the element a. This correspondence is one-to-one between the elements of $\mathfrak{G}$ and the non-negative integers, and we have a recursive function $g_2(x, y) = \mu z[f(h(z), g_1(h(x), h(y))) = 1]$ which gives the integer corresponding to the element of G which is the product of the elements corresponding to x and y. Hence G is computable.

In this approach to computability, there are two stages at which the process may fail. First, the representation as words may not be possible; no way to do this for E^* or for E under symmetric difference is known. If this difficulty is overcome, it may still be impossible to obtain the recursive function $f(x, y)$; this would be the case with any possible noncomputable groups of the type of Example 1.

19. **Conclusion.** The original motivation for the investigations reported in this paper was a desire to gain insight into the nature of recursive functions (as mathematical rather than as metamathematical objects) by a study of what they do. One of the principal things which they do is to produce sequences. Because of their metamathematical importance, the sets enumerated by such sequences have come to form the basis of a well-developed field of study, but other aspects of the behavior of sequences have received only incidental attention. It was hoped that a systematic investigation of these other aspects might add to the basic understanding of recursive functions, as well as perhaps have applications to the classical problems in the theory of r.e. sets.

Indeed, this may yet come about, when some of the questions which we have had to leave open are answered. And although a complete theory of recursive functions and their sequences will have to include an analysis of repetition properties, both set and order theory will undoubtedly supply essential tools for this work.

References

1. Alonzo Church and S. C. Kleene, *Formal definitions in the theory of ordinal numbers*, Fund. Math. vol. 28 (1936) pp. 11–21.

2. J. C. E. Dekker, *Two notes on recursively enumerable sets*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 495–501.

3. ———, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 791–796.

4. S. C. Kleene, *Recursive predicates and quantifiers*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 41–73.

5. ———, *Introduction to metamathematics*, New York, Amsterdam and Gröningen, 1952.

6. W. Markwald, *Zur Theorie der konstruktiven Wohlordnungen*, Math. Ann. vol. 127 (1954) pp. 135–149.

7. E. L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316.

University of New Hampshire,
Durham, N. H.