THE CRITICAL POINTS OF PEANO-INTERIOR FUNCTIONS DEFINED ON 2-MANIFOLDS(1)

BY

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INTRODUCTION

It is common practice to interpret the level curve family of a harmonic function as the family of stream lines of a certain flow. Such an interpretation leads one to attribute the critical points of the harmonic function to the presence of various components in the boundary of the domain of harmonicity and to the manner in which the various boundary curves "merge" with the level curve family. The first discovery of a quantitative relation between the number of critical points and the connectivity of the domain of harmonicity appears to have been made by Felix Klein in 1882. On page 39 of his book [1](2), he indicates an argument in support of the fact tht if a function on a compact Riemann surface (without boundary) of genus g is harmonic everywhere except for n logarithmic poles then the sum of the multiplicities of the critical points is precisely $2g+n-2$. The level curve family of such a function is geometrically identical with that of a harmonic function with constant boundary values on a compact surface (with boundary) of genus g and n boundary curves. One notes that $2g+n-1$ is the first Betti number, p_1, of the surface so that the sum of the multiplicities of the critical points is simply p_1-1 in the case of constant boundary values. This discovery by Klein seems to have gone unnoticed. For example, Nevanlinna [2, 1936] computes the sum of the multiplicities of the critical points of the harmonic measure h by using the argument principle on the derivative of an analytic function whose real part is h. Also J. L. Walsh [3, 1946] proves the Klein relation for the case $g=0$ in the course of studying the location of the critical points of a harmonic function on a plane domain.

The first complete proof of the Klein relation for harmonic functions is contained in the general critical point theory of Marston Morse as developed in 1934 [4; 5, Theorem 1.4, p. 145]. However, no specific mention of this corollary (the Klein relation) is made there. Within a few years Morse's interest in the special cases of harmonic and pseudo-harmonic functions quickened and his collaboration with M. Heins produced in 1945 and 1946

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(2) Numbers in square brackets refer to the bibliography at the end of the paper.

the basic papers [6; 7; 8]. These, together with his own work, formed the basis of Morse's book [9], which is devoted primarily to the theory and applications of pseudo-harmonic functions on plane domains. Here, Klein's relation ($g=0$) for pseudoharmonic functions is first established. Then, in order to shift the study of light-interior transformations f to the pseudo-harmonic functions $\log |f|$, Morse seeks to relax the requirements of constant boundary values by the imposition of various conditions both geometric (only finitely many points bearing local maximum or minimum values allowed on the boundary) and analytic (the boundary values may be extended to be C' in a neighborhood of the boundary) in nature. In these cases, and others, Morse obtains connectivity-multiplicity relations after introducing multiplicities for boundary points as well as for interior points.

The present paper is concerned with those real continuous functions on 2-manifolds which are interior (i.e., send open sets onto open sets) and the components of whose level curves are locally finite linear graphs. This class contains (properly) the class of pseudo-harmonic functions. In Chapter I the class of functions under consideration is shown to consist of those real valued interior mappings whose level curves are locally connected (and hence are necessarily Peano spaces, whence the name Peano-interior). In Chapter II sufficient conditions are obtained that the number of critical points be finite and, in Chapter III, the Klein relation is obtained for plane domains. After a certain combinatorial formula is developed in Chapter IV, the Klein formula is established in Chapter V for Peano-interior functions (with constant boundary values) on compact metric orientable 2-manifolds (with boundary). The case of more general boundary values will be treated in a second paper whose results are now summarized. General sufficient conditions of a geometric nature are stated which, imposed on the boundary values, imply that the given Peano-interior function is the restriction, to a sub-manifold, of another Peano-interior function whose boundary values are constant. The containing surface is fabricated from the given one by addition of "cuffs" (perhaps with holes) to each boundary curve. The sum of the multiplicities of those critical points which *are not* interior points of the original surface is predicted exactly. Since the over-all sum is given by Klein's relation, the sum of the multiplicities of those critical points which *are* interior points of the original surface may be computed. Moreover, the extended function is pseudo-harmonic provided that the given one is pseudo-harmonic and that the hypotheses on the boundary values are "uniformized." These hypotheses contain all the various boundary values considered by Morse in [9] to obtain his connectivity-multiplicity relations.

In what follows, the convention is adopted that words or phrases being defined are to be italicized. Inter-chapter references will be preceded by the (Roman numeral) number of the chapter, e.g. II: 2.4. Intra-chapter references will consist only of the identifying number, e.g., 2.4.

Chapter I. The local level curve structure of Peano-interior functions

1. Definitions, notation, and examples. A space M is a *2-manifold with boundary* if M is a connected Hausdorff space every point of which is a point of a set, open in M, whose closure is homeomorphic with the closed unit disk, i.e., a closed 2-cell. An *interior point* of M is a point which is an interior point of such a 2-cell neighborhood. The set of interior points of M is called the *interior of* M. Points of M which are not interior points are called *boundary points*. If M has no boundary points then M is simply a *2-manifold* or equivalently a *2-manifold without boundary*. Unfortunately, every 2-manifold without boundary is also a 2-manifold with boundary.

A mapping $f: A \rightarrow B$ is *interior* provided that $f(U)$ is open in B whenever U is open in A. If V is any subset of A and if p is any point of A for which $f(p)=c$ then the symbols V^p and V_c will be used to denote the set of all points x in V for which $f(x)=c$. Thus $V^p = V_c$ is the intersection with V of the point-inverse $f^{-1}(c)=f^{-1}(f(p))$. The notation V_c agrees with that of Morse in [9], and the notation V^p is used here to avoid the rather cumbersome alternative $V_{f(p)}$.

Let M be a 2-manifold with boundary. A real function $f: M \rightarrow E^1$ is *Peano-interior* if

a. f is continuous on M,

b. f is interior on the interior of M, and

c. each level curve M_c of f is locally connected at those of its points which are interior points of M.

Briefly a Peano-interior function is a real interior mapping whose level curves are Peano-spaces. The study of Peano-interior functions was suggested to the author by G. S. Young, Jr., to whom the author is indebted for insight and encouragement. This study is motivated by the expectation that the topological properties of their level curve systems and of their critical points and their multiplicities will at least reflect those of pseudo-harmonic functions. A *pseudo-harmonic function* is defined by Morse [9] as a continuous real function $f: M \rightarrow E^1$ such that for every interior point p of M there is a homeomorphism g of the open unit disk into M with the properties that $g(0, 0)=p$ and that the composition fg is harmonic on the unit disk. The connection between pseudo-harmonic functions and Peano-interior functions is made more evident by a theorem of Tôki [10], which characterizes the class of functions pseudo-harmonic on M as the class of all real functions $f: M \rightarrow E^1$ with the properties that

a′. f is continuous on M,

b′. f is interior on the interior of M,

c′. for each interior point p of M (with the exception of a discontinuum) and each neighborhood U of p, there is a neighborhood V of p for which

$\bar{V} \subset U$ and such that V^x is contained in a component of U^x whenever x is a point of V.

If this last condition is relaxed to apply only to the point $x=p$, then it becomes simply the requirement that the level curves be locally connected at every interior point of M with the exception of a discontinuum. Moreover, reference to this exceptional set may now be deleted in view of the fact, proved as Theorem 12.3 of Chapter I in [11], that the set of points at which a locally compact connected metric space fails to be locally connected must contain a nondegenerate continuum. Thus every pseudo-harmonic function is a Peano-interior function. There exist, however, Peano-interior functions which are not pseudo-harmonic, as in the following example.

Let A_n be the topological open interval consisting of the straight line segment between the points $(1/(2n-1), 0)$ and $(1/2n, 1)$ together with the vertical half lines $x=1/(2n-1)$, $y>0$ and $x=1/2n$, $1>y$. Let I_n be the strip bounded by A_n and A_{n+1}. Let $f_n: I_n \to E^1$ be the harmonic function which takes the value $1/(2n-1)$ on A_n and $1/(2n+1)$ on A_{n+1}. Each level curve of f_n is a topological open interval separating I_n. Let $f: E^2 \to E^1$ be defined as follows:

$$f(x, y) = f_n(x, y) \text{ if } (x, y) \text{ is in } I_n \text{ for } n > 0,$$

and

$$f(x, y) = x \quad \text{for all other points } (x, y).$$

The function $f: E^2 \to E^1$ is Peano-interior because it is interior and each level curve is a topological open interval. However it is not pseudo-harmonic for at every point of the interval extending from the point $(0, 1)$ to the origin it fails to satisfy Tôki's condition c′.

Examples of Peano-interior functions which illustrate the possible behavior of the level curve families at the boundary may be constructed by restricting the domain of definition of simple functions whose level curve families are known (for example, the absolute value function) to a domain whose boundary is wild. An instance of such an example will be discussed in a subsequent article.

The first step in the study of Peano-interior functions is to show that they share with harmonic functions the classical property that each level curve M^p consists locally of an even number of arcs or spokes, disjoint from each other except for one common end point, which separate a small disk into sectors so that adjacent sectors bear functional values which are greater than $f(p)$ on one and less than $f(p)$ on the other. It is convenient to refer to a proposition asserting that a certain class of functions has this property as a *spoke theorem*; the primary purpose of this chapter is to prove a spoke theorem (Theorem 5.1) for Peano-interior functions at interior points of M. Thus the class of Peano-interior functions will be characterized as the class of real continuous functions on M for which a spoke theorem is true at every interior point of M.

2. **End points of level curves.** A *continuum* is a compact, connected set. A *generalized continuum* is a locally compact, connected set. A point p of a topological space K is an *end point of* K if there exist arbitrarily small neighborhoods of p, each with the property that its boundary consists of a single point. In order to show that level curves of Peano-interior functions $f: M \rightarrow E^1$ have no end points except possibly in the boundary of M, it is necessary to know a characteristic property of end points of locally connected generalized continua such as the level curves M^p. The next theorem has probably appeared in the literature, but no reference is known to the author. The proof uses the notion of a point being a *cut-point of a set* A, i.e., the complement in A of the point is not connected.

THEOREM 2.1. *Let K be a closed locally connected generalized continuum in the plane which contains no simple closed curve nor an open set of the plane. If p is an end point of K then there exists an arbitrarily small Jordan domain D which contains p and has the properties that K meets the boundary of D in exactly one point and that $D-K$ is connected.*

Proof. Let U be a bounded open set in the plane containing p and let K_i, $i=1, 2, 3$, be connected neighborhoods in K of p such that(3) $\mathrm{Cl}\ (K_i) \subset K_{i+1}$ and $\mathrm{Cl}\ K_3 \subset U$ where the boundary of K_i consists of the single point q_i. Thus q_i is a cut point of K_{i+1}. Because K is closed, it follows that $\mathrm{Cl}\ (K_i) = K_i \cup (q_i)$ which implies that $\mathrm{Cl}\ (K_i)$ is locally connected. This is a consequence of the fact already used in §1 that a generalized metric continuum cannot fail to be locally connected only at the points of a discontinuum. Therefore, $\mathrm{Cl}\ (K_i)$ does not separate the plane since every locally connected continuum which separates the plane must contain a Jordan curve [11, 2.51, Chapter VI]. From this it follows that D^*, the complement in the 2-sphere (i.e., the plane compactified by the addition of one point) of $\mathrm{Cl}\ (K_3)$, is a domain; in addition K contains no open set of the plane since K contains no simple closed curve so that the boundary of D^* is exactly equal to $\mathrm{Cl}\ (K_3)$. Thus q_2 is a cut point of the boundary of a domain in the 2-sphere, and a result of Wilder's [12, 6.9, Chapter IV] may be applied to yield the existence of a Jordan curve $J_2 \subset D^* \cup (q_2)$ which separates K_2 from $\mathrm{Cl}\ (K_3) - \mathrm{Cl}\ (K_2)$. Let D_2 be the complementary domain of J_2 which contains K_2. By the same argument there is a Jordan curve $J_1 \subset D_2 \cup (q_1)$ which separates K_1 from $\mathrm{Cl}\ (K_2) - \mathrm{Cl}\ (K_1)$ in D_2. Let D_1 be the bounded complementary domain of J_1; $D_1 \subset D_2$. The domain D_1 may contain points of K other than points in K_1; if so, let V be a neighborhood of q_1 whose closure does not meet $K-K_2$ and such that $V \cap J_1$ is connected. There exists a Jordan curve L in $D_1 \cup V$ which separates $\mathrm{Cl}\ (K_1)$ from $K - \mathrm{Cl}\ (K_2)$ in D_1. Let w be a point, if any, in $(K - \mathrm{Cl}\ (K_2)) \cap D_1$; by construction, that component L' of $L \cap D_1$ which separates K_1 from w also separates K_1 from $K-K_2$. Of course, L' is a cross cut of

(3) If K is any set then $\mathrm{Cl}(K)$ denotes the closure of K.

D_1; hence there is an arc in J_1 which contains q_1 and whose union with L' is a simple closed curve J which separates K from $K - \mathrm{Cl}\ (K_2)$. But, since $J \subset \mathrm{Cl}\ (D_1)$, if follows that J separates K_1 from $K_2 - \mathrm{Cl}\ (K_1)$ also. Therefore if D is the bounded complementary domain of J then $D \cap K = K_1$ and $J \cap K = q_1$.

It remains to prove that K does not separate D. As was shown above, $\mathrm{Cl}\ (K_1)$ does not separate the plane. Let x and y be two points in $D - K_1$ and let A be an arc from x to y in the complement of $\mathrm{Cl}\ (K_1)$. It is possible to find subarcs A_x and A_y of $A \cap \overline{D}$ which extend in D from x to a point u of J and from y to a point v of J. Let T be the component of $J - ((u) \cup (v))$ which does not contain q_1; $A_x \cup T \cup A_y$ is an arc in the complement of the compact set $\mathrm{Cl}\ (K_1)$. It follows that T can be replaced by an arc in D which also connects A_x with A_y and does not meet $\mathrm{Cl}\ (K_1)$. Thus x and y are not separated in $D - K_1$.

LEMMA 2.2. *Let M be a 2-manifold without boundary and let $f: M \rightarrow E^1$ be a Peano-interior function. For each point p of M let $A_p(B_p)$ be the set of points x for which $f(x)$ is larger (smaller) than $f(p)$. Then A_p and B_p are not empty and are open, so that $M - M^p$ is not connected.*

Proof. Because f is interior, $f(M)$ is an open interval containing $f(p)$, so that A_p and B_p are not empty. Moreover both these sets are inverse images of open sets so that each is open in M by the continuity of f.

THEOREM 2.3. *Let M be a 2-manifold without boundary, let p be a point of M and let $f: M \rightarrow E^1$ be a Peano-interior function. Then p is not an end point of the component of M^p which contains p.*

Proof. Let K be the component of M^p which contains p. The level curve M^p is locally connected at p so that there are arbitrarily small two dimensional Euclidean neighborhoods U of p such that $U \cap M^p$ is connected. Let V be the component of U which contains p; V is a domain. Because the connected set $U \cap M^p$ is a subset of V, it follows that $U \cap M^p = V \cap M^p$. Thus the set $V \cap M^p$ is both closed in V and is a locally connected, generalized continuum contained in K. In addition, $V \cap M^p$ does not contain a simple closed curve, for if it did it would be possible to find a component of $V - M^p$ whose boundary is included in M^p. This would contradict the interiority of the function f. The interiority of f also implies that M^p contains no open set of M. It remains to show only that p is not an end point of $V \cap M^p$. If p is an end point of $V \cap M^p$, then 2.1 applied to the point p contradicts 2.2 applied to the Peano-interior function f restricted to the 2-manifold V.

3. **Separation of 2-cells by continua.** The proof of the spoke theorem is initiated by studying how the level curve M^p separates certain neighborhoods of p. It is complicated by the fact that p will be permitted to be either an interior point or a boundary point of the manifold.

The significance of the following lemma lies in its corollary which will be applied to show that some neighborhoods of p whose intersections with M^p are connected may be taken to be simply connected domains. Throughout, the phrase "a domain of x" is used to mean a connected open subset of x.

LEMMA 3.1. *Let E be the closed unit disk with boundary circle L, and let D be a domain of E such that $D\cap L$ is connected. If B is the component of $E-D$ which contains $L-D$, then $E-B$ is a simply connected domain of E.*

Proof. The set D is open in E so that B, as a component of the closed set $E-D$, is itself closed and $E-B$ is open. To show that $E-B$ is connected one first notes that $E-B$ is the union of D together with all components, different from B, of $E-D$. Thus it is sufficient to find a connected subset of $E-B$ which meets both D and each component of $E-D$, except for B. Such a subset is $\overline{D}-B$ which is certainly connected for it is contained between D and $\overline{D}$. Evidently $\overline{D}-B$ meets D. If Q is any component, except B, of $E-D$, then $\overline{Q}$ does not meet L since $L\subset D\cup B$. Therefore (by the Zoretti theorem [11, p. 109]) there exists a simple closed curve C, arbitrarily close to Q, which does not meet the compact set Cl $(E-D)$ of which Q is a component, i.e., C meets D. Therefore Q contains a limit point of $\overline{D}$. Since Q is disjoint from B, one can conclude that Q meets $\overline{D}-B$, which set therefore has the properties claimed for it so that the connectedness of $E-B$ is proved.

Finally, it must be shown that $E-B$ is simply connected, i.e., if J is a Jordan curve in $E-B$ and A is the bounded complementary domain (in the plane) of J, then A is a subset of $E-B$. Since B contains $E-(L\cap(E-B))$, it therefore contains points of the plane exterior to J. Because J is given to be disjoint from B, it follows that the connected set B is contained in the exterior of J, so that $A\cap B$ is empty. This means that $A\subset E-B$, which was to be proved.

COROLLARY 3.2. *If K is an arcwise connected set in the plane such that $K\cap D$ is connected and not empty, then $K\cap(E-B)$ is also connected.*

Proof. Let p be a point of $K\cap D$ and x a point of $K\cap(E-B)$. There is an arc R in K from x to p. If R is in $E-B$, the proposition is proved. If not, then in the order from x to p there exists a first point q in $R\cap(\overline{D}-D)$. Let T be the subarc of R from x to q, not including q. It will be sufficient to show that $T\cap D$ is not empty, for then T will connect x to a point of the component of $K\cap(E-B)$ which contains p. If $T\cap D$ is empty, then T is a connected subset of $E-D$ and so is contained in that component Q of $E-D$ which contains x. Because Q is closed, it follows that q is in Q; in addition q is in $\overline{D}-D$ which is a subset of B. Therefore $Q=B$, so that x is in B. This contradicts the original choice of x as a point of $K\cap(E-B)$.

These two results will not be used until the next section where they serve to verify that the main theorem of this section can be applied there.

The proof of the theorem to follow uses the notion of a contracting family. The components in the metric space B of a set $B-A$ constitute a contracting family provided that for each positive number ϵ there are no more than a finite number of sets in the family of diameter greater than ϵ. A theorem due to Schoenflies [12, Theorem 7.7, Chapter IV] asserts in part that the complementary domains in the 2-sphere of a locally connected continuum form a contracting family. Thus the complementary domains in the plane of a locally connected closed generalized continuum K also form a contracting family, for if K is not compact, this class of sets is exactly the class of complementary domains in the 2-sphere of $\overline{K}$ since $\overline{K}$ coincides with K plus the point at infinity. Schoenflies' theorem applies here because $\overline{K}$ is necessarily locally connected. If not, then $\overline{K}$ fails to be locally connected at exactly one point, which is impossible for a continuum.

THEOREM 3.3. *Let E be the closed unit disk with boundary curve L, let D be a simply connected domain of E, let K be a locally connected continuum in E, and let p be a point of $D\cap K$. Further, suppose that* (a) *$D\cap L$ is a proper connected subset of L, and is nonempty only if p is in L,* (b) *$K\cap D$ is connected,* and (c) *the number of components of $E-(K\cup L)$ whose boundaries are included in $(K\cup L)\cap D$ is finite. Then there is a set V open in D, containing p, which is disjoint from all but a finite number of the components of $D-(K\cup L)$.*

Proof. If p is in L then $D\cap L$ may be identified with an open arc in the boundary of a topological open disk D^*. Under this identification $D'=D\cup D^*$ is homeomorphic with the plane and $K'=(K\cup L)\cap D$ is a closed locally connected generalized continuum in D'. If p is not in $D\cap L$ then by hypothesis $D\cap L$ is empty so that D is already homeomorphic with the plane, and of course $D\cap K$ is a closed locally connected generalized continuum. Thus in either event the components of $D-(K\cup L)$ form a contracting family. Let U and V be open sets in D containing p such that $\overline{V}\subset U$ and $\overline{U}\subset D$. Then the number of components of $D-(K\cup L)$ which meet V and also meet $D-\overline{U}$ is finite since these components constitute a contracting family in view of the fact that the diameter of each exceeds the distance from V to $D-\overline{U}$. But the number of components of $D-(K\cup L)$ which meet V and do not meet $D-\overline{U}$ is also finite because such a component is included in U and so is a component of $E-(K\cup L)$, whose boundary is necessarily contained in $D\cap(K\cup L)$. By hypothesis, the number of such components is finite. Therefore, the number of components of $D-(K\cup L)$ which meet V is finite.

4. **Canonical neighborhoods.** The proof of the spoke theorem consists first of the construction of a certain canonical neighborhood of the given point. The results of the previous section are instrumental in this construction. So is the following theorem.

THEOREM 4.1. *If R and T are locally connected generalized metric continua* *and S is open with a compact closure such that $\overline{R}\subset S$ and $\overline{S}\subset T$, then there exists a locally connected continuum K such that $\overline{R}\subset K\subset S$.*

Proof. A theorem due to Wilder [12, 3.3 in Chapter III] states that each point in a locally connected metric generalized continuum has arbitrarily small connected, uniformly locally connected neighborhoods. Theorem 3.6 in Chapter III of [12] asserts that the closure of any uniformly locally connected set is locally connected. Therefore every point p of $\overline{R}$ is contained in a connected open set U whose compact closure is locally connected and contained in S. Then $\overline{R}$ is covered by the union, K of a finite number of such sets.

The notion of a dendrite snd some of its elementary properties are used frequently in the rest of this chapter. The following summary is taken from Whyburn [11]. A *dendrite* is a locally connected metric continuum which contains no simple closed curve. If K is a dendrite, each point of K is either a cut point or an end point of K. In addition, not only is every connected subset of K necessarily arcwise connected, but between every two points of K there exists exactly one arc of K.

The spoke theorem will be formulated for certain boundary points of the manifold as well as for all interior points. Let $f: M^*\rightarrow E^1$ be a Peano-interior function, let M be the interior of M^*, and let p be a point in the boundary of M^*. The point p is said to be f-normal if (a) there is a boundary arc A containing p as an interior point such that M^{*p} is locally connected at every point of $A\cap M^{*p}$ and (b) the number of components of $M-M^p$ whose closures meet A is finite.

THEOREM 4.2. *Let M^* be a 2-manifold with boundary whose interior is M and let $f: M^*\rightarrow E^1$ be a Peano-interior function. Let p be a point of M or an f-normal point of M^*-M. Then there exists an arbitrarily small simply connected domain D of M^* containing p such that* (a) *$\overline{D}$ is compact,* (b) *$\overline{D}\cap M^{*p}$ is a dendrite none of whose end points is in $D\cap M$,* (c) *there is a set V, open in M^*, containing p which is disjoint from all but a finite number of the components of $D-M^{*p}$, and* (d) *$D\cap(M^*-M)$ is connected.*

Proof. Let U be an open set of M^* containing p such that $U\cap M^{*p}$ is locally connected and connected. Let E be the closure of a 2-cell neighborhood of p in U, let the Jordan curve L be the boundary circle of E, and suppose E is chosen so that the set $L\cap(M^*-M)$ is an arc. Let I denote this arc less its end points. Also let $E'=(E-L)\cup I$; E' is now a domain in M^*. Because $T=U\cap M^{*p}$ is locally connected there exists a domain D in M^* containing p such that the compact set $\overline{D}$ is included in E' and such that $R=D\cap M^{*p}$ is connected. By 3.1, D may be taken to be simply connected, and, by 3.2, R remains connected; in addition $E-\overline{D}$ is now connected and it remains true that $\overline{D}\subset E'$. Then $\overline{R}\subset E'\cap M^{*p}=S$, an open subset of T. Therefore by 4.1 there is a locally connected continuum K such that $\overline{R}=\mathrm{Cl}\ (D\cap M^{*p})\subset K \subset E'\cap M^{*p}=S$. The set K is a dendrite, because if K contains a Jordan curve then a domain of E' may be found whose boundary is in K and hence in M^{*p}. This will contradict the interiority of f restricted to M. Thus Cl $(D\cap K)$ $=$ Cl $(D\cap M^{*p})$ is a subcontinuum of a dendrite and so is a dendrite itself. By 2.3, Cl $(D\cap K)$ has no end points in $D\cap M$, which establishes conclusion (b).

To show that $D\cap L$ is connected, one notes that if $D\cap L$ is not connected then $L-D\cap L$ is not connected and there is a cross-cut of E which lies entirely in $E-\overline{D}$, since that set is a domain in E. This cross-cut must separate E into two components each of which contains at least one of the components of $D\cap L$, which would contradict the connectedness of D. Moreover this implies that $D\cap L=D\cap(M^*-M)$ as follows. First, D cannot contain an end point of I because D is open in both E and M^*. Therefore the connected set $D\cap L$ is a subset of I, so that $D\cap L\subset D\cap(M^*-M)$. It is equally simple to show the reverse inclusion by verifying that $D\cap(M^*-M)\subset D\cap\bar{I}\subset D\cap L$. This establishes conclusion (d).

Conclusion (c) will follow as an application of 3.3 once hypothesis (c) of that theorem has been proved to hold in this case. Thus it remains to show that the boundaries of at most finitely many components of $E-$(Cl $(D\cap K)$ $\cup L$) are included entirely in $D\cap$(Cl $(D\cap K)\cup L$). There are two cases. If p is in the boundary of M^*, the fact that p is f-normal shows that this finiteness condition holds. On the other hand, if p is in M then $D\cap L=D\cap(M^*-M)$ is empty and no component of the open set (in M) $E-$(Cl $(D\cap K)\cup L$) has its boundary entirely in Cl $(D\cap K)\subset M^{*p}$, for otherwise the interiority of f on M would be contradicted.

5. **The spoke theorem.**

THEOREM 5.1. *Let M^* be a 2-manifold with boundary whose interior is M and let $f: M^*\to E^1$ be a Peano-interior function. Let p be a point of M^*; if p is in the boundary of M^* then suppose further that p is f-normal. Then either p is a component of M^{*p} or there exists an arbitrarily small simply connected domain N of M^* which contains p and whose closure is compact such that* (a) $(\overline{N}\cap M^{*p})-(p)$ *has a finite number of components,* (b) *if C is a component of $(\overline{N}\cap M^{*p})-(p)$ then $\overline{C}$ is an arc whose interior is in N, one of whose end points is p, the other end point being a point of the boundary of N,* (c) *$N-M^{*p}$ has a finite number of components each of which has p as a limit point,* (d) *if p is in M^*-M, then $N\cap(M^*-M)$ is an open arc such that each component of $(N\cap(M^*-M))-(p)$ is either in M^{*p} or is disjoint from M^{*p}, and* (e) *if p is in M then the number of spokes, or components of $(\overline{N}\cap(M^{*p}))-(p)$, is even.*

Of course, this means that a level curve in the interior of M is a locally finite linear graph.

Proof. Let D be the canonical domain of p given by 4.2, let K be the dendrite $\overline{D}\cap M^{*p}$, and let the components of $D-K=B$ of which p is a limit point be denoted as $B_1, \cdots, B_n$. Conclusion (c) of 4.2 implies that there are at most finitely many such components. Let G be the set of points in

$M^{*p} \cap D$ which are not limit points of the set $B^* = B - (\bigcup_i B_i)$. The set G is chosen this way so as to turn out to be the spokes of M^{*p} which separate the sets B_i from one another. Finally let $N = G \cup (\bigcup_i B_i)$. The proof that N has the properties claimed for it in the statement of this theorem will be broken up into a sequence of propositions.

(1) The point p is in N.

By conclusion (c) of 4.2, there is an open set V of M^* containing p which meets only those components of $D-K$ of which p is a limit point. Thus V does not meet B^* so that p is not a limit point of B^*. Hence p is in N.

(2) The set N is open in D and hence in M^*.

If a point x of D is a limit point of $D-N = (B^* \cup K) - G$, then either x is a limit point of $B^* - G$ or x is a limit point of $K-G$. If x is a limit point of B^*-G, then, by definition of N, x must be in $D-N$. If x is a limit point of $K-G$, then, by definition of G, x is the limit of a sequence of points each of which is a limit point of B^*. Then x itself is a limit point of B^* and so, as above, x must be in $D-N$. Therefore, $D-N$ is closed in D.

(3) The sets N and $(p) \cup (N-K) = (p) \cup (\bigcup_i B_i)$ are connected.

The set $(p) \cup (N-K)$ is connected because it is the union of connected sets $(B_i \cup (p))$ which have a point, p, in common. The set N is connected because it is included between the connected set $(p) \cup (N-K)$ and its closure.

(4) The domain N is simply connected.

Let J be a Jordan curve in N. Then $D-J$ has a component Q whose boundary is J and whose closure is compact and included in D. It is required to show that $Q \subset N$. Suppose that $Q \cap (D-N)$ is not empty; then also $Q \cap (D - \overline{N})$ is not empty, because Q is open in D. Since M^{*p} does not contain an open set of M, and hence of D, it follows that $Q \cap (D - (\overline{N} \cup K))$ is not empty. Let y be a point in that set and let U be the component of $B = D-K$ which contains y. Because y is in $U-N$, the definition of N implies that U is not one of the sets $\{B_i\}$, so that $U \cap N$ is empty. Therefore $U \cap J$ is also empty since $J \subset N$. Moreover U is a subset of Q, because U meets Q and does not meet J, the boundary of Q. Let T be the boundary of D; that is, $T = \overline{D} - (D \cap M)$. The definition of U implies that the boundary of U is included in the set $(T \cup K) \cap \overline{Q}$. However the boundary of U cannot be entirely contained in K, for this would contradict the interiority of f on U. Therefore, there is a point x in the boundary of U which is common to $\overline{Q}$ and T, and which is not in K. Such a point is then in $M^* - M$ because $\overline{Q} \cap T \subset (M^* - M)$ and hence x is in J also. This implies that x is already a point of U because $U \cup (x)$ is a connected set in $D-K$. Thus the original supposition that $Q \cap (D-N)$ is empty leads to a contradiction, for x is in J and U is disjoint from J.

(5) Each point of a component A of $(N \cap K) - (p)$ is a cut point of $\overline{A}$; if A meets the boundary of M^* then $\overline{A}$ is an arc which is either one of the two components of $D \cap (M^* - M) - (p)$ or it is the union of two arcs, one in M

from p to a point b of $D\cap(M^*-M)$ and the other a segment of $D\cap(M^*-M)$ from b to an end point of $D\cap(M^*-M)$.

For the set $\overline{A}$ is a continuum in the dendrite K and so it is a dendrite itself. Moreover, A is open in K, so that any end point which $\overline{A}$ may have in A is also an end point of K; but K has no end points in M. Therefore, if A does not meet the boundary of M^*, then every point of A is a cut point of $\overline{A}$ since every point of a dendrite is either a cut point or an end point. If A does meet the boundary of M^*, then either A is included in $D\cap(M^*-M)=I$, in which case A is an open arc since (by 4.2) I is an open arc, or A meets $N-I$. The set A must in the latter case meet exactly one component I^* of $I-(p)$, because if A had points in both components of $I-(p)$ then A would contain a cross-cut in $K-(p)$ of the simply connected domain N which would contradict the definition of N since every component of $N-K$ must have p as a limit point. Moreover, since A meets $N-I$, no component H of $A\cap I^*$ has p as an end point, for if this were to occur then an arc (from the other end point of H to a point of $\overline{A}$ in the boundary of N) in the dendrite $\overline{A}$ could be found which would be a cross-cut in $K-(p)$ of N, another contradiction. Let q be the end point of I^* which is different from p. If I^* is ordered from p to q, then there is a first point b of I^*, different from p, which is in A, so that the open arc A_1 in A from p to q is in $N-I$. If $A-A_1$ is empty, then A is an open arc, which was to be proved. If $A-A_1$ is not empty, then it must be a subset of I^*. If not, then a cross-cut of N can be constructed from a point of $A-A_1$ to a point of $\overline{A}$ in the boundary of N which, as before, leads to a contradiction. Thus A consists of A_1 plus a segment of I^* from b to q.

Note that this completes the argument concerning the component A when A meets the boundry of M^*. The cases when A does not meet the boundary of M^* are treated below.

(6) If the component A of $(N\cap K)-(p)$ does not meet the boundary of M^*, then $\overline{A}$ meets the boundary of N in at most one point.

For, let S be the boundary of N and let x and y be different points of $\overline{A}$. Because $\overline{A}$ is a dendrite, x and y are end points of an arc R in $\overline{A}$. If R is a subset of $\overline{A}\cap S$, then, since x and y are limit points of A, there are disjoint arcs X and Y in $\overline{A}$ which have x and y respectively as an end point, the other two end points, x' and y', being in A. Then $X\cup R\cup Y$ contains an arc R' from x' to y'; but A is arcwise connected so that there exists another arc in A from x' to y'. This contradicts the uniqueness of R' as an arc in the dendrite $\overline{A}$ from x' to y'. The other alternative is that R is not a subset of $\overline{A}\cap S$. In this event, R meets N, so that R must contain a cross-cut in $K-(p)$ of N which leads to another contradiction. Therefore, it must be the case that x and y are the same point.

(7) If A is a component of $N\cap K-(p)$ which does not meet the boundary of M^*, then A is an open arc with p as one end point and the other in the boundary of N.

By (5) and (6), $\bar{A}$ is a locally connected continuum with at most two non-cut points. This is the classical characterization of the arc.

(8) The set $K-(p)$ has a finite number of components.

Because each pair of components of $K-(p)$ forms a cross-cut of N, one sees that if there are infinitely many components in $K-(p)$ then $N-K$ would have infinitely many components also. But the components of $N-K$ are exactly the sets $B_1, \cdots, B_n$, which are finite in number.

It remains to check that if p is in M then the number of spokes or components of $N \cap K-(p)$ is even. This proposition follows from the facts that this number is also the number of sectors or components of $N-K$ and that the interiority of f on N requires that for each sector on which the f-values are less than $f(p)$ there be a sector on which the f-values are greater than $f(p)$. This in turn depends on the fact that each spoke is in the boundary of exactly two different sectors.

The fact that conclusion (5) in this proof does not quite accord with assertion (d) in the statement of this theorem can be adjusted by a slight modification in the domain N.

COROLLARY 5.2. *If $f: M \rightarrow E^1$ is a real function continuous on the 2-manifold (without boundary) M, then f is Peano-interior if and only if each component of each level curve of f is a locally finite linear graph.*

CHAPTER II. COMPLEMENTARY DOMAINS OF LEVEL CURVES AND THE NUMBER OF CRITICAL POINTS

1. **Definitions.** Let M be a 2-manifold without boundary and let $f: M \rightarrow E^1$ be Peano-interior. If x is a point of M, the *order of x relative to f* is the order of x in the locally finite linear graph M^x. (The *order of a point in a locally finite linear graph* is the number of different edges of the graph which have the point as an end point.) By the spoke theorem the order of x relative to f is an even positive integer. If the order of x relative to f is $2m$, the *multiplicity of x relative to f* is $m-1$; if the multiplicity of x relative to f is positive then x is called a *critical point of f*. The sum of the multiplicities of the critical points is evaluated in Chapter V. This chapter is concerned primarily with showing that, under suitable conditions, this sum is finite. The argument depends on properties characteristic of domains complementary to level curves. In what follows, if L is a sub-linear-graph of M^x the phrases "the order of x relative to f" and "the order of x in L" will in general denote different numbers and should not be confused.

Here, and also later, some use will be made of the rudiments of cyclic element theory as developed in [11]. A summary of the needed information is stated for the sake of completeness. Let K be a space; two points of K are *conjugate* if no point of K separates them from each other. *A cyclic element* of K is either a cut point, an end point, or a point p together with all other points q which are conjugate to p. In the third case, the cyclic element is called a *true cyclic element*. If K is a generalized continuum, its cyclic elements cover K. A set X is *cyclicly connected* if every two points of X lie on some Jordan curve in X. If K is a locally connected continuum, then every true cyclic element of K is cyclicly connected.

2. The number of level curves with critical points and their complementary domains.

LEMMA 2.1. *Let M be a compact 2-manifold with boundary and let $f: M \to E^1$ be Peano-interior. If M^x is a locally connected level curve then M^x contains only a finite number of components. Each component of M^x which does not meet the boundary of M contains at most a finite number of critical points.*

Proof. The set M^x is closed because it is a level curve and so it is compact. Because components of a locally connected space are open, it then follows that finitely many components of the compact locally connected space M^x cover M^x. If K is a component of M^x which does not meet the boundary of M, then K can be covered by sets U open in M such that each set U contains at most one critical point, by the spoke theorem. But K, a closed set, is compact and so is covered by a finite number of the sets U. Thus K contains at most a finite number of critical points.

THEOREM 2.2. *Let M be a compact 2-manifold with boundary and let $f: M \to E^1$ be Peano-interior. If M^x is a level curve which does not meet the boundary of M and if the order relative to f of x is $2m$, then there are m Jordan curves in M^x such that the intersection of any two of them has a finite number of components, one of which is the point x.*

Proof. Let K be the component of M^x which contains x and let L_1 be a component of $K-(x)$. The set $\mathrm{Cl}\,(L_1) = L_1 \cup (x)$ is then a finite linear graph by 2.1 and the spoke theorem, so that the number of points of odd order in $\mathrm{Cl}\,(L_1)$ is even. In view of the fact that each point of L_1 is of even order in $\mathrm{Cl}\,(L_1)$, it follows that x is also of even order in $\mathrm{Cl}\,(L_1)$. This implies that x is not an end point of $\mathrm{Cl}\,(L_1)$. Moreover x is not a cut point of $\mathrm{Cl}\,(L_1)$, since $\mathrm{Cl}\,(L_1)-(x)=L$, so that there is a true cyclic element C of $\mathrm{Cl}\,(L_1)$ containing x. Since $\mathrm{Cl}\,(L_1)$ is locally connected, C must be cyclicly connected so that a Jordan curve J_1 containing x can be found in C.

Each point of $\mathrm{Cl}\,(K-J_1)$ is of even order since the removal of J_1 decreased the order of each point by exactly two. Therefore, the same argument may be used again to construct a Jordan curve J_2 in $\mathrm{Cl}\,(K-J_1)$ which also contains x. By the spoke theorem every component of $J_1 \cap J_2$ either is a critical point or is an arc whose end points are critical points. Thus the number of components of $J_1 \cap J_2$ is finite, by 2.1. It is now clear how to continue the process until m such Jordan curves have been constructed.

COROLLARY 2.3. *Let M be a compact 2-manifold with boundary and $f: M \to E^1$ a Peano-interior function. If K is a component of a level curve, if K does* *not meet the boundary, and if K contains no critical points, then K is a Jordan curve.*

Proof. By 2.2, K contains a Jordan curve C. If K contains a point p not in this Jordan curve, then there is an arc A in K from the point p to a point q in C. In the order from p to q, the first point of $A \cap C$ is necessarily a critical point, by the spoke theorem.

This result shows that if D is a complementary domain of a level curve which contains no critical points, then $\bar{D}$ is a 2-manifold with boundary. However, almost the full force of this statement is true whether or not $\bar{D}-D$ contains critical points, as in the next theorem. Of course, for domains D of genus zero this property follows directly from the classical characterization of plane domains in terms of the number of components in their boundary.

THEOREM 2.4. *Let M be a compact 2-manifold with boundary and let $f: M \to E^1$ be Peano-interior. Suppose that M^p is a level curve of f and that D is a component of $M-M^p$ with the property that D has a finite number n of boundary components each of which is either a boundary circle of M or is in M^p and does not meet the boundary of M. Then the interior of D is homeomorphic with the interior of a compact metric 2-manifold with n boundary curves.*

Proof. The first step of the proof is to construct a set of n pairwise disjoint Jordan curves $J_1, J_2, \cdots, J_n$ in D^*, the interior of D, whose union $J=\bigcup_i J_i$ separates D^* into $n+1$ components $D_1, D_2, \cdots, D_{n+1}$ with the properties that (a) for i less than $n+1$, the boundary of D_i consists of two components, one being the curve J_i and the other being a boundary component of D, and (b) the boundary of D_{n+1} is the set J, so that $\mathrm{Cl}\,(D_{n+1}) \subset D^*$.

The construction of J_1 will be described for the boundary component A_1 of D. The other Jordan curves are constructed in the same way. If A_1 is a boundary curve of the compact manifold M or is a component of M^p which contains no points of order (in M^p) greater than two, then, since M is compact, A_1 is a Jordan curve. But every boundary curve of a compact 2-manifold (in this case, D) is a boundary curve of a topological annulus contained in the manifold. Let D_1 be this annulus and let its boundary curve in D^* be denoted as J_1. If, on the other hand, A_1 is in M^p (and is thus necessarily disjoint from the boundary of M) with some points of order greater than two, then by 2.1 and the spoke theorem it follows that A_1 is a one-dimensional complex whose vertices include each of the critical points of M^p. By an application of the spoke theorem to each vertex v, let one additional spoke be constructed from v to an interior point of each sector which is in D^*. Let each such additional spoke S be associated with those two edges of A_1 which meet the boundary of the sector containing S.

Let E be an edge of A_1; E is an arc which has been associated with exactly two of the additional spokes, say S_1 and S_2. One end point of E is also an end point of S_1, the other is an end point of S_2. Let E' be an arc in D^* joining the end points of S_1 and S_2 (which are in D^*) such that $E \cup S_1 \cup E' \cup S_2$ is a Jordan curve in $D^* \cup A_1$, and is the boundary of the Jordan domain Q in D^*. The arcs E', one for each edge E, can be constructed with their interiors disjoint from each other and from the additional spokes (such as S_1 and S_2). Therefore the union of the arcs E' forms a Jordan curve J_1 in D^*. The set $D^* - J_1$ has exactly one component D_1, the union of the domains Q, whose boundary in M is $A_1 \cup J_1$.

Now that the construction has been completed it will be shown that D_{n+1} is homeomorphic with D^*. Let Q be a Jordan domain in D^* bounded by $E \cup S_1 \cup S_2 \cup E'$ where E is an edge of a boundary component, S_1 and S_2 are two additional spokes, and E' is an arc in one of the curves $\{J_i\}$; Q and $Q \cup E' \cup D_{n+1}$ are homeomorphic. But D^* may be obtained by a finite number of such steps; thus D^* is homeomorphic with D_{n+1} whose closure Cl $(D_{n+1}) = D_{n+1} \cup J$ is a compact 2-manifold with exactly n boundary curves.

LEMMA 2.5. *If A and B are different sets each of which separates the space M into two domains with a common boundary, then $M-(A \cup B)$ has at least three components.*

Proof. Let the components of $M-A$ be Q and R and let the components of $M-B$ be S and T. At least one of the sets Q and R meets $B-A$. Suppose that $Q \cap (B-A)$ is not empty; this implies that Q meets both S and T. Therefore it is sufficient to obtain a nonempty set X such that X is separated from both $Q \cap S$ and $Q \cap T$ by $A \cup B$. If R meets $B-A$, let X be either $R \cap S$ or $R \cap T$. If R does not meet $B-A$, let X be R.

THEOREM 2.6. *Let M be a compact 2-manifold with boundary and let $f: M \rightarrow E^1$ be a Peano-interior function which is constant on the boundary curves on M. Then the number of level curve components which are contained in the interior of M and which contain critical points is finite.*

Proof. Let $\{K_i\}$ be the (indexed) set of all level curve components in the interior of M which contain critical points, and let c_i be a critical point in K_i. By 2.2, there are two Jordan curves ϵ_i and δ_i in K_i such that the point c_i is a component of $\epsilon_i \cap \delta_i$. The first step is to reduce the situation to that in which M is separated by each ϵ_i and by each δ_i, as follows.

Let $\{\sigma_1, \sigma_2, \cdots, \sigma_t\}$ be a finite subset of $\{\epsilon_i\}$ with the property that $\bigcup_i \sigma_i$ does not separate M but that its union with any other curve ϵ_i does separate M. Such a set may be constructed by successive inspection of each curve ϵ_i; the number t is necessarily not greater than the genus of M, so that t is finite. Let each curve δ_j for which ϵ_j is one of the chosen σ_i's be dropped from the collection $\{\delta_i\}$. The same process may be applied to this collection to augment the set $\{\sigma_1, \sigma_2, \cdots, \sigma_t\}$ with a finite subset $\{\sigma_{t+1}, \sigma_{t+2}, \cdots, \sigma_s\}$ of $\{\delta_i\}$ so that the set $\sigma = \bigcup_i \sigma_i$ fails to separate M although $M-\sigma$ is separated by each curve δ_i and ϵ_i which does not meet σ. Let ϵ be such a curve; necessarily $(M-\sigma)-\epsilon$ has exactly two components. By regarding the closures of these components as distinct topological spaces with no points in common and then identifying only the boundary curve ϵ in one with its duplicate in the other, one obtains a new 2-manifold M^* with $2s$ more boundary curves than those of M. Each curve ϵ_i and δ_i in M^* separates M^*, and the Peano-interior function $f: M^* \to E^1$ is constant on each boundary curve of M^*. Since M^* contains all but s of the sets $\{K_i\}$, it is sufficient to show that these are finite in number. Therefore, let the indexing of $\{K_i\}$ be changed to exclude these which meet the boundary of M^*.

Let N denote the decomposition space of M^* relative to f in which the points of N are the level curve components of f, and let $p: M^* \to N$ be the continuous monotone transformation which maps each point x of M^* onto the component of M^{x*} which contains it. The function p has the property [11, Theorem 2.2, Chapter VIII] that if A is connected in N then $p^{-1}(A)$ is connected in M^*. Moreover, N is a locally connected continuum since these properties are all preserved by continuous transformations.

If t is a cut point of N and A is a component of $N-t$ then A must contain a point $p(J)$ where J is a boundary curve of M^*. This follows because $p^{-1}(A)$ is a domain of M^* whose boundary is included in the level curve component $p^{-1}(t)$. The interiority of f on the interior of M^* is contradicted unless $p^{-1}(A)$ meets a boundary curve. But, by the definition of p, if $p^{-1}(A)$ meets a boundary curve it must include that boundary curve. Therefore, if $\{A_i\}$ is a class of pairwise disjoint components of complements of cut points of A, then the number of its elements does not exceed the number of boundary curves of M^*, which is finite.

It was established above that both ϵ_i and δ_i separate M^*. Then, by 2.5, K_i must separate M^* into at least three components, which implies that $p(K_i)$ separates N into at least three components. Suppose there are infinitely many of the sets $\{K_i\}$. Let B_1 be a component of $N-p(K_1)$ which contains infinitely many of the points $p(K_i)$, let A_1 be some other component of $N-p(K_1)$, and let n_1 be the least index for which $p(K_{n_1})$ is in B. The set B is separated by $p(K_{n_1})$ into at least three components. Let B_2 be one which contains infinitely many of the points $p(K_i)$ and let A_2 be another whose closure does not contain $p(K_1)$. Then A_2 is a component of $N-p(K_{n_1})$. This process may be repeated to yield an arbitrarily large class $\{A_i\}$ of disjoint components of complements of cut points of N which contradicts the result of the previous paragraph and completes the proof.

COROLLARY 2.7. *Let M be a compact 2-manifold and $f: M \to E^1$ a Peano-interior function. If each boundary curve of M is a component of a level curve of f, then the sum of the critical point multiplicities in M is finite.*

Proof. Under these conditions the hypotheses of 2.4 are satisfied and the result then follows from 2.1.

3. **An example.** Unless special conditions are imposed, there is no reason to expect the number of critical points to be finite. Indeed, there may be a sequence of them converging to an interior point which itself is not a critical point, as in the following example.

Let M denote the rectangle whose vertices have the coordinates $(1, 0)$, $(1, 2)$, $(-1, 2)$, and $(-1, 0)$, let x_n be the point $(1/n, 0)$ and y_n the point $(1/n, 2)$, and let L_n be the line segment from x_{2n} to y_{2n+1} and L_n^* the segment between x_{2n+1} and y_{2n}. Finally let X_n be the segment from x_{n+1} to x_n and Y_n the segment from y_{n+1} to y_n.

A function f will be defined first on the segment L_n, L_n^*, X_n, and Y_n, and also on the boundary of M. For those points (u, v) with $0>u$, $0=u$, or $1=u$, $f(u, v)$ will be defined to be u. Any other point (u, v) of M lies in a component C of $M-\{(\bigcup_n L_n)\cup(\bigcup_n L_n^*)\}$. On C let f be defined as that harmonic function which agrees with f as defined on the boundary of C.

The boundary values are as follows:

(1) On X_{2n}, f takes the value $1/2n$ at x_{2n+1} and rises monotonically to a maximum value of $1/n$ from which it falls monotonically to the value $1/2n$ at x_{2n}.

(2) On Y_{2n}, let f be defined as the negative of its values at corresponding points of X_{2n}.

(3) On X_{2n-1}, let f take the value $1/2n$ at x_{2n} and rise monotonically to the value $1/(2n-1)$ at x_{2n-1}.

(4) On Y_{2n-1}, f is defined exactly as on X_{2n-1}.

(5) On L_n and L_n^*, f takes the constant value $1/2n$.

One can check that the function $f: M\to E^1$ is interior on the interior of M. Since each level curve is either an arc or a figure "X," the function f is necessarily Peano-interior. Let p_n be the intersection of L_n and L_n^*. Then each point p_n is a critical point of multiplicity one, and the sequence $\{p_n\}$ converges to the point $(0, 1)$ which is an interior point of the compact 2-manifold M.

Chapter III. The sum of critical point multiplicities on plane domains with constant boundary values

1. **Introduction.** Throughout this chapter the following assumptions will be made. Let M be a compact 2-manifold of genus zero with boundary curves $J_1, \cdots, J_n$, each of which is necessarily a simple closed curve, and let J be the union of these curves, so that the interior of M is $M-J$. Let $f: M\to E^1$ be a Peano-interior function which takes the constant value zero on J_1 and the constant value one on each of the other boundary curves. It then follows that a level curve M^x meets J if and only if $f(x)=0$ or $f(x)=1$. Since the genus of M is zero, it is no loss of generality to suppose that M is embedded in the plane in such a way that J_1 is its exterior boundary curve. Also it is the case that $n>1$, because $f(M-J)$ is an open interval both of whose end points

are necessarily values of f on J since $f(M)$ is compact.

The main result of this chapter is that the sum of the critical point multiplicities in M is $n-2$, one less than the first Betti number of M.

2. **The topological structure of level curves and their complementary domains.** This section is devoted to the theorem that each, with one exception, of the components of $M-M^p$ is a domain whose boundary consists of disjoint Jordan curves. Many of the results preliminary to this theorem are of some interest in their own right insofar as they characterize the level curves structure.

Throughout, the symbol K_x is used to denote the component of M^x which contains the point x. Also, it will be convenient to adopt in the sequel the notation $I(C)$ for the intersection with $M-J$ of the bounded complementary domain of the Jordan curve C in view of the fact that most of the components of $M-M^x$ will be shown to be domains of the form $I(C)$ for some curve C.

LEMMA 2.1. *If C is a Jordan curve in K_x, a component of the level curve $M^x \subset M-J$, then $I(C)$ does not meet M^x.*

Proof. The set $f(I(C))$ is an open interval. Because Cl $(I(C))$ is compact, both end points of $f(I(C))$ are values taken on by f on the boundary of $I(C)$. But these values are precisely the numbers $f(x)$ and one since the boundary of $I(C)$ is included in $M^x \cup (J-J_1)$, so that $f(x)$ cannot also be in $f(I(C))$, which would be the case unless $I(C) \cap M^x$ is empty.

COROLLARY 2.2. *If C and C' are two different Jordan curves in K_x, then $I(C)$ and $I(C')$ are disjoint.*

Proof. If the domains $I(C)$ and $I(C')$ are different and yet have one point at least in common, then a boundary point of one is an interior point of the other. Such a point could not be in J (which does not meet the interior of M) and so it must be in K_x. But this would contradict 2.2.

It is as yet unknown whether any of the components mentioned in II: 2.2 can be nondegenerate or even nonempty, unless the manifold has genus zero when the following lemma is applicable.

LEMMA 2.3. *If C and C' are two different Jordan curves in K_x, a component of a level curve in $M-J$, then they can have no more than one point in common.*

Proof. By 2.2, C is exterior to C'. If $C \cap C'$ contains more than one point, then C contains a cross-cut T of the domain $(M-J)-$Cl $(I(C'))$. The end points of T separate C' into two arcs A and B, each disjoint from T. It then follows, from the fact that $A \cup C$ separates the plane into exactly three components [11, Theorem 1.6, Chapter VI], that either $I(T \cup A)$ contains B or $I(T \cup B)$ contains A. In either event 2.1 is contradicted.

THEOREM 2.4. *If the order of p in K_p is $2k$, then $K_p-(p)$ consists of exactly k components. The closure of each component of $K_p-(p)$ contains a Jordan curve which contains p and which separates p from all other points in that component.*

Proof. By II:2.2 there are k Jordan curves $C_1, C_2, \cdots, C_k$ in K_p such that p is a component of the intersection of any two of them. Then, together with the spoke theorem, this implies that the number N of components of $K_p-(p)$ is not greater than k. On the other hand if $k>N$ then, for some i and j, $C_i-(p)$ and $C_j-(p)$ are not separated in $K_p-(p)$. Hence there is an arc A in $K_p-(p)$ connecting $C_i-(p)$ and $C_j-(p)$. But in $A\cup(C_i\cup C_j)$ it is possible to find a pair of Jordan curves which contradicts 2.3.

COROLLARY 2.5. *A point of K_p is a cut point of K_p if and only if it is a critical point.*

It is now possible to prove the main result of this section.

THEOREM 2.6. *If x is a point in $M-J$, then each component of $(M-J)-M^x$ whose boundary does not contain J_1 is a domain whose boundary components are Jordan curves exactly one of which is in M^x with the remainder (of which there is at least one) in $J-J_1$.*

Proof. Let Q be a component of $(M-J)-M^x$ whose boundary does not contain J_1. No boundary component in M^x of Q can be a single point by the spoke theorem, so that any such component contains a noncritical point p of K_x which by 2.4 lies in a simple closed curve C of K_x. If Q coincides with the component $I(C)$ of $(M-J)-M^x$, the theorem is proved. Otherwise Q is necessarily disjoint from $I(C)$, although they share the noncritical boundary point p. Both $f(Q)$ and $f(I(C))$ are open intervals which do not contain the number $f(x)$; $f(I(C))$ has the numbers $f(x)$ and one as end points because the boundary of $I(C)$ is included in $J-J_1$. An application of the spoke theorem to the noncritical point p implies that the values of f on Q must be less than $f(x)$ since the values of f on $I(C)$ are all greater than $f(x)$. By the interiority of f on the interior of M, it follows that the boundary of Q cannot be contained in M^x. Therefore $\overline{Q}$ meets J. Since Q was chosen so that $\overline{Q}\cap J_1$ is empty, it follows that $f(Q)$ is an open interval with the number one as an end point. This contradicts the fact that the values of f on Q are all less than $f(x)$.

3. **The sum of the critical point multiplicities in a single level curve.**

THEOREM 3.1. *Let K_x be a component of M^x and let $T(x)+1$ be the number of components in $(M-J)-K_x$. Then the sum of the critical point multiplicities in K_x is $T(x)-1$.*

Proof. Let the components of $(M-K_x)-J$ be $Q_1, \cdots, Q_{T(x)+1}$, where Q_1 is the one containing J_1 in its boundary. The set $f(Q_i)$ is an open interval whose end points are values assumed by f on the boundary of Q_i. If $i>1$, the boundary of Q_i is a subset of $M^x\cup(J-J_1)$, so that the end points of $f(Q_i)$ are necessarily the numbers $f(x)$ and one. Therefore Q_i does not meet M^x, so that Q_i is a component of $(M-M^x)-J$ for all $i>1$, and, by 2.6, is of the form $I(C_i)$, where the Jordan curve C_i is the only boundary component in M^x of Q_i. In particular, this implies that $T(x)$ is greater than or equal to one.

The proof will be carried out by induction on $T(x)$. If $T(x)=1$, it follows that K_x is the single Jordan curve C_2 and the desired formula is valid for there can be no critical points in K_x. (In view of 2.4, K_x-C_2 must be empty because the assumption $T(x)=1$ implies that K_x contains exactly one Jordan curve.) Suppose now that the formula holds for any continuum A exhibiting the properties attributed by the previous theorems to level curves such that the number of domains complementary to A in M is less than or equal to $m+1$. and suppose that $T(x)=m+1$. Let the components of the closure of K_x-C_2 be $K_1, \cdots, K_k$, where the number of components in $(M-K_i)-J$ is T_i+1. Necessarily T_i+1 is less than or equal to $T(x)=m+1$. Now the inductive hypothesis may be applied to the continua $\{K_i\}$ individually, yielding the relation $S^*=(T_1-1)+\cdots+(T_k-1)$, where S^* is the sum of the critical point multiplicities in the sets $\{K_i\}$. Each set K_i meets C_2 in exactly one point, for if $K_i\cap C_2$ contains more than one point a cross-cut in K_x of the exterior domain of C_2 can be constructed, contradicting 2.3. Thus the Jordan curve C_2 contributes a multiplicity of one to each point $K_i\cap C_2$. It then follows that $S^*+k=T_1+\cdots+T_k$ is the sum of the critical point multiplicities over K_x.

It has already been shown that each set K_i is simply a union of certain of the Jordan curves $C_3, \cdots, C_{T(x)+1}$. Thus the only complementary domains of K_n which meet any of those of K_m is that containing Q_1, provided that $n\neq m$. Therefore, the sum $(T_1+1)+\cdots+(T_k+1)$ counts each domain complementary to K_x exactly once except for Q_1 which is counted k times and for $Q_2=I(C_2)$ which is not counted at all. This means that $T(x)+1 =(T_1+1)+\cdots+(T_k+1)-(k-1)+1$ or that $T(x)-1=T_1+\cdots+T_k$, which was to be proved.

COROLLARY 3.2. *Let the number of components in M^x be $N(x)$ and let $(M-M^x)-J$ have $T(x)+1$ components. Then the sum of the multiplicities of the critical points in M^x is equal to $T(x)-N(x)$.*

Proof. Let T_i be the number of domains complementary to the ith component of M^x. By an argument identical with that used in the conclusion of the proof of 3.1 it can be established that the sum $(T_1+1)+\cdots+(T_{N(x)}+1)$ counts each complementary domain of M^x exactly once except for the exterior one which is counted $N(x)$ times. Therefore, this sum is equal to $T(x)+N(x)$. This fact, together with the result obtained by applying 3.1 to each component of M^x and then adding, establishes the corollary.

4. **The sum of the critical point multiplicities.** In this section it will be shown that the sum of the critical point multiplicities of f is given by $n-2$ where n is the number of boundary curves in M. It is convenient to formulate the proof in a series of lemmas. For each number c between zero and one, let

$N(c)$ be the number of components of M_c and let $T(c)+1$ be the number of components of $M-M_c$, or equivalently of $(M-M_c)-J$. (Note that $N(c)=N(x)$ and $T(c)=T(x)$ whenever $f(x)=c$, where $N(x)$ and $T(x)$ are symbols used in the previous section.) Finally let S denote the sum of the critical point multiplicities of f on M; by II:2.7, S is finite.

LEMMA 4.1. *If $S=0$ and $N(c)=1$ for all non-negative c smaller than one, then the number, n, of boundary curves in M is equal to two.*

Proof. This follows directly from the fact that the components of the level curves form an upper semi-continuous collection. Let c_n be a monotone increasing sequence of positive numbers with limit one. Let Q_n be the component of $M-M_{c_n}$ which contains the boundary curve J_2, and let x_n be chosen in $Q_n \cap M_{c_{n+1}}$ in such a way that the sequence $\{x_n\}$ is convergent with limit x in J_2. By hypothesis and 3.2, $N(c)=T(c)=1$ for all c, so that Q_n contains all boundary curves of M except J_1. If a third boundary curve exists, then a sequence $\{y_n\}$ of points in $Q_n \cap M_{c_{n+1}}$ converging to the point y in J_3 can be constructed. But the fact that $N(c)=1$ for all c implies that M_{c_n} is connected for each n. Thus the upper semi-continuity of the components of the level curves of f would imply that x and y are in the same component of M^x. Therefore $J_2=J_3$, and M necessarily possesses exactly two boundary curves.

LEMMA 4.2. *If $S=0$ and if $n>2$, then there exists a positive number c smaller than one such that $N(c)$ is less than $n-1$.*

Proof. One notes first that, by 3.2, $N(c)=T(c)$ for all c, and that, by 2.6, the number $T(c)$ is necessarily less than or equal to $n-1$. Suppose now that this theorem is false; that is, suppose that $N(c)$ is greater than or equal to $n-1$ for all c. This implies that $N(c)=T(c)=n-1$ for all c. Let K be an arc in the interior of M except for its end points in the boundary curves J_2 and J_3. From the equality $T(c)+1=n$, it follows that each component of $M-M_c$ contains exactly one boundary curve of M which implies that J_2 is separated from J_3 in M by M_c. Therefore the compact set K meets every level curve M_c, and so contains points whose f-values are arbitrarily close to zero. However, K is separated from J_1, the set of all points whose f-value is zero. This contradiction establishes that the original supposition is false.

LEMMA 4.3. *If $S=0$, it follows that $T(c)=1$ for all positive numbers c less than one, and then that $n=2$.*

Proof. The proof will consist of showing by induction on n that $T(c)=1$ for all c. When this is proved, then 4.1 implies the second conclusion that the number of boundary curves in M is equal to two.

In the first step of the inductive argument, when $n=2$, it is necessarily the case that $T(c)+1$ is less than or equal to $n=2$. Therefore the positive integer $T(c)$ must be equal to one, for all c.

In the second step of the inductive argument, let n be an integer greater than two and suppose that the desired conclusion $T(c)=1$ is valid whenever the number of boundary curves does not exceed n. It is known that $T(c)$ is an integer between one and $n-1$; it will now be shown that $T(c)$ must equal one of these two values. If not, then there is a value c and a component Q of $M-M_c$ such that $\overline{Q}$ contains more than one of the curves J_i for which $i>1$ and no more than $n-2$ of them. By the inductive hypothesis, every level curve $M_{c'}$ (for $c'>c$) separates Q into exactly two components. But 3.2 applied to $\overline{Q}$ then implies that every level curve in Q is connected so that the hypotheses of 4.1 are satisfied for Q. Therefore $\overline{Q}$ has exactly two boundary curves, at most one of which is in M_c by 2.6. This contradiction proves that, for each value c, either $T(c)=1$ or $T(c)=n$.

Let A be the set of numbers c for which $T(c)=1$. By 3.2 and 4.2, A contains a positive number and so has a positive maximum which is denoted here by k. Moreover, A has the property that if c is in A and if c^* is positive but less than c, then c^* is also in A because M_{c^*} is a subset of that component R of $M-M_c$ which contains J_1. The domain R is an annulus with two boundary circles because its boundary consists of J_1 and the connected level curve M_c, which contains no critical points and so is necessarily a Jordan curve. Therefore, the inductive hypothesis applies to R yielding the fact that $T(c^*)=1$. It will therefore be sufficient to show that k, the maximum of A, is equal to one, for this property just demonstrated implies that A contains all positive numbers c less than k. Let T be the component of $M-M_k$ which contains J_1. By 3.2, $\overline{T}$ is the closure of a domain with $T(k)+1$ boundary circles. By 3.2 and 4.1, it follows that $T(k)=1$. Therefore, if k is less than one, the set $M-T$ is actually a 2-manifold with n boundary circles with the property that every level curve in $M-T$ separates it into n components so that $N(c)=n-1$ for all $c>k$. (This presumes the fact that every level curve M_c with $c>k$ is a subset of $M-T$.) This conclusion however contradicts 4.2 as applied to $M-T$ and thus concludes the proof of this lemma.

LEMMA 4.4. *If k is the smallest critical value, then* $N(k)=1$.

Proof. The existence of a positive number k as the smallest critical value, provided that there are any critical values at all, is guaranteed by II:2.6, in which it is shown that S is finite. Let Q be the component of $(M-J)-M_k$ whose boundary contains J_1. Although the boundary curves of $\overline{Q}$ may not be Jordan curves, nevertheless, by II:2.4, the interior of Q is homeomorphic with a domain Q^* bounded by $N(k)$ Jordan curves. This homeomorphism defines f^* on Q^*; then f^* may be extended to $\overline{Q}^*$ to be continuous by being given the constant boundary value k on all boundary curves except the one corresponding to J_1 on which f^* is necessarily zero. By 4.3, it follows that Q^* is an annulus with two boundary curves. Therefore M_k must have been connected, for the number of boundary curves in Q^* is $N(k)+1$.

THEOREM 4.5. *If M is a plane domain whose boundary consists of n simple closed curves and $f:\overline{M}\rightarrow E^1$ is a Peano-interior function which takes the constant value one on each boundary curve of M except for one on which it takes the value zero, then the sum of the critical point multiplicities of f in M is $n-2$.*

Proof. As has been pointed out before, $T(c)+1$ is always less than or equal to n. Therefore, if $n=2$, it follows that $T(c)=1$ for all c, and by 3.2 that $S(c)=1-N(c)$, where $S(c)$ is the sum of the critical point multiplicities in M_c. Since $N(c)$ is greater than or equal to one and $S(c)$ is not negative, this implies that $S(c)=0$, for all c. Hence there are no critical points when $n=2$.

To proceed with the inductive step, let n be greater than two, let k be the smallest critical value. Then, by 3.2 and 4.4, $S(k)=T(k)-1>0$, so that $T(k)>1$. Let the components of $M-M_k$ be $Q_1, \cdots, Q_{T(k)+1}$, numbered so that the last one contains J_1, and let S_i be the sum of the critical point multiplicities in Q_i. This means that $S_{T(k)+1}=0$. For each i with $T(k)+1>i$, Q_i has one boundary circle in M_k and a_i boundary circles in J (by 2.6), so that $a_1+\cdots+a_{T(k)}=n-1$. Moreover, (a_i+1) is not greater than $n-1$ because $T(k)>1$. This means that each of the manifolds $\overline{Q}_i$ with $T(k)+1>i$ has no more than $n-1$ boundary curves. The inductive hypothesis therefore implies that $S_i=a_i-1$, for each i less than $T(k)+1$. Summing, one obtains $S-S(k)=S_1+\cdots+S_{T(k)}=(n-1)-T(k)$. But $S(k)$ has already been evaluated as $T(k)-1$, so that the formula is proved.

CHAPTER IV. ON THE SEPARATION OF 2-MANIFOLDS BY JORDAN CURVES

1. **Some preliminary results.** It is convenient to state here some facts concerning those 2-manifolds obtained by identifying certain boundary curves of a pair of given 2-manifolds. These results will be applied not only in the main theorem of this chapter but also in the constructions to be described in the next chapter. One should recall that, for 2-manifolds with boundary, compactness implies both metricity and triangulability.

The relation between the genus g, the number n of boundary curves, and the first Betti number p_1 (with integral coefficients) of a compact 2-manifold with boundary is known and is stated for example in [13, p. 144]. In the orientable case, the formula is simply $p_1=2g+n-1$ when $n>0$. The following theorems are restricted to orientable manifolds in order to permit the use of this relation.

THEOREM 1.1. *If M is a compact, orientable 2-manifold (with boundary) of genus g, X the union of N pairwise disjoint curves in the interior of M, and $M-X$ is connected, then $M-X$ has genus $g-N$.*

Proof. A (homological) *base* of a compact 2-manifold Z (with boundary) is the union of a class of pairwise disjoint simple closed curves in the interior of Z which does not separate Z and which is maximal with respect to this property. The *genus* of Z is the number of components in the largest base. Evidentally the complement of a base has genus zero. The classic fact that the number of components in any base is always equal to the genus may be derived by considerations of the following sort. To each base B there corresponds a triangulation such that each component of the base is a closed polygonal path in the edges of the triangulation. Each such path determines two independent one-dimensional cycles (of opposite orientation) and these cycles are not only linearly independent (with integral coefficients) but in fact they (together with any $n-1$ of the one-cycles arising from the boundary of n components) generate the entire one-dimensional homology group since every one-cycle is a linear combination of one-cycles carried by simple closed polygonal paths which are either in the boundary, J, of Z, or are in B, or else separate the complement of $B \cup J$ and are therefore bounding one-cycles. The topological invariance of the rank $(=2g+n-1)$ of the homology group thus assures one that the number of components of the base is in fact equal to the genus g.

Now, let Y be a base for $M-X$. Evidently $M-(X\cup Y)$ is of genus zero, for if J is a simple closed curve in the interior of $M-(X\cup Y)$ then J is in the interior of $M-X$ and J must separate $M-X$ for J does not meet Y. If J separates $M-X$ then J must separate M since J does not meet X. Therefore $X\cup Y$ is a base for M, which proves the theorem.

THEOREM 1.2. *If M_1 and M_2 are compact, orientable 2-manifolds (with boundary) of genus g_1 and g_2 respectively and J_1 and J_2 are boundary curves of the two manifolds, then the manifold M^* obtained by identifying J_1 with J_2 has genus g_1+g_2.*

Proof. Because the two manifolds are compact, they are triangulable. Let J denote the common boundary curve. By taking suitable refinements of given triangulations, one can obtain triangulations of the two manifolds which agree on J. Using the representatives of the respective (one-dimensional) homology groups induced by these triangulations, one sees that every (bounding) one-cycle in M^* is the sum of (bounding) one-cycles on M_1 and on M_2. Therefore, the homology group of M^* is the direct sum of the homology groups of M_1 and of M_2 with the one-cycles on J identified, so that the relation

$$p^* = p_1 + p_2 - 1$$

between the first Betti number p^* of M^* and those of M_1 and M_2 is established. Moreover, if the number of boundary curves in M_i is n_i, $i=1, 2$, then

$$p_1 = 2g_1 + n_1 - 1,$$
$$p_2 = 2g_2 + n_2 - 1,$$

and

$$p^* = 2g^* + (n_1 + n_2 - 2) - 1.$$

If the last equality is subtracted from the sum of the previous two equalities, and the first relation is taken into account, one then finds that the desired formula has been proved.

THEOREM 1.3. *If D is a bounded plane domain whose boundary contains n simple closed curves $J_1, \cdots, J_n$, and these are identified with n of the boundary curves of a compact orientable 2-manifold M of genus g, the resulting manifold M^* has genus $g^* = g + n - 1$.*

Proof. The case for $n=1$ is a special instance of 1.2. Suppose that $n=2$ and let S denote the manifold obtained by identifying just one of the two boundary curves of D with a boundary curve of M. By 1.2, the genus s of S is equal to g. Because the remaining curve, J_2, does not separate M^* it is possible to infer that $g^* - 1 = s$ from 1.1. Hence $g^* = g + 1$ and the desired equality holds when $n=2$. If it holds whenever the number of boundary curves to be identified is less than n, then let S denote the manifold obtained by identifying $J_1, \cdots, J_{n-1}$ with $(n-1)$ boundary curves of M. By the inductive hypothesis, the genus s of S is equal to $g + n - 2$ if n exceeds 2, and, if $n=2$, the theorem has already been proved. Moreover, as above, $g^* - 1 = s$, so that $g^* - 1 = g + n - 2$, which establishes the induction.

THEOREM 1.4. *If M is a compact 2-manifold with boundary and X is the union of n pairwise disjoint simple closed curves in the boundary of M, then there exists a simple closed curve C in the interior of M one of whose complementary domains has genus zero and meets the boundary of M in the set X.*

Proof. Let a triangulation of M be given. In this triangulation each component of X is necessarily a polygonal path. Let n be the number of components of X. If $n=1$, then the polygonal path determined by the union of the edges of those triangles in which the opposite vertices are in X is the desired simple closed curve. If the theorem is true for the first $(n-1)$ members of X, then there is a domain D in X of genus zero, whose boundary in the interior of X is a simple closed curve A, such that D contains all but one of the components of X. Thus $X - D$ is a compact 2-manifold whose boundary consists of A and $B = X - D$. Evidently it is sufficient to treat the case $n=2$, where the two components of X are A and B. Let a triangulation of X be selected which is the refinement (by barycentric subdivision) of one in which no edge has vertices on both A and B. Let C be a simple polygonal path in $X - D$ with one end point on B and the other on A. The union of all edges of those triangles in which the opposite vertices are on A, B, or C is the desired simple closed curve. (The properties of the triangulation are chosen to ensure that this path is simple.)

2. **The main theorem.** Although this result may possibly be inferred from

modern work in algebraic topology, it is, in this instance, as economical to give a proof along classical lines.

THEOREM 2.1. *If M is a compact, orientable 2-manifold with boundary and X is the union of N pairwise disjoint simple closed curves in the interior of M which separates M into $T+1$ components whose closures are $M_1, \cdots, M_{T+1}$, where M_i has genus g_i and M has genus g, then $g = g_1 + \cdots + g_{T+1} + N - T$.*

Proof. The argument will be an induction on T. When $T=0$ the formula follows from 1.1. A special notation is needed for the proof of the inductive step. Let the curves of X which are in M_1 be $J_1, \cdots, J_p$ and let their union be X^*. Let the closures of the components of $M - X^*$ be $M_2^*, \cdots, M_q^*$ where M_i^* has genus g_i^*. Let p_i be the number of curves of X^* in M_i^*; then

$$p_2 + \cdots + p_q = p.$$

Let the number of curves of $X - X^*$ in M_i^* be N_i; then

$$N_2 + \cdots + N_q = N - p.$$

Let t_i be the number of manifolds M_j which are in M_i^*; then

$$t_2 + \cdots + t_q = T,$$

and each number t_k is not greater than T. Finally let g^n be the sum of the genuses of the manifolds M_j which are in M_n^*; then

$$g^2 + \cdots + g^q = g_2 + \cdots + g_{T+1}.$$

If the inductive hypothesis is applied to M_n^* separated by the N_n curves in $M_n^* \cap (X - X^*)$ into t_n components, one obtains

$$g_n^* = g^n + N_n - (t_n - 1) = g^n + N_n + 1 - t_n. \tag{1}$$

If (1) is summed through the range $n=2$ to $n=q$, one obtains

$$g_1 + g_2^* + \cdots + g_q^* = g_1 + \cdots + g_{T+1} + (N - p) + (q - 2) - T. \tag{2}$$

Let $A_1 = M_1$, $A_2 = M_1 \cup M_2^*, \cdots, A_q = A_{q-1} \cup M_q^*$ and let the genus of A_i be G_i. It is already known that $G_1 = g_1$. A method will now be given for evaluating G_{s+1} if G_s is known. The formula to be determined is

$$G_s + g_{s+1}^* + p_{s+1} - 1 = G_{s+1}. \tag{3}$$

Because $A_q = M$, it follows that $G_q = g$. Thus successive application of this recursion relation will complete the argument for the inductive step as follows:

$$\begin{aligned} g = G_q &= \{g_1 + g_2^* + \cdots + g_q^*\} + p_1 + \cdots + p_q - (q - 2) \\ &= \{g_1 + \cdots + g_{T+1} + (N - p) + (q - 2) - T\} + p - (q - 2) \\ &= g_1 + \cdots + g_{T+1} + N - T. \end{aligned}$$

The second step above involves the use of (2). It now remains to demonstrate the recursion relation (3) itself. The set A_{s+1} consists of A_s and $M_{s+1}{}^*$ separated by p_{s+1} curves of X^*. By 1.4, there is a Jordan curve C in A_s and a component Q of $A_s - C$ such that Q has genus zero and contains only the boundary curves of A_s which are in M_{s+1}. If these p_{s+1} boundary curves in Q are identified with their duplicates in $M_{s+1}{}^*$, then by 1.3,

$$\text{genus } \{Q \cup M_{s+1}{}^*\} = g_{s+1}{}^* + p_{s+1} - 1.$$

The set A_{s+1} can also be thought of as consisting of $(A_s - Q)$ and $(Q \cup M_{s+1}{}^*)$, the two separated by C. By 1.2,

$$G_{s+1} = \text{genus } (A_s - Q) + \text{genus } (Q \cup M_{s+1}{}^*).$$

But $A_s \cap Q = C$, so that the two sets are separated by exactly one curve. Therefore, by 1.2, genus $\{A_s - Q\} =$ genus $\{A_s\} = G_s$. The last three equalities combine to prove the recursion relation (3) and thus end the proof of this theorem.

Chapter V. The sum of critical point multiplicities on 2-manifolds with constant boundary values

1. **Preliminary remarks.** Let M be a compact metric orientable 2-manifold with boundary curves $J_1, \cdots, J_n$, and let $f: M \to E^1$ be Peano-interior. The function f will be said to be *canonical* if each boundary curve of M is a component of a level curve. It will be seen at the end of this chapter that there is no loss of generality in considering only the *strongly canonical* case in which $f(J_1)$ is the minimum value of f and $m = f(J_i) = f(J_j)$ for i and $j > 1$, is the maximum value of f on M. If the genus of M is g, let $S\{f: M, g, n\}$ denote the sum of the multiplicities of the critical points of f on M; it is the purpose of this chapter to show that for canonical functions $S\{f: M, g, n\} = 2g + n - 2$ which is simply one less than the first Betti number of M.

2. **Summing the critical point multiplicities of strongly canonical functions.** In this section, the main burden of proof will rest on the case when $n = 2$. It is no loss of generality to suppose that the minimum value of f is zero. In this event, for each value c, $m > c > 0$, M is separated by M_c into exactly two components, each of which is a noncompact 2-manifold with one boundary curve. The components of $M - M_c$ will be denoted by $Q_1(c)$ and $Q_2(c)$; the number of components of M_c will be denoted by $N(c)$. Let the genus of $Q_i(c)$ be $g_i(c)$. For each value of c, critical or not, the symbol Cl $(Q_i(c))$ will be used to denote the compact metric orientable 2-manifold with $N(c)+1$ boundary curves and of genus $g_i(c)$, given by II:2.4, whose interior is homeomorphic with the interior of $Q_i(c)$. Of course f can be extended from the interior of $Q_i(c)$ to Cl $(Q_i(c))$ to be strongly canonical.

LEMMA 2.1. *If the function f is strongly canonical, then $N(c)$ is finite for every value c.*

LEMMA 2.2. *$S\{f:M, g, 2\}$ is finite for all strongly canonical functions f.*

These two lemmas are merely applications of II:2.1 and II:2.7 respectively. The main theorem of the chapter will be proved by induction on the genus of M. Its proof will consist of choosing a separation of M by some level curve M_c into components to each of which the following theorem can be applied. In other words the next theorem constitutes a strengthening of the inductive hypothesis which is more conveniently proved by being removed from the main body of the argument.

THEOREM 2.3. *If $S\{f:M, g, 2\}=2g$ for all strongly canonical f and all g not greater than g^*, then $S\{f:M, g, n\}=2g+n-2$ whenever $g+n-2\leqq g^*$.*

Proof. Let D be the bounded domain in the plane bounded by the unit circle C and by $n-1$ pairwise disjoint circles $J_2, \cdots, J_n$ in the unit disk. There is a harmonic function $f:D\rightarrow E^1$ which takes the value m on all boundary curves of D except on C where f takes the value $m+1$; the function f is Peano-interior on the interior of D. Let M' denote the compact 2-manifold obtained by identifying the interior boundary curves of D with the $n-1$ boundary curves of M on which f takes the value m. The function f is strongly canonical on the interior of M^*. By IV:1.3, the genus g' of M' is equal to $g+n-2$ which is assumed to be not greater than g^*, so that $S\{f:M', g', 2\}=2g+2n-4$. Moreover $S\{f:D, 0, 2\}=n-2$ by III:4.5, so that $S\{f:M, g, n\}=2g+2n-4-(n-2)=2g+n-2$, which was to be proved.

THEOREM 2.4. *If $S\{f:M, g, 2\}=2g$ for all strongly canonical f and all g not greater than g^*, then $S\{f:M, g^*+1, 2\}=2(g^*+1)$ for all strongly canonical f.*

Proof. Let the sets $Q_i(c)$ be numbered so that $g_1(c)\geqq g_2(c)$ for all c, and so that, whenever they are of equal genus, $Q_1(c)$ is the one which contains J_1. First one notes that if there is a noncritical value c such that, for each value (1 and 2) of i, either $g_i(c)=0$ or $g_i(c)+N(c)-1$ is less than or equal to g^*, then by III:4.5, or by 2.3, one can obtain

$$S\{f:\mathrm{Cl}\,(Q_i(c)), g_i(c), N(c)+1\} = 2g_i(c)+N(c)-1. \tag{1}$$

Hence $S\{f:M, g^*+1, 2\}=2\{g_1(c)+g_2(c)+N(c)-1\}$ provided that c is not a critical value. But in this event, by II:2.3 it is possible to apply IV:2.1 to M separated by M_c and obtain

$$g^*+1 = g_1(c)+g_2(c)+N(c)-1. \tag{2}$$

By substituting relation (2) into (1), one obtains the formula to be proved.

Therefore, suppose that

$$g_1(c)+N(c)-1 > g^* \tag{3}$$

for all noncritical c, and that

$$g_1(c) > 0 \tag{4}$$

for all noncritical c. These assumptions will ultimately lead to a contradiction which will include the proof of the theorem. From (2) it follows that $g_1(c) + N(c) - 1 - (g^* + 1) = -g_2(c)$ is less than or equal to zero. From this and (3) one now infers that

$$g_1(c) + N(c) - 1 = g^* + 1 \tag{5}$$

and

$$g_2(c) = 0. \tag{6}$$

These two facts will be used extensively in the following paragraphs. The rest of the proof will be broken up into a sequence of lettered propositions.

(a) There is at least one critical value.

If this is not the case, then by III:4.5,

$$S\{f:\mathrm{Cl}\ (Q_2(c)),\ 0,\ N(c) + 1\} = N(c) - 1 = 0 \tag{7}$$

for all c. Some value c_0 may be found sufficiently close to zero so that for every c such that $c_0 > c$, the set $f\{Q_2(c)\}$ is the number interval from zero to c, $(0, c)$. This will follow from the fact that there is a domain of M which contains J_1 and which has genus zero. Let values c_1 and c_2 be chosen so that $c_0 > c_2 > c_1$. It will now be shown that the supposition $f\{Q_2(c_i)\} = (0, c_i)$ leads to a contradiction. By (7), $N(c) = 1$ for all c, so that $Q_1(c_1)$ is separated by M_{c_2} into two components $Q_1(c_2)$ and $Q = Q_1(c_1) - \mathrm{Cl}\ (Q_1(c_2))$ where Q has genus zero because $Q \subset Q_2(c_2)$. Then III:2.1, applied to $Q_1(c_1)$ separated by M_{c_2} yields $g_1(c_1) = N(c_2) + 1 - 2 = 0$ which contradicts (4). Hence (a) is proved.

Let the critical values be numbered as $c_{r+1} = m > c_r > \cdots > c_2 > c_1 > 0 = c_0$. There are only finitely many by II:2.7.

(b) $S\{f:\mathrm{Cl}\ (Q_2(c)),\ 0,\ N(c)+1\} = N(c) - 1$ for all noncritical values c.

This is a direct result of III:4.5 and (6).

(c) If $c_{i+1} > b > a > c_i$, then $N(a) = N(b)$.

To prove this fact it is necessary to consider four cases.

(c1) $f\{Q_2(a)\} = (a, m)$ and $f\{Q_2(b)\} = (b, m)$,

(c2) $f\{Q_2(a)\} = (0, a)$ and $f\{Q_2(b)\} = (0, b)$,

(c3) $f\{Q_2(a)\} = (a, m)$ and $f\{Q_2(b)\} = (0, b)$, and

(c4) $f\{Q_2(a)\} = (0, a)$ and $f\{Q_2(b)\} = (b, m)$.

In the first two cases one of the sets $Q_2(a)$ and $Q_2(b)$ includes the other in such a way that $S\{f:\mathrm{Cl}\ (Q_2(a)),\ 0,\ N(a)+1\} = S\{f:\mathrm{Cl}\ (Q_2(b)),\ 0,\ N(b)+1\}$. In this event the desired result follows from (b). In case (c3), $f\{Q_1(a)\} = (0, a)$, so that $Q_1(a) \subset Q_2(b)$ which implies that $g_1(a) = 0$ and so contradicts (4). Finally in (c4) a more special argument is needed. Let $U = \{x: b > x > a$ and $f(Q_2(x)) = (x, 1)\}$, and let V be the complement of U in the interval (a, b). It is clear that if $b > y > x$ and x is in U then y is in U also; similarly y in V implies that x is in V. Thus U and V constitute a Dedekind cut of the interval (a, b), determining a cut-value c, $b>c>a$. It will now be shown that neither U nor V can contain c, a contradiction which will establish (c4). If c is in U, then there is a monotone sequence $x_n \to c$ with $c>x_n>a$, i.e., x_n is in V. Now let J be a Jordan curve in $Q_1(c)$; since c is supposed to be in U, $f\{Q_1(c)\}=(0, c)$. Because J is compact, there is an x_n such that $f(J)\subset(a, x_n)$. Thus $J\subset Q_2(x_n)$ since x_n is in V, and J therefore separates $Q_2(x_n)$; in particular J separates M_{x_n} from M_a in M, so that J separates $Q_1(c)$ which therefore has genus zero contradicting (4). In the same way it is impossible for c to be in V.

(d) If $c_1>a>0$, then $f\{Q_2(a)\}=(0, a)$.

As in the argument proving (a), there is a value a^*, $c_1>a^*>0$, such that $f\{Q_2(a^*)\}=(0, a^*)$. Then by (b), $S\{f:\mathrm{Cl}\ (Q_2(a^*)), 0, N(a^*)+1\}=N(a^*)-1=0$. Hence $N(a^*)=1$. By (c), $N(a)=1$ for all a such that $c_1>a>0$, so that $S\{f:\mathrm{Cl}\ (Q_2(a)), 0, 2\}=0$ for all a such that $c_1>a>0$. If $f\{Q_2(a)\}=(a, m)$, then $Q_2(a)$ contains M_{c_1} and so contains at least one critical point, which proves (d).

(d*) If $c_{r+1}=m>a>c_r$, then $f\{Q_2(a)\}=(a, m)$.

This fact can be proved by the same argument used to prove (d).

Now let numbers a_i be selected, $c_{i+1}>a_i>c_i$ and let k be the smallest index such that $f\{Q_2(a_i)\}=(0, a_i)$ for $i=1, 2, \cdots, k$ and $f\{Q_2(a_{k+1})\}=(a_{k+1}, m)$. Note that, by (d) and (d*), such a k must exist.

(e) $f\{Q_2(a_m)\}=(a_m, 1)$ for $m=k+1, \cdots, r$.

If this is false, let m be the smallest index not smaller than $k+1$ for which $f\{Q_2(a_m)\}=(0, a_m)$. Then $Q_1(a_{m-1})\subset Q_2(a_m)$, so that $g_1(a_{m-1})=0$, contradicting (4).

(f) $M_{c_{k+1}}$ separates M into two components, each of which has genus zero. This follows by exactly the argument used to finish the proof of (c4), since in both instances (in this one by (e)) it is known that if $m>a_j>c_{k+1}>a_i>0$, then $f\{Q_2(a_i)\}=(0, a_i)$ and $f\{Q_2(a_j)\}=(a_j, m)$.

(g) $N(a_k)=N(a_{k+1})=N(c_{k+1})$.

By the numbering convention adopted for the components of $M-M_c$, it follows that $f\{Q_1(c_{k+1})\}=(0, c_{k+1})$ and $f\{Q_2(c_{k+1})\}=(c_{k+1}, m)$. Then $Q_2(a_k)\subset Q_1(c_{k+1})$ and $Q_2(a_{k+1})\subset Q_2(c_{k+1})$ and all critical points in $Q_i(c_{k+1})$ are in $Q_2(a_k)$ or $Q_2(a_{k+1})$. Therefore, by (f),

$$S\{f:\mathrm{Cl}\ (Q_2(a_{k+1})), 0, N(a_{k+1})+1\} = S\{f:\mathrm{Cl}\ (Q_2(c_{k+1})), 0, N(c_k)+1\}$$

and

$$S\{f:\mathrm{Cl}\ (Q_2(a_k)), 0, N(a_k)+1\} = S\{f:\mathrm{Cl}\ (Q_1(c_{k+1})), 0, N(c_{k+1})+1\}.$$

An application of III:4.5 completes the argument for (g).

Now let $M^*=M-(Q_2(a_k)\cup Q_2(a_{k+1}))$, and let a_k and a_{k+1} be chosen so close to c_{k+1} that each component Q_i of M^* contains exactly one component of $M_{c_{k+1}}$. Moreover let the genus of Q_i be g_i, let Q_i have p_i boundary curves in M_{a_k} and q_i boundary curves in $M_{a_{k+1}}$. Then

$$p_1 + \cdots + p_t = N(a_k) \tag{8}$$

and

$$q_1 + \cdots + q_t = N(a_{k+1}), \tag{9}$$

where $t = c_{k+1}$. By (g) it now follows that

$$(p_1 + q_1 - 2) + \cdots + (p_t + q_t - 2) = 0 \tag{10}$$

so that

$$p_i + q_i - 2 = 0, \tag{11}$$

or

$$p_i = q_i = 1. \tag{12}$$

Let S_i denote the sum of the critical point multiplicities in Q_i. Then $S_1 + \cdots + S_t$ is the sum of the critical point multiplicities in $M_{c_{k+1}}$ and therefore is not zero. It will now be shown that, for each i, $g_i = 0$. This will imply that $S_i = 0$, by (12) and III:4.5, and so will constitute the desired contradiction.

By using the spoke theorem, one can cover the critical points in Q_i by pairwise disjoint disks D whose intersection with the continuum $K_i = Q_i \cap M_{c_{k+1}}$ is an even number of arcs running from the critical point to the boundary of D. The set K_i separates Q_i into exactly two components A_i and B_i. Let Q_i^* denote the 2-manifold formed by removing the disks D from Q_i; let $A_i^* = A_i \cap Q_i^*$ and $B_i^* = B_i \cap Q_i^*$. The removal of the disks D removed "triangles" attached to the boundary of A_i and B_i from A_i and B_i so that A_i^* is homeomorphic with A_i and B_i^* is homeomorphic with B_i. By II:2.4, A_i and A_i^* have the same number of boundary components. The same is true of B_i and B_i^*. But Cl (A_i^*) and Cl (B_i^*) are simply plane domains whose boundary components are Jordan curves. If each component of $D_i^* \cap K_i$ is added to the union $A_i^* \cup B_i^*$ one obtains a set which is not only homeomorphic with D_i^* but which can be represented as a pair of plane domains which have been connected by identifying certain arcs in their boundaries, in each instance all of these arcs being included in a single boundary circle. It is possible to prove by induction that such a connection between two domains yields a new domain also of genus zero. Thus Q_i^* has genus zero and Q_i can be formed from Q_i^* by adding a finite number of disks. By repeated applications of IV:1.2, one can conclude therefore that the genus of Q_i is zero. This completes the proof.

THEOREM 2.5. *$S\{f: M, g, n\} = 2g + n - 2$ for all strongly canonical functions f and all g.*

Proof. By III:4.5 and 2.4, the formula for $n = 2$ is established by induction on g. Then it is extended for all n by 2.3.

3. The sum of critical point multiplicities of canonical functions.

THEOREM 3.1. *If $f:M\rightarrow E^1$ is a Peano-interior function on the compact orientable 2-manifold M, and if each boundary curve of M is a component of a level curve, then* $S\{f:M, g, n\}=2g+n-2$.

Proof. For each boundary curve J_i there is a domain D_i in M such that $D_i\cap\{\bigcup_{j=1}^n J_j\}=J_i$. Let the boundary curves of M be numbered so that for some k, $f(D_i-J_i)$ has $f(J_i)$ as its smaller end point whenever $k\geqq i$ and as its larger end point whenever $i>k$. Then let a and b be chosen so that $b>f(J_i)>a$ for all i. Let A and B be bounded plane domains bounded by $k+1$ and by $n-(k+1)$ circles respectively, and let $f:A\rightarrow E^1$ be the harmonic function which takes the value a on one boundary curve and takes the values $f(J_i)$ $(k\geqq i\geqq 1)$, on the others. Similarly let $f:B\rightarrow E^1$ be the harmonic function which takes the value b on one boundary curve and takes the values $f(J_i)$ $(i>k)$ on the others. Then, by IV:1.3, a new manifold M^* of genus $g+n-2$ with two boundary curves may be constructed by a suitable identification of the boundary curves of A and of B with those of M in such a way that f may be extended to M^* to be strongly canonical. Then 2.5 may be applied to it, and also to the restrictions of that function to A and to B. Subtraction of the latter two quantities from the first yields the desired formula.

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