ON THE PROJECTION GEOMETRY OF A FINITE AW*-ALGEBRA

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It is well-known that the projection lattice of a finite AW*-algebra is a continuous geometry. At the time this was proved [7, Theorem 6.5], AW*-algebras had just been invented, while continuous geometries had already been studied extensively. It was therefore natural that basic results were occasionally quoted from continuous geometry, rather than proved anew for the projection geometry, such as the existence of a dimension function [7, §6], strong semi-simplicity [18, Theorem 2.7], the existence of a suitably related regular ring, and the extension of projection ortho-isomorphisms (see [4]). Nevertheless there are valid reasons for repeating this work in the context of AW*-algebras: (1) the subject of AW*-algebras is made more accessible by bypassing continuous geometry; (2) the AW* proofs are easier; notably, the projection lattice is orthocomplemented, and the relation of equivalence more tractable (see [13]); (3) continuous geometries are too general; there is more information to be squeezed out of the AW* case.

The dimension function can already be read out of [2, Lemme 6.13], an easy argument (implicit in [2]) for strong semi-simplicity is given in §1, the regular ring is reconstructed by AW* methods in [1], and the projection ortho-isomorphism theorem is proved in [4] (in a generality greater than is afforded by continuous geometry). The contribution of this paper to the program is a description of the reduction of the projection geometry into a complete set of irreducible continuous geometries [12], together with the attached regular rings.

The organization of the paper is roughly as follows: §§1 and 2 are expository, §3 is trivial algebra, and §§4 and 5 are partly a continuation of [1]. The expository material has to do with the A/M theorem [18, Theorems 4.1, 5.1], (the assumption of trace is removed in a forthcoming paper by T. Yen); to exhibit certain technical details (Lemma 2.4) which are needed later (in Lemma 5.2), we have found it expedient to give a complete proof of this theorem. In §3 we discuss a class of rings with involution whose basic feature is

(a) the right annihilator of any element is the principal right ideal generated by a projection.

The Baer *-rings of Kaplansky [11, Chapter III] are special cases:

(b) the right annihilator of any subset is the principal right ideal generated by a projection.

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C^*-algebras satisfying condition (a) are the B_p^*-algebras (with unity) of Rickart [16, Definition 2.1]; condition (b) gives the AW*-algebras. For a regular ring with involution, condition (a) is a reformulation of *-regularity [15, Theorem 4.5; 10, §2], condition (b) is complete *-regularity. We have not attempted a study of *-rings satisfying (a) alone, but have added a rather artificial set of axioms; still, there are a number of nontrivial examples (see §§4, 5). In §5, a new class of Baer *-rings is exhibited, and related to the reduction of the projection geometry of a finite AW*-algebra; there are connections and divergencies with the A/M theorem.

Throughout the paper A is a finite AW*-algebra (though occasionally greater generality is indicated), thought of as a self-adjoint subalgebra of its regular enlargement C [1]. None of the actual details of the construction of C will be needed, and the reduction theory (Theorem 5.3; Theorem 5.7, parts (1), (2), (3)) does not depend on C at all. General notations are the same as in [1], with the following exception: $e \prec f$ means there is a projection g such that $e \sim g$ and $g \leqq f$. The notion of p-ideal [18, Definition 2.1] will occur frequently, and a substantial part of [18] is quoted during the proof of Lemma 2.4.

1. **Dimensions in closed ideals.** Strong semi-simplicity of A [18, Theorem 2.7], which is relevant for the reduction theory, can be proved as follows. Assume to the contrary that the intersection N of all maximal ideals is not 0. Then N contains a nonzero projection e (spectral theory), which we may assume to be "simple" [2, Lemme 4.9, Lemme 6.4]. The central cover h of e, being the sum of finitely many orthogonal projections equivalent to e, also lies in N [18, Lemma 2.1]. But there is a maximal ideal excluding h, namely any maximal ideal containing $(1-h)A$.

The rest of the section is devoted to proving: if I is a closed ideal of A, and e is a projection in I, its dimension $D(e)$ is also in I; and if M is a maximal ideal containing $D(e)$, it necessarily contains e.

When A is type I, the center-valued trace is available, and since $\mathrm{Tr}(e) = D(e)$, these results can be quoted from [18, Theorem 3.1]; but it may be of interest to see that continuous geometry can be circumvented. First, Dixmier's approximation theorem [2, Théorème 7] is valid in AW*-algebras (and somewhat more generally; see [6, Theorem 1]. Consequently $\mathrm{Tr}(a)$ is the limit (in norm) of convex combinations $\sum \lambda_i u_i a u_i^*$ of unitary transforms of a [2, Théorème 3]. In particular $a \in I$ implies $\mathrm{Tr}(a) \in I$. If M is a maximal ideal, the discussion in [5, Lemme 15] is applicable: $\{a \in A : \mathrm{Tr}\,(a^*a) \in M\}$ is a proper ideal containing M, thus it coincides with M.

For the remainder of the section, it will be assumed that A is type II; Z is the center of A.

LEMMA 1.1. *If $c \in Z$ and $0 \leqq c \leqq 1$, there exists a projection e such that $D(e) = c$.*

Proof. Let (e_i) be a maximal family of orthogonal projections such that

$\sum_1^n D(e_i) \leqq c$ for every finite sum. Setting $e = \text{LUB } e_i$, we have $D(e) = \sum D(e_i) \leqq c$. If $c - D(e) \leqq 0$, we are through. Assume to the contrary; choose a nonzero central projection h and a number $\epsilon > 0$ such that $[c - D(e)]h \geqq \epsilon h$. Let n be an integer such that $2^{-n} \leqq \epsilon$, and construct a projection f such that $D(f) = 2^{-n}h$ [2, Lemme 6.2]. Since $D(f) \leqq \epsilon h \leqq [c - D(e)]h \leqq c - D(e) \leqq 1 - D(e) = D(1-e)$, we have $f < 1 - e$; passing to an equivalent projection, it can be assumed that $f \leqq 1 - e$. Then $D(f) \leqq c - D(e)$, $D(f) + D(e) \leqq c$, and maximality is contradicted.

Let the center Z of A be expressed as the algebra $C(\Sigma)$ of continuous functions on a Stone space Σ; the maximal ideals of Z are the ideals $N_\sigma = \{z \in Z: z(\sigma) = 0\}$, $\sigma \in \Sigma$. For $\sigma \in \Sigma$, we shall denote by $\mathfrak{J}_\sigma$ the set of projections $e \in A$ for which $D(e)(\sigma) = 0$. Since $D(e \cup f) \leqq D(e) + D(f)$, and since $e < f$ implies $D(e) \leqq D(f)$, it is immediate that $\mathfrak{J}_\sigma$ is a proper (excludes 1) p-ideal [18, Definition 2.1]. Proper p-ideals give rise to proper ideals of Z:

Lemma 1.2. *Let $\mathfrak{J}$ be a p-ideal in A, and define $P = \{\alpha D(e): \alpha > 0, e \in \mathfrak{J}\}$. Then $N = P - P + iP - iP$ is an ideal of Z, whose positive part is P, and whose projections are $\mathfrak{J} \cap Z$. In particular if $\mathfrak{J}$ is proper, N is a proper ideal.*

Proof. We will show that along with each of its elements, P contains all smaller positive elements of Z, and that P is closed under addition. Suppose $0 \leqq z \leqq \alpha D(e)$, $e \in \mathfrak{J}$, $\alpha > 0$. Then $0 \leqq \alpha^{-1} z \leqq D(e) \leqq 1$; by Lemma 1.1 there is a projection f such that $D(f) = \alpha^{-1} z$. Since $D(f) \leqq D(e)$, $f < e$, and since $\mathfrak{J}$ is a p-ideal, $f \in \mathfrak{J}$. Then $z = \alpha D(f)$ shows that $z \in P$. If $\alpha D(e)$ and $\beta D(f)$ are elements of P, then $e \cup f \in \mathfrak{J}$, $\alpha D(e) + \beta D(f) \leqq \alpha D(e \cup f) + \beta D(e \cup f) = (\alpha + \beta) D(e \cup f)$; thus $\alpha D(e) + \beta D(f) \in P$ by the first part of the proof.

It is now straightforward to show that N is an ideal, with positive part P, and of course $\mathfrak{J} \cap Z \subset P$. Suppose conversely that h is a projection in N, say $h = \alpha D(e)$. Choose an integer n such that $2^n \geqq \alpha$, and a projection f such that $D(f) = 2^{-n} h$. Then $D(f) \leqq \alpha^{-1} h = D(e)$, $f < e$, $f \in \mathfrak{J}$; since h is the sum of 2^n projections equivalent to f, $h \in \mathfrak{J}$.

Theorem 1.3. *Every maximal p-ideal $\mathfrak{J}$ of A has the form $\mathfrak{J} = \mathfrak{J}_\sigma$ for some $\sigma \in \Sigma$.*

Proof. Use $\mathfrak{J}$ to define the proper ideal N as in Lemma 1.2. Say $N \subset N_\sigma$, $\sigma \in \Sigma$. If $e \in \mathfrak{J}$, then $D(e) \in N$, $D(e) \in N_\sigma$, $D(e)(\sigma) = 0$, $e \in \mathfrak{J}_\sigma$; thus $\mathfrak{J} \subset \mathfrak{J}_\sigma$, and since $\mathfrak{J}$ is maximal, $\mathfrak{J} = \mathfrak{J}_\sigma$.

Conversely, every $\mathfrak{J}_\sigma$ is maximal. For, let $\mathfrak{J}$ be a maximal p-ideal such that $\mathfrak{J}_\sigma \subset \mathfrak{J}$. By Theorem 1.3, $\mathfrak{J} = \mathfrak{J}_\tau$ for some $\tau \in \Sigma$, thus $\mathfrak{J}_\sigma \subset \mathfrak{J}_\tau$. Necessarily $\sigma = \tau$; otherwise a central projection h could be found such that $h(\sigma) = 0$ and $h(\tau) = 1$, in other words $h \in \mathfrak{J}_\sigma$ but $h \notin \mathfrak{J}_\tau$. Thus $\mathfrak{J}_\sigma = \mathfrak{J}_\tau = \mathfrak{J}$ is maximal.

Corollary 1.4. *If M is a maximal ideal of A, and $e \in A$ is a projection, then $e \in M$ if and only if $D(e) \in M$.*

Proof. Let $\mathfrak{J}$ be the set of projections of M, so that $\mathfrak{J}$ is a maximal p-ideal

[18, Theorem 2.4]. By Theorem 1.3, $\mathfrak{J}=\mathfrak{J}_\sigma$, $\sigma\in\Sigma$. On the other hand, $M\cap Z$ is a maximal ideal of Z, say $M\cap Z=N_\tau$ [18, Theorem 2.6]. If h is a central projection, the following are equivalent: $h(\tau)=0$, $h\in N_\tau$, $h\in M$, $h\in\mathfrak{J}$, $h\in\mathfrak{J}_\sigma$, $h(\sigma)=0$; thus $\sigma=\tau$, $M\cap Z=N_\sigma$. Then for a projection $e\in A$, the following are equivalent: $e\in M$, $e\in\mathfrak{J}$, $e\in\mathfrak{J}_\sigma$, $D(e)(\sigma)=0$, $D(e)\in N_\sigma$, $D(e)\in M$.

LEMMA 1.5. *If P is a primitive ideal of A, there is exactly one maximal ideal M such that $P\subset M$; one has $P\cap Z=M\cap Z$.*

Proof. If h is a central projection, either $h\in P$ or $1-h\in P$; for, A/P has scalar center [9, Lemma 9]. Let M be any maximal ideal containing P; of course the projections in $P\cap Z$ belong to $M\cap Z$. Conversely if $h\in M$ is a central projection, necessarily $h\in P$; otherwise $1-h\in P\subset M$, $1\in M$. Thus the closed ideals $P\cap Z$, $M\cap Z$ of Z contain the same projections; since they are generated (as closed ideals) by their projections, $P\cap Z=M\cap Z$. Uniqueness of M follows from weak centrality [18, Theorem 2.5].

THEOREM 1.6. *If I is a closed ideal of A, $e\in I$ implies $D(e)\in I$.*

Proof. Since the C*-algebra A/I is semi-simple, I is the intersection of primitive ideals, say $I=\bigcap P_\alpha$. By Lemma 1.5 there is a maximal ideal M_α such that $P_\alpha\subset M_\alpha$ and $P_\alpha\cap Z=M_\alpha\cap Z$. If $e\in I$, then $e\in P_\alpha\subset M_\alpha$, $D(e)\in M_\alpha$ by Corollary 1.4, $D(e)\in M_\alpha\cap Z=P_\alpha\cap Z\subset P_\alpha$; since α is arbitrary, $D(e)\in I$.

The following two corollaries have to do with defining dimension in homomorphic images:

COROLLARY 1.7. *If I is a closed ideal of A, and $e, f\in A$ are projections such that $e-f\in I$, then $D(e)-D(f)\in I$.*

Proof. There are orthogonal decompositions $e=e'+e''$, $f=f'+f''$, such that $e'\sim f'$ and $e''f=ef''=0$ [17, Theorem 1]; specifically, $e'=\mathrm{LP}(ef)$, $f'=\mathrm{RP}(ef)$, $e''=e-e'$, $f''=f-f'$. Since I contains $e-f$, it contains $(e-f)e''=e''$, as well as $(e-f)f''=f''$. By Theorem 1.6, $D(e'')\in I$, $D(f'')\in I$. Since $D(e')=D(f')$, $D(e)-D(f)=D(e'')-D(f'')\in I$.

COROLLARY 1.8. *Let M be a maximal ideal of A, $e, f\in A$ projections such that $D(e)-D(f)\in M$. There exist subprojections $e_1\leqq e$, $f_1\leqq f$ such that $e_1\sim f_1$ and $e\equiv e_1$, $f\equiv f_1$ (mod M).*

Proof. By generalized comparability, write $e=e_1+e_2$, $f=f_1+f_2$, where $e_1\sim f_1$ and e_2, f_2 have orthogonal central covers. Say $h=C(e_2)$. By assumption M contains $D(e)-D(f)=D(e_2)-D(f_2)$, hence it contains $h[D(e_2)-D(f_2)]=D(e_2)$; hence it also contains $D(f_2)=D(e_2)-[D(e_2)-D(f_2)]$. By Corollary 1.4, M contains e_2, f_2, thus $e\equiv e_1$ and $f\equiv f_1$.

2. **The A/M theorem.** Throughout this section, M is a maximal ideal of A. We give a proof (due mainly to Wright and Kaplansky) of the Wright-Yen theorem: A/M is AW*. Say $M\cap Z=N_\sigma$, σ a suitable point in the spec-

trum of the center (see §1 for notation). The canonical mapping $A \to A/M$ will be written $a \to \bar{a}$. Any projection $u \in A/M$ can be expressed in the form $u = \bar{e}$, e a projection in A [18, Theorem 3.2]. We define $\overline{D}(u) = D(e)(\sigma)$; by Corollary 1.7, $\overline{D}(u)$ depends only on u, and by Corollary 1.4, $\overline{D}(u) = 0$ if and only if $u = 0$. Thus $\overline{D}$ is a well-defined "definite" numerical function on the set of projections of A/M; we proceed to show that $\overline{D}$ has the usual properties of a dimension function. The following lemma is valid for any closed ideal in any AW*-algebra, in view of the generality of [18, Theorem 3.2].

LEMMA 2.1. *Let u, v be equivalent projections in A/M. Say $u = \bar{e}$, $v = \bar{f}$, where e and f are projections in A. There exist subprojections $e_0 \leqq e$, $f_0 \leqq f$, and a partial isometry $w \in A$ such that: $w^*w = e_0$, $ww^* = f_0$, $\bar{e}_0 = u$, $\bar{f}_0 = v$, and $\bar{w}$ is the given partial isometry implementing $u \sim v$.*

Proof. By assumption there is an $x \in A$ such that $x^*x \equiv e$ and $xx^* \equiv f$ (mod M). Replacing x by fxe, one can assume $fx = x = xe$. Let $e_0 = \text{RP}(x) \leqq e$, $f_0 = \text{LP}(x) \leqq f$; let $x = wr$ be the canonical factorization [19, Lemma 2.1], $r \geqq 0$, $r^2 = x^*x$, $w^*w = e_0$, $ww^* = f_0$. Multiplying through $x^*x \equiv e$ by e_0, we have $x^*x \equiv e_0$, $e \equiv e_0$. Similarly $f \equiv f_0$. Since $r^2 = x^*x \equiv e_0$, $\bar{r}$ and $\bar{e}_0$ are positive square roots of $\bar{e}_0$; since positive square roots are unique in a C*-algebra, $\bar{r} = \bar{e}_0$. Thus $\bar{x} = \bar{w}\bar{r} = (we_0)^- = \bar{w}$.

In particular, if u, $v \in A/M$ are equivalent projections, they are unitarily equivalent; for, in the notation of the lemma, e_0 and f_0 are unitarily equivalent [7, Theorem 5.7], and we may pass to quotients. Consequently $u \sim v$ implies $1 - u \sim 1 - v$; and $u \leqq v$, $u \sim v$ imply $u = v$. This already shows that if A/M is indeed AW*, it is necessarily finite.

LEMMA 2.2. *Let u, $v \in A/M$ be projections. Then $u \sim v$ if and only if $\overline{D}(u) = \overline{D}(v)$.*

Proof. Say $u = \bar{e}$, $v = \bar{f}$. If $\overline{D}(u) = \overline{D}(v)$, then $D(e) - D(f) \in M$; applying Corollary 1.8 and passing to quotients, we have $u \sim v$. Conversely if $u \sim v$, Lemma 2.1 provides subprojections $e_0 \leqq e$, $f_0 \leqq f$ such that $e_0 \sim f_0$, $\bar{e}_0 = u$, $\bar{f}_0 = v$; then $\overline{D}(u) = D(e_0)(\sigma) = D(f_0)(\sigma) = \overline{D}(v)$.

LEMMA 2.3. *If u_1, u_2, u_3, $\cdots$ is a sequence of orthogonal projections in A/M, one can write $u_n = \bar{e}_n$ with the e_n orthogonal projections in A.*

Proof. Let $u_1 = \bar{e}_1$, $e_1 \in A$ a projection. If the canonical mapping $A \to A/M$ is restricted to $(1 - e_1)A(1 - e_1)$, the image is $(1 - u_1)A/M(1 - u_1)$; since the latter contains u_2, there is a projection $e_2 \leqq 1 - e_1$ such that $\bar{e}_2 = u_2$. The proof continues by induction.

If u, v are orthogonal projections in A/M, then $\overline{D}(u+v) = \overline{D}(u) + \overline{D}(v)$; for by Lemma 2.3 one can write $u = \bar{e}$, $v = \bar{f}$ with $ef = 0$, hence $\overline{D}(u+v) = \overline{D}(\bar{e} + \bar{f}) = \overline{D}[(e+f)^-] = D(e+f)(\sigma) = D(e)(\sigma) + D(f)(\sigma) = \overline{D}(u) + \overline{D}(v)$. The properties of $\overline{D}$ noted so far ensure that as soon as A/M is shown to be AW* (finite, scalar center), $\overline{D}$ will be the unique normalized numerical dimension function. From the additivity of $\overline{D}$, one sees that $\overline{D}(u) \leqq \overline{D}(v)$ when $u \leqq v$ (or when $u \prec v$). In particular if $u_1, \cdots, u_n$ are orthogonal, $\sum_1^n \overline{D}(u_i) = \overline{D}(u_1 + \cdots + u_n) \leqq 1$. It follows at once that A/M is of denumerable type: every family of orthogonal nonzero projections is countable. In the next lemma, the passage to subprojections is crucial for later arguments (Lemma 5.2).

LEMMA 2.4. *Let $u_1, u_2, \cdots$ be a sequence of orthogonal projections in A/M; say $u_n = \bar{e}_n$, with the e_n orthogonal. There exist subprojections $f_n \leqq e_n$ such that:* (1) $\bar{f}_n = u_n$; (2) *on setting $f = LUB\ f_n$ and $u = \bar{f}$, one has $u = LUB\ u_n$. One has $\overline{D}(u) = \sum_1^\infty \overline{D}(u_n)$.*

Proof. Since the type I case can be read out of [18, Theorem 5.1], we assume A is type II. Set $\alpha_n = \overline{D}(u_n) = D(e_n)(\sigma)$. By Lemma 1.1, there is a projection $g_n \in A$ such that $D(g_n) = \alpha_n$ (the constant function). In particular $D(g_n)(\sigma) = \alpha_n = D(e_n)(\sigma)$, $D(g_n) - D(e_n) \in M$; by Corollary 1.8, there are subprojections $h_n \leqq g_n$, $f_n \leqq e_n$, such that $h_n \sim f_n$ and $\bar{f}_n = \bar{e}_n = u_n$. Set $\alpha = \sum_1^\infty \alpha_n$ (convergent and $\leqq 1$), and $f = \mathrm{LUB}\ f_n$. Then $D(f) = \sum D(f_n) = \sum D(h_n) \leqq \sum D(g_n) = \sum \alpha_n = \alpha$, and in particular $D(f)(\sigma) \leqq \alpha$; thus $\overline{D}(u) \leqq \alpha$, where $u = \bar{f}$. On the other hand, $f \geqq f_n$, $u = \bar{f} \geqq \bar{f}_n = u_n$, $u \geqq \sum_1^m u_n$, $\overline{D}(u) \geqq \sum_1^m \overline{D}(u_n) = \sum_1^m \alpha_n$, $\overline{D}(u) \geqq \alpha$. Thus $\overline{D}(u) = \alpha$.

We now show that $u = \mathrm{LUB}\ u_n$. Already $u \geqq u_n$ for all n. Suppose conversely that $v \in A/M$ is a projection such that $u_n \leqq v$ for all n, and assume to the contrary that u is not $\leqq v$, that is, $u(1-v) \neq 0$. Let $v = \bar{g}$, and set $a = f(1-g)$. We are assuming $\bar{a} \neq 0$, equivalently $\bar{a}\bar{a}^* \neq 0$, $aa^* \notin M$. By spectral theory, aa^* is the limit in norm of linear combinations of projections, each a multiple of aa^*; not all of these projections can lie in M, since this would imply $aa^* \in M$ (M is closed). Thus let $h \in A$ be a projection such that $h \notin M$ and $aa^*b = h$ for suitable $b \in A$. Since $fa = a$, clearly $h \leqq f$. Since $u_n\bar{a} = u_n u(1-v) = u_n(1-v) = u_n - u_n v = 0$, also $u_n \bar{h} = 0$. Thus u_n, $\bar{h}$ are orthogonal projections $\leqq u$, hence $\sum_1^\infty \overline{D}(u_n) + \overline{D}(\bar{h}) \leqq \overline{D}(u)$, $\alpha + \overline{D}(\bar{h}) \leqq \alpha$, $\overline{D}(\bar{h}) = 0$, $\bar{h} = 0$, $h \in M$, contradiction.

To prove that A/M is AW*, let $\{x_j\}$ be any family of elements of A/M; it must be shown that the right annihilator of the family is the principal right ideal generated by a projection u. Let (u_n) be a maximal family (necessarily countable) of nonzero orthogonal projections such that for each j, $x_j u_n = 0$ for all n. Set $u = \mathrm{LUB}\ u_n$. Assertion: $x_j u = 0$. If not, there is a nonzero projection v such that $v = yux_j^* x_j u$ for suitable $y \in A/M$ (see the proof of Lemma 2.4). Clearly $vu = v$; but $(x_j u)u_n = x_j u_n = 0$ implies $vu_n = 0$, $u_n \leqq 1 - v$, $u \leqq 1 - v$, $uv = 0$, thus $v = uv = 0$, contradiction.

Consequently the right annihilator of the family $\{x_j\}$ contains $u(A/M)$. Conversely if $x_j z = 0$ for all j, necessarily $uz = z$. For if $(1-u)z \neq 0$, then for suitable $y \in A/M$, $(1-u)zz^*(1-u)y$ is a nonzero projection v. Evidently

$v \leqq 1-u$; also $x_j(1-u)z=(x_j-x_ju)z=x_jz=0$ implies $x_jv=0$ for all j. Since $vu=0$, maximality is contradicted.

3. Some preliminary algebra. Throughout this section, B is a ring with involution $*$, with axioms to be added from time to time. The axioms are satisfied by A and by its regular enlargement C; our reason for introducing B is to discuss simultaneously A, C, and certain of their homomorphic images. The subset of B right annihilating an element x will be denoted $R(x)$.

AXIOM (i). *For every $x \in B$, there is a projection e such that $R(x)=eB$ (see [7, Theorem 2.3], and [1, Corollary 7.3]).*

Putting $x=0$, one sees that B has a unity element. The projection e is unique; $f=1-e$ is the smallest projection such that $xf=x$, in the sense that if g is any other, necessarily $f=fg$. Notation: $f=\mathrm{RP}(x)$, the right projection of x; $xy=0$ is equivalent to $fy=0$. Similarly for the left projection, $\mathrm{LP}(x)$.

If $x^*x=0$, necessarily $x=0$; for if $e=\mathrm{LP}(x)$, $x^*e=0$, $x=ex=0$. It follows that $R(y^*y)=R(y)$, hence $\mathrm{RP}(y^*y)=\mathrm{RP}(y)$. The relation $e=ef$ defines a partial ordering $e \leqq f$ in the set of all projections. To get a lattice, we introduce

AXIOM (ii). *If $x^*x+y^*y=0$, then $x=y=0$ (see [1, Lemma 3.4]).*

LEMMA 3.1. *The projections of B form a lattice, specifically $e \cup f=RP(e+f)$. More generally, if $e=RP(x)$ and $f=RP(y)$, then $e \cup f=RP(x^*x+y^*y)$.*

Proof. Let $g=\mathrm{RP}\ (x^*x+y^*y)$. Since $(x^*x+y^*y)(1-g)=0$, $(1-g)(x^*x+y^*y)(1-g)=0$, $[x(1-g)]^*[x(1-g)]+[y(1-g)]^*[y(1-g)]=0$, axiom (ii) gives $x(1-g)=y(1-g)=0$, $e(1-g)=f(1-g)=0$, $e \leqq g$ and $f \leqq g$. On the other hand if $e \leqq h$ and $f \leqq h$, then $1-h$ right annihilates x, y, and therefore x^*x+y^*y; hence $g(1-h)=0$, $g \leqq h$. Thus $g=e \cup f$.

AXIOM (iii). *Given any $x, y \in B$, there is a $z \in B$ such that $x^*x+y^*y=z^*z$ (see [1, Corollary 6.2]).*

By induction any finite sum $x_1^*x_1+ \cdots +x_n^*x_n$ can be expressed in the form z^*z; Axiom (ii) can be written for finite sums, and yields $\mathrm{RP}\ (x_1^*x_1+ \cdots +x_n^*x_n)=\mathrm{RP}\ (x_1) \cup \cdots \cup \mathrm{RP}\ (x_n)$.

An element $x \in B$ is *positive*, notation $x \geqq 0$, in case $x=y^*y$ for some $y \in B$. By Axiom (iii), $x \geqq 0$ and $y \geqq 0$ imply $x+y \geqq 0$. If $x \geqq 0$, $z^*xz \geqq 0$ for all z. If $x \geqq 0$ and $-x \geqq 0$, then $x=0$ by Axiom (ii). If $x, y \in B$ are self-adjoint, $x \leqq y$ means $y-x \geqq 0$; the usual properties of an order relation are verified. There is no conflict with the earlier ordering for projections:

LEMMA 3.2. *For projections e, f, the following are equivalent:*

(1) $$e = ef,$$

(2) $$f - e \text{ is a projection},$$

(3) $$f - e \geqq 0.$$

Proof. The relations (1)⇒(2)⇒(3) are obvious. Suppose $f-e=x^*x$. Then $f=e+x^*x$, $f=\mathrm{RP}(f)=\mathrm{RP}(e^*e+x^*x)=e\cup\mathrm{RP}(x)$, hence $e=ef$; that is, $e\leqq f$ in the earlier sense.

An element $w\in B$ is a *partial isometry* if w^*w is a projection. Say $w^*w=e$; since $(we-w)^*(we-w)=0$, $we=w$, hence $f=ww^*$ is a projection. Thus w^* is also a partial isometry. Projections e, f are called *equivalent* if there is a partial isometry w such that $w^*w=e$, $ww^*=f$; notation: $e\sim f$.

Axiom (iv). *Any $x\in B$ can be written $x=wr$, $r\geqq 0$, $r^2=x^*x$ (see* [19, *Lemma* 2.1], *and* [1, *Corollary* 7.4]).

Assume $x=wr$ as above, and let $e=\mathrm{RP}(x)$; then $e=\mathrm{RP}(x^*x)=\mathrm{RP}(r^2)=\mathrm{RP}(r)$. Thus $x=wr=(we)r$; replacing w by we, we can assume $we=w$. Necessarily $w^*x=r$; for, $(x^*w-r)r=x^*wr-r^2=x^*x-r^2=0$, hence $(x^*w-r)e=0$, $x^*w-r=0$. It follows that $w^*w=e$; for, $(w^*w-e)r=w^*wr-er=w^*x-r=0$, hence $(w^*w-e)e=0$, $w^*w-e=0$. Let $f=\mathrm{LP}(x)$. Then $ww^*=f$. For, $ww^*x=wr=x$ shows that $f\leqq ww^*$; on the other hand $(1-f)x=0$, $(1-f)wr=0$, $[(1-f)w]e=0$, $(1-f)w=0$, $(1-f)ww^*=0$, $ww^*\leqq f$. Thus:

Lemma 3.3. *Any $x\in B$ can be factored $x=wr$, $r\geqq 0$, $r^2=x^*x$, $w^*x=r$, $w^*w=RP(x)$, $ww^*=LP(x)$. In particular $RP(x)\sim LP(x)$.*

All that is really needed in the above is $r^*=r$, but $r\geqq 0$ will be used in Lemma 3.9. Since $w^*xw^*=rw^*=(wr)^*=x^*$, every (two-sided) ideal I of B is self-adjoint; passing to quotients, B/I admits an involution.

Just as in [7, Lemma 5.3], one computes that $\mathrm{LP}[e(1-f)]=e-e\cap f$ and $\mathrm{RP}[e(1-f)]=e\cup f-f$, hence

Corollary 3.4. *For any pair of projections $e, f\in B$,*

$$e\cup f-f\sim e-e\cap f.$$

Corollary 3.5. *If I is any ideal in B, its projections form a p-ideal.*

Proof. The same as [18, Lemma 2.1].

Lemma 3.6. *If $\mathfrak{J}$ is a p-ideal, then $I=\{x\in B\colon RP(x)\in\mathfrak{J}\}$ is an ideal, the smallest containing $\mathfrak{J}$, and $\mathfrak{J}$ is the set of projections in I.*

Proof. If $x\in I$, then $\mathrm{RP}(x^*)=\mathrm{LP}(x)\sim\mathrm{RP}(x)\in\mathfrak{J}$ shows that $x^*\in I$. If $x, y\in I$, then $\mathrm{RP}(x+y)\leqq\mathrm{RP}(x)\cup\mathrm{RP}(y)\in\mathfrak{J}$, thus $x+y\in I$. If $x\in I$ and $y\in B$, then $\mathrm{RP}(yx)\leqq\mathrm{RP}(x)\in\mathfrak{J}$, $yx\in I$. Suppose I' is any other ideal containing $\mathfrak{J}$; if $x\in I$ and $e=\mathrm{RP}(x)$, then $x=xe$ shows that $x\in I'$, thus $I\subset I'$. For a projection e, $\mathrm{RP}(e)=e$, so I contains no new projections.

An ideal I is said to be *restricted* if it is generated by its projections [3, Definition 3.3]; in other words, $x\in I$ if and only if $\mathrm{RP}(x)\in I$ (Lemma 3.6).

For use in the next proof, we observe that if $x, y\in B$ and $\mathrm{RP}(x)=e$, then $\mathrm{RP}(xy)=\mathrm{RP}(ey)$. For, the following relations are equivalent: $(xy)z=0$, $x(yz)=0$, $e(yz)=0$, $(ey)z=0$; thus $R(xy)=R(ey)$.

THEOREM 3.7. *If I is a restricted ideal of B, the *-ring B/I satisfies axioms* (i) *through* (iv). *If $x \to \bar{x}$ is the canonical mapping $B \to B/I$, then*:

(1) $RP(\bar{x}) = [RP(x)]^{-}$,

(2) *all projections of B/I are of the form $\bar{e}$, e a projection of B,*

(3) $(e \cup f)^{-} = \bar{e} \cup \bar{f}$,

(4) $x \geqq 0$ *implies* $\bar{x} \geqq 0$; *and if* $\bar{x} \geqq 0$, *one can write* $\bar{x} = \bar{y}$ *with* $y \geqq 0$.

Proof. Let $\bar{x} \in B/I$, $e = \mathrm{RP}(x)$; by the preceding remarks, the following relations are equivalent: $\bar{x}\bar{y} = 0$, $xy \in I$, $\mathrm{RP}(xy) \in I$, $\mathrm{RP}(ey) \in I$, $ey \in I$, $\bar{e}\bar{y} = 0$. This proves Axiom (i) and (1). Axiom (ii): if $\bar{x}^*\bar{x} + \bar{y}^*\bar{y} = 0$, then $x^*x + y^*y \in I$, thus I contains $\mathrm{RP}(x^*x + y^*y) = \mathrm{RP}(x) \cup \mathrm{RP}(y)$, $\mathrm{RP}(x)$ and $\mathrm{RP}(y)$, hence x and y; thus $\bar{x} = \bar{y} = 0$. Axioms (iii), (iv), and assertion (4) are obvious. If $u = \bar{x}$ is a projection in B/I, and $e = \mathrm{RP}(x)$, then $u = \mathrm{RP}(u) = \mathrm{RP}(\bar{x}) = [\mathrm{RP}(x)]^{-} = \bar{e}$. If $e, f \in B$ are projections, then $\bar{e} \cup \bar{f} = \mathrm{RP}(\bar{e} + \bar{f}) = [\mathrm{RP}(e+f)]^{-} = (e \cup f)^{-}$.

THEOREM 3.8. *Let u, v be equivalent projections in B/I, I a restricted ideal. Say $u = \bar{e}$, $v = \bar{f}$. Then there exist subprojections $e_0 \leqq e$, $f_0 \leqq f$, and a partial isometry $w \in B$ such that: $w^*w = e_0$, $ww^* = f_0$, $\bar{e}_0 = u$, $\bar{f}_0 = v$, and $\bar{w}$ is the given partial isometry implementing $u \sim v$.*

Proof. The argument proceeds as in Lemma 2.1, until we have $\bar{r}$, $\bar{e}_0$ positive square roots of $\bar{e}_0$. Since $re_0 = r$, $\bar{r}$ and $\bar{e}_0$ commute. Then $\bar{r} = \bar{e}_0$ results from the following lemma, which might as well (Theorem 3.7) be proved in B:

LEMMA 3.9. *If $x \geqq 0$, $y \geqq 0$, $xy = yx$, and $x^2 = y^2$, then $x = y$.*

Proof. Same as in [1, Corollary 6.2].

LEMMA 3.10. *If I is a restricted ideal of B, and $u_1, u_2, u_3, \cdots$ is a sequence of orthogonal projections in B/I, one can write $u_n = \bar{e}_n$, with $e_1, e_2, e_3, \cdots$ a sequence of orthogonal projections in B.*

Proof. By Theorem 3.7, $u_1 = \bar{e}_1$, $e_1 \in B$ a projection. Since $(1-e_1)B(1-e_1)$ maps canonically onto $(1-u_1)B/I(1-u_1)$, suppose $x \in (1-e_1)B(1-e_1)$ with $\bar{x} = u_2$. Set $e_2 = \mathrm{RP}(x) \leqq 1 - e_1$; then $u_2 = \mathrm{RP}(u_2) = [\mathrm{RP}(x)]^{-} = \bar{e}_2$. The proof proceeds by induction.

Up to now the discussion is relevant for either C or A (or any AW*-algebra) in the role of B. What follows has content only for C; see the definition following

AXIOM (v). *B is an algebra over the complex numbers, and * is conjugate linear.*

An element $a \in B$ is *bounded* if there exists a real number $\alpha \geqq 0$ such that $a^*a \leqq \alpha$. The set of bounded elements is denoted B_0. (If $B = C$, then $B_0 = A$ by [1, Lemma 5.1].)

LEMMA 3.11. *B_0 is a self-adjoint subalgebra of B, containing all partial isometries (hence all projections). B_0 satisfies all axioms to date, and for $a\in B_0$, $RP(a)$ is the same whether computed in B_0 or in B.*

Proof. Suppose $w^*w=e$, e a projection; then $1-w^*w=1-e=(1-e)^*(1-e)\geqq 0$, $w^*w\leqq 1$, $w\in B_0$. Let $a, b\in B_0$, say $a^*a\leqq\alpha$ and $b^*b\leqq\beta$. Then $(a+b)^*(a+b)\leqq(a+b)^*(a+b)+(a-b)^*(a-b)=2a^*a+2b^*b\leqq 2\alpha+2\beta$, thus $a+b\in B_0$. Also $(ab)^*(ab)=b^*(a^*a)b\leqq\alpha b^*b\leqq\alpha\beta$, hence $ab\in B_0$. Clearly B_0 is a subalgebra of B. If $a\in B_0$, also $a^*\in B_0$; for if $a^*a\leqq\alpha$, and $a=wr$ as in Lemma 3.3, then $aa^*=wr^2w^*=w(a^*a)w^*\leqq\alpha ww^*=\alpha f\leqq\alpha$.

Since B_0 is a $*$-subalgebra containing all projections, it is clear that B_0 satisfies Axiom (i), and RP (a) is unambiguous for $a\in B_0$. Axiom (ii): a fortiori. Axiom (iii): suppose $a^*a\leqq\alpha$, $b^*b\leqq\beta$, and $a^*a+b^*b=z^*z$ with $z\in B$; then $z^*z\leqq\alpha+\beta$, $z\in B_0$. Axiom (iv): if $a\in B_0$ and $a=wr$ as above, then $r^2=a^*a\leqq\alpha$, thus w, $r\in B_0$.

In particular, for projections e, f, $e\sim f$ means the same thing in B_0 and B, and the lattice operations in B_0 are the same as those in B. We shall refer to B_0 as the *bounded subalgebra* of B.

AXIOM (vi). *If $x, y\in B$, $x\geqq 0$, $y\geqq 0$, and $x^2\leqq y^2$, there is a bounded element v such that $x=vy$. (See [1, Corollary 7.6].)*

If I is a restricted ideal of B, B/I has its own bounded subalgebra (Theorem 3.7). Perhaps Axiom (vi) does not survive passage to quotients, but we shall not need it.

THEOREM 3.12. *If I is a restricted ideal of B, the bounded subalgebra of B/I is the canonical image of the bounded subalgebra of B.*

Proof. Since $x\geqq 0$ implies $\bar{x}\geqq 0$, clearly $a\in B_0$ implies that $\bar{a}$ is in the bounded subalgebra of B/I. Suppose conversely $\bar{x}^*\bar{x}\leqq\alpha$; we can assume $\alpha=1$, $\bar{x}^*\bar{x}\leqq 1$. Say $1-\bar{x}^*\bar{x}=\bar{y}^*\bar{y}$, thus $x^*x+y^*y\equiv 1$ (mod I). Let $x=wr$ as in Lemma 3.3. Suppose $x^*x+y^*y=s^2$, $s\geqq 0$ (Axioms (iii), (iv)). Since $s^2\equiv 1$, one has $s\equiv 1$ by Lemma 3.9. Since $r^2=x^*x\leqq x^*x+y^*y=s^2$, Axiom (vi) provides $v\in B_0$ with $r=vs$. Then $\bar{x}=\bar{w}\bar{r}=\bar{w}\bar{v}\bar{s}=(wv)^-$, where w, $v\in B_0$.

REMARKS. (1). If I is a restricted ideal of B, $I\cap B_0$ is a restricted ideal of B_0; since both are determined by the same p-ideal, $I\rightarrow I\cap B_0$ is a one-one correspondence between the restricted ideals of B and B_0.

(2). If B is regular, and therefore $*$-regular [10, §2], an element of B and its right projection are multiples of each other; hence all ideals I of B are restricted, and $I\rightarrow I\cap B_0$ is a one-one correspondence between all ideals of B and all restricted ideals of B_0.

(3). If I is a restricted ideal of B, there is a natural mapping $B_0/I\cap B_0\rightarrow B/I$; it is a $*$-isomorphism into, and the image is the bounded subalgebra of B/I (Theorem 3.12). In particular the projection geometries of $B_0/I\cap B_0$ and B/I are identical.

(4). Eventually we will have occasion to discuss instances of B in connection with [10]. The notion of equivalence of projections employed there is slightly different; for convenience let us call it "algebraic equivalence": projections $e, f \in B$ are algebraically equivalent if there exist elements $x \in fBe$, $y \in eBf$ such that $yx = e$ and $xy = f$. But then $e \sim f$. For, let $e_0 = \mathrm{RP}(x) \leqq e$; then $x(e-e_0)=0$, $yx(e-e_0)=0$, $e(e-e_0)=0$, $e=e_0$, thus $e=\mathrm{RP}(x)$. Similarly $f=\mathrm{LP}(x)$, and we know that $\mathrm{RP}(x) \sim \mathrm{LP}(x)$. Thus, using only Axioms (i) through (iv), we see that the two notions of equivalence coincide in B.

4. **Restricted ideals of A and C.** Let $\mathfrak{J}$ be a p-ideal of A, I the ideal of A generated by $\mathfrak{J}$, namely $I = \{a \in A : \mathrm{RP}\,(a) \in \mathfrak{J}\}$. Denote by $a \rightarrow \tilde{a}$ the canonical mapping $A \rightarrow A/I$. It is well-known that Axioms (i) through (iv) of §3 are verified in A (and in any AW*-algebra), hence they are verified in A/I, by Theorem 3.7. Consequently A/I possesses all properties through Lemma 3.10 (the rest being relevant only for C). Specifically: the projections of A/I form a lattice; any projection $u \in A/I$ has the form $u = \tilde{e}$, e a projection in A; if $a \in A$ and $e = \mathrm{RP}(a)$, the right annihilator of $\tilde{a}$ is $(1-\tilde{e})A/I$; if $e, f \in A$ are projections, $(e \cup f)^{\sim} = \tilde{e} \cup \tilde{f}$; for any pair of projections $u, v \in A/I$, $u \cup v - u \sim v - u \cap v$; any sequence of orthogonal projections in A/I lifts to a sequence of orthogonal projections in A; any pair of equivalent projections in A/I can be lifted (along with the partial isometry) to equivalent projections in A. Because of the finiteness of A, we have further:

LEMMA 4.1. *The projection lattice of A/I is modular. Equivalent projections are unitarily equivalent.*

Proof. if $u \sim v$, one can write $u = \tilde{e}$, $v = \tilde{f}$ with $e \sim f$; since e, f are unitarily equivalent [7, Theorem 5.7], so are u, v. In particular $u \sim v$ implies $1-u \sim 1-v$; also $u \leqq v$, $u \sim v$ imply $u = v$. Modularity can now be proved exactly as in [7, Theorem 6.3]. Alternatively, suppose $u, v, w \in A/I$ are projections such that $u \leqq w$. Since $u(1-w)=0$, one can write $u=\tilde{e}$, $w=\tilde{g}$ with $e(1-g)=0$; if $v=\tilde{f}$, the modular law in A gives $(e \cup f) \cap g = e \cup (f \cap g)$, and passage to quotients gives the desired $(u \cup v) \cap w = u \cup (v \cap w)$.

For projections $u, v \in A/I$, $u \prec v$ means that u is equivalent to a subprojection of v. An easy consequence of finiteness (Lemma 4.1) is that $u \prec v$ and $v \prec u$ imply $u \sim v$. Generalized comparability holds in A/I: if u, v are any projections, there is a central projection h such that $hu \prec hv$ and $(1-h)v \prec (1-h)u$ (lift u, v to A, apply generalized comparability in A, pass to quotients).

Thus the projections of A/I form an orthocomplemented modular lattice; following Dye's terminology [4], we call it the projection geometry of A/I. By [15, Theorems 14.1, 4.3, 4.5], there is a *-regular ring R [10, §2] whose projection geometry is that of A/I; in this circle of ideas one must assume that A/I contains at least 4×4 matrix units. However, this regular ring is already at hand, with no assumptions needed about matrix units, and A/I is embedded in it:

THEOREM 4.2. *Let $\mathfrak{J}$ be a p-ideal of A, and I (resp. J) the ideal of A (resp. C) generated by $\mathfrak{J}$. The projections of A/I form an orthocomplemented modular lattice; the natural mapping $A/I \rightarrow C/J$ is a *-isomorphism onto the bounded subalgebra of the *-regular ring C/J, and in particular A/I and C/J have ortho-isomorphic projection geometries.*

Proof. Denote by $x \rightarrow \hat{x}$ the canonical mapping $C \rightarrow C/J$. If $x \in C$ and $e = \mathrm{RP}(x)$, one has $Cx = Ce$; hence $(C/J)\hat{x} = (C/J)\hat{e}$, thus C/J is *-regular. The bounded subalgebra of C is A, hence the bounded subalgebra of C/J is the set of all $\hat{a}$, $a \in A$ (Theorem 3.12). By Remarks 2, 3 and at the end of §3, $I = A \cap J$, and the mapping $\tilde{a} \rightarrow \hat{a}$ $(a \in A)$ is a *-isomorphism into; the image is the bounded subalgebra of C/J. By Theorem 3.7, the projections of A/I (resp. C/J) are all of the form $\tilde{e}$ (resp. $\hat{e}$), with e a projection of A; thus the correspondence $\tilde{e} \leftrightarrow \hat{e}$ identifies the projection geometries of A/I, C/J.

5. **Reduction of the projection geometry of** A. Let M be a maximal ideal of A, I the restricted ideal of A generated by the projections of M. Thus $I = \{a \in A : \mathrm{RP}(a) \in M\}$, and M is the closure of I. Without reference to the regular ring C, we shall prove that A/I is a Baer *-ring, and that its projections form an irreducible continuous geometry. The regular ring for this continuous geometry is C/J, where J is the ideal of C generated by the projections of M. The AW*-algebra A/M will play an auxiliary role. The notations for canonical mappings: $A \rightarrow A/I$ $(a \rightarrow \tilde{a})$, $A \rightarrow A/M$ $(a \rightarrow \bar{a})$, and $C \rightarrow C/J$ $(x \rightarrow \hat{x})$.

Since $I \subset M$, there is a natural mapping $\phi: A/I \rightarrow A/M$, namely $\phi(\tilde{a}) = \bar{a}$; ϕ is a *-homomorphism onto. Since every projection of A/I (resp. A/M) has the form $\tilde{e}$ (resp. $\bar{e}$) with e a projection in A, ϕ maps the projection geometry of A/I onto that of A/M, preserving orthogonal complements, order, equivalence. Moreover since I and M contain the same projections, $\bar{e} = 0$ is equivalent to $\tilde{e} = 0$; thus $\phi(\tilde{e}) = 0$ if and only if $\tilde{e} = 0$, ϕ is "faithful at the origin" on projections. But it is conceivable that $\bar{e} = \bar{f}$ without $\tilde{e} = \tilde{f}$.

LEMMA 5.1. *A/I is of denumerable type.*

Proof. That is, if $\{u_i\}$ is a family of orthogonal nonzero projections in A/I, the family is countable; this follows from the fact that $\{\phi(u_i)\}$ is a family of orthogonal nonzero projections in A/M, and A/M is of denumerable type (remarks preceding Lemma 2.4).

LEMMA 5.2. *Every sequence of orthogonal projections in A/I has a LUB.*

Proof. Let $u_1, u_2, \cdots$ be the given orthogonal sequence. By Lemma 3.10, write $u_n = \tilde{e}_n$, with the e_n orthogonal projections of A. Then $\bar{e}_n$ is an orthogonal sequence in A/M. By Lemma 2.4, there are subprojections $f_n \leqq e_n$ such that $\bar{f}_n = \bar{e}_n$, and such that on setting $f = \mathrm{LUB}\, f_n$ one has $\bar{f} = \mathrm{LUB}\ \bar{e}_n$. Our choice for LUB u_n is $u = \tilde{f}$. First, $u_n \leqq u$. For, $e_n - f_n \in M$, $e_n - f_n$ is a projection, hence $e_n - f_n \in I$; thus $\tilde{f}_n = \tilde{e}_n = u_n$; since $f_n(1-f) = 0$, passage to quotients in A/I

gives $u_n(1-u)=0$. Suppose $v\in A/I$ is a projection such that $u_n\leqq v$ for all n; it must be shown that $u(1-v)=0$. Say $v=\tilde{g}$. Assume to the contrary that $u(1-v)\neq 0$, that is $f(1-g)\notin I$. Let $f_0=\mathrm{LP}\ [f(1-g)]\leqq f$; then $f_0\notin I, \tilde{f}_0\neq 0$. We have $u_n[u(1-v)]=u_n(1-v)=u_n-u_nv=0$; since $\tilde{f}_0=\mathrm{LP}[u(1-v)]$ it follows that $u_n\tilde{f}_0=0$. Then $\phi(u_n)\phi(\tilde{f}_0)=0$, $\bar{e}_n\bar{f}_0=0$ for all n, $\bar{f}\bar{f}_0=0$, $(ff_0)^-=0$, $\bar{f}_0=0$, $\tilde{f}_0=0$, contradiction. We mention again for later use that $\overline{D}(\bar{f})=\sum_1^\infty \overline{D}(\bar{e}_n)$ (Lemma 2.4).

A ring with involution is a *Baer *-ring* if the right annihilator of any subset is the principal right ideal generated by a projection [11, Chapter III, Definition 2]; it follows at once that the projections form a complete lattice [11, Chapter III, Proposition 1].

THEOREM 5.3. *A/I is a Baer *-ring; in particular its projections form a complete lattice.*

Proof. Given a subset $\{x_j\}$ of A/I, let (u_n) be a maximal orthogonal family (necessarily countable) of nonzero projections such that for each j, $x_ju_n=0$ for all n. Set $u=\mathrm{LUB}\ u_n$ (Lemma 5.2). Necessarily $x_ju=0$; for if $v_j=\mathrm{RP}\ (x_j)$, then $v_ju_n=0$ for all n, $u_n\leqq 1-v_j$, $u\leqq 1-v_j$, $v_ju=0$, $x_ju=0$. Thus the right annhilator of the family $\{x_j\}$ includes $u(A/I)$. Conversely suppose $x_jy=0$ for all j. Assertion: $uy=y$. If not, let $u_0=\mathrm{LP}\ [(1-u)y]\neq 0$; clearly $u_0u=0$. For any j, $x_j[(1-u)y]=(x_j-x_ju)y=x_jy=0$, hence $x_ju_0=0$. This contradicts maximality of the family (u_n), since $u_0u_n=0$ for all n.

Thus the projections of A/I form an orthocomplemented complete modular lattice, hence a continuous geometry [10]. We shall see shortly that [10] can be circumvented entirely. Let us temporarily allow the regular ring C to intrude. The embedding $A/I\rightarrow C/J$, mapping projections onto projections, shows that C/J is a complete *-regular ring (Theorem 5.3). Thus we can already quote [10] at Theorem 3. A further shortcut: the proof of finiteness [10, Theorem 2] is covered by Lemma 4.1 and Remark 4 at the end of §3. After the next lemma, we can bypass [10], and refer the matter back to [7].

The next remarks will be used in Lemma 5.4. Suppose (u_n) is a sequence of projections in A/I, $u_n=\tilde{e}_n$, $e=\mathrm{LUB}\ e_n$, $u=\tilde{e}$, and suppose that $u=\mathrm{LUB}\ u_n$. Then it is permissible to pass to subprojections, in the following sense: if $f_n\leqq e_n$ and $\tilde{f}_n=\tilde{e}_n$, then $f=\mathrm{LUB}\ f_n$ also satisfies $\tilde{f}=\mathrm{LUB}\ u_n$. For, $f\leqq e$ implies $\tilde{f}\leqq\tilde{e}=u$; and $f_n\leqq f$ implies $\tilde{f}_n\leqq\tilde{f}$, $u_n\leqq\tilde{f}$, hence $u\leqq\tilde{f}$.

LEMMA 5.4. *If (u_n), (v_n) are sequences of orthogonal projections in A/I such that $u_n\sim v_n$ for all n, then LUB $u_n\sim$LUB v_n. More specifically, let $u=\mathrm{LUB}u_n$, $v=\mathrm{LUB}v_n$, and let x_n be a partial isometry such that $x_n^*x_n=u_n$, $x_nx_n^*=v_n$. Then there exists a partial isometry x such that $x^*x=u$, $xx^*=v$, and $xu_n=x_n$ for all n.*

Proof. Write $u_n=\tilde{e}_n$ with the e_n orthogonal projections in A; similarly $v_n=\tilde{f}_n$, the f_n orthogonal projections in A. Dropping down to subprojections of the e_n (resp. f_n), we can assume from the proof of Lemma 5.2 that $e = \text{LUB } e_n$ satisfies $\tilde{e} = u$ (resp. $f = \text{LUB } f_n$ satisfies $\tilde{f} = v$). Dropping down still further, we can assume $e_n \sim f_n$ (by Theorem 3.8, and the remarks preceding this lemma). At the same time we have partial isometries $w_n \in A$ such that $w_n^* w_n = e_n$, $w_n w_n^* = f_n$, and $\tilde{w}_n = x_n$. By [8, Lemma 20], there is a partial isometry $w \in A$ such that $w^* w = e$, $ww^* = f$, and $we_n = w_n$. Then $x = \tilde{w}$ meets all requirements.

Let us return to the proof that the projection lattice is a continuous geometry. This lattice is complemented, modular (Lemma 4.1), and complete (Theorem 5.3), thus only the continuity axioms need to be verified. First note that for any pair of projections $u, v \in A/I$, either $u \prec v$ or $v \prec u$. For, if $h \in A$ is a central projection, either $h \in M$ or $1 - h \in M$ (A/M is simple), thus $h \in I$ or $1 - h \in I$; our assertion follows on lifting u, v to A, applying generalized comparability, and passing to quotients. The proof of the continuity axioms in [7, Theorem 6.5] can now be used as it stands; its main ingredients are comparability, the "parallelogram law" (Corollary 3.4), finiteness (Lemma 4.1), and additivity of equivalence (Lemma 5.4). Thus

THEOREM 5.5. *The projections of A/I form an irreducible continuous geometry.*

Proof. Observe that 0 and 1 are the only central projections in A/I. For suppose $u = \tilde{e}$ is central; then $\phi(u) = \bar{e}$ is central in A/M, $\bar{e} = 0$ or 1 (A/M is simple), $e \in M$ or $1 - e \in M$, $e \in I$ or $1 - e \in I$. Thus by [7, Theorem 6.6, part (a)], 0 and 1 are the only projections in A/I with unique complements; in other words the projection geometry is irreducible in the sense of continuous geometry.

The dimension function for this continuous geometry is already at hand. If $u = \tilde{e}$, set $\tilde{D}(u) = \overline{D}[\phi(u)]$, that is, $\tilde{D}(\tilde{e}) = \overline{D}(\bar{e}) = D(e)(\sigma)$, where σ is the character of the center determined by M. Since ϕ is a $*$-homomorphism, $\overline{D}$ is a dimension function on A/M, and $\phi(u) = 0$ only when $u = 0$, we have immediately: $0 \leqq \tilde{D}(u) \leqq 1$, $\tilde{D}(1) = 1$, $\tilde{D}(u) = 0$ only when $u = 0$, $\tilde{D}(u + v) = \tilde{D}(u) + \tilde{D}(v)$ when $uv = 0$, $u \sim v$ implies $\tilde{D}(u) = \tilde{D}(v)$. It follows that if $\tilde{D}(u) = \tilde{D}(v)$, then $u \sim v$; for, supposing by comparability that $u \sim w \leqq v$, one has $\tilde{D}(v - w) = \tilde{D}(v) - \tilde{D}(w) = \tilde{D}(v) - \tilde{D}(u) = 0$, $v - w = 0$. Similarly $u \prec v$ if and only if $\tilde{D}(u) \leqq \tilde{D}(v)$. If (u_n) is a sequence of orthogonal projections and $u = \text{LUB } u_n$, then $\tilde{D}(u) = \sum_1^\infty \tilde{D}(u_n)$; choice of proofs: (a) the same proof as [14, Lemma 8.3.2], or (b) the last line of the proof of Lemma 5.2. Thus if (u_n) is an increasing sequence of projections with LUB u, then $\tilde{D}(u) = \text{LUB } \tilde{D}(u_n)$ [14, Lemma 8.3.3]. The same result holds for increasingly directed families:

LEMMA 5.6. *If $u_j \uparrow u$, then $\tilde{D}(u_j) \uparrow \tilde{D}(u)$.*

Proof. The notation $u_j \uparrow u$ means: the indices form an ordered set; for any indices j_1, j_2, there is a j_3 such that $j_1 \leqq j_3$ and $j_2 \leqq j_3$; $j \leqq k$ implies $u_j \leqq u_k$;

finally $u=\text{LUB } u_j$. Set $\Delta_j=\tilde{D}(u_j)$; then $\Delta_j\leqq\tilde{D}(u)$, and $\Delta_j\uparrow$. Let $\Delta=\text{LUB } \Delta_j$, so that $\Delta_j\uparrow\Delta\leqq\tilde{D}(u)$. We must show $\Delta=\tilde{D}(u)$. Choose a sequence (Δ_{j_n}) such that $\Delta_{j_n}\to\Delta$; while we are at it, we can require $j_1\leqq j_2\leqq j_3\leqq\cdots$. Then if $v=\text{LUB } u_{j_n}$, we have $u_{j_n}\uparrow v$; by remarks preceding the lemma, $\tilde{D}(v)=\text{LUB } \tilde{D}(u_{j_n})=\text{LUB } \Delta_{j_n}=\Delta$. Thus it will suffice to prove $v=u$. Of course $v\leqq u$, so it is enough to show that if k is any fixed index, $u_k\leqq v$; equivalently, $u_k\cup v=v$. By associativity of the LUB operation, we have $u_k\cup u_{j_n}\uparrow u_k\cup v$, hence $\tilde{D}(u_k\cup u_{j_n})\uparrow\tilde{D}(u_k\cup v)$. For each n, choose an index k_n such that $k\leqq k_n$ and $j_n\leqq k_n$. Then $u_k\cup u_{j_n}\leqq u_{k_n}$, hence $\Delta_{j_n}=\tilde{D}(u_{j_n})\leqq\tilde{D}(u_k\cup u_{j_n})\leqq\tilde{D}(u_{k_n})\leqq\Delta$; letting $n\to\infty$, we have $\tilde{D}(u_k\cup u_{j_n})\uparrow\Delta$. But already $\tilde{D}(u_k\cup u_{j_n})\uparrow\tilde{D}(u_k\cup v)$, thus $\tilde{D}(u_k\cup v)=\Delta=\tilde{D}(v)$. Finally $\tilde{D}(u_k\cup v-v)=\tilde{D}(u_k\cup v)-\tilde{D}(v)=\Delta-\tilde{D}(v)=0$, $u_k\cup v-v=0$.

On the basis of Lemma 5.6, we give still another proof of the continuity axioms in A/I. Suppose $\{u_j\}$ is an increasingly directed family of projections with LUB u, in other words $u_j\uparrow u$. If v is any other projection, we must show $u_j\cap v\uparrow u\cap v$. Let $w=\text{LUB } (u_j\cap v)$; it is clear that $u_j\cap v\uparrow w\leqq u\cap v$. Since $u\cap v-u_j\cap v=u\cap v-u_j\cap(u\cap v)\sim(u\cap v)\cup u_j-u_j\leqq u-u_j$, we have $\tilde{D}(u\cap v-u_j\cap v)\leqq\tilde{D}(u)-\tilde{D}(u_j)$. Since $\tilde{D}(u)-\tilde{D}(u_j)\downarrow 0$ by Lemma 5.6, $\tilde{D}(u\cap v)=\text{LUB } \tilde{D}(u_j\cap v)$. In other words (Lemma 5.6), $\tilde{D}(u\cap v)=\tilde{D}(w)$, $\tilde{D}(u\cap v-w)=0$, $u\cap v-w=0$.

Finally, we restate the above results in terms of a reduction theory [12]:

THEOREM 5.7. *Let* $(\mathfrak{J}_\lambda)$ *be the family of all maximal* p*-ideals of the finite* AW^**-algebra* A, I_λ *the ideal of* A *generated by* $\mathfrak{J}_\lambda$, M_λ *the closure of* I_λ, G *the projection geometry of* A, G_λ *the projection geometry of* A/I_λ, $\theta_\lambda: G\to G_\lambda$ *the restriction to* G *of the canonical mapping* $A\to A/I_\lambda$, $\phi_\lambda: A/I_\lambda\to A/M_\lambda$ *the natural *-homomorphism onto. Then*:

(1) G_λ *is an irreducible continuous geometry*;

(2) θ_λ *maps* G *onto* G_λ, $\theta_\lambda(1-e)=1-\theta_\lambda(e)$, $\theta_\lambda(e\cup f)=\theta_\lambda(e)\cup\theta_\lambda(f)$, $\mathfrak{J}_\lambda$ *is the subset of* G *mapped onto* 0;

(3) *if* $\tilde{D}_\lambda$ *(resp.* $\overline{D}_\lambda$*) is the unique normalized dimension function for* G_λ *(resp.* A/M_λ*), one has* $\tilde{D}_\lambda=\overline{D}_\lambda\circ\phi_\lambda$.

If C *is the regular ring of* A, *and* J_λ *is the ideal of* C *generated by* $\mathfrak{J}_\lambda$, *then*:

(4) C/J_λ *is the regular ring of* G_λ, *and is a simple complete *-regular ring*;

(5) *the natural mapping* $\iota_\lambda: A/I_\lambda\to C/J_\lambda$ *is a *-isomorphism whose image is the bounded subalgebra of* C/J_λ.

Proof. (1), (2), (3) were noted above. Since p-ideals are in one-one correspondence with the ideals of C (Remark 2 at the end of §3), J_λ is a maximal ideal; thus C/J_λ is simple. The rest of (4), (5) is contained in Theorem 4.2. This system of irreducible representations of G is complete in the following sense: if $\theta_\lambda(e)=\theta_\lambda(f)$ for all λ, then $e=f$; for, $e-f$ lies in every I_λ, and the intersection of the I_λ (even the M_λ) is 0.

For the following remarks, adopt the notation prior to Theorem 5.7.

(1). The proof of continuity axioms given following Lemma 5.6 could just as well be used for A; this reverses the point of view in [7] (first continuity axioms, then dimension).

(2). The trio A/I, A/M, C/J illustrates the appropriateness of Baer $*$-ring as a unifying concept.

(3). Since the dimension functions of A/I and A/M have the same range of values, their projection geometries are jointly type II or type I; and if type I, the orders are the same. Thus [18, Theorem 5.1] provides a construction for Baer $*$-rings of type II. See Remark 4.

(4). If I is contained in M properly (the usual case), then A/M and A/I cannot be isomorphic. For, A/M is simple, while A/I has the nontrivial ideal M/I. Since M/I is the only maximal ideal of A/I, A/I is an example of a finite Baer $*$-ring which is not strongly semi-simple. The key to the situation is that M/I contains no projections other than 0 (M and I contain the same projections), thus the "EP-axiom" is violated with a vengeance [11, Chapter VII, Theorem 7].

(5). Let $I \subset M$ properly as in 4. Let C_M be the regular ring of A/M as provided by [1]. The bounded subalgebra (of C_M or of C/J) is characterized as the set of all x such that $x^*x \leqq n$ for some integer n. This shows that C_M and C/J cannot be isomorphic as rings with involution (if they were, their bounded subalgebras would be isomorphic; but A/M and A/I are not isomorphic even as rings). Still, it is conceivable that C_M and C/J might be isomorphic as rings. To put it another way, the projection geometries of A/M and A/I might be isomorphic as lattices ([15, Theorem 4.2] with appropriate assumptions about orders). We have not been able to resolve this question.

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