

ON FREE PRODUCTS OF GROUPS

BY

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1. **Introduction.** The homomorphisms between two free groups have been characterized by Herbert Federer and Bjarni Jónsson (see [FJ]⁽¹⁾) as being retractive, i.e., if G and H are free groups and if f is a homomorphism on G onto H , then $G = S * Z$ with f mapping S isomorphically onto H and Z into the identity. Their proof consists of a transfinite convergence process making strong use of Jakob Nielsen's procedure (see [NL] or [FJ]) for reducing a finite subset of a free group. In a footnote, they have expressed belief that their methods could be used to extend a theorem due to I. Grushko (see [GK] or [K2]) which characterizes the homomorphisms on a finitely generated free group F onto a free product as mapping some free factorization of F onto the given free factors. It is desired to remove the finitary restriction, the resulting statement amounting to a generalization of the above Federer-Jónsson result.

With this goal in mind, a procedure is given in §3 for reducing a subset of a free product. These reduction transformations are motivated by Nielsen's procedure and possess similar properties. Some important known theorems on the rank and the subgroups of a free product are obtained somewhat as by-products of this development.

The conjecture of Federer and Jónsson is verified in §4 (their methods apply quite well). The main result is Theorem 4.4 which can be stated in more familiar notation: if f is a homomorphism on the free group F onto

$$\prod_{i \in I}^* H_i,$$

then there exists J such that

$$F = \prod_{i \in I}^* J_i \quad \text{and} \quad f(J_i) = H_i \quad \text{for } i \in I.$$

An example is given in 4.5 of an indecomposable group with a decomposable homomorphic image.

In §5 some computability assumptions are made within the factors of a free product, and the reduction procedure thereby becomes effective for decision procedure purposes. This fact is applied to prescribe a recursive pro-

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(1) We designate references to the Bibliography at the end by symbols in brackets.

cedure for deciding whether or not a given element of a free product belongs to the subgroup generated by a given finite subset.

This paper is a revised version of the writer's thesis, written under the direction of his esteemed teachers, Professors Jónsson and Federer, and submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at Brown University, April 23, 1951. The revision⁽²⁾ has included simplifying the reduction procedure and the addition of §5. The writer is also indebted to Professor H. W. Kuhn for his constructive criticism of this paper.

2. Preliminaries. If X is a set, we denote the cardinal number of X by $\text{card } X$. In any numerical inequality, the numbers we compare are integers. When we write $b \in X^n$, we mean that $n \geq 0$ and b is an n -termed sequence, $\langle b_1, \dots, b_n \rangle$, with $b_1, \dots, b_n \in X$.

Suppose \mathcal{E} is an equivalence relation over the set X . For $x, y \in X$, according as x does or does not stand in the relation \mathcal{E} to y , we write

$$x \mathcal{E} y \quad \text{or} \quad x \text{ not } \mathcal{E} y.$$

We designate the \mathcal{E} -equivalence class of each $x \in X$ by

$$(x/\mathcal{E}) = \{y \mid x \mathcal{E} y\}.$$

In proving Theorem 3.5, we shall compare the fineness of certain equivalence relations over *finite* sets as measured by

$$\text{gauge } \mathcal{E} = \prod_{A \in \Delta} (\text{card } A + 1),$$

where

$$\Delta = \{(y/\mathcal{E}) \mid y \in X\}.$$

If G is a group, we define $[A]$ to be the subgroup of G generated by A for $A \subset G$, and by the *rank* of G we mean the minimum $\text{card } A$ such that $[A] = G$. Thus $\text{rank } \{e\} = 0$.

The properties of free groups that we use are quite elementary and can be found, for example, in §2 of [FJ].

Suppose P is a subset of the group G , τ is a function on P into G ,

$$C = \{r \mid \tau(r) = r\},$$

and

$$\tau(p) \in [C]p^{\pm 1}[C] \quad \text{for } p \in P.$$

We then say that τ is an *elementary transformation*⁽³⁾ of P . It is clear that

⁽²⁾ Substantially completed while the writer was on leave from the Operations Evaluation Group, Navy Department.

⁽³⁾ This is a special kind of the elementary transformations of sequential forms considered in [FJ].

$[\text{range } \tau] = [P]$. In fact, if $[P]$ is a free group freely generated by P , then $\text{range } \tau$ freely generates $[P]$.

Suppose G is a group and ϕ is a family of subgroups of G such that

$$G = \left[\bigcup_{H \in \phi} H \right]$$

and such that if K is any group and f_H is a homomorphism on H into K for $H \in \phi$, then there exists a homomorphism g on G into K such that

$$g(x) = f_H(x) \quad \text{for } x \in H \in \phi.$$

In this event, G is said to be the *free product* of the members of ϕ , in symbols,

$$G = \prod_{H \in \phi}^* H.$$

If $\phi = \{H_1, \dots, H_k\}$ with H_1, \dots, H_k distinct, this is also written

$$G = H_1 * \dots * H_k.$$

For $H \in \phi$, H is called a *free factor* of G . We call ϕ a *free factorization* of G .

It is clear that if ϕ and ϕ' are respectively free factorizations of the groups G and G' and are in one-one correspondence with corresponding factors isomorphic, then G is isomorphic to G' .

Suppose ϕ is a set of groups intersecting pairwise in a common identity e . We construct a group G as follows so that ϕ is a free factorization of G . Let G be the set of finite sequences over

$$\bigcup_{H \in \phi} H - \{e\},$$

every pair of consecutive terms of which belong to different members of ϕ . The product of two elements of G is obtained from the sequence they form in juxtaposition by amalgamating into one term the neighboring end terms if they belong to the same member of ϕ , deleting the term if the pair cancels, and repeating until neither cancellation nor amalgamation is possible. Identifying e with the vacuous sequence and u with $\langle u \rangle$ whenever $e \neq u \in H \in \phi$, one can check that this construction meets the requirements. (For an easy proof of associativity, see B. L. van der Waerden [WD].)

In view of the preceding two paragraphs, corresponding to each free factorization ϕ of a group G , each $x \in G$ is uniquely expressible in the form

$$x = \prod_{i=1}^n u_i,$$

where $e \neq u_i \in H_i \in \phi$ for $1 \leq i \leq n$ and $H_i \neq H_{i+1}$ for $1 \leq i < n$. Obviously this unique representation property implies the homomorphism property stated

in the definition of the free product. Another characterization of ϕ as a free factorization of G is the property that⁽⁴⁾

$$G = \left[\bigcup_{H \in \phi} H \right] \quad \text{and} \quad \prod_{i=1}^m v_i \neq e$$

whenever $m > 0$, $e \neq v_i \in H_i \in \phi$ for $1 \leq i \leq m$, and $H_i \neq H_{i+1}$ for $1 \leq i < m$.

The unique representation described in the preceding paragraph gives rise to a notion of length: n is called the *length* of x with respect to ϕ . In particular, e has zero length.

Throughout the remainder of this paper, we shall assume that ϕ is a free factorization of the group G . Most of our subsequent definitions will be made with respect to ϕ , but we shall neglect to denote this explicitly.

For $x \in G$, we denote the length of x with respect to ϕ by $L(x)$. We fix

$$\Sigma = \bigcup_{H \in \phi} H = \{u \mid L(u) \leq 1\},$$

$$\Gamma = \{xu x^{-1} \mid u \in \Sigma \text{ and } x \in G\}.$$

Suppose $x \in G$ and u is the unique member of $\Sigma^{L(x)}$ for which

$$x = \prod_{i=1}^{L(x)} u_i.$$

If $0 < \mu \leq L(x)$, then by $\pi_\mu(x)$ we mean the member H of ϕ for which $u_\mu \in H$, and by $I_\mu(x)$ we mean the μ -initial segment of x , i.e.,

$$u_\mu \in \pi_\mu(x) \in \phi \quad \text{and} \quad I_\mu(x) = \prod_{i=1}^{\mu} u_i.$$

If $\mu = 0$ or $L(x) < \mu$, we agree that

$$\pi_\mu(x) = \{e\} \quad \text{and} \quad I_\mu(x) = e.$$

We follow [K1] and [NM] in defining, for $x \in G$,

$$s(x) = I_\mu(x),$$

$$t(x) = I_\mu(x^{-1})^{-1},$$

$$c(x) = s(x)^{-1} x t(x)^{-1},$$

where μ is $L(x)/2$ or $(L(x)-1)/2$ according as $L(x)$ is even or odd. Thus

$$\Gamma = \{x \mid s(x) = t(x)^{-1}\},$$

and, for $x \in G$, $L(x)$ is even if and only if $c(x) = e$.

We find it convenient to fix an equivalence relation \mathcal{K} over $G - \{e\}$ suggested by the theorem of A. G. Kurosch on subgroups of a free product (see

⁽⁴⁾ Free products of groups have usually been defined in this way.

[K1]). For $x, y \in G - \{e\}$, we say that $x \mathcal{K} y$ if and only if

$$x = y \quad \text{or} \quad x = y^{-1} \quad \text{or} \quad x, y \in s(x)\pi_1(c(x))s(x)^{-1}.$$

If $b \in G^n$ and there exist no j, k with $1 \leq j \leq k \leq n$,

$$b_j \mathcal{K} b_{j+1} \mathcal{K} \cdots \mathcal{K} b_k, \quad \text{and} \quad \prod_{i=j}^k b_i = e,$$

then we say that b is *irredundant*. Obviously for $a \in G^m$ there exists a (possibly vacuous) subsequence b of a such that b is irredundant and

$$\prod_{i=1}^m a_i = \prod_j b_j.$$

If $b \in G^n$, b is irredundant, and $1 \leq j \leq k \leq n$, then $\langle b_j, \dots, b_k \rangle$ is irredundant.

We conclude this section with some lemmas (which can be proved by straightforward checking) regarding length in products of elements x, y, z of G . We shall apply Lemma 2.1 frequently but without reference.

It may help the reader in visualizing statements of this nature to use a graphical description such as that employed by F. W. Levi (see [LV]), i.e., represent $s(x)$ and $t(x)$ as sides of an isosceles triangle or trapezoid with $c(x)$ (if different from e) at the apex. The reader may also find useful the following summary of Lemma 2.1:

	$L(y)$ even	$L(y)$ odd
$L(x) > L(xy)$	more than half cancelled or amalgamated	center cancelled
$L(x) = L(xy)$	exactly half cancelled, no amalgamation	center amalgamated
$L(x) < L(xy)$	less than half cancelled	center neither cancelled nor amalgamated

2.1. LEMMA. Suppose $L(s(y)) = \mu$ and $L(s(y)c(y)) = \nu > 0$. Then

(i) $L(x) > L(xy)$ if and only if

$$I_\nu(x^{-1}) = s(y)c(y) \quad \text{and} \quad \pi_{\mu+1}(x^{-1}) = \pi_{\mu+1}(y);$$

(ii) $L(x) = L(xy)$ if and only if

$$I_\mu(x^{-1}) = s(y)$$

and

$$(I_\mu(x^{-1})^{-1}I_\nu(x^{-1}) \neq c(y) \in \pi_\nu(x^{-1}) \quad \text{or} \quad \pi_{\mu+1}(x^{-1}) \neq \pi_{\mu+1}(y)),$$

which in turn holds if and only if

$$I_\mu((xy)^{-1}) = t(y)^{-1} \quad \text{and} \quad \pi_{\mu+1}((xy)^{-1}) = \pi_{\mu+1}(x^{-1});$$

(iii) $L(x) < L(xy)$ if and only if

$$I_\mu(x^{-1}) \neq s(y) \text{ or } c(y) \notin \pi_\nu(x^{-1}),$$

which in turn holds if and only if

$$I_\nu((xy)^{-1}) = t(y)^{-1}c(y)^{-1} \text{ and } \pi_{\mu+1}((xy)^{-1}) = \pi_{\mu+1}(y^{-1}).$$

2.2. LEMMA. If $L(x) \leq L(xy)$ and $L(y) > L(z)$, then

- (i) $L(xy) > L(xyz)$ implies $L(y) > L(yz)$;
- (ii) $L(xy) = L(xyz)$ implies $L(y) = L(yz)$.

2.3. LEMMA. If $L(x) \leq L(xy) \leq L(xyz)$, then

- (i) $L(y) \geq L(z)$ implies $L(y) \geq L(yz)$;
- (ii) $L(y) \leq L(z)$ implies $L(yz) \leq L(z)$.

2.4. LEMMA. If $L(x) = L(xy) \geq L(y)$ and $L(xyz) < L(z)$, then $L(yz) = L(z)$.

2.5. LEMMA. If $L(x) < L(xy)$ and $L(yz) \geq L(z)$, then

- (i) $L(y) < L(yz)$ implies $L(xy) < L(xyz)$;
- (ii) $L(y) \leq L(z)$ implies $L(xy) \leq L(xyz)$.

2.6. LEMMA. If $L(x) \leq L(xy)$, $L(y) \geq L(z)$, and $L(y) \geq L(yz)$, then $L(xy) \geq L(xyz)$.

2.7. LEMMA. If $L(x) \leq L(xy) \leq L(y)$ and $xy \in \Gamma$, then $L(xy) \geq L(xyx)$.

3. A reduction procedure for free products. In this section we develop properties of the reduction procedure and the associated notion of irreducibility stated in Definition 3.1. We shall make strong use in §4 of Theorems 3.3, 3.5, and 3.11 which correspond to key properties of Nielsen's reductions discussed below. In §5, we shall apply these results again and in addition Theorem 3.17. The latter theorem describes the structure of elements of a subgroup of a free product in terms of irreducible generators. Some known results are provided by Theorems 3.13, 3.15, and 3.20.

Let us digress, at this point, to discuss the relationship of Definition 3.1 to various choices of generators elsewhere in the literature.

Our motivation comes from Nielsen's procedure (as applied in [FJ]) for reducing a finite subset A of a free group in a finite number of steps to a set which freely generates $[A]$. Nielsen's basic methods appear here as reductions of type (i) and (ii). A reduction of type (ii) must be effected by a multiplier not belonging to Γ , lest it be repeatable indefinitely. For this reason we include type (iii) reductions, and in making up a multiplier to effect a reduction of type (i) in general we add to x^α , in Definition 3.1(i), appropriate factors from Γ . Our reductions are illustrated in Example 3.2.

Because of the different notions of length, Nielsen's reductions do not coincide with the present reductions applied to a free group (which is, of course, a free product of infinite cyclic groups). Indeed, in case G is freely

generated by X with x, y, z distinct members of X ,

$$\begin{aligned}\phi &= \{[\{w\}] \mid w \in X\}, \\ P &= \{xy^2, y^{-2}z\}, & Q &= \{xy^2, xz\}, \\ P' &= \{x^3y, y^{-1}x^{-1}z^2\}, & \text{and } Q' &= \{x^3y, x^2z^2\},\end{aligned}$$

then Q is an X reduction of P and P is irreducible with respect to ϕ (i.e., in the sense of Definition 3.1), while Q' is a ϕ reduction of P' and P' is irreducible with respect to X .

Any subsequent mention of reductions or irreducibility refers to Definition 3.1.

From Theorem 3.6 we find that every finitely generated subgroup of G has an irreducible set of generators. This is not true in general for groups which are not finitely generated, because there exist groups which have no independent set of generators. However, it can be shown that each subgroup of G possesses essentially irreducible generators in the sense of Definition 3.14. (We do not use this fact except in proving known results.) For example, one can choose as generators the set K defined by Marshall Hall, Jr. (see [HL]) as those elements not generated by their predecessors in his semi-alphabetical well-ordering. Applying Lemma 2 of [HL] one easily checks that each finite subset of K can be made irreducible by eliminating superfluous transforms.

There exist finite irreducible sets which cannot be realized as Hall's set K . For example, if u, v, w, x, y, z differ from e and belong to distinct members of ϕ with $v^2 \neq e \neq y^2$ and

$$P = \{uvw, uv^2x, uyx, zy^2x\},$$

then P is such a set as is any Q with $Q \cup Q^{-1} = P \cup P^{-1}$, $Q \cap Q^{-1} = 0$.

H. W. Kuhn's method (see [KN]) of choosing generators using a system of coset functions called a Kurosch system does not in general yield an irreducible set. For example, if ϕ consists of four groups of order two generated by x, y, z, w respectively, then $\{xy, xzwy\}$ can be obtained by Kuhn's method and is reducible (type (ii)) to $\{xy, y^{-1}zwy\}$. We believe that it can be shown that any irreducible set can be obtained by suitably modifying Kuhn's methods, it being necessary to permit his fixed index α_0 to vary from coset to coset for the purpose^(*). Evidently Theorem 3.10 is required to prove this^(*).

The conditions imposed by Kurosch, B. H. Neumann, and M. Takahasi in their respective choices of generators in [K1], [NM], and [TK] do not insure irreducibility, although they do insure the property of irreducibility given by Theorem 3.11.

Let us now return to our development.

(*) Professor Kuhn joins in this conjecture.

3.1. DEFINITION. Suppose P is a subset of G , $q \in [P]$, and τ is a function on P into G . If (i)(a), (ii)(a), or (iii)(a) holds, we say that P is *reducible by the multiplier q* . If furthermore (i)(b), (ii)(b), or (iii)(b) respectively holds, we say that τ is a *reduction transformation of P effected by q* .

(i) There exist $x, z \in P$ and $\alpha, \gamma = \pm 1$ such that

(a) we have

$$q \in x^\alpha [t(x^\alpha)^{-1} \pi_1(c(x)) t(x^\alpha) \cap P - \{z\}],$$

$$x \neq z, \quad L(x) = L(q) \leq L(z), \quad \text{and} \quad L(qz^\gamma) < L(z);$$

(b) $\tau(z) = qz^\gamma$ and $\tau(p) = p$ for $z \neq p \in P$.

(ii) There exist $x, y, z \in P$ and $\alpha, \beta, \gamma = \pm 1$ such that

(a) we have

$$q = y^\beta, \quad y \notin \Gamma,$$

$$x \neq y \neq z, \quad L(x) \geq L(y) \leq L(z),$$

$$L(x) \geq L(x^\alpha q), \quad \text{and} \quad L(qz^\gamma) \leq L(z);$$

(b) for $p \in P$, if $p = y$ or $L(p) < L(q)$, then $\tau(p) = p$, otherwise

$$\tau(p) = \begin{cases} qpq^{-1} & \text{if } L(qp) \leq L(p) \leq L(pq^{-1}), \\ qp & \text{if } L(qp) \leq L(p) < L(pq^{-1}), \\ pq^{-1} & \text{if } L(qp) > L(p) \geq L(pq^{-1}), \\ p & \text{if } L(qp) > L(p) < L(pq^{-1}). \end{cases}$$

(iii) There exist $x, z \in P$, $\alpha, \gamma = \pm 1$, and $\mu > 0$ such that

(a) we have

$$L(s(x)) \geq \mu \leq L(s(z)),$$

$$q \in [I_\mu(z^\gamma) \pi_\mu(z^\gamma) I_\mu(z^\gamma)^{-1} \cap P] - \{e\}, \quad \text{and} \quad I_\mu(x^\alpha) = q I_\mu(z^\gamma);$$

(b) for $p \in P$, if $L(p) \leq L(q)$, then $\tau(p) = p$, otherwise

$$\tau(p) = \begin{cases} qpq^{-1} & \text{if } I_\mu(p) = I_\mu(z^\gamma) = I_\mu(p^{-1}), \\ qp & \text{if } I_\mu(p) = I_\mu(z^\gamma) \neq I_\mu(p^{-1}), \\ pq^{-1} & \text{if } I_\mu(p) \neq I_\mu(z^\gamma) = I_\mu(p^{-1}), \\ p & \text{if } I_\mu(p) \neq I_\mu(z^\gamma) \neq I_\mu(p^{-1}). \end{cases}$$

We shall refer to reducibility and reduction transformations as being of type (i), (ii), or (iii). By an *irreducible* set we mean a subset of G which is not reducible.

3.2. EXAMPLE. The need for each of the three types of reduction transformations in Definition 3.1 may be illustrated as follows. Assume that

a, b, c, u, v belong to distinct members of ϕ and have infinite order. Let

$$\begin{aligned} p_1 &= uav^{-3}b^{-1}a^{-1}, \\ p_2 &= abvb^{-1}a^{-1}, \\ p_3 &= abvu^{-2}c^{-1}, \\ p_4 &= cuc^{-1}, \\ p_5 &= cuvua, \\ P &= \{p_1, p_2, p_3, p_4, p_5\}. \end{aligned}$$

Let τ_1, τ_2 , and τ_3 be the mappings which leave p_1, p_2, p_3 , and p_4 fixed with

$$\begin{aligned} \tau_1(p_5) &= p_4p_5 = cu^2vua, \\ \tau_2(\tau_1(p_5)) &= p_3p_4p_5 = abv^2ua, \\ \tau_3(\tau_2(\tau_1(p_5))) &= p_1p_2p_3p_4p_5 = uaua. \end{aligned}$$

Then τ_1, τ_2 , and τ_3 are respectively reduction transformations of types (iii), (ii), and (i) effected by the respective multipliers p_4, p_3 , and p_1p_2 , and $\tau_3(\tau_2(\tau_1(P)))$ is irreducible. Each of the three types of reduction is necessary in that the other two do not suffice to obtain the property of irreducibility stated in Theorem 3.11 (with $x = uaua$).

3.3. THEOREM. *If $P \subset G$ and τ is a reduction transformation of P , then*

- (i) τ is an elementary transformation of P ;
- (ii) $L(\tau(p)) \leq L(p)$ for $p \in P$;
- (iii) if τ is of type (i), then $L(\tau(p)) < L(p)$ for some $p \in P$.

3.4. LEMMA. *If $k \geq h$ and $j \geq 0$, then*

$$(h + j + 1)(k + 1) \geq (h + 1)(j + k + 1),$$

with equality holding only if $h = k$ or $j = 0$.

3.5. THEOREM⁽⁶⁾. *A sequence of reduction transformations applied to a finite set must terminate. Precisely, if P_0, \dots, P_m are finite subsets of G , τ_i is a reduction transformation of P_i and range $\tau_i = P_{i+1}$ for $0 \leq i < m$, card $P_0 = n$, and $\lambda = \sum_{p \in P_0} L(p)$, then*

$$m + 1 \leq (\lambda + 1)2^{4\lambda n}.$$

Proof. For $0 \leq i \leq m$ and $\rho > 0$ we define $\mathfrak{A}_i^\rho, \mathfrak{B}_i^\rho$, and \mathfrak{C}_i^ρ to be the equivalence relations over $P_i \times \{1, -1\}$ such that if $p, r \in P_i$ and $\delta, \epsilon = \pm 1$, then

$$\begin{aligned} (p, \delta) \mathfrak{A}_i^\rho (r, \epsilon) &\text{ if and only if } I_\rho(s(p^\delta)) = I_\rho(s(r^\epsilon)), \\ (p, \delta) \mathfrak{B}_i^\rho (r, \epsilon) &\text{ if and only if } \pi_\rho(s(p^\delta)c(p)) = \pi_\rho(s(r^\epsilon)c(r)), \\ (p, \delta) \mathfrak{C}_i^\rho (r, \epsilon) &\text{ if and only if } (p, \delta) \mathfrak{A}_i^\rho (r, \epsilon) \text{ and } (p, \delta) \mathfrak{B}_i^{\rho+1} (r, \epsilon). \end{aligned}$$

⁽⁶⁾ We follow the pattern of 3.8(11) and 3.9 of [FJ].

Next we define Ξ to be the set of $(2\lambda+1)$ -termed sequences ξ of integers such that

$$0 \leq \xi_1 \leq \lambda \quad \text{and} \quad 1 \leq \xi_i \leq 2^{2^n} \quad \text{for} \quad 1 < i \leq 2\lambda + 1.$$

We order Ξ lexicographically so that if $\xi, \eta \in \Xi$, then $\xi < \eta$ if and only if there exists $j > 0$ such that

$$\xi_i = \eta_i \quad \text{for} \quad 1 \leq i < j \quad \text{and} \quad \xi_j < \eta_j.$$

For $0 \leq i \leq m$ we define

$$\xi^{(i)} = \langle \sum_{p \in P_i} L(p), \text{gauge } \mathcal{C}_i^\lambda, \text{gauge } \mathcal{A}_i^\lambda, \text{gauge } \mathcal{C}_i^{\lambda-1}, \text{gauge } \mathcal{A}_i^{\lambda-1}, \dots, \\ \text{gauge } \mathcal{C}_i^1, \text{gauge } \mathcal{A}_i^1 \rangle.$$

As a consequence of the inequality of the geometric and arithmetic means, we note that if $\text{card } X = 2n < \infty$ and \mathcal{E} is an equivalence relation over X , then $\text{gauge } \mathcal{E} \leq 2^{2^n}$. Accordingly, $\xi^{(i)} \in \Xi$ for $0 \leq i \leq m$. In order to show that

$$m + 1 \leq \text{card } \Xi = (\lambda + 1)2^{4\lambda n},$$

we assume that $0 \leq i < m$ and show that $\xi^{(i+1)} < \xi^{(i)}$. This is evident if

$$\sum_{p \in P_i} L(\tau_i(p)) < \sum_{p \in P_i} L(p), \text{ i.e., } \xi_1^{(i+1)} < \xi_1^{(i)}.$$

By Theorem 3.3 we may therefore assume that τ_i is not of type (i) and

$$L(\tau_i(p)) = L(p) \quad \text{for } p \in P_i, \text{ i.e., } \xi_1^{(i+1)} = \xi_1^{(i)}.$$

Assuming further that τ_i is effected by the multiplier q and $L(s(q)) = \nu$, we complete the proof with the following statements:

- (i) $\text{gauge } \mathcal{A}_i^p \geq \text{gauge } \mathcal{A}_{i+1}^p$ and $\text{gauge } \mathcal{C}_i^p \geq \text{gauge } \mathcal{C}_{i+1}^p$ for $p > \nu$;
- (ii) if τ_i is of type (iii), then $\text{gauge } \mathcal{A}_i^{\nu+1} > \text{gauge } \mathcal{A}_{i+1}^{\nu+1}$;
- (iii) if τ_i is of type (ii) and $c(q) = e$, then

$$\text{gauge } \mathcal{A}_i^\nu > \text{gauge } \mathcal{A}_{i+1}^\nu \quad \text{and} \quad \text{gauge } \mathcal{C}_i^\nu \geq \text{gauge } \mathcal{C}_{i+1}^\nu;$$

- (iv) if τ_i is of type (ii) and $c(q) \neq e$, then $\text{gauge } \mathcal{C}_i^\nu > \text{gauge } \mathcal{C}_{i+1}^\nu$.

In proving these statements, we let $x, y, z, \alpha, \beta, \gamma, \mu$ play the same roles as in Definition 3.1(ii) or (iii), according to the type of reduction transformation under consideration.

For $p \in P_i$ and $\delta = \pm 1$ we have

$$s(\tau_i(p)^\delta) \neq s(p^\delta)$$

if and only if either τ_i is of type (iii) with

$$I_{r+1}(s(p^\delta)) = I_\mu(z^\gamma),$$

or τ_i is of type (ii) with

$$p \neq y, \quad t(q)^{-1} = I_\nu(s(p^\delta)), \quad \text{and} \quad c(q) \in \pi_{r+1}(s(p^\delta)c(p));$$

furthermore, if $s(\tau_i(p)^\delta) \neq s(p^\delta)$, then, since $L(\tau_i(p)) = L(p)$,

$$I_\nu(s(\tau_i(p)^\delta)) = s(q),$$

$L(p) > L(q)$ implies

$$I_\nu(s(\tau_i(p)^\delta))^{-1}s(\tau_i(p)^\delta) = c(q)I_\nu(s(p^\delta))^{-1}s(p^\delta),$$

and

$$\pi_\rho(s(\tau_i(p)^\delta)c(\tau_i(p))) = \pi_\rho(s(p^\delta)c(p)) \quad \text{for } \rho > r.$$

We thereby obtain, for $p, r \in P_i$, $\delta, \epsilon = \pm 1$, and $\rho > r$,

$$(p, \delta) \mathfrak{B}_i^\rho(r, \epsilon) \text{ implies } (\tau_i(p), \delta) \mathfrak{B}_{i+1}^\rho(\tau_i(r), \epsilon)$$

and

$$(p, \delta) \mathfrak{A}_i^\rho(r, \epsilon) \text{ implies } (\tau_i(p), \delta) \mathfrak{A}_{i+1}^\rho(\tau_i(r), \epsilon),$$

whence

$$(p, \delta) \mathfrak{C}_i^\rho(r, \epsilon) \text{ implies } (\tau_i(p), \delta) \mathfrak{C}_{i+1}^\rho(\tau_i(r), \epsilon).$$

We now can apply Lemma 3.4 (with $h=0$) to obtain (i).

To prove (ii), we note that if τ_i is of type (iii), then

$$(x, \alpha) \text{ not } \mathfrak{A}_i^{r+1}(z, \gamma) \quad \text{and} \quad (\tau_i(x), \alpha) \mathfrak{A}_{i+1}^{r+1}(\tau_i(z), \gamma),$$

and we apply Lemma 3.4 with

$$h = 0, \quad j = \text{card}((x, \alpha)/\mathfrak{A}_i^{r+1}), \quad \text{and} \quad k = \text{card}((z, \gamma)/\mathfrak{A}_{i+1}^{r+1}).$$

To prove (iii), we assume that τ_i is of type (ii) and $c(q) = e$. We then have, for $p, r \in P_i$ and $\delta, \epsilon = \pm 1$,

$$\begin{aligned} (y, -\beta) \neq (p, \delta) \mathfrak{A}_i^r(y, -\beta) &\text{ implies } (\tau_i(p), \delta) \mathfrak{A}_{i+1}^r(\tau_i(y), \beta), \\ (y, -\beta) \text{ not } \mathfrak{A}_i^r(p, \delta) \mathfrak{A}_i^r(r, \epsilon) &\text{ implies } (\tau_i(p), \delta) \mathfrak{A}_{i+1}^r(\tau_i(r), \epsilon), \\ &\text{and } (y, \beta) \text{ not } \mathfrak{A}_i^r(y, -\beta). \end{aligned}$$

These statements also hold with $\mathfrak{C}_i^r, \mathfrak{C}_{i+1}^r$ substituted for $\mathfrak{A}_i^r, \mathfrak{A}_{i+1}^r$ respectively. Letting

$$h = 1, \quad i = \text{card}((y, -\beta)/\mathfrak{C}_i^r) - 1, \quad \text{and} \quad k = \text{card}((y, \beta)/\mathfrak{C}_i^r),$$

we apply Lemma 3.4 to obtain gauge $\mathfrak{C}_i^* \geq \text{gauge } \mathfrak{C}_{i+1}^*$. Since also

$$(y, \beta) \neq (x, -\alpha) \mathfrak{A}_i^* (y, \beta) \quad \text{and} \quad (y, -\beta) \neq (z, \gamma) \mathfrak{A}_i^* (y, -\beta),$$

we obtain gauge $\mathfrak{A}_i^* > \text{gauge } \mathfrak{A}_{i+1}^*$ by substituting \mathfrak{A}_i^* for \mathfrak{C}_i^* in the preceding choice of h, j, k and applying Lemma 3.4. This proves (iii).

We establish (iv) with a proof similar to the foregoing proof of (iii). This completes the proof.

3.6. THEOREM. *If P is a finite subset of G , then there exists Q such that $[Q] = [P]$, $\text{card } Q \leq \text{card } P$, and Q is irreducible.*

Proof. Apply Theorems 3.3(i) and 3.5.

3.7. LEMMA. *If P is irreducible, $b \in (P \cup P^{-1})^3$, b is irredundant,*

$$L(b_1) \geq L(b_1 b_2), \quad \text{and} \quad L(b_2 b_3) \leq L(b_3),$$

then $b_2 \in \Gamma$.

Proof. Since b is irredundant, $b_1 \neq b_2^{-1} \neq b_3$. If $b_1 = b_2$ or $b_2 = b_3$, then $L(b_2 b_2) \leq L(b_2)$, so $b_2 \in \Gamma$. In the alternative case, $L(b_1) \geq L(b_2)$, else $L(b_1 b_2) < L(b_2)$, whence P would be reducible (type (i)) by the multiplier b_1 ; similarly $L(b_2) \leq L(b_3)$. Hence in this case also $b_2 \in \Gamma$, else P be reducible (type (ii)) by the multiplier b_2 .

3.8. LEMMA. *If P is irreducible, $n > 1$, $b \in (P \cup P^{-1})^n$, b is irredundant, and*

$$L(b_1) = L\left(\prod_{i=1}^j b_i\right) = L(b_j) \quad \text{and} \quad b_j \in \Gamma \quad \text{for } 1 < j \leq n-1,$$

then

$$L\left(\prod_{i=1}^n b_i\right) \geq L(b_n).$$

Proof. We deny the conclusion. We have, for $1 < j < n$,

$$t(b_1) = s(b_j)^{-1} = t(b_j) \quad \text{and} \quad \pi_1(c(b_1)) = \pi_1(c(b_j)).$$

Thus P is reducible (type (i)) by the multiplier $\prod_{i=1}^{n-1} b_i$ unless

$$L(b_1) > L(b_n) \quad \text{or} \quad b_n^{\pm 1} \in \{b_1, \dots, b_{n-1}\}.$$

If $L(b_1) > L(b_n)$, then $L(b_{n-1}) > L(b_{n-1} b_n)$ by Lemma 2.2(i), whence P is reducible (type (i)).

Alternatively, $b_1 \neq b_j \neq b_1^{-1}$ for $1 < j \leq n$ and $b_k^{\pm 1} = b_n$ for some k with $1 < k < n$, since b is irredundant. Again P is reducible (type (i)), in contradiction, in this case by the multiplier

$$\left(\prod_{i=2}^n b_i \right)^{-1}.$$

3.9. LEMMA. *If P is irreducible, $n > 1$, $b \in (P \cup P^{-1})^n$, b is irredundant, and*

$$L(b_1) = L\left(\prod_{i=1}^j b_i\right) \quad \text{for } 1 < j \leq n-1,$$

then

- (i) $L(b_{j-1}) \geq L(b_j)$ for $1 < j \leq n-1$;
- (ii) if $L(b_1) \geq L(\prod_{i=1}^n b_i)$, then $L(b_{n-1}) \geq L(b_n)$;
- (iii) $L(\prod_{i=1}^n b_i) \geq L(b_n)$.

Proof. Denying the conjunction of (i) and (ii), we let

$$m = \min \{j \mid L(b_j) < L(b_{j+1})\}$$

and

$$k = \min \{j \mid j \leq m \text{ and } L(b_i) = L(b_m) \text{ for } j \leq i \leq m\}.$$

By Lemma 2.3

$$L(b_{j-1}) \geq L(b_{j-1}b_j) \quad \text{and} \quad L(b_jb_{j+1}) \leq L(b_{j+1}) \quad \text{for } k < j \leq m.$$

Therefore $b_j \in \Gamma$ for $k < j \leq m$ by Lemma 3.7. Should $k=1$, we would have

$$L\left(\prod_{i=1}^{m+1} b_i\right) \leq L(b_1) = L(b_m) < L(b_{m+1})$$

in contradiction to Lemma 3.8. Therefore $k > 1$ and by Lemmas 2.3 and 3.7, as above, $b_k \in \Gamma$. We have

$$L(b_{k-1}) > L(b_k) = \cdots = L(b_m) < L(b_{m+1}).$$

Let $\mu = L(s(b_k)c(b_k))$. Then for $k \leq j \leq m$, by Lemma 2.2(ii) and induction,

$$L(b_{k-1}) = L\left(\prod_{i=k-1}^j b_i\right),$$

$$b_j \in I_\mu(b_{k-1}^{-1})\pi_\mu(b_{k-1}^{-1})I_\mu(b_{k-1}^{-1})^{-1},$$

and

$$b_k \mathcal{K} b_j.$$

By Lemma 2.3,

$$L\left(\prod_{i=k-1}^m b_i\right) \geq L\left(\prod_{i=k-1}^{m+1} b_i\right) \quad \text{or} \quad L\left(\prod_{i=k-1}^{m+1} b_i\right) \leq L(b_{m+1});$$

in both of these cases,

$$I_\mu(b_{k-1}^{-1})^{-1} \prod_{i=k}^m b_i = I_\mu \left(\left(\prod_{i=k-1}^m b_i \right)^{-1} \right)^{-1} = I_\mu(b_{m+1}).$$

Since b is irredundant,

$$\prod_{i=k}^m b_i \neq e.$$

Thus P is reducible (type (iii)), in contradiction, proving (i) and (ii).

If (iii) is false, then

$$L\left(\prod_{i=1}^n b_i\right) < L(b_n)$$

and, by Lemma 2.4 and induction,

$$L\left(\prod_{i=j}^n b_i\right) = L\left(\prod_{i=j+1}^n b_i\right) = L(b_n) \quad \text{for } 1 < j < n.$$

By (i) and (ii) applied to $\langle b_n^{-1}, \dots, b_1^{-1} \rangle$, $L(b_j) \leq L(b_{j+1})$ for $1 \leq j < n$, so that $L(b_1) = L(b_j)$ for $1 < j < n$. Moreover, by Lemmas 2.3 and 3.7, $b_j \in \Gamma$ for $1 < j < n$. We have contradicted Lemma 3.8. This proves (iii) and completes the proof.

3.10. THEOREM. *If P is irreducible, $b \in (P \cup P^{-1})^n$, and b is irredundant, then*

$$L\left(\prod_{i=h}^{h'} b_i\right) \leq L\left(\prod_{i=1}^n b_i\right) \quad \text{for } 1 \leq h \leq h' \leq n.$$

Proof. The theorem is trivial if $n=1$. We assume that $n>1$ and the theorem is true for smaller n . We need only show that

$$L\left(\prod_{i=1}^{n-1} b_i\right) \leq L\left(\prod_{i=1}^n b_i\right).$$

Let

$$k = \max \left\{ j \mid 1 \leq j < n \text{ and } L\left(\prod_{i=1}^{j-1} b_i\right) < L\left(\prod_{i=1}^j b_i\right) \right\},$$

$$m = \max \left\{ j \mid k \leq j < n \text{ and } L(b_k) = L\left(\prod_{i=k}^h b_i\right) \text{ for } k \leq h \leq j \right\}.$$

Then $1 \leq k \leq m < n$. We have, using the inductive hypothesis,

$$L\left(\prod_{i=1}^{k-1} b_i\right) < L\left(\prod_{i=1}^k b_i\right) \leq L\left(\prod_{i=1}^m b_i\right)$$

and

$$L(b_k) = L\left(\prod_{i=k}^j b_i\right) \geq L(b_j) \quad \text{for } k \leq j \leq m;$$

thus also, by Lemma 3.9(iii),

$$L\left(\prod_{i=k}^{m+1} b_i\right) \geq L(b_{m+1}).$$

Suppose $m+1 < n$. Then by inductive hypothesis and definition of m ,

$$L\left(\prod_{i=k}^m b_i\right) < L\left(\prod_{i=k}^{m+1} b_i\right).$$

By Lemma 2.5(i), with $x = \prod_{i=1}^{k-1} b_i$, $y = \prod_{i=k}^m b_i$, and $z = b_{m+1}$, we have

$$L\left(\prod_{i=1}^m b_i\right) < L\left(\prod_{i=1}^{m+1} b_i\right),$$

in contradiction to the definition of k . Therefore $m = n - 1$.

In case $L(b_k) \leq L(b_n)$, we apply Lemma 2.5(ii) with x, y, z as above to obtain

$$L\left(\prod_{i=1}^{n-1} b_i\right) \leq L\left(\prod_{i=1}^n b_i\right).$$

Suppose that $L(b_k) > L(b_n)$ and $L(\prod_{i=1}^{n-1} b_i) > L(\prod_{i=1}^m b_i)$. Let

$$l = \max \{j \mid k \leq j < n \text{ and } L(b_j) > L(b_n)\}.$$

Then by Lemma 2.2

$$L(b_l) = L\left(\prod_{i=l}^{j-1} b_i\right) = L\left(\prod_{i=l}^j b_i\right) > L(b_j) \quad \text{for } l < j < n,$$

and

$$L(b_l) = L\left(\prod_{i=l}^{n-1} b_i\right) > L\left(\prod_{i=l}^n b_i\right).$$

Since $L(b_j) = L(b_n)$ for $l < j < n$, by Lemma 2.3(ii), the inductive hypothesis, and induction we have

$$L\left(\prod_{i=j}^n b_i\right) \leq L\left(\prod_{i=j+1}^n b_i\right),$$

whence

$$L\left(\prod_{i=j}^n b_i\right) = L(b_n) \quad \text{for } l < j < n.$$

By Lemma 3.9(iii) applied to $\langle b_n^{-1}, \dots, b_1^{-1} \rangle$ we thus have, in contradiction,

$$L(b_i) \leq L\left(\prod_{i=1}^n b_i\right).$$

3.11. THEOREM. *If P is irreducible and $x \in [P]$, then*

$$x \in [P \cap \{p \mid L(p) \leq L(x)\}].$$

Proof. Choose $b \in (P \cup P^{-1})^n$ such that $x = \prod_{i=1}^n b_i$, and b is irredundant. Apply Theorem 3.10.

3.12. THEOREM. *If P is irreducible and $[P] = G$, then $P \subset \Sigma$.*

Proof. By Theorem 3.11 we have $[P \cap \Sigma] = G$. Therefore $P - \Sigma = 0$, else P be reducible (type (i)).

3.13. THEOREM⁽⁷⁾. *If rank G is finite, then rank $G = \sum_{H \in \phi} \text{rank } H$.*

Proof. Apply Theorem 3.6 to obtain an irreducible set Q such that $[Q] = G$ and $\text{card } Q = \text{rank } G$. Then $Q \subset \Sigma$ by Theorem 3.12. Therefore

$$\text{rank } H \leq \text{card } (Q \cap H) \text{ for } H \in \phi,$$

whence

$$\sum_{H \in \phi} \text{rank } H \leq \text{card } Q = \text{rank } G.$$

The reverse inequality is obvious.

3.14. DEFINITION. We say that a subset Q of G is *essentially irreducible* if Q is reducible only by transformations of type (i) with

$$q \mathcal{K} x \mathcal{K} z \in \Gamma \quad \text{and} \quad qz^\gamma = e,$$

referring to the notation of Definition 3.1(i).

Thus if R is a finite subset of an essentially irreducible set, then there exists $P \subset R$ such that $[P] = [R]$, P is irreducible, and such that if $x \in R - P$, then

$$x \in [P \cap \Gamma \cap (x/\mathcal{K})].$$

3.15. THEOREM⁽⁸⁾. *If Q is essentially irreducible and*

$$\psi = \{[(x/\mathcal{K}) \cap Q] \mid x \in Q\},$$

(7) Proved independently by Grushko (see [GK]) and B. H. Neumann (see [NM]). By elementary cardinality arguments (see Levi, [LV], or [NM]) this theorem also holds if rank G is infinite.

(8) In view of the sixth paragraph of this section, this theorem provides no less than the sixth published proof of the Kurosch subgroup theorem (we cite [K1], [BL], [TK], [KN], and [HL]), although, except in the finitely generated case, we borrow heavily from a previous proof to obtain essentially irreducible generators.

then ψ is a free factorization of $[Q]$. Moreover, each member of ψ is either infinite cyclic or conjugate to a subgroup of some member of ϕ .

Proof. Suppose $n > 0$, $a \in (Q \cup Q^{-1})^n$, and a is irredundant. To prove that ψ is a free factorization of $[Q]$, we must show that

$$\prod_{i=1}^n a_i \neq e.$$

Referring to Definition 3.14, we obtain $P \subset Q$ such that P is irreducible and

$$a_i \in P \cup P^{-1} \cup [P \cap \Gamma \cap (a_i/\mathcal{K})] \quad \text{for } 1 \leq i \leq n.$$

We further obtain $m \geq n$ and $b \in (P \cup P^{-1})^m$ such that b is irredundant and

$$\prod_{i=1}^m b_i = \prod_{i=1}^n a_i.$$

The nonvanishing of this product then follows from Theorem 3.10. The remainder of the theorem follows from the fact that only members of Γ can have finite order.

3.16. DEFINITION. Suppose $b \in G^n$ and $1 \leq k \leq n$. We say that k is *isolated leftwards* {*rightwards*} in b if

$$\begin{aligned} L\left(\prod_{i=1}^{k-1} b_i\right) < L\left(\prod_{i=1}^k b_i\right) \quad \text{and} \quad L\left(\prod_{i=k}^n b_i\right) \geq L\left(\prod_{i=k+1}^n b_i\right) \\ \left\{ L\left(\prod_{i=1}^{k-1} b_i\right) \leq L\left(\prod_{i=1}^k b_i\right) \quad \text{and} \quad L\left(\prod_{i=k}^n b_i\right) > L\left(\prod_{i=k+1}^n b_i\right) \right\}. \end{aligned}$$

We say that k is *isolated in* b if k is isolated leftwards in b or k is isolated rightwards in b .

3.17. THEOREM. If P is irreducible, $b \in (P \cup P^{-1})^n$, b is irredundant, and $1 < k \leq n$, then

- (i) 1 is isolated leftwards and n is isolated rightwards in b ;
- (ii) k is not isolated leftwards in b if and only if

$$L\left(\prod_{i=1}^{k-1} b_i\right) = L\left(\prod_{i=1}^k b_i\right);$$

- (iii) if k is not isolated leftwards in b , then $L(b_{k-1}) \geq L(b_k)$;
- (iv) if $1 \leq m < k$, then k is not isolated leftwards in b if and only if $k-m+1$ is not isolated leftwards in $\langle b_m, \dots, b_n \rangle$;
- (v) if k is not isolated leftwards in b , $m < k$, and m is isolated rightwards in b , then

$$L\left(\prod_{i=m}^{k-1} b_i\right) > L(b_k).$$

(vi) if k is not isolated in b , then $b_k \in \Gamma$.

Proof. Both (i) and (ii) follow from Theorem 3.10.

Suppose k is not isolated leftwards in b . Let

$$l = \max \{j \mid 1 \leq j < k \text{ and } j \text{ is isolated leftwards in } b\}.$$

Then for $l < j \leq k$ by (ii) and Theorem 3.10

$$L\left(\prod_{i=1}^{l-1} b_i\right) < L\left(\prod_{i=1}^l b_i\right) \leq L\left(\prod_{i=1}^{j-1} b_i\right) = L\left(\prod_{i=1}^j b_i\right),$$

so that

$$L(b_l) = L\left(\prod_{i=1}^{j-1} b_i\right) = L\left(\prod_{i=1}^j b_i\right),$$

else we contradict Lemma 2.5(i). Therefore $L(b_{k-1}) \geq L(b_k)$ by Lemma 3.9(i), proving (iii).

Assuming $1 \leq m < k$, we note that if k is not isolated leftwards in b or if $k-m+1$ is not isolated leftwards in $\langle b_m, \dots, b_n \rangle$, then by Theorem 3.10 and (iii)

$$L\left(\prod_{i=m}^{k-1} b_i\right) \geq L(b_{k-1}) \geq L(b_k).$$

Consequently we can apply Theorem 3.10 and Lemmas 2.3(i) and 2.6 to prove (iv).

Under the hypothesis of (v), applying (iv) and its dual, $k-m+1$ is not isolated leftwards in $\langle b_m, \dots, b_n \rangle$, and m is isolated rightwards in $\langle b_1, \dots, b_k \rangle$; hence

$$L\left(\prod_{i=m}^{k-1} b_i\right) = L\left(\prod_{i=m}^k b_i\right) > L\left(\prod_{i=m-1}^k b_i\right) \geq L(b_k).$$

Finally, if k is not isolated in b , then by (iv) and (ii)

$$L(b_{k-1}) = L(b_{k-1}b_k) \quad \text{and} \quad L(b_k b_{k+1}) = L(b_{k+1}),$$

so $b_k \in \Gamma$ by Lemma 3.7. This proves (vi) and completes the proof.

3.18. LEMMA⁽⁹⁾. If Q is essentially irreducible and

$$w \in [Q] \cap \Gamma,$$

then there exist an equivalence class A in the \mathcal{K} -partitioning of Q and $d \in [Q]$ such that $A \subset \Gamma$ and $w \in d[A]d^{-1}$.

Proof. Choose $P \subset Q$, $n > 0$, and $b \in (P \cup P^{-1})^n$ such that P is irreducible, b is irredundant, and

⁽⁹⁾ This lemma corresponds to Hilfssatz 6 of [K1].

$$w = \prod_{i=1}^n b_i.$$

Let λ be the length of w with respect to the free factorization of $[Q]$ given by Theorem 3.15. If $\lambda = 1$, i.e.

$$b_1 \mathcal{K} b_2 \mathcal{K} \cdots \mathcal{K} b_n,$$

then $s(b_1) = t(b_n)^{-1}$, whence $b_1 \in \Gamma$, so we take $A = (b_1/\mathcal{K}) \cap Q$ and $d = e$. Let us assume that $\lambda > 1$ and the lemma holds for smaller λ .

If $b_1 \mathcal{K} b_n$, we find m such that $1 \leq m < n$ and

$$b_1 \mathcal{K} \cdots \mathcal{K} b_m \text{ not } \mathcal{K} b_{m+1},$$

and we apply the inductive hypothesis to

$$\left(\prod_{i=1}^m b_i \right)^{-1} w \prod_{i=1}^m b_i.$$

Suppose b_1 not $\mathcal{K} b_n$. Then $\langle b_1, \dots, b_n, b_1, \dots, b_n \rangle$ is irredundant. By Theorem 3.10 and Lemma 2.7 and its dual

$$L\left(\prod_{i=1}^n b_i\right) = L\left(\prod_{i=1}^n b_i b_1\right) = L\left(\prod_{i=2}^n b_i b_1\right).$$

Therefore by Theorem 3.17(ii), (iv), (vi), $L(b_n) = L(b_n b_1)$ and $b_1 \in \Gamma$. Dually $b_n \in \Gamma$ and $L(b_n b_1) = L(b_1)$. Therefore $b_1 \mathcal{K} b_n$, in contradiction.

3.19. LEMMA. *If Q is essentially irreducible, $u, v \in Q \cap \Gamma$, $d \in [Q]$, and $u \mathcal{K} bvd^{-1}$, then $u \mathcal{K} v$.*

Proof. We choose $P \subset Q$, $n, m, l > 0$,

$$b \in (P \cup P^{-1})^n, \quad x \in (P \cup P^{-1})^m, \quad y \in (P \cup P^{-1})^l$$

such that P is irreducible, b, x, y are irredundant,

$$d = \prod_{i=1}^n b_i, \quad u = \prod_{i=1}^m x_i, \quad v = \prod_{i=1}^l y_i,$$

$$u \mathcal{K} x_i \text{ for } i = 1, \dots, m \text{ and } v \mathcal{K} y_i \text{ for } i = 1, \dots, l.$$

It may be assumed that u not $\mathcal{K} b_1$ and b_n not $\mathcal{K} v$ for by conjugation we could arrive at this case or obtain the conclusion. Then

$$\langle x_1, \dots, x_m, b_1, \dots, b_n, y_1, \dots, y_l, b_n^{-1}, \dots, b_1^{-1} \rangle$$

is irredundant. Since $u \mathcal{K} bvd^{-1} \in \Gamma$, by Theorem 3.10 we have

$$L(u) \leq L\left(u \prod_{i=1}^{n-1} b_i\right) \leq L(ud) \leq L(udv) \leq L(udvb_n^{-1}) \leq L(udvd^{-1}) \leq L(u).$$

Consequently by Theorem 3.17(ii), (iii), (iv)

$$L(b_n) \geq L(v) \geq L(b_n^{-1}) \quad \text{and} \quad L(v) = L(vb_n^{-1}).$$

Since $v \in \Gamma$, we also have $L(b_nv) = L(v)$. Therefore $b_n \in \Gamma$ by Theorem 3.17(ii), (vi). It follows that $b_n \mathcal{K} v$, in contradiction.

3.20. THEOREM⁽¹⁰⁾. *If Q is essentially irreducible, $H \in \phi$, $x \in G$, and $[Q] \cap xHx^{-1} \neq \{e\}$, then there exists a unique equivalence class A in the \mathcal{K} -partitioning of Q such that*

$$[Q] \cap xHx^{-1} = d[A]d^{-1} \text{ for some } d \in [Q].$$

Proof. From Lemmas 3.18 and 3.19 we see that there exists at least one A with the required properties. If A' is another such set, then $A = A'$, else $[A] * [A']$ be a free factor of $[Q]$ by Theorem 3.15, with $[A]$ conjugate in $[Q]$ to $[A']$, which, as observed by Baer and Levi, is impossible.

4. Homomorphisms on a free group onto a free product. This section is devoted chiefly to proving our main result, Theorem 4.4, which removes the finite rank restriction on Grushko's theorem characterizing the homomorphisms on a free group onto a free product in a simple natural way. The conjecture of Federer and Jónsson in [FJ] is thereby verified—we generally follow their methods in 4.1, 4.2, and 4.3. Example 4.5 prohibits removing the proviso in Theorem 4.4 that F be free.

It may be noted that with Theorems 3.3(i), 3.5, and 3.12 available, an elementary proof of Theorem 4.4 is at hand for the case where rank F is finite. Choose a set X which freely generates F , carry out a succession of reduction transformations beginning on $f(X)$ until an irreducible set is obtained and carry out corresponding elementary transformations beginning on X .

Once Theorems 4.3 and 4.4 are established, there is no difficulty in obtaining the related results of Federer and Jónsson (Theorems 6.3 and 6.4 of [FJ]) which inspired this investigation. One easily proves, for this purpose, that if f is a homomorphism on a free group F onto a cyclic group, then there exist X and $x \in X$ such that F is freely generated by X and

$$f(y) = e \text{ for } x \neq y \in X.$$

4.1. LEMMA. *If f is a homomorphism on the group F onto G , P' is a finite subset of F , $P = f(P')$, P is reducible, and*

$$p \notin [P - \{p\}] \text{ for } p \in P \cap \Sigma,$$

then there exist τ, τ', Q, Q' such that, setting $T = [[P] \cap \Sigma]$,

(i) τ is a reduction transformation of P and τ' is an elementary transformation of P' ;

⁽¹⁰⁾ This theorem provides the improvement of Reinhold Baer and F. W. Levi (see [BL]) to the Kurosch subgroup theorem (see footnote (*)).

- (ii) $Q = \text{range } \tau$ and $Q' = \text{range } \tau'$;
- (iii) $P' \cap f^{-1}(\Sigma) \subset Q'$;
- (iv) $r \in P'$ implies $r \in [Q' \cap \{d \mid L(f(d)) \leq L(f(r))\}]$;
- (v) $r \in P'$ and $f(r) \in T$ implies

$$r \in [Q' \cap \{d \mid L(f(d)) < L(f(r)) \text{ or } (f(d) \in T \text{ and } L(f(d)) \leq L(f(r)))\}].$$

Proof. First we observe that if $x \in [P]$ and $s(x), t(x) \in T$, then $x \in T$, and if $x \in T$, then $I_\mu(x) \in T$ for $\mu \geq 0$.

We choose τ to be a reduction transformation of P effected by a multiplier q subject to the following conditions:

Case I. If possible, $q \in T$.

Case II. If Case I does not apply and τ is of type (ii), then $t(q) \in T$.

Fixing $C = \{p \mid \tau(p) = p\}$, we choose $b \in (C \cup C^{-1})^n$ such that

$$q = \prod_{i=1}^n b_i, \quad s(q) = s(b_1), \quad t(q) = t(b_1)$$

and

$$t(b_1) = s(b_j)^{-1} = t(b_j) \text{ for } 1 < j \leq n.$$

We further choose $b' \in (P' \cup P'^{-1})^n$ and $q' \in [P']$ such that

$$f(b'_j) = b_j \text{ for } 1 \leq j \leq n \text{ and } q' = \prod_{i=1}^n b'_i.$$

For $r \in P'$ we define $\tau'(r)^\epsilon$ to be $q'rq'^{-1}$, $q'r$, rq'^{-1} , or r according as $\tau(f(r))^\epsilon$ is $qf(r)q^{-1}$, $qf(r)$, $f(r)q^{-1}$, or $f(r)$ with $\epsilon = \pm 1$. We define $Q = \text{range } \tau$ and $Q' = \text{range } \tau'$. Then (i) and (ii) are immediate.

Supposing (iii) is false, choose $r \in P'$ such that $f(r) \in \Sigma$ and $\tau'(r) \neq r$. Then r is of type (i) and $\tau(f(r)) = e$, whence

$$f(r) \in [P - \{f(r)\}],$$

in contradiction to the hypothesis. Therefore (iii) holds.

To check (iv) and (v) we suppose that $r \in P'$ and $p = f(r)$. We have

$$r \in [\{b'_1, \dots, b'_n, \tau'(r)\}].$$

Clearly $\tau'(r) \in Q'$ and, for $1 \leq j \leq n$, $b'_j \in Q' \cup Q'^{-1}$, since $b_j \in C \cup C^{-1}$.

Assuming that $\tau'(r) \neq r$, we have $L(b_j) \leq L(p)$ for $1 \leq j \leq n$ and, by Theorem 3.3(ii), $L(\tau(p)) \leq L(p)$. Therefore (iv) holds.

Assume further that $p \in T$. Then $t(q) \in T$. Therefore if Case II is in effect, τ is of neither type (ii) nor (iii) and $L(\tau(p)) < L(p)$. In any case,

$$L(\tau(p)) < L(p) \text{ or } \tau(p) \in T.$$

Clearly $q \in T$ implies $b_j \in T$ for $1 \leq j \leq n$. Therefore to complete the proof of

(v) we assume Case II (whence τ is of type (i)) and show that $L(q) < L(p)$. Supposing $L(q) = L(p)$, we choose $\delta = \pm 1$ such that

$$L(b_1) = L(q) = L(p) > L\left(\prod_{i=1}^n b_i p^\delta\right),$$

whence, for $1 < j \leq n$, $s(b_j) = t(b_j)^{-1} = s(p^\delta) \in T$, so that $b_j \notin T$. Then $b_1 \notin T$, else Case I would apply, so $b_1, b_1^{-1} \notin \{b_2, \dots, b_n\}$, and P is reducible (type (i)) by the multiplier

$$\left(\prod_{i=2}^n b_i p^\delta\right)^{-1} \in T.$$

Thus Case I applies, in contradiction. This completes the proof.

4.2. LEMMA. *If f is a homomorphism on the group F into G and W is a finite subset of F freely generating $[W]$, then there exists a set D freely generating $[W]$ such that*

$$W \cap f^{-1}(\Sigma) \subset D$$

and for each $a \in W$ for which $f(a) \in [[f(W)] \cap \Sigma]$ we have

$$a \in [D \cap \{d \mid f(b) \in \Sigma \text{ or } L(f(b)) < L(f(a))\}].$$

Proof. Letting $Q'_0 = W$ and $Q_0 = f(W)$, we apply Lemma 4.1 and Theorem 3.5 to obtain $m \geq 0$ and, for $1 \leq j \leq m$, sets $P_j, P'_j, Q_j, Q'_j, E_j, E'_j$ such that Q_m is irreducible and, for $1 \leq j \leq m$,

$$\begin{aligned} Q'_{j-1} &= P'_j \cup E'_j, & P'_j \cap E'_j &= 0, \\ f(Q'_j) &= Q_j, & f(P'_j) &= P_j, & f(E'_j) &= E_j \subset \Sigma, \\ [Q'_{j-1}] &= [P_j], & p &\notin [P_j - \{p\}] \text{ for } p \in P_j \cap \Sigma, \end{aligned}$$

and Q'_j and Q_j are related to P'_j and P_j just as Q' and Q are related to P' and P in Lemma 4.1.

For $1 \leq j \leq m$ we see that

$$[W] \text{ is freely generated by } Q'_j \cup \bigcup_{i=1}^j E'_i,$$

$$W \cap f^{-1}(\Sigma) \subset Q'_j \cup \bigcup_{i=1}^j E'_i, \quad [Q_0] = [Q_j],$$

and for each $a \in W$ for which $f(a) \in [[f(W)] \cap \Sigma]$ we have

$$a \in \left[(Q'_j \cap \{b \mid f(b) \in \Sigma \text{ or } L(f(b)) < L(f(a))\} \text{ or } \right.$$

$$\left. (f(b) \in [[Q_j] \cap \Sigma] \text{ and } L(f(b)) \leq L(f(a))) \right) \cup \bigcup_{i=1}^j E'_i \Big].$$

Now $[Q_m] \cap \Sigma \subset [Q_m \cap \Sigma]$ by Theorem 3.11. Consequently

$$Q_m \cap [[Q_m] \cap \Sigma] \subset \Sigma,$$

else Q_m would be reducible (type (i)). We take

$$D = Q'_m \cup \bigcup_{i=1}^m E'_i.$$

4.3. THEOREM. *If the group F is freely generated by X and f is a homomorphism on F onto G , then there exists a set Y freely generating F such that*

$$X \cap f^{-1}(\Sigma) \subset Y \quad \text{and} \quad f(Y) \subset \Sigma.$$

Proof. Let Ω be the class of all three-termed sequences A such that

A_1, A_2, A_3 are disjoint subsets of F ,

F is freely generated by $A_1 \cup A_2 \cup A_3$,

$X \cap f^{-1}(\Sigma) \subset A_1, \quad f(A_1) \subset \Sigma,$

$X - f^{-1}(\Sigma) \supset A_3, \quad A_2$ is finite,

and $[A_1 \cup A_2] = [X - A_3]$.

We observe that $\langle X \cap f^{-1}(\Sigma), 0, X - f^{-1}(\Sigma) \rangle \in \Omega$.

We partially order Ω by the relation \leq in such a way that for $A, B \in \Omega$ we have $A < B$ if and only if

$$A \neq B, \quad A_1 \subset B_1, \quad A_3 \supset B_3,$$

and $a \in A_2$ implies

$$a \in [B_1 \cup \{b \mid b \in B_2 \text{ and } L(f(b)) < L(f(a))\}].$$

We can follow the proof of Theorem 6.3 of [FJ] quite closely to show that if S is a nonvacuous simply-ordered subclass of Ω containing no maximal element, then

$$\left\langle \bigcup_{A \in S} A_1, 0, \bigcap_{A \in S} A_3 \right\rangle$$

is an upper bound of S in Ω . Since $\Omega \neq 0$, we can therefore apply the inductive principle known as Zorn's Lemma to obtain a maximal element A of Ω . To complete the proof it suffices to show that $A_2 \cup A_3 = 0$, for then we can take $Y = A_1$.

Supposing $A_2 \cup A_3 \neq 0$, we choose finite nonvacuous sets R and W such that

$$A_2 \subset R \subset A_2 \cup A_3, \quad R \subset W \subset A_1 \cup A_2 \cup A_3,$$

$$f(R) \subset [[f(W)] \cap \Sigma].$$

Applying Lemma 4.2, we obtain a set D with the properties stated there.

Let B be the three-termed sequence given by

$$B_1 = A_1 \cup (D \cap f^{-1}(\Sigma)), \quad B_2 = D - B_1, \quad \text{and} \quad B_3 = A_3 - W.$$

We continue to follow 6.3 of [FJ] to show successively that $[A_1 \cup W] = [B_1 \cup B_2]$, $B \in \Omega$, $A \leq B$, and $A \neq B$, in contradiction to the maximality of A .

4.4. THEOREM. *If f is a homomorphism on the free group F onto G , then there exists a free factorization ψ of F in one-one correspondence with ϕ such that $f(J) \in \phi$ and $f(J)$ corresponds to J for $J \in \psi$.*

Proof. Apply Theorem 4.3 to obtain X such that F is freely generated by X and $f(X) \subset \Sigma$. Choose $H_0 \in \phi$ such that if $\{e\} \in \phi$, then $H_0 = \{e\}$. Let

$$\psi = \{[X \cap f^{-1}(H - \{e\})] \mid H_0 \neq H \in \phi\} \cup \{[X \cap f^{-1}(H_0)]\}.$$

4.5. EXAMPLE. The following example demonstrates that the requirement that F be free may not be deleted from Theorem 4.4.

Consider the case where ϕ consists of two groups of order two. Suppose the group E is freely generated by $\{x, y\}$ and $x \neq y$. Let N be the least normal subgroup of E containing x^2y^{-2} . We assert that in this case:

- (i) there exists a homomorphism on E/N onto G ;
- (ii) there exists no proper free factorization of E/N .

Let f be a homomorphism on E onto G such that $f(x), f(y) \in \Sigma$. Then (i) follows from the fact that $N \subset \ker f$.

Assuming E/N can be decomposed, apply Theorem 4.4 (or Gruschko's theorem) to obtain a set $\{z, w\}$ which freely generates E and such that zN and wN generate the free factors. Then zN or wN , say zN , has finite order, so that either $f(z) = e$ or $L(f(z))$ is odd, which can be shown to be false⁽¹⁾.

5. Decision procedures in free products. In this section we consider the decidability of the questions of reducibility of finite subsets of G and membership in finitely generated subgroups of G .

For each $H \in \phi$ we make the following assumptions:

- (i) Multiplication and inversion in H are computable.
- (ii) A procedure is at hand, given a finite subset A of H and $u \in H$, for deciding whether or not $u \in [A]$ and for producing, in the event of an affirmative answer, $a \in (A \cup A^{-1})^n$ such that $u = \prod_{i=1}^n a_i$.

5.1. THEOREM. *If P is a finite subset of G , then we can decide whether or not P is reducible and, if so, we can carry out a reduction transformation.*

Proof. To test for reducibility of type (i), assuming $P \cap P^{-1} = 0$, we examine all $x, z \in P \cup P^{-1}$ such that

⁽¹⁾ Details omitted, since the indecomposability of E/N has become obvious from the test given by A. Shenitzer (see [SN]). Professor Kuhn has proposed a classical approach by showing that each element of E/N can be uniquely represented as $(xy)^m y^n N$ with m, n arbitrary integers, and deducing that E/N has no elements of finite order. To prove uniqueness his proof requires reference to the Schreier representation of a free product with identified subgroups (see [SR]), but unlike the other two proofs does not require Gruschko's theorem.

$$x \neq z, \quad L(x) \leq L(z), \quad t(x)^{-1} = I_\mu(z), \quad \text{and} \quad \pi_{\mu+1}(x^{-1}) = \pi_{\mu+1}(z),$$

where $\mu = L(t(x))$, resolving the question into the question of whether or not, in some such case, $c(x) = e$ or

$$I_1(I_\mu(z)^{-1}z)c(x) \in [\{c(p) \mid p \in t(x)^{-1}\pi_1(c(x))t(x) \cap P - \{z\}\}],$$

which can be decided by assumption (ii). We test for reducibility of type (iii) in a similar manner. Testing for reducibility of type (ii) and carrying out reduction transformations of any type are straightforward.

5.2. THEOREM. *Suppose we are given a finite subset P of G and $x \in G$. Then we can prescribe a procedure for deciding whether or not $x \in [P]$ and for producing, in the event of an affirmative answer, $b \in (P \cup P^{-1})^n$ such that $x = \prod_{i=1}^n b_i$.*

Proof. In view of Theorems 5.1, 3.3(i), and 3.5, we may just as well assume that P is irreducible. If $L(x) = 1$, then the question is decidable by hypothesis and Theorem 3.11. We assume that $L(x) > 1$ and that we have prescribed the procedure for smaller values of $L(x)$.

The answer is affirmative if and only if we can find $b \in (P \cup P^{-1})^n$ such that b is irredundant and $x = \prod_{i=1}^n b_i$. We shall consider candidates for the roles of b_h and b_k in such a sequence where

$$h = \max \{i \mid i \text{ is isolated leftwards in } b\},$$

$$k = \min \{i \mid h \leq i \text{ and } i \text{ is isolated rightwards in } b\};$$

thus $1 \leq h \leq k \leq n$. Such b , h , k must satisfy the subsequent conditions. From Theorem 3.17 we deduce that

$$L(b_h) = L(b_k), \quad L\left(\prod_{i=1}^{h-1} b_i\right) < L\left(\prod_{i=1}^h b_i\right),$$

$$L(b_h) = L\left(\prod_{i=h}^j b_i\right) \geq L(b_j) \quad \text{for } h < j \leq n,$$

$$L(b_k) = L\left(\prod_{i=k}^j b_i\right) > L(b_j) \quad \text{for } k < j \leq n,$$

$$L(b_h) = L(b_j) \quad \text{and} \quad b_j \in \Gamma \quad \text{for } h < j < k.$$

We therefore see, where $\mu = L(s(b_k))$ and $\nu = L(s(b_k)c(b_k))$, that

$$t(b_h) = s(b_k)^{-1}, \quad \pi_1(c(b_h)) = \pi_1(c(b_k)),$$

$$t(b_h) = s(b_j)^{-1} = t(b_j) \quad \text{and} \quad \pi_1(c(b_h)) = \pi_1(c(b_j)) \quad \text{for } h < j < k,$$

$$t(b_k) \prod_{i=k+1}^n b_i = I_\mu(x^{-1})^{-1},$$

$$\prod_{i=h}^k c(b_i) = I_\nu(x^{-1})^{-1} I_\mu(x^{-1}), \quad \prod_{i=1}^{h-1} b_i s(b_h) = x I_\nu(x^{-1}),$$

$$\pi_\nu(b_k^{-1}) = \pi_\nu(x^{-1}), \quad \text{and} \quad \pi_\nu(b_h) = \pi_{\mu+1}(x^{-1}).$$

We now describe the test to which we put each pair $p, r \in P \cup P^{-1}$ as candidates for the respective roles of b_h and b_k in the preceding paragraph. In view of Theorem 3.10 and the preceding paragraph, we reject the candidates unless

$$L(p) = L(r) \leq L(x),$$

$$l(p) = s(r)^{-1}, \quad \text{and} \quad \pi_1(c(p)) = \pi_1(c(r)).$$

Letting $\mu = L(s(r))$ and $\nu = L(s(r)c(r))$, assuming this test met, we determine whether or not

$$\pi_\nu(r^{-1}) = \pi_\nu(x^{-1}), \quad \pi_\nu(p) = \pi_{\mu+1}(x^{-1}),$$

and

$$c(p)^{-1}I_\nu(x^{-1})^{-1}I_\mu(x^{-1})c(r)^{-1} \in [\{c(q) \mid s(r)c(r)s(r)^{-1} \mathcal{K} q \in P\}];$$

if the last of these three fails and $p=r$, we make an alternative test for the possibility that $h=k$ by determining whether or not

$$c(p)^{-1}I_\nu(x^{-1})^{-1}I_\mu(x^{-1}) = e.$$

If negative, we reject the candidates. If affirmative, we find $z \in (P \cup P^{-1})^l$ such that $z_1=p$, $z_i=r$,

$$l(p) = s(z_i)^{-1} = l(z_i) \quad \text{and} \quad \pi_1(c(p)) = \pi_1(c(z_i)) \quad \text{for } 1 < i < l$$

and

$$\prod_{i=1}^l c(z_i) = I_\nu(x^{-1})^{-1}I_\mu(x^{-1}).$$

Then

$$s(p)^{-1} \left(\prod_{i=1}^l z_i \right) l(r)^{-1} = I_\nu(x^{-1})^{-1}I_\mu(x^{-1}).$$

If $e(r)=e$, then $\mu=\nu$,

$$\pi_\mu(x^{-1}) = \pi_1(l(r)), \quad \text{and} \quad \pi_\nu(s(p)) = \pi_{\nu+1}(x^{-1}).$$

It follows that, whether or not $c(r)=e$,

$$L(I_\mu(x^{-1})l(r)) < L(r) \leq L(x) \quad \text{and} \quad L(s(p)I_\nu(x^{-1})^{-1}x^{-1}) < L(x),$$

from which we are equipped by inductive hypothesis to determine whether or not

$$I_\mu(x^{-1})l(r) \in [P] \quad \text{and} \quad s(p)I_\nu(x^{-1})^{-1}x^{-1} \in [P],$$

which we next proceed to do. If negative, we reject the candidates. If affirmative, we obtain $w \in (P \cup P^{-1})^j$ and $y \in (P \cup P^{-1})^m$ such that

$$l(r) \prod_{i=1}^j w_i = I_\mu(x^{-1})^{-1},$$

and

$$\prod_{i=1}^m y_i s(p) = x I_\nu(x^{-1}).$$

If no pair meets the foregoing tests, we conclude that the roles of b_h and b_k cannot be filled and $x \notin [P]$. If one pair meets the test, then while this pair may not fill the roles of b_h and b_k , we nevertheless have

$$x = \prod_{i=1}^m y_i \prod_{i=1}^l z_i \prod_{i=1}^j w_i \in [P],$$

with y, z, w as in the preceding paragraph.

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