STRUCTURE THEORY OF FAITHFUL RINGS
II. RESTRICTED RINGS[1]

BY

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The first paper of this series[2] concerned itself mainly with closure operations on a lattice. This paper applies these results to the global structure theory of a faithful ring and its modules.

A ring R is called faithful if $aR=0$ implies $a=0$. Let $\mathfrak{L}$ ($\mathfrak{L}^{t}$) be the lattice of right (two-sided) ideals of R, and $\mathfrak{F}''=\{A;\ A\in\mathfrak{L}^{t},\ A\cap A^{l}=0,\ A=A^{ll}\}$. The set $\mathfrak{F}''$ becomes a Boolean algebra with the obvious definition of the union operation. In case $\mathfrak{F}''$ is complete, R is called a restricted ring. Being complete, $\mathfrak{F}''$ induces a closure operation f on $\mathfrak{L}$ and $\mathfrak{L}^{t}$. It is shown that if f is homogeneous, then there exist irreducible rings A_i such that $\sum_i A_i\subset R \subset\sum_i^* \overline{A}_i$, where $\sum$ ($\sum^*$) designates the discrete (full) direct sum and $\overline{A}_i$ is a universal extension ring of A_i. A reducible ring is shown to be restricted, where R is reducible if and only if for every pair A, B of ideals of R with zero intersection, there exist ideals $A'\supset A$ and $B'\supset B$ also with zero intersection such that $A'^{l}\cap B'^{l}=0$.

Modules of a faithful ring are studied in the fourth section. It is shown that every suitably restricted closure operation on $\mathfrak{L}$ induces a closure operation on $\mathfrak{M}$, the lattice of submodules of a R-module M. If ${}^{l}A$, $A\subset R$, designates the annihilator of A in M, then it is shown in the fifth section that $\mathfrak{K}=\{{}^{l}A;\ A\in\mathfrak{F}''\}$ is a Boolean algebra.

In the final two sections, it is assumed that the ring R and the R-module M have the property that for every nonzero element x in R or M there exists a nonzero $A\in\mathfrak{L}$ such that $xa\neq 0$ for every nonzero $a\in A$. It is shown that R is restricted, and that every A in $\mathfrak{L}$ ($\mathfrak{M}$) has a unique maximal essential extension A^{s} (A^{t}). For the closure operation s on $\mathfrak{L}$ so defined, $\mathfrak{F}''$ is proved to be the center of the lattice $\mathfrak{L}^{s}$, and similarly for $\mathfrak{K}$ in $\mathfrak{M}^{t}$. Imbedding M in its unique minimal injective extension $\hat{M}$, it is proved that the lattice $\mathfrak{M}^{t}$ is isomorphic to the lattice of principal right ideals of the centralizer $\mathfrak{C}$ of R over $\hat{M}$. If s is atomic, $\mathfrak{C}$ is a full direct sum of primitive rings with minimal right ideals.

1. **Faithful rings.** If R is a ring, then $\mathfrak{L}(R)(\mathfrak{L}^{t}(R))$ will designate the lattice of all right (two-sided) ideals of R. We shall upon occasion write $\mathfrak{L}$ or $\mathfrak{L}^{t}$ if the ring in question is obvious. For each subset A of R, A^{l} (A^{r}) will

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(2) *Structure theory of faithful rings*, I. *Closure operations on lattices.* This paper is referred to henceforth as (FI). The bibliography of both papers is contained at the end of (FI).

designate the left (right) annihilator of A in R. Clearly $A^r \in \mathfrak{L}$ for every $A \subset R$, and $A^r \in \mathfrak{L}'$ if $A \in \mathfrak{L}$. The mapping $lr: A \rightarrow A^{lr}$ is a closure operation on both $\mathfrak{L}$ and $\mathfrak{L}'$.

A ring R will be called (left) *faithful* if $R^l = 0$. One could define right faithful rings analogously. We shall investigate in this section the general properties of faithful rings. Thus each ring R considered in this section is assumed to be faithful.

The ideals S of R for which $S^l = 0$ play an important part in our discussion. Let us designate by $\mathfrak{J}(R)$ the set of all such ideals. By assumption, $R \in \mathfrak{J}(R)$; and if $S, T \in \mathfrak{J}(R)$, then $ST \in \mathfrak{J}(R)$. Thus it is evident that $\mathfrak{J}(R)$ is a (not necessarily complete) sublattice of $\mathfrak{L}'(R)$.

If $A \in \mathfrak{L}(R)$, A is called *prime* [10, §2] if and only if

$$rS \subset A, \quad \text{for} \quad r \in R \quad \text{and} \quad S \in \mathfrak{J}(R), \text{ implies } r \in A.$$

We shall designate by $\mathfrak{L}^p(R)$ the set of all prime right ideals of R. It is proved in (FI, §5) that $p \in C_m^0(\mathfrak{L})$ and that $p \leq lr$.

If $A \in \mathfrak{L}^p(R)$ and B is the bound of A, so that B is the largest ideal of R contained in A, then $B \in \mathfrak{L}^p(R)$ also. Thus, if $rS \subset B$ for some $r \in R$ and $S \in \mathfrak{J}(R)$, then $(r)S \subset B$ and $(r) \subset A$, where (r) designates the ideal of R generated by r. Clearly, then, $(r) \subset B$ and $B \in \mathfrak{L}^p(R)$. Consequently, for each $A \in \mathfrak{L}'(R)$, $A^p \in \mathfrak{L}'(R)$ and $p \in C_m^0(\mathfrak{L}')$. It is easily seen that for $A \in \mathfrak{L}'(R)$, $A^p = R$ if and only if $A \in \mathfrak{J}(R)$.

We shall need to consider those ideals of R that as rings are faithful. We separate such ideals into three classes as follows:

$$\mathfrak{F}(R) = \{A; A \in \mathfrak{L}'(R), A \cap A^l = 0\},$$
$$\mathfrak{F}'(R) = \{A; A, A^l \in \mathfrak{F}(R)\},$$
$$\mathfrak{F}''(R) = \{A; A \in \mathfrak{F}(R), A = A^{ll}\}.$$

Clearly $\mathfrak{F}''(R) \subset \mathfrak{F}'(R) \subset \mathfrak{F}(R)$. Also

$$A^l \subset A^r, \qquad A \subset A^{ll} \qquad \text{for each } A \in \mathfrak{F}(R).$$

It is evident that 0, $R \in \mathfrak{F}''(R)$, that $\mathfrak{J}(R) \subset \mathfrak{F}'(R)$, and that if $A \in \mathfrak{F}'(R)$ then $A \in \mathfrak{J}(A^{ll})$ and $A^l \in \mathfrak{F}''(R)$. Also

$$A \cap B = 0 \text{ if and only if } AB = 0; \qquad A \in \mathfrak{L}(R) \text{ and } B \in \mathfrak{F}(R).$$

If $r \in R$, $S \in \mathfrak{J}(R)$, and $A \in \mathfrak{F}(R)$, and if $rS \subset A^l$, then $rAS \subset A \cap A^l = 0$ and $rA = 0$. Thus $r \in A^l$, and we conclude that $A^l \in \mathfrak{L}^p(R)$. In particular, $\mathfrak{F}''(R) \subset \mathfrak{L}^p(R)$.

1.1. LEMMA. *If $A \in \mathfrak{L}'(R)$ and $B \in \mathfrak{F}(R)$, then $(A \cap B)^l \cap B = A^l \cap B$.*

Proof. Clearly $(A \cap B)^l \cap B \supset A^l \cap B$. Since $[(A \cap B)^l \cap B]AB \subset [(A \cap B)^l \cap B](A \cap B) = 0$, we have $[(A \cap B)^l \cap B]A \subset B^l \cap B = 0$. Hence $(A \cap B)^l \cap B \subset A^l \cap B$, and 1.1 is proved.

If, in 1.1, also $A\in\mathfrak{F}(R)$, then $(A\cap B)^l\cap(A\cap B)=A\cap A^l\cap B=0$, and $A\cap B\in\mathfrak{F}(R)$. If A, $B\in\mathfrak{F}''(R)$, then so is $A\cap B$ since $A\cap B\subset(A\cap B)^{ll}\subset A^{ll}\cap B^{ll}$. We have proved the following theorem.

1.2. THEOREM. *Both $\mathfrak{F}(R)$ and $\mathfrak{F}''(R)$ are closed under the finite intersection operation.*

The $\mathfrak{F}$-sets are not generally closed under the union operation, although we do have the following result.

1.3. THEOREM. *If $\{A_i\}\subset\mathfrak{F}'(R)$, then $\bigcup_i A_i\in\mathfrak{F}(R)$.*

Proof. If $A=\bigcup_i A_i$ then $A^l=\bigcap_i A_i^l$. If $A\cap A^l\neq 0$, there would exist elements $A_1, \cdots, A_n$ in $\{A_i\}$ such that, if $B=A_1\cup\cdots\cup A_n$, $B\cap B^l\neq 0$. Since each $A_i^l\in\mathfrak{F}(R)$, $B^l\in\mathfrak{F}(R)$ and therefore $BB^l\neq 0$. However, this is contrary to the fact that $A_iA_i^l=0$ for each i. This contradiction proves 1.3.

The set $\mathfrak{F}''(R)$ can be made into a lattice by defining the union operation $\vee$ as follows:

$$A\vee B=(A+B)^{ll},\qquad A,B\in\mathfrak{F}''(R).$$

Since $(A+B)^{ll}=(A^l\cap B^l)^l$, $A\vee B\in\mathfrak{F}''(R)$ by 1.2. Clearly $A\vee B$ is the least element of $\mathfrak{F}''(R)$ containing both A and B. The mapping $l{:}A\to A^l$ is a dual automorphism of $\mathfrak{F}''(R)$, since

$$(A\vee B)^l=A^l\cap B^l,\qquad (A\cap B)^l=A^l\vee B^l;\qquad A,B\in\mathfrak{F}''(R).$$

The unique complement of A in $\mathfrak{F}''(R)$ is A^l. Thus the following theorem is a consequence of [2, Theorem 17, p. 171].

1.4. THEOREM. *The lattice $\mathfrak{F}''(R)$ is a Boolean algebra.*

For subsets A and B of R, let $AB^{-1}=\{r;r\in R, rB\subset A\}$, and similarly for $B^{-1}A$. Thus $AB^{-1}(B^{-1}A)$ is the largest subset of R such that $(AB^{-1})B\subset A$ $(B(B^{-1}A)\subset A)$. If $A\in\mathfrak{L}(R)$, $B^{-1}A\in\mathfrak{L}(R)$; if $A\in\mathfrak{L}(R)$ and $B\in\mathfrak{L}^{\text{`}}(R)$, then $AB^{-1}\in\mathfrak{L}(R)$.

Each $S\in\mathfrak{L}^{\text{`}}(R)$ defines a mapping

$$\phi_S{:}\phi_SA=A\cap S,\qquad A\in\mathfrak{L}(R),$$

of $\mathfrak{L}(R)$ into $\mathfrak{L}(S)$. Clearly $\phi_SR=S$, ϕ_S is an $\cap$-homomorphism, and $\phi_S(\bigcup_i A_i)=\bigcup_i\phi_SA_i$ for each chain $\{A_i\}\subset\mathfrak{L}(R)$. Thus ϕ_S is an $\cap'$-map (FI, §4) of $\mathfrak{L}(R)$ into $\mathfrak{L}(S)$.

Another mapping associated with $S\in\mathfrak{L}^{\text{`}}(R)$ is

$$\theta_S{:}\theta_SB=BS^{-1},\qquad B\in\mathfrak{L}(S).$$

Since $(BS^{-1})RS\subset(BS^{-1})S\subset B$, evidently $BS^{-1}\in\mathfrak{L}(R)$. Also, $SS^{-1}=R$ and $(\bigcap_i B_i)S^{-1}=\bigcap_i B_iS^{-1}$ for every $\{B_i\}\subset\mathfrak{L}(S)$. Thus θ_S is an $\cap$-map of $\mathfrak{L}(S)$ into $\mathfrak{L}(R)$.

Associated with each subset $\mathfrak{A}$ of $\mathfrak{L}(R)$ ($\mathfrak{B}$ of $\mathfrak{L}(S)$) is a subset $\phi_S\mathfrak{A}$ of

$\mathfrak{L}(S)$ ($\theta_S\mathfrak{B}$ of $\mathfrak{L}(R)$). If $\mathfrak{A}$ ($\mathfrak{B}$) is an inset (FI, §2), then so is $\phi_S\mathfrak{A}$ ($\theta_S\mathfrak{B}$).

If $\mathfrak{A}$ is a partially ordered set, we shall use the notation $\mathfrak{A}/S=\{A; A\in\mathfrak{A}, A\supset S\}$.

1.5. Theorem. *If $S\in\mathfrak{F}'(R)$, ϕ_S is a homomorphism of the inset $\mathfrak{L}^p(R)$ onto $\mathfrak{L}^p(S)$. Actually, ϕ_S is an isomorphism of $\mathfrak{L}^p(R)/S^l$ onto $\mathfrak{L}^p(S)$ with inverse θ_S.*

Proof. If $A\in\mathfrak{L}^p(R)$ and $cP\subset\phi_S A$ for some $c\in S$ and $P\in\mathfrak{J}(S)$, then $c(SPR+S^l)\subset A$. Since $SPR+S^l\in\mathfrak{J}(R)$, $c\in A\cap S$ and therefore $\phi_S A\in\mathfrak{L}^p(S)$.

On the other hand, if $B\in\mathfrak{L}^p(S)$ and $cP\subset\theta_S B$ for some $c\in R$ and $P\in\mathfrak{J}(R)$, then $cSPS\subset B$ and, since $PS\in\mathfrak{J}(S)$, $cS\subset B$. Thus $c\in BS^{-1}$ and evidently $\theta_S B\in\mathfrak{L}^p(R)$. It is clear that $\theta_S B\supset S^l$ and that $\theta_S B\cap S=B$ for each $B\in\mathfrak{L}^p(S)$. Hence $\phi_S\theta_S$ is the identity mapping on $\mathfrak{L}^p(S)$, and ϕ_S is a homomorphism of $\mathfrak{L}^p(R)$ onto $\mathfrak{L}^p(S)$.

If $A\in\mathfrak{L}^p(R)/S^l$, then $\theta_S\phi_S A=(A\cap S)S^{-1}\supset A$. Since $(\theta_S\phi_S A)S\subset A\cap S$, $(\theta_S\phi_S A)(S+S^l)\subset A$ and, since $S+S^l\in\mathfrak{J}(R)$, $\theta_S\phi_S A\subset A$. Thus $\theta_S\phi_S$ is the identity mapping on $\mathfrak{L}^p(R)/S^l$, and 1.5 follows.

In case $S\in\mathfrak{J}(R)$, ϕ_S is an isomorphism of $\mathfrak{L}^p(R)$ onto $\mathfrak{L}^p(S)$ with inverse θ_S according to this theorem.

We mentioned above that ϕ_S maps insets of $\mathfrak{L}(R)$ onto insets of $\mathfrak{L}(S)$. Thus ϕ_S defines a mapping of the set $I(\mathfrak{L}(R))$ of all insets of $\mathfrak{L}(R)$ into the set $I(\mathfrak{L}(S))$ of all insets of $\mathfrak{L}(S)$. Similar remarks may be made for θ_S.

Since ϕ_S defines a mapping of $I(\mathfrak{L}(R))$ into $I(\mathfrak{L}(S))$, it also defines a mapping of the set $C(\mathfrak{L}(R))$ of all closure operations on $\mathfrak{L}(R)$ into the set $C(\mathfrak{L}(S))$ of all closure operations on $\mathfrak{L}(S)$ in a natural way: for $a\in C(\mathfrak{L}(R))$, $b\in C(\mathfrak{L}(S))$,

$$\phi_S a = b \text{ if and only if } \phi_S\mathfrak{L}^a(R) = \mathfrak{L}^b(S).$$

If $A\in\mathfrak{L}(S)$, then A is not necessarily in $\mathfrak{L}(R)$. However, we may still define A^a to be the least element of $\mathfrak{L}^a(R)$ containing A. If $a\in C(\mathfrak{L}(R))$ and $b=\phi_S a$, then clearly

$$A^b = A^a \cap S \qquad \text{for every } A \in \mathfrak{L}(S).$$

In a similar way, θ_S defines a mapping of $C(\mathfrak{L}(S))$ into $C(\mathfrak{L}(R))$.

If $S\in\mathfrak{F}'(R)$ and $a\in C(\mathfrak{L}(R))/p$, then $\phi_S a\in C(\mathfrak{L}(S))/p$ in view of 1.5, and similarly for θ_S. Thus, if $a\geqq p$ and $b=\phi_S a$, then $b\geqq p$ and therefore $A^{pb}=A^b$ for every $A\in\mathfrak{L}(S)$. Hence

$$A^b = A^{pa} \cap S \qquad \text{for every } A \in \mathfrak{L}(S).$$

The advantage of this formula for A^b over the previous one is that $A^p\in\mathfrak{L}(R)$ for every $A\in\mathfrak{L}(S)$.

1.6. Theorem. *If $S\in\mathfrak{F}'(R)$, ϕ_S is an $\cap$-homomorphism of $C(\mathfrak{L}(R))/p$ onto $C(\mathfrak{L}(S))/p$, whereas θ_S is a homomorphism of $C(\mathfrak{L}(S))/p$ into $C(\mathfrak{L}(R))/p$. The mapping ϕ_S carries $C_m(\mathfrak{L}(R))/p$ onto $C_m(\mathfrak{L}(S))/p$. If $S\in\mathfrak{J}(R)$, ϕ_S is an isomorphism of $C(\mathfrak{L}(R))/p$ onto $C(\mathfrak{L}(S))/p$ with inverse θ_S.*

Proof. For $a\in C(\mathfrak{L}(R))/p$, $\phi_S a=b$ where $\phi_S\mathfrak{L}^a(R)=\mathfrak{L}^b(S)\subset\mathfrak{L}^p(S)$. Evidently $\phi_S\theta_S\mathfrak{L}^b=\mathfrak{L}^b$ by 1.5. Since $\phi_S(\bigcup_i \mathfrak{L}^{a_i})=\bigcup_i\phi_S\mathfrak{L}^{a_i}$ for every set $\{a_i\}\subset C(\mathfrak{L}(R))/p$, ϕ_S is an $\cap$-homomorphism of $C(\mathfrak{L}(R))/p$ onto $C(\mathfrak{L}(S))/p$ by (FI, 2.2). Clearly θ_S is a homomorphism of $C(\mathfrak{L}(S))/p$ into $C(\mathfrak{L}(R))/p$.

If $a\in C_m(\mathfrak{L}(R))/p$ and $b=\phi_S a$, then $(A\cap B)^b=(A\cap B)^{pa}\cap S=(A^p\cap B^p)^a\cap S=A^{pa}\cap B^{pa}\cap S=A^b\cap B^b$ for every A, $B\in\mathfrak{L}(S)$, and consequently $b\in C_m(\mathfrak{L}(S))/p$. Next, let us assume that $b\in C_m(\mathfrak{L}(S))/p$, and let us prove that $a=\theta_S b$ is in $C_m(\mathfrak{L}(R))/p$. Since $\phi_S a=b$, this will prove that ϕ_S carries $C_m(\mathfrak{L}(R))/p$ onto $C_m(\mathfrak{L}(S))/p$. Let C be a maximal element of $(\mathfrak{L}(R); A, B)$ (FI, §2) where $A\in\mathfrak{L}(R)$ and $B\in\mathfrak{L}^a(R)$. Then $C\cap A\subset B$ and $\phi_S C\cap\phi_S A\subset\phi_S B$. Since $\phi_S B\in\mathfrak{L}^b$ and $\mathfrak{L}^b$ is an m-inset of $\mathfrak{L}(S)$, there exists some $D\in\mathfrak{L}^b$ such that $D\supset\phi_S C$ and $D\cap\phi_S A\subset\phi_S B$. Hence $\theta_S D\cap\theta_S\phi_S A\subset\theta_S\phi_S B=B$ and, since $\theta_S\phi_S A\supset A$, $\theta_S D\cap A\subset B$. However, $\theta_S D\supset C$, and therefore $\theta_S D=C$ due to the maximality of C in $(\mathfrak{L}(R); A, B)$. Thus $C\in\mathfrak{L}^a$, and $\mathfrak{L}^a$ is an m-inset of $\mathfrak{L}(R)$ by (FI, 2.3). Consequently $a\in C_m(\mathfrak{L}(R))/p$.

If $S\in\mathfrak{I}(R)$, $\phi_S a=b$ if and only if $b=\theta_S a$, and the last part of the theorem follows readily. This completes the proof of 1.6.

1.7. THEOREM. *If $S\in\mathfrak{F}'(R)$, ϕ_S is a homomorphism of $\mathfrak{F}''(R)$ onto $\mathfrak{F}''(S)$. Furthermore, ϕ_S is an isomorphism of $\mathfrak{F}''(R)/S^l$ onto $\mathfrak{F}''(S)$ with inverse θ_S.*

Proof. If $A\in\mathfrak{F}''(R)$, then $\phi_S A\in\mathfrak{F}''(S)$. If $B\in\mathfrak{F}''(S)$, then $B\in\mathfrak{L}^p(S)$ so that $B\in\mathfrak{L}(R)$. Actually, $B\in\mathfrak{L}^i(R)$, since $RBB^l=0$ and $RB\subset B$. Let $A=\theta_S B=BS^{-1}$. Then $AS\subset A\cap S=B$. Now $(A\cap S)^l\cap S=A^l\cap S=B^l\cap S$ and $[(A\cap S)^l\cap S]^l\cap S=(A^l\cap S)^l\cap S=A^{ll}\cap S=B^{ll}\cap S=B$. Thus $A^{ll}S\subset B$ and $A^{ll}\subset A$. This proves that $A\in\mathfrak{F}''(R)$. The rest of the theorem follows from 1.5.

The symbols $\sum$ and $\sum^*$ are used to designate the discrete and the full direct sum respectively of the rings of a given set. It may be shown that if $\{R_i\}$ is a set of faithful rings and if $R=\sum_i R_i$, then

$$\mathfrak{L}^p(R)=\left\{\sum_i A_i;\, A_i\in\mathfrak{L}^p(R_i)\right\},\qquad \mathfrak{F}''(R)=\left\{\sum_i A_i;\, A_i\in\mathfrak{F}''(R_i)\right\}.$$

The concept of the universal extension of a faithful ring was introduced in [9]. Since we shall have occasion to use this extension ring frequently, we will sketch its construction.

For the (left) faithful ring R, let $\mathfrak{C}(R^+, R)$ be the centralizer of the additive group R^+ of R (considered as a right R-module) over the ring R. Then we may consider R^+ as a $(\mathfrak{C}, R)$-module, in which case R may be considered a subring of $\mathfrak{C}$ due to the faithfulness of R. Let the normalizer of R in $\mathfrak{C}$, that is, the largest subring of C containing R as an ideal, be designated by $\overline{R}$. Since $\overline{R}$ has a unit element, it is faithful, and since $R^l=0$ in $\mathfrak{C}$, $R\in\mathfrak{I}(\overline{R})$.

If S is a faithful ring and $R\in\mathfrak{I}(S)$, and if the mapping a^* of R^+ is defined by $a^*: a^*x=ax$, $x\in R^+$, $a\in S$, then $S^*=\{a^*;\, a\in S\}\subset\mathfrak{C}(R^+, R)$. Since R is an ideal in S, $S^*\subset\overline{R}$. Thus, up to an isomorphism, $\overline{R}$ is the universal faithful ring containing R as an ideal such that $R^l=0$ in $\overline{R}$. If R has a unit element, then $\overline{R}=R$; thus, if $S=\overline{R}$, $\overline{S}=\overline{R}$ for every ring R.

1.8. THEOREM. *If $\{R_i\}$ is a set of faithful rings and $R=\sum_i R_i$, then $\overline{R}=\sum_i^* \overline{R}_i$.*

Proof (See [10, 4.4]). Clearly $R\in\mathfrak{Z}(\sum_i^* \overline{R}_i)$, and therefore $\overline{R}\supset\sum_i^* \overline{R}_i$. Since $R_j(R_i\overline{R})=(R_jR_i)\overline{R}=0$ if $i\neq j$, $R_i\overline{R}\subset R_i$. Also, $R_j(\overline{R}R_i)=(R_j\overline{R})R_i\subset R_jR_i=0$ if $i\neq j$, and hence $\overline{R}R_i\subset R_i$. Thus each R_i is an ideal of $\overline{R}$. Now for each $r\in\overline{R}$, $rR_i\subset R_i$ and r has the same effect on R_i as some $r_i\in\overline{R}_i$. Evidently $r=\sum_i^* r_i$ and $\overline{R}\subset\sum_i^* \overline{R}_i$. This proves 1.8.

2. **Restricted rings.** A semi-prime ring is a ring having no nonzero nilpotent ideals [10]. If R is a semi-prime ring, then $\mathfrak{L}^l(R)=\mathfrak{F}(R)=\mathfrak{F}'(R)$, and clearly $\mathfrak{F}''(R)$ is a complete lattice. This leads us to define a *restricted ring* as a faithful ring R satisfying the following condition:

$$\text{If } \{A_i\}\subset\mathfrak{F}''(R), \quad \text{then} \quad \bigcap_i A_i\in\mathfrak{F}''(R).$$

No example is known to us of a faithful ring that is not restricted.

If it is known that $\bigcap_i A_i\in\mathfrak{F}(R)$, where $\{A_i\}\subset\mathfrak{F}''(R)$, then it may easily be proved that $\bigcap_i A_i\in\mathfrak{F}''(R)$. Thus the definition of a restricted ring is not weakened if the conclusion is just that $\bigcap_i A_i\in\mathfrak{F}(R)$. Defining the union operation $\vee$ on $\mathfrak{F}''(R)$ as previously, $\bigvee_i A_i=(\bigcup_i A_i)^{ll}$, it is evident that $\mathfrak{F}''(R)$ is a complete Boolean algebra for a restricted ring.

Unless otherwise stated, each ring R considered in this section is assumed to be restricted.

2.1. THEOREM. *Each ideal $S\in\mathfrak{F}'(R)$ is a restricted ring.*

Proof. If $\{A_i\}\subset\mathfrak{F}''(S)$, then each $A_i=B_i\cap S$ for some $B_i\in\mathfrak{F}''(R)$ according to 1.7. Since $\bigcap_i A_i=(\bigcap_i B_i)\cap S$ and $\bigcap_i B_i\in\mathfrak{F}''(R)$, $\bigcap_i A_i\in\mathfrak{F}''(S)$. This proves 2.1.

2.2. THEOREM. *If S is a restricted ring and $S\in\mathfrak{Z}(R)$, then R also is restricted.*

Proof. If $\{A_i\}\subset\mathfrak{F}''(R)$ and $A=\bigcap_i A_i$, then $A\cap S=\bigcap_i(A_i\cap S)$ is in $\mathfrak{F}''(S)$ by assumption. Thus $(A\cap S)^l\cap(A\cap S)=0$ and, by 1.1, $A^l\cap A\cap S=0$. Hence $A^l\cap A=0$ and $A\in\mathfrak{F}(R)$. This proves 2.2.

2.3. THEOREM. *If $\{S_i\}$ is a set of restricted rings and $R=\sum_i S_i$, then R also is a restricted ring.*

Proof. As previously remarked, $\mathfrak{F}''(R)=\{\sum_i A_i;\ A_i\in\mathfrak{F}''(S_i)\}$. If $\{B_j\}\subset\mathfrak{F}''(R)$, then $B_j=\sum_i B_j\cap S_i$ and $\bigcap_j B_j=\sum_i(\bigcap_j(B_j\cap S_i))$. Thus $\bigcap_j B_j\in\mathfrak{F}''(R)$, and 2.3 follows.

The following theorem gives one condition on a faithful ring that insures it to be restricted.

2.4. THEOREM. *If for the faithful ring R we have $\mathfrak{F}(R)=\mathfrak{F}'(R)$, then R is a restricted ring.*

Proof. If $\{A_i\}\subset\mathfrak{F}''(R)$, then $\bigcup_i A_i^l\in\mathfrak{F}(R)$, and hence $\mathfrak{F}'(R)$, by 1.3. Thus $(\bigcup_i A_i^l)^l=\bigcap_i A_i\in\mathfrak{F}''(R)$.

It is not true that $\mathfrak{F}(R)=\mathfrak{F}'(R)$ for every restricted ring, as the following example shows.

2.5. EXAMPLE. Let F be a finite field, and the following rings be subrings of the total matrix ring F_4:

$$S_1 = Fe_{21},\qquad S_2 = Fe_{33}+Fe_{43},\qquad S=S_1+S_2.$$

Also let

$$\alpha = e_{11}+e_{22}+e_{44},\qquad R=S+F\alpha.$$

Clearly S is an ideal of R and R is a faithful restricted ring since its ideal lattice is finite. Now $S_1=S_2^l$, and therefore $S_2\in\mathfrak{F}(R)$. However, $S_2\notin\mathfrak{F}'(R)$ since $S_2^{rr}\supset S_1$. We note that $\mathfrak{F}''(R)=\{0, R\}$.

Since $\mathfrak{F}''(R)$ is closed under infinite intersections for a restricted ring R, $\mathfrak{F}''(R)$ induces a closure operation f on $\mathfrak{L}(R)$ and $\mathfrak{L}^c(R)$. Thus, for $A\in\mathfrak{L}(R)$, A^f is the least element of $\mathfrak{F}''(R)$ containing A. Clearly $f\in C^0(\mathfrak{L}^c)$, although f is not generally in $C^0_m(\mathfrak{L}^c)$ as is shown by 2.5.

2.6. LEMMA. *If $a\in C^0_m(\mathfrak{L})$, then $a\leq f$.*

Proof. For every $A\in\mathfrak{L}$, $A^f\cap A^{fl}=0$; hence $A\cap A^{fl}=0$ and $A^a\cap A^{fl}=0$. Thus $A^aA^{fl}=0$ and $A^a\subset A^f$. This proves 2.6.

2.7. THEOREM. *If $a\in C^0_m(\mathfrak{L})$ is atomic (homogeneous) then so is f and A^f is an atom of $\mathfrak{L}^f$ for every atom A of $\mathfrak{L}^a$.*

Proof. Let A be an atom of $\mathfrak{L}^a$. If $0\neq B\subset A^f$ for some $B\in\mathfrak{L}^f$, then $B\in\mathfrak{L}^a$ by 2.6. If $A\cap B\neq 0$, then $A\subset B$ due to the atomicity of A, and $B=A^f$. If $A\cap B=0$, $A\subset B^l$ and therefore $A\subset B^l\cap A^f\neq A^f$ in contradiction to the assumption that A^f is the least element of $\mathfrak{L}^f$ containing A. Thus, always, $A^f=B$ and A^f must be an atom of $\mathfrak{L}^f$. The rest of the theorem is obvious.

If a is an atomic closure operation on $\mathfrak{L}(R)$, let us define

$$\mathfrak{B}_a = \{A;\, A \text{ an atom of } \mathfrak{L}^a\}.$$

The *base* B_a of R relative to a is defined to be the ring union of all the elements of $\mathfrak{B}_a$ [10, §4]. If f is atomic, then $B_f^l\in\mathfrak{F}''(R)$ and therefore $B_f\in\mathfrak{F}'(R)$.

2.8. THEOREM. *If f is atomic, the base B_f of R relative to f is the discrete direct sum of the atoms of $\mathfrak{L}^f(R)$.*

Proof. If $\{A_i\}\subset\mathfrak{B}_f$ and $S=\bigcup_i A_i$, then $S^l=\bigcap_i A_i^l$, an element of $\mathfrak{F}''(R)$. If $A\cap S\neq 0$ for some $A\in\mathfrak{B}_f$, then $AS\neq 0$ and $AA_i\neq 0$ for some i. Hence

$A \cap A_i \neq 0$, and $A = A_i$ since both are atoms of $\mathfrak{L}^f$. Consequently $S = \sum_i A_i$ and 2.8 is proved.

If f is atomic and S is an atom of $\mathfrak{L}^f(R)$ $(=\mathfrak{F}''(R))$, then $\mathfrak{F}''(S) = \{0, S\}$ according to 1.8. Let us call a faithful ring *irreducible* if $\mathfrak{F}''(R) = \{0, R\}$. Thus the atoms of $\mathfrak{L}^f(R)$ are irreducible rings.

We are now in a position to prove the main structure theorems for a restricted ring R having an atomic m-closure operation on its lattice of right ideals. We recall from the first section that associated with each faithful ring S is its universal extension ring $\overline{S}$. The following theorem is analogous to [10, 4.5].

2.9. THEOREM. *If f is atomic and $C = B_f^l$, then $B_f \oplus C \subset R \subset \overline{B}_f + \overline{C}$.*

Proof. Since $B_f + C_f^l \in \mathfrak{I}(R)$, the completion, $\overline{B}_f + \overline{C}_f^l$, contains R. This proves 2.9.

The restricted ring B_f^l appearing in the theorem above can have no atomic m-closure operation on its lattice of right ideals, since clearly f is nonatomic on B_f^l.

If f is homogeneous, then $B_f^l = 0$ and we have the following corollary of 2.9 and 1.8.

2.10. COROLLARY. *If f is homogeneous, there exist irreducible rings A_i such that*

$$\sum_i A_i \subset R \subset \sum_i{}^* \overline{A}_i.$$

The converse of 2.10 which is analogous to [10, 4.6] is the following result.

2.11. THEOREM. *If $\{A_i\}$ is a set of irreducible rings and if R is a ring such that*

$$\sum_i A_i \subset R \subset \sum_i{}^* \overline{A}_i,$$

then R is a restricted ring and f is a homogeneous closure operation on $\mathfrak{L}^r(R)$ for which $\mathfrak{B}_f = \{\overline{A}_i \cap R\}$.

Proof. Each A_i is a restricted ring, and hence $S = \sum_i A_i$ is a restricted ring by 2.3. Since $R \subset \overline{S}$, $S \in \mathfrak{I}(R)$ and R is a restricted ring according to 2.2. By 1.6, θ_S is an isomorphism of $\mathfrak{F}''(S)$ onto $\mathfrak{F}''(R)$. Since f is homogeneous on $\mathfrak{L}^r(S)$, f must be homogeneous on $L^r(R)$. The atoms of $\mathfrak{F}''(R)$ have the form $\theta_S A_i = A_i S^{-1} = \overline{A}_i \cap R$. This proves 2.11.

3. **Reducible rings.** A completely reducible ring is a ring R such that $\mathfrak{L}^r(R)$ is a complemented modular lattice. It is well-known (see, for example, [3, Theorem 2]) that a completely reducible ring is a discrete direct sum of simple rings.

As a generalization of complete reducibility, let us call a faithful ring R *reducible* if for every pair A, B of ideals of R for which $A \cap B = 0$, there exist ideals $A' \supset A$ and $B' \supset B$ such that $A' \cap B' = 0$ and $A' + B' \in \mathfrak{I}(R)$. A semi-prime ring is reducible since $A \cap A^l = 0$ and $A + A^l \in \mathfrak{J}(R)$ for every ideal A of R [10, §1]. We give now another example of a reducible ring.

3.1. Example. Let F be a field and the rings described below be subrings of F_4.

$$A = Fe_{11} + Fe_{21}, \qquad B = Fe_{33} + Fe_{43}, \qquad S = A + B.$$

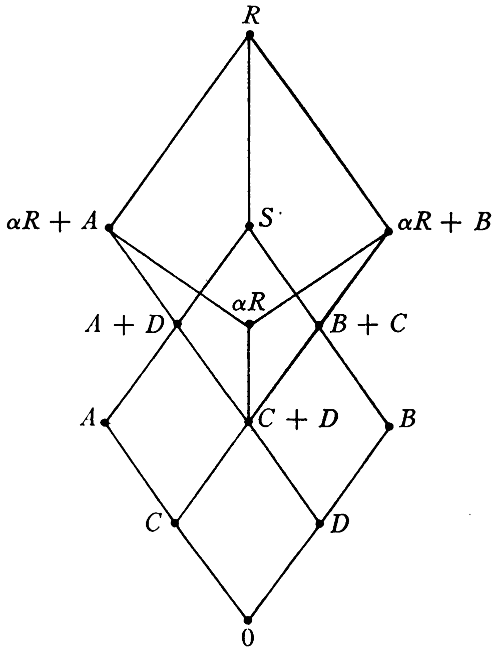
Also let

$$\alpha = e_{22} + e_{44}, \qquad R = S + F\alpha.$$

Since R has a unit element, R is faithful. Evidently $S \in \mathfrak{J}(R)$, and $S^r = C + D + F\alpha$ where

$$C = Fe_{21}, \qquad D = Fe_{43}.$$

The ideal lattice of R may be verified to be as in the figure. Since $A + B \in \mathfrak{J}(R)$,



$\mathfrak{L}^l(R)$

it is evident from this figure that R is a reducible ring. Also, $\mathfrak{F}''(R) = \{0, A, B, R\}$. An interesting feature of this example is that R is not reducible as a right faithful ring, since $S^r \neq 0$.

If $H = Fe_{11} + Fe_{21} \subset F_2$, then H is isomorphic to rings A and B. The centralizer of H, considered as a right H-module, is just F_2. It is clear that the universal extension ring $\overline{H}$ is given by

$$\overline{H} = Fe_{11} + Fe_{21} + Fe_{22}.$$

The structure of R in 2.10 has the form

$$A \oplus B \subset R \subset \overline{A} \oplus \overline{B},$$

where $\overline{A}$ and $\overline{B}$ are isomorphic to $\overline{H}$.

3.2. THEOREM. *If R is reducible, then $\mathfrak{F}(R)=\mathfrak{F}'(R)$.*

Proof. Let $A\in\mathfrak{F}(R)$, so that $A\cap A^{l}=0$. Since R is reducible, there exist ideals $B\supset A$ and $B'\supset A^{l}$ such that $B\cap B'=0$ and $B+B'\in\mathfrak{J}(R)$. Now $B'\subset B^{l}$ and $B^{l}\subset A^{l}$, and therefore $A^{l}=B'$. Since $B\cap A^{l}=0$, $A^{l}\subset B^{l}$ and, in view of our previous remarks, $A^{l}=B^{l}$. Thus $(B+A^{l})^{l}=B^{l}\cap A^{ll}=0$, $A^{l}\cap A^{ll}=0$, and $A\in\mathfrak{F}'(R)$. This proves 3.2.

If R is reducible, R is restricted in view of 3.2 and 2.4. Thus $\mathfrak{F}''(R)=\mathfrak{L}^{f}(R)$, where $f\in C^{0}(\mathfrak{L}^{i}(R))$. We shall now determine the nature of the closure operation f.

We recall that if A, $B\in\mathfrak{L}^{i}(R)$, B is called an essential extension (FI, §6) of A in $\mathfrak{L}^{i}(R)$ if $A\subset B$ and for every $C\in\mathfrak{L}^{i}(R)$ such that $C\cap B\neq 0$ also $C\cap A\neq 0$. The set $\mathcal{E}(A)$ of all essential extensions of A has maximal elements.

3.3. THEOREM. *For every ideal A of the reducible ring R, A^{f} is the unique maximal essential extension of A in $\mathfrak{L}^{i}(R)$.*

Proof. Let B be a maximal element of $(\mathfrak{L}^{i};A,0)$ and C be a maximal element of $\mathcal{E}(A)$. Since C is also a maximal element of $(\mathfrak{L}^{i};B,0)$ and since R is reducible, $C+B\in\mathfrak{J}(R)$, $B=C^{l}$, and $C=B^{l}$. Thus $C\in\mathfrak{F}''(R)$ and $A^{f}\subset C$ since A^{f} is the least element of $\mathfrak{F}''(R)$ containing A. On the other hand, $A^{f}\in\mathfrak{F}''(R)$ and $A^{fl}\cap A^{f}=0$. Since $A\cap A^{fl}=0$, necessarily $C\cap A^{fl}=0$ and $C\subset A^{fll}=A^{f}$. Thus $C=A^{f}$, and the theorem is proved.

3.4. THEOREM. *If R is a reducible ring, the closure operation f is the maximal element of $C^{0}_{m}(\mathfrak{L}^{i})$.*

Proof. Let us first prove that $f\in C^{0}_{m}(\mathfrak{L}^{i})$. To this end, let $A\in\mathfrak{L}^{i}$, $B\in\mathfrak{L}^{f}$, and C be a maximal element of $(\mathfrak{L}^{i};A,B)$. If $D\supset C$, then $D\cap A\not\subset B$ and $E=D\cap A\cap B^{l}\neq 0$. Since $E\cap D\neq 0$ whereas $E\cap C\subset B\cap B^{l}=0$, $D\notin\mathcal{E}(C)$. Hence $\mathcal{E}(C)=\{C\}$, and $C=C^{f}$ by 3.3. Thus $f\in C^{0}_{m}(\mathfrak{L}^{i})$ by (FI, 2.3).

The closure operation f is reducible in the sense of (FI, §6). Hence, by (FI, 6.3), f is the $\cap$-identity element of $C^{0}_{m}(\mathfrak{L}^{i})$. This completes the proof of 3.4.

3.5. THEOREM. *If $\{S_i\}$ is a set of reducible rings and $R=\sum_i S_i$, then R also is a reducible ring.*

Proof. Let A and B be maximal ideals of R such that $A\cap B=0$. We wish to prove that $A+B\in\mathfrak{J}(R)$. Since A, $B\in\mathfrak{L}^{p}(R)$, $A=\sum_i A_i$ and $B=\sum_i B_i$ for some A_i, $B_i\in\mathfrak{L}^{p}(S_i)$. Clearly A_i and B_i are maximal ideals of S_i such that $A_i\cap B_i=0$. Thus $A_i+B_i\in\mathfrak{J}(S_i)$ for each i. If $c(A+B)=0$, where $c=\sum_i c_i$, $c_i\in S_i$, then $c_i(A_i+B_i)=0$ and $c_i=0$. Thus $c=0$, $A+B\in\mathfrak{J}(R)$, and 3.5 is proved.

3.6. Theorem. *If S is a reducible ring and $S\in\mathfrak{J}(R)$, then R also is a reducible ring.*

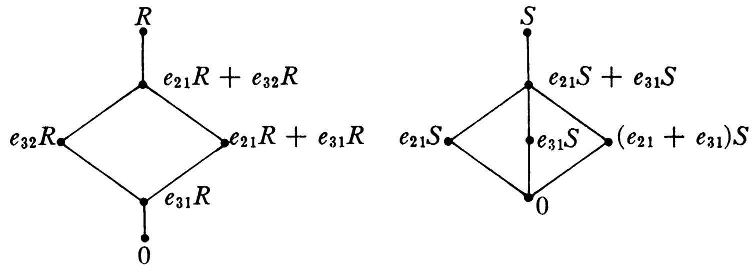
Proof. Let A and B be ideals of R for which $A\cap B=0$, and let $A'=A\cap S$ and $B'=B\cap S$. By assumption, there exist ideals $C\supset A'$ and $D\supset B'$ of S such that $C\cap D=0$, $C^l\cap S=D$, and $D^l\cap S=C$. Since C, $D\in\mathfrak{L}^p(S)$, C, $D\in\mathfrak{L}(R)$ and SC, $SD\in\mathfrak{L}^l(R)$. If $a(SC+SD)=0$, then $aS(C+D)=0$, $aS=0$, and, finally, $a=0$. Thus $SC\cap SD=0$ and $SC+SD\in\mathfrak{J}(R)$. Now $(A+SC)\cap S\subset C$ and $(B+SD)\cap S\subset D$, and therefore $(A+SC)\cap(B+SD)\cap S=0$. Since $S\in\mathfrak{J}(R)$, $(A+SC)\cap(B+SD)=0$, and R is a reducible ring.

The converse of the preceding theorem does not hold, as is shown by the following example.

3.7. Example. Let F be the field of integers modulo 2, and S and R be the following subrings of F_3:

$$S = Fe_{11} + Fe_{21} + Fe_{31}, \qquad R = S + Fe_{32}.$$

Clearly R and S are left faithful rings, and $S\in\mathfrak{J}(R)$. The ideal lattices of R and S are shown in the figure. We see that trivially R is reducible, whereas S is not reducible since $e_{21}S+e^{0}_{31}S$ is not in $\mathfrak{J}(S)$.



$\mathfrak{L}^l(R)$ $\qquad$ $\mathfrak{L}^l(S)$

The fact that R and S are one-sided faithful rings in the example above is important. Thus, it can be shown that if R is both left and right faithful and left and right reducible, then each ideal of R that is faithful is reducible.

Let us call a ring R *strongly irreducible* (s-irreducible) if $A\cap B\neq 0$ for every pair A, B of nonzero ideals of R. The ring R of 3.7 is s-irreducible. Every s-irreducible ring is trivially reducible. It is not known if the atoms of $\mathfrak{L}^f(R)$ need be s-irreducible if R is reducible. However, the following result is true.

3.8. Theorem. *If $\{A_i\}$ is a set of s-irreducible rings and if R is a ring such that*

$$\sum_i A_i \subset R \subset \sum_i{}^* \overline{A}_i,$$

then R is a reducible ring and f is a homogeneous closure operation on $\mathfrak{L}^l(R)$ with $\mathfrak{B}_f=\{\overline{A}_i\cap R\}$.

Proof. The ring $\sum_i A_i$ is reducible by 3.5, and therefore R is reducible by 3.6. The rest of the theorem follows from 2.11.

4. **Modules of a faithful ring.** The ring R encountered in this section is assumed to be faithful. For each $x \in R$ and $A \in \mathfrak{L}(R)$, $x^{-1}A \in \mathfrak{L}(R)$ and x^{-1} is an $\cap$-endomorphism of $\mathfrak{L}(R)$. Since $x^{-1}(\bigcup_i A_i) = \bigcup_i x^{-1}A_i$ for every chain $\{A_i\} \subset \mathfrak{L}(R)$, x^{-1} is an $\cap'$-map (FI, §4) of $\mathfrak{L}(R)$ into $\mathfrak{L}(R)$.

A closure operation a on $\mathfrak{L}(R)$ commutes with x^{-1} if $ax^{-1} = x^{-1}a$, that is, if $(x^{-1}A)^a = x^{-1}A^a$ for every $A \in \mathfrak{L}(R)$. Let

$$C'(\mathfrak{L}) = \{a;\ a \in C(\mathfrak{L}),\ ax^{-1} = x^{-1}a \text{ for every } x \in R\}.$$

By (FI, 4.9), $C'(\mathfrak{L})$ is a complete sublattice of $C(\mathfrak{L})$. That $p \in C'(\mathfrak{L})$ follows from (FI, 5.2).

4.1. Lemma. *If $s \in C'(\mathfrak{L})/p$, then $A^s = \{r;\ r \in R,\ (r^{-1}A)^s = R\}$ for every $A \in \mathfrak{L}$.*

Proof. If $(r^{-1}A)^s = R$ then $r^{-1}A^s = R$, $rR \subset A^s$, and, since $s \geqq p$, $r \in A^s$. Conversely, if $r \in A^s$ then $(r^{-1}A)^s = r^{-1}A^s = R$.

4.2. Lemma. *If $s \in C'(\mathfrak{L})/p$, then $s \in C(\mathfrak{L}^\iota)$.*

Proof. If $A \in \mathfrak{L}^\iota(R)$, $r \in R$ and $c \in A^s$, then $(rc)^{-1}A = c^{-1}(r^{-1}A) \supset c^{-1}A$. Hence $((rc)^{-1}A)^s = R$ and $rc \in A^s$. Thus $A^s \in \mathfrak{L}^\iota(R)$.

An immediate consequence of this lemma is that the bound of every element of $\mathfrak{L}^s(R)$ also is in $\mathfrak{L}^s(R)$.

Let us turn now to a discussion of modules of a faithful ring R. Associated with each (right) R-module M is its lattice of submodules $\mathfrak{M}(M)$. An $N \in \mathfrak{M}(M)$ is called *prime* if and only if

$$xS \subset N, \qquad x \in M \text{ and } S \in \mathfrak{I}(R), \text{ implies } x \in N.$$

We shall again designate by p the closure operation on $\mathfrak{M}(M)$ induced by the set of all prime submodules of M. It is proved in (FI, §5) that $p \in C_m(\mathfrak{M})$. The module M itself is called *prime* if 0 is a prime submodule, that is, if $p \in C_m^0(\mathfrak{M})$.

For subsets A and B of the R-module M, let $B^{-1}A = \{r;\ r \in R,\ Br \subset A\}$. If $A \in \mathfrak{M}(M)$, then $B^{-1}A \in \mathfrak{L}(R)$; if $A \in \mathfrak{M}^p$, $B^{-1}A \in \mathfrak{L}^p(R)$. This last statement follows from the observation that if $rS \subset B^{-1}A$ for some $r \in R$ and $S \in \mathfrak{I}(R)$, then $BrS \subset A$, $Br \subset A$, and finally $r \in B^{-1}A$.

If $s \in C(\mathfrak{L})$, an R-module M is called *s-admissible* if and only if $x^{-1}0 \in \mathfrak{L}^s(R)$ for every $x \in M$. For example, if M is prime then M is p-admissible.

It will be assumed henceforth in this section that R is a ring and s is a closure operation on $\mathfrak{L}(R)$ such that

$$s \in C_m^0(\mathfrak{L})/p,$$

and that M is a prime s-admissible module.

If $N\in\mathfrak{M}(M)$ and x, $y\in M$ such that $(x^{-1}N)^s=(y^{-1}N)^s=R$, and if $r\in R$, then $[(x-y)^{-1}N]^s\supset[(x^{-1}N)\cap(y^{-1}N)]^s=R$, and $[(xr)^{-1}N]^s=[r^{-1}(x^{-1}N)]^s=r^{-1}(x^{-1}N)^s=r^{-1}R=R$. Hence, if for each $N\in\mathfrak{M}(M)$ we define

$$N^t = \{x;\ x\in M,\ (x^{-1}N)^s = R\},$$

then $N^t\in\mathfrak{M}(M)$ and $t=t(s)$ is at least a quasi-closure operation on $\mathfrak{M}(M)$.

4.3. THEOREM. *For every* $x\in M$, $sx^{-1}=x^{-1}t$.

Proof. We must prove that for every $N\in\mathfrak{M}(M)$, $(x^{-1}N)^s=x^{-1}N^t$. If $r\in(x^{-1}N)^s$, then $[r^{-1}(x^{-1}N)]^s=R$ by 4.1. Hence $[(xr)^{-1}N]^s=R$, $xr\in N^t$, and $r\in x^{-1}N^t$. Thus $(x^{-1}N)^s\subset x^{-1}N^t$. A reversal of this argument proves that $x^{-1}N^t\subset(x^{-1}N)^s$, and establishes 4.3.

Since $(x^{-1}N^t)^s=(x^{-1}N)^{ss}=(x^{-1}N)^s$, $(x^{-1}N^t)^s=R$ if and only if $(x^{-1}N)^s=R$. Consequently $N^{tt}=N^t$ for every $N\in\mathfrak{M}(M)$, and t is a closure operation on $\mathfrak{M}(M)$. Actually $t\in C^0(\mathfrak{M})$, since $(x^{-1}0)^s=x^{-1}0$ and $x^{-1}0=R$ if and only if $x=0$.

4.4. LEMMA. *If* K, $N\in\mathfrak{M}^p(M)$ *and* $x^{-1}K=x^{-1}N$ *for every* $x\in M$, *then* $K=N$.

Proof. If $x\in K$, then $x^{-1}N=R$, $xR\subset N$, and $x\in N$. Hence $K\subset N$. Similarly $N\subset K$, and 4.4 follows.

4.5. THEOREM. $t\in C'^0_m(\mathfrak{M})/p$; *and if* $s=p$ *then* $t=p$ *also.*

Proof. If $xS\subset N^t$ for some $x\in M$ and $S\in\mathfrak{Z}(R)$, then $S\subset(x^{-1}N)^s$ and, since $S^p=R$ and $p\leqq s$, $(x^{-1}N)^s=R$ and $x\in N^t$. Hence $N^{tp}=N^t$ and $p\leqq t$ by (FI, 1.6).

To prove that $t\in C_m(\mathfrak{M})$, let K, $N\in\mathfrak{M}(M)$. Then for every $x\in M$, $x^{-1}(K\cap N)^t=[x^{-1}(K\cap N)]^s=(x^{-1}K\cap x^{-1}N)^s=(x^{-1}K)^s\cap(x^{-1}N)^s=x^{-1}K^t\cap x^{-1}N^t=x^{-1}(K^t\cap N^t)$. Hence $(K\cap N)^t=K^t\cap N^t$ by 4.4.

Finally, we know from (FI, 5.2) that $(x^{-1}N)^p=x^{-1}N^p$ for every $x\in M$ and $N\in\mathfrak{M}(M)$. Since $(x^{-1}N)^s=x^{-1}N^t$ by 4.3, $t=p$ if $s=p$ by 4.4.

4.6. THEOREM. *If* s *is atomic and if* $xA\neq 0$ *for some* $x\in M$ *and atom* $A\in\mathfrak{L}^s(R)$, *then* t *is atomic and* $(xA)^t$ *is an atom of* $\mathfrak{M}^t(M)$.

Proof. This follows from (FI, 4.10).

If $N\in\mathfrak{M}(M)$ and $K\in\mathfrak{M}^t(M)$ with $K\subset N$, then the R-module $N-K$ is prime and $\mathfrak{M}(N-K)=\{A-K;\ A\in\mathfrak{M}(M),\ K\subset A\subset N\}$. Since $(x+K)^{-1}\cdot(A-K)=x^{-1}A$, $(x+K)^{-1}0\in\mathfrak{L}^s(R)$ and $N-K$ is s-admissible. The closure operation induced on $N-K$ by s is the obvious one, as stated in the following theorem.

4.7. THEOREM. *If* $N\in\mathfrak{M}(M)$ *and* $K\in\mathfrak{M}^t(M)$ *with* $K\subset N$, *then the closure operation* u *on* $\mathfrak{M}(N-K)$ *induced by* s *is given by*

$$\mathfrak{M}^u(N - K) = \{(A \cap N) - K; A \in \mathfrak{M}^t(M), A \supset K\}.$$

Proof. If $x \in N$ and $A \in \mathfrak{M}^t(M)$ with $A \supset K$, then $(x+K)^{-1}[(A \cap N)-K] = x^{-1}(A \cap N) = x^{-1}A = R$ if and only if $x \in A$. Thus $(A \cap N) - K \in \mathfrak{M}^u(N-K)$. On the other hand, if $B - K \in \mathfrak{M}^u(N-K)$, where $K \subset B \subset N$, and $x \in B^t \cap N$, then $(x^{-1}B)^s = R$ and $[(x+K)^{-1}(B-K)]^s = R$. Hence $x + K \in B - K$ and $x \in B$. This proves that $B^t \cap N = B$, and completes the proof of 4.7.

This theorem allows us to construct examples of the modules considered in this section. We need only select $M = R^+$, $K \in \mathfrak{L}^s(R)$, and $N \in \mathfrak{L}(R)$ in 4.7 to get a module $N - K$ of the type being considered.

Each of the sets $\mathfrak{L}^s(R)$ and $\mathfrak{M}^t(M)$ can be made into a lattice by defining the union operation $\vee$ in the obvious way. Thus, for $\{N_i\} \subset \mathfrak{M}^t(M)$, $\vee_i N_i = (\cup_i N_i)^t$. The lattices so formed are modular by (FI, 6.1).

5. **Annihilating submodules.** In this section, M is assumed to be a prime R-module of a faithful ring R. For each subset A of R, let ${}^lA$ designate the (left) annihilator of A in M.

5.1. LEMMA. *If $A \in \mathfrak{F}'(R)$, then ${}^lA = (MA^l)^p$.*

Proof. We first prove that ${}^lA \in \mathfrak{M}^p(M)$. If $xS \subset {}^lA$ for some $x \in M$ and $S \in \mathfrak{J}(R)$, then $xSA = 0$, $xASA = 0$, and, finally, $xAS(A + A^l) = 0$. Since $A + A^l \in \mathfrak{J}(R)$, $xAS = 0$ and $xA = 0$ by the primeness of M. Hence $x \in {}^lA$ and ${}^lA \in \mathfrak{M}^p(M)$.

Clearly $MA^l \subset {}^lA$, and therefore $(MA^l)^p \subset {}^lA$. If $x \in {}^lA$, then $x(A + A^l) = xA^l \subset (MA^l)^p$ and $x \in (MA^l)^p$. Thus ${}^lA \subset (MA^l)^p$, and the lemma is proved.

5.2. LEMMA. *If $A \in \mathfrak{F}''(R)$ and $N \in \mathfrak{M}(M)$, then ${}^lA \cap N = 0$ if and only if $N \subset {}^l(A^l)$.*

Proof. By 5.1, ${}^lA = (MA^l)^p$ and ${}^l(A^l) = (MA)^p$. Since $(MA^l \cap MA) \cdot (A + A^l) = 0$, $MA^l \cap MA = 0$; and since $p \in C^0_m(\mathfrak{M})$, ${}^lA \cap {}^l(A^l) = 0$. Now if ${}^lA \cap N = 0$ for some $N \in \mathfrak{M}(M)$, then $NA^l \subset {}^lA \cap N = 0$ and $N \subset {}^l(A^l)$. This proves 5.2.

5.3. LEMMA. *If $A, B \in \mathfrak{F}''(R)$, then ${}^l(A \vee B) = {}^lA \cap {}^lB$.*

Proof. Clearly ${}^l(A \vee B) \subset {}^lA \cap {}^lB$. If $C = A + B$, then $C^l = A^l \cap B^l = (A \vee B)^l$. Now $[{}^l(C^l) \cap {}^lA \cap {}^lB](C^l + C) = 0$, and therefore (since $C^l + C \in \mathfrak{J}(R)$) ${}^l(C^l) \cap {}^lA \cap {}^lB = 0$. Thus ${}^lA \cap {}^lB \subset {}^l(C^{ll}) = {}^l(A \vee B)$, and the lemma is proved.

The set

$$\mathfrak{K}(\mathfrak{M}(M)) = \{{}^lA; A \in \mathfrak{F}''(R)\}$$

is closed under the intersection operation by 5.3. It can be made into a lattice by defining ${}^lA \vee {}^lB = {}^l(A \cap B)$, ${}^lA$, ${}^lB \in \mathfrak{K}(\mathfrak{M})$. It may be verified that ${}^lA \vee {}^lB$ actually is the least element of $\mathfrak{K}(\mathfrak{M})$ containing both ${}^lA$ and ${}^lB$.

5.4. THEOREM. *The lattice $\mathfrak{K}(\mathfrak{M})$ is a Boolean algebra. If ${}^lA \neq M$ for every* *nonzero* $A \in \mathfrak{F}''(R)$, *then* $\mathfrak{F}''(R)$ *and* $\mathfrak{K}(\mathfrak{M})$ *are dual isomorphic under the correspondence* $A \to {}^{\iota}A$.

Proof. It is evident that ${}^{\iota}A \cap {}^{\iota}(A^{\iota}) = 0$ and ${}^{\iota}A \vee {}^{\iota}(A^{\iota}) = M$. If ${}^{\iota}A \cap {}^{\iota}B = 0$ and ${}^{\iota}A \vee {}^{\iota}B = M$ for some $A, B \in \mathfrak{F}''(R)$, then ${}^{\iota}B \subset {}^{\iota}(A^{\iota})$. If we let $C = A^{\iota} \vee B$, then ${}^{\iota}C = {}^{\iota}B$ so that ${}^{\iota}A \cap {}^{\iota}C = 0$ and ${}^{\iota}A \vee {}^{\iota}C = M$. Now $C = A^{\iota} \vee (A \cap C)$, and therefore ${}^{\iota}C = {}^{\iota}(A^{\iota}) \cap ({}^{\iota}A \vee {}^{\iota}C) = {}^{\iota}(A^{\iota})$. Hence each ${}^{\iota}A \in \mathfrak{K}(\mathfrak{M})$ has a unique complement ${}^{\iota}(A^{\iota})$, and $\mathfrak{K}(\mathfrak{M})$ is a Boolean algebra.

If ${}^{\iota}A \neq M$ for every nonzero $A \in \mathfrak{F}''(R)$ and if ${}^{\iota}A = {}^{\iota}B$ for some $A, B \in \mathfrak{F}''(R)$, then ${}^{\iota}(A \cap B^{\iota}) = {}^{\iota}A \vee {}^{\iota}(B^{\iota}) = {}^{\iota}B \vee {}^{\iota}(B^{\iota}) = M$ and $A \cap B^{\iota} = 0$. Thus $A \subset B$. Similarly one proves that $B \subset A$, and therefore $A = B$. Hence $\mathfrak{F}''(R)$ and $\mathfrak{K}(\mathfrak{M})$ are dual isomorphic, and 5.4 is proved.

If $M = R^{+}$, then it is clear that ${}^{\iota}A = A^{\iota}$ for each $A \in \mathfrak{F}''(R)$, and the mapping $A \to {}^{\iota}A$ is essentially the dual automorphism of R defined in the Boolean algebra $\mathfrak{F}''(R)$.

The lattice $\mathfrak{K}(\mathfrak{M})$ is quite easily shown to be a complete Boolean algebra in case R is a restricted ring. Also, $\mathfrak{K}(\mathfrak{M})$ is dual isomorphic with $\mathfrak{F}''(R)/C$ (and also $\mathfrak{F}''(C^{\iota})$) if R is restricted, where C is the maximal annihilator of M in $\mathfrak{F}''(R)$. In case R is restricted, the lattice $\mathfrak{K}(\mathfrak{M})$ induces a closure operation g on $\mathfrak{M}(M)$; for $N \in \mathfrak{M}(M)$, N^{g} is the least element of $\mathfrak{K}(\mathfrak{M})$ containing N.

6. **Rings with nonsingular elements.** For $A, B \in \mathfrak{L}(R)$, we shall write $A \subset' B$ if B is an essential extension of A. The elements of $\mathfrak{L}(R)$ having R as an essential extension are of particular importance in the following discussion. Thus we shall let

$$\mathfrak{L}^{\blacktriangle}(R) = \{A;\, A \in \mathfrak{L}(R),\, A \subset' R\}.$$

If M is an R-module, then we shall also let

$$\mathfrak{M}^{\blacktriangle}(M) = \{N;\, N \in \mathfrak{M}(M),\, N \subset' M\}.$$

Clearly $\mathfrak{L}^{\blacktriangle}$ and $\mathfrak{M}^{\blacktriangle}$ are lattices, although not usually complete lattices.

Contained in R and M are the subsets $R^{\blacktriangle}$ and $M^{\blacktriangle}$ defined by

$$R^{\blacktriangle} = \{r;\, r \in R,\, r^{-1}0 \in \mathfrak{L}^{\blacktriangle}(R)\}, \qquad M^{\blacktriangle} = \{x;\, x \in M,\, x^{-1}0 \in \mathfrak{L}^{\blacktriangle}(R)\}.$$

It may be shown that $R^{\blacktriangle}$ is an ideal of R and $M^{\blacktriangle}$ is a submodule of M. We call $R^{\blacktriangle}$ the *singular ideal* [6, p. 894] of R and $M^{\blacktriangle}$ the *singular submodule* of M.

If $N \in \mathfrak{M}(M)$, evidently $N^{\blacktriangle} = M^{\blacktriangle} \cap N$. Therefore, if $N \subset' M$, $N^{\blacktriangle} = 0$ if and only if $M^{\blacktriangle} = 0$. Also, if $R^{\blacktriangle} \neq 0$ and $MR^{\blacktriangle} \neq 0$, then $M^{\blacktriangle} \supset MR^{\blacktriangle} \neq 0$.

We shall assume henceforth in this section that R and M are so chosen that

$$R^{\blacktriangle} = 0 \quad \text{and} \quad M^{\blacktriangle} = 0.$$

Under these restrictions, $A^{\iota} = 0$ and ${}^{\iota}A = 0$ for every $A \in \mathfrak{L}^{\blacktriangle}(R)$.

If $B \in \mathfrak{L}(R)$ and B' is a complement of B, then $B + B' \in \mathfrak{L}^{\blacktriangle}(R)$ and $(B + B')^{\iota} = 0$. In particular, if $A \in \mathfrak{F}(R)$, then A^{ι} is the unique complement of

A and $(A+A^l)^l=A^l\cap A^{ll}=0$. Therefore $\mathfrak{F}(R)=\mathfrak{F}'(R)$, and we conclude from 2.4 that R is a restricted ring if $R^\blacktriangle=0$.

6.1. THEOREM. *If $N\in\mathfrak{M}(M)$, $N\subset' M$ if and only if $x^{-1}N\in\mathfrak{L}^\blacktriangle(R)$ for every $x\in M$.*

Proof. If $N\subset' M$ and $x\in M$, and if $A\cap x^{-1}N=0$ for some $A\in\mathfrak{L}(R)$, then $xA\cap N=0$, $xA=0$, and $A\subset x^{-1}N$. Thus $A=0$, and we conclude that $x^{-1}N \in\mathfrak{L}^\blacktriangle(R)$.

Conversely, if $K\in\mathfrak{M}(M)$ and $K\cap N=0$, and if $x^{-1}N\in\mathfrak{L}^\blacktriangle(R)$ for every $x\in K$, then $x(x^{-1}N)=0$ and $x\in M^\blacktriangle$. Hence $K=0$ and $N\subset' M$.

6.2. THEOREM. *If $\{N_i\}\subset\mathfrak{M}(M)$ and, for each i, K_i is selected so that $N_i\subset' K_i$, then $\bigcup_i N_i\subset'\bigcup_i K_i$.*

Proof. If $P\in\mathfrak{M}(M)$ such that $P\subset\bigcup_i K_i$ and $P\cap(\bigcup_i N_i)=0$, then each $k\in P$ has the form $k=\sum_i k_i$, $k_i\in K_i$. If $A=\bigcap_i k_i^{-1}N_i$ (a finite intersection), then $A\in\mathfrak{L}^\blacktriangle(R)$ by 6.1 and $kA=0$. Hence $k=0$, and $P=0$. This proves 6.2.

6.3. THEOREM. *If $N\in M(M)$ and $x\in M$ such that $x^{-1}N\in\mathfrak{L}^\blacktriangle(R)$, then $N\subset' N+(x)$.*

Proof. Since $(y+xr)^{-1}N=(xr)^{-1}N=r^{-1}(x^{-1}N)\in\mathfrak{L}^\blacktriangle(R)$ for every $y\in N$ and $r\in R$ (and similarly for $(y+nx)^{-1}N$, n an integer), 6.3 follows from 6.1.

6.4. THEOREM. *Each $N\in\mathfrak{M}(M)$ has a unique maximal essential extension N^t given by $N^t=\{x;\, x\in M,\, x^{-1}N\in\mathfrak{L}^\blacktriangle(R)\}$.*

Proof. The union of all essential extensions of N is the unique maximal essential extension by 6.2. The rest of the theorem follows from 6.3.

This theorem applies equally well to R. Thus each $A\in\mathfrak{L}(R)$ has a unique maximal essential extension $A^s=\{r;\, r\in R,\, r^{-1}A\in\mathfrak{L}^\blacktriangle(R)\}$.

That 6.4 is not true in general may be seen from the example that follows.

6.5. EXAMPLE. Let $N\subset' M$ and $P=M\oplus(M-N)$. Define the submodule K of P as follows: $K=\{(x, x+N);\, x\in M\}$. Then both M and K are maximal essential extensions of N in P. If $M^\blacktriangle=0$, then $P^\blacktriangle=M-N$.

6.6. THEOREM. *$s\in C_m'^0(\mathfrak{L})/p$ and $t\in C_m^0(\mathfrak{M})/p$.*

Proof. If A, $B\in\mathfrak{M}(M)$, $A\subset B$, then $A+B\subset' A^t+B^t$ by 6.2. Thus $B\subset' A^t+B^t$ and $A^t\subset B^t$. Therefore it is clear that $t\in C^0(\mathfrak{M})$. Since $A\cap B \subset' A^t\cap B^t\subset'(A\cap B)^t$ and $(A\cap B)^t\subset A^t\cap B^t$, we have $(A\cap B)^t=A^t\cap B^t$ and $t\in C_m^0(\mathfrak{M})$. Clearly each A^t is prime, and hence $t\geqq p$.

We need only prove that $s\in C'(\mathfrak{L})$ to complete the proof of 6.6. Let us actually prove the stronger result that $sx^{-1}=x^{-1}t$ for every $x\in M$. Let $A\in\mathfrak{L}(R)$ with $A\neq 0$, $x\in M$ and $N\in\mathfrak{M}(M)$ such that $A\subset x^{-1}N^t$. Then $xA\subset N^t$ and either $xA=0$ or $xA\cap N\neq 0$. In either case, we have $A\cap x^{-1}N\neq 0$. Thus $x^{-1}N\subset' x^{-1}N^t$ and therefore $x^{-1}N^t\subset'(x^{-1}N)^s$. On the other hand, if

$r \in (x^{-1}N)^s$ and $A = r^{-1}(x^{-1}N)$, then $A \in \mathfrak{L}^\blacktriangle(R)$. Hence $xrA \subset N$, $xr \in N^t$ and $r \in x^{-1}N^t$. Thus $(x^{-1}N)^s \subset x^{-1}N^t$ and, in view of the earlier inclusion, $(x^{-1}N)^s = x^{-1}N^t$. This completes the proof of 6.6.

It is clear now that the assumptions $R^\blacktriangle = 0$ and $M^\blacktriangle = 0$ lead to realizations of the closure operations s and t discussed in §4. Actually, the closure operations s and t are reducible in the sense of (FI, §6). Thus, for each $A \in \mathfrak{L}^s(R)$, $A \neq R$, there exists some nonzero $B \in \mathfrak{L}^s(R)$ such that $A \cap B = 0$, and similarly for t. Hence $\mathfrak{L}^s(R)$ and $\mathfrak{M}^t(M)$ are complemented modular lattices by (FI, 6.2).

Let us now show that the complete Boolean algebra $\mathfrak{M}^g(M)(=\mathfrak{K}(\mathfrak{M}))$ is a sublattice of $\mathfrak{M}^t(M)$. Analogous remarks will hold for $\mathfrak{L}^f(R)$ in $\mathfrak{L}^s(R)$.

If $\{A_i\} \subset \mathfrak{L}^f(R)$ and $A = \bigcap_i A_i$, then by definition $(\bigcup_i {}^lA_i)^g = {}^lA$. For every $x \in {}^lA$, $x^{-1}(\bigcup_i {}^lA_i) \supset A \cup (\bigcup_i A_i^l) \in \mathfrak{L}^\blacktriangle(R)$. Hence $\bigcup_i {}^lA_i \subset' {}^lA$ by 6.1, and $(\bigcup_i {}^lA_i)^t = {}^lA$. Thus $\mathfrak{M}^g(M)$ is a sublattice of $\mathfrak{M}^t(M)$. We shall presently prove that $\mathfrak{M}^g(M)$ is the center of $\mathfrak{M}^t(M)$. First, let us prove the corresponding result for R.

6.7. THEOREM. *The Boolean algebra $\mathfrak{L}^f(R)$ is the center of $\mathfrak{L}^s(R)$.*

Proof. Clearly $\mathfrak{L}^f$ is contained in the center of $\mathfrak{L}^s$. Let $A \in \mathfrak{L}^s$ have the unique complement A', so that A is also the unique complement of A'. We shall first prove that $A, A' \in \mathfrak{L}^i$. This will be done by showing that $rA \cap A' = 0$ for every $r \in R$, which will imply that $rA \subset A$ for every $r \in R$. If $ra \in A'$ for some $r \in R$ and $a \in A$, then $(ra+a)R \cap A' = 0$ since $rax + ax \in A'$ implies $ax \in A'$ and $ax = rax = 0$. Hence $(ra+a)R \subset A$, $raR \subset A$, and $ra \in A$. Since $ra \in A'$ by assumption, $ra = 0$. This proves that A is an ideal of R. Similarly, $A' \in \mathfrak{L}^i$.

If $rA = 0$ for some $r \in A$, then $r(A+A') = 0$ and $r = 0$; hence $A \in \mathfrak{F}(R)$. Since $A^f = A^s$ for $A \in \mathfrak{F}(R)$, $A \in \mathfrak{L}^f(R)$ and 6.7 follows.

6.8. THEOREM. *Let $N \subset' M$ and, for every $K \in \mathfrak{M}(N)$, K^u be the maximal essential extension of K in N. Then $(K \cap N)^u = K^t \cap N$, $K \in \mathfrak{M}(M)$, and $\mathfrak{M}^t(M) \cong \mathfrak{M}^u(N)$ under the correspondence $K \to K \cap N$, $K \in \mathfrak{M}^t(M)$.*

Proof. If $x \in (K \cap N)^u$, then $x \in N$ and $x^{-1}(K \cap N) \in \mathfrak{L}^\blacktriangle(R)$. Hence $x^{-1}K \in \mathfrak{L}^\blacktriangle(R)$ and $x \in K^t$. On the other hand, if $x \in K^t \cap N$, then $x \in N$ and $x^{-1}K \in \mathfrak{L}^\blacktriangle(R)$. Hence $x^{-1}(K \cap N) \in \mathfrak{L}^\blacktriangle(R)$ and $x \in (K \cap N)^u$. This proves that $(K \cap N)^u = K^t \cap N$ for every $K \in \mathfrak{M}(M)$.

For $K_i \in \mathfrak{M}(M)$, $(K_1 \cap N)^u = (K_2 \cap N)^u$ implies $(K_1^t \cap N)^t = (K_2^t \cap N)^t$ and $K_1^t = K_2^t$. Clearly $(K_1 \cap N)^u \subset (K_2 \cap N)^u$ if and only if $K_1^t \subset K_2^t$. This proves 6.8.

If $N \in \mathfrak{M}(M)$ and N' is a complement of N, then for every $K \supset N$ such that $K \cap N' = 0$, necessarily $N \subset' K$ (FI, §6).

If K, $N \in \mathfrak{M}^t(M)$ and $K \not\subset N$, then K is not an essential extension of $K \cap N$. Thus there exists a nonzero $A \in \mathfrak{M}(M)$ such that $A \subset K$, $A \cap N = 0$.

We shall further restrict ourselves in the remainder of this section by assuming that s is homogeneous. By 4.6, it is clear that t also is homogeneous. In view of our remarks of the previous paragraph, if K and N are distinct elements of $\mathfrak{M}^t(M)$, there exists an atom of $\mathfrak{M}^t(M)$ contained in one and only one of K and N.

The following theorem was proved for prime rings in [7, 4.11].

6.9. THEOREM. *If $x \in M$, $x \neq 0$, then $(xR)^t$ is an atom of $\mathfrak{M}^t(M)$ if and only if $x^{-1}0$ is a maximal element of $\mathfrak{L}^s(R)$.*

Proof. If $(xR)^t$ is an atom of $\mathfrak{M}^t(M)$ and $x^{-1}0$ is not maximal in $\mathfrak{L}^s(R)$, then $x^{-1}0 \subset C \subset R$ for some $C \in \mathfrak{L}^s$, $C \neq x^{-1}0$, $C \neq R$. Hence there exist atoms A, $B \in \mathfrak{L}^s(R)$ for which $x^{-1}0 \cap A = 0$, $A \subset C$, and $B \cap C = 0$. Evidently $xA \neq 0$ and $xB \neq 0$, and therefore $xA \cap xB \neq 0$ since $(xR)^t$ is an atom. Hence $x^{-1}0 \cap (A+B) \neq 0$, contrary to the choice of A and B. Thus $x^{-1}0$ must be maximal in $\mathfrak{L}^s$.

Conversely, if $x^{-1}0$ is maximal in $\mathfrak{L}^s$ and N is an atom of $\mathfrak{M}^t$ such that $N \subset (xR)^t$, then $N \cap xR \neq 0$, $x^{-1}N \supset x^{-1}0$, and therefore $x^{-1}N = R$. Hence $x \in N$ and $(xR)^t$ is an atom of $\mathfrak{M}^t$.

From this theorem, R cannot be an integral domain unless $\mathfrak{L}^s(R) = \{0, R\}$. And if R is an integral domain, our assumptions are such as to insure that R necessarily has a right quotient division ring.

The atoms of $\mathfrak{M}^g(M)$ may be characterized by use of the concept of perspectivity. Two elements A and B of $\mathfrak{M}^t(M)$ are called *perspective* [2, p. 118], and we write $A \sim B$, if and only if A and B have a common complement. It is easily shown that perspectivity is an equivalence relation on $\mathfrak{B}_t$, the set of all atoms of $\mathfrak{M}^t(M)$. As in [2, p. 120, Lemma 3], we may show that two distinct elements of $\mathfrak{B}_t$ are perspective if and only if their union contains a third atom. A ring-theoretic version of this result is as follows.

6.10. THEOREM. *For $A, B \in \mathfrak{B}_t$, $A \sim B$ if and only if there exist nonzero $a \in A$ and $b \in B$ such that $a^{-1}0 = b^{-1}0$.*

Proof. The theorem is obvious if $A = B$, so let us assume that $A \neq B$. If $a^{-1}0 = b^{-1}0$ for some nonzero $a \in A$ and $b \in B$, let $C = [(a+b)R]^t$. Then C is an atom by 6.9, $C \subset A \vee B$, and $A \sim B$.

Conversely, if $A \sim B$, select $C \subset A \vee B$, $C \neq A$ and $C \neq B$. Each nonzero $c \in C$ has the form $c = a + b$ for some nonzero $a \in A$ and $b \in B$. Since $c^{-1}0$ is a maximal element of $\mathfrak{L}^s$ and $c^{-1}0 = a^{-1}0 \cap b^{-1}0$, we have $a^{-1}0 = b^{-1}0$. This proves 6.10.

Our remarks on perspective elements apply equally well to $\mathfrak{L}^s(R)$. The relations of perspectivity on $\mathfrak{B}_s$ and $\mathfrak{B}_t$ are connected by the following result.

6.11. THEOREM. *If A, $B \in \mathfrak{B}_s$ and x, $y \in M$ with $xA \neq 0$ and $yB \neq 0$, then $A \sim B$ if and only if $(xA)^t \sim (yB)^t$.*

Proof. If $A \sim B$, so that $a^{-1}0 = b^{-1}0$ for some nonzero $a \in A$ and $b \in B$ then

$(xa)^{-1}0=(yb)^{-1}0$ and $(xA)^t \sim (yB)^t$. Conversely, if $(xA)^t \sim (yB)^t$, there exist nonzero $u \in (xA)^t$ and $v \in (yB)^t$ such that $u^{-1}0=v^{-1}0$. Since $uR \cap xA \neq 0$, $ur=xa\neq 0$ for some $r \in R$ and $a \in A$, and $(xa)^{-1}0=(vr)^{-1}0$. Similarly, $vrr' = yb \neq 0$ for some $r' \in R$ and $b \in B$, and $(xar')^{-1}0=(yb)^{-1}0$. Since $A \cap x^{-1}0 = B \cap y^{-1}0 = 0$, evidently $(ar')^{-1}0=b^{-1}0$ and $A \sim B$.

For each $A \in \mathfrak{B}_t$, let A^* designate the union in $\mathfrak{M}^t(M)$ of all $B \sim A$. It is easily demonstrated that an atom $B \subset A^*$ if and only if $B \sim A$.

6.12. Theorem. *If $A \in \mathfrak{B}_t$, then A^* is an atom of the center of $\mathfrak{M}^t(M)$.*

Proof. If B is a complement of A^* and B is not unique, then there exists some atom C such that $C \cap A^* = C \cap B = 0$. Since $C \cap (A^*+B) \neq 0$, $c=a+b$ for some nonzero $a \in A$, $b \in B$, $c \in C$. Clearly $a^{-1}0=b^{-1}0=c^{-1}0$, $C \sim (aR)^t$, and $C \sim A$ contrary to assumption. Hence A^* has a unique complement and is in the center of $\mathfrak{M}^t$.

If N is an atom of the center of $\mathfrak{M}^t$ and $A \subset N$, $A \in \mathfrak{B}_t$, then $N \subset A^*$. If $N \neq A^*$, there exists $B \in \mathfrak{B}_t$ such that $B \subset A^*$ and $B \cap N = 0$. Since $A \sim B$, $C \subset A \vee B$ for some atom C distinct from A and B. If $C \subset N$, then $C \vee A \subset N$ and $B \subset N$ contrary to our choice of B. If $C \subset N'$, the complement of N, then $C \vee B \subset N'$ and $A \subset N'$ contrary to our choice of A. Hence $N=A^*$, and 6.12 is proved.

Letting $M=R^+$ in the theorem above, we have that the atoms of $\mathfrak{L}^f(R)$, the center of $\mathfrak{L}^s(R)$, are of the form A^* for $A \in \mathfrak{B}_s$. This is in keeping with the classical result for a ring R with minimal right ideals, in which case each foot of the socle is a union of isomorphic (as R-modules) minimal right ideals.

If A^* is an atom of $\mathfrak{L}^f(R)$, $A \in \mathfrak{B}_s$, then $(MA^*)^t$ is in $\mathfrak{M}^g(M)$ and hence in the center of $\mathfrak{M}^t(M)$. Let N be an atom of the center of $\mathfrak{M}^t$, $N \subset (MA^*)^t$. If $x \in MA^*$, $x \neq 0$, and $B \in \mathfrak{B}_s$ with $xB \neq 0$, then necessarily $B \subset A^*$. Since $NA^* \neq 0$, $yC \neq 0$ for some $y \in N$ and $C \sim B$. Thus $(xB)^t \sim (yC)^t$ by 6.11, and $xB \subset N$ by 6.12. Hence $xB \subset N$ for every atom B of $\mathfrak{L}^s$ and $x^{-1}N \in \mathfrak{L}^{\blacktriangle}$. Therefore $x \in N^t = N$, and $(MA^*) = N$. We have proved that the atoms of the center of $\mathfrak{M}^t$ are just the atoms of $\mathfrak{M}^g$. Each element of the center of $\mathfrak{M}^t$ is a union of atoms of the center, and the following theorem is established.

6.13. Theorem. *The Boolean algebra $\mathfrak{M}^g(M)$ is the center of $\mathfrak{M}^t(M)$.*

The well-known result that every complete atomic modular lattice is a discrete direct sum of atomic projective geometries [2, p. 131] applies to the lattice $\mathfrak{M}^t(M)$. In our case, the projective geometries are the submodules N_i^* where $N_i \in \mathfrak{B}_t$, and $M = \bigvee_i N_i^*$.

7. **Injective modules.** The R-module M is called *injective* [4] if and only if for every pair A, B of R-modules with $A \subset B$, each homomorphism of A into M can be extended to one of B into M. If R is a ring with unity 1 and if 1 is the identity operator of an R-module M, then it has been shown by Baer [1] and more recently by Eckmann and Schopf [4] that M has a mininal injective

extension $\hat{M}$, unique up to an isomorphism. At the same time, $\hat{M}$ is a maximal essential extension of M. These results are true for any ring R and any R-module, as we shall now prove.

7.1. THEOREM. *If R is a ring and M is an R-module, there exists a minimal injective extension $\hat{M}$ of M that is unique up to an isomorphism.*

Proof. We may imbed R in a ring S with unity in a standard way. Thus let $S=\{(a, n); a\in R, n\in I\}$, where I is the ring of integers, with the operations defined as if $(a, n)=a+n$ [12, p. 87]. If we identify R with the subring $\{(a, 0); a\in R\}$ of S, then R is an ideal of S.

Now M becomes an S-module if we define $x(a, n)=xa+nx$ for each $x\in M$ and $(a, n)\in S$. The identity element $(0, 1)$ of S acts as the identity operator on M. Therefore, as an S-module, M has a minimal injective extension $\hat{M}$.

In order to prove that $\hat{M}$ is an injective extension of M as an R-module, let A and B be R-modules with $A\subset B$ and let ϕ be a R-homomorphism of A into $\hat{M}$. Then ϕ also is an S-homomorphism of A into $\hat{M}$ if we define $\phi[x(a, n)]=\phi(xa+nx)$, $x\in A$, $(a, n)\in S$. Thus ϕ can be extended to an S-homomorphism ϕ' of B into $\hat{M}$. Clearly ϕ' is an R-homomorphism of B into $\hat{M}$, and we conclude that M is an injective extension of the R-module M.

Each maximal essential extension of M in $\hat{M}$ will be a minimal injective extension of M. The uniqueness of this minimal injective extension may be shown as in [4].

Let us assume henceforth in this section that R is a ring and M is an R-module such that

$$R^{\blacktriangle} = 0 \quad \text{and} \quad M^{\blacktriangle} = 0.$$

Again, $\hat{M}$ designates the minimal injective extension of M. We know that $\hat{M}^{\blacktriangle}=0$. For N in $\mathfrak{M}(\hat{M})(\mathfrak{M}(M))$, let N^v (N^t) be the maximal essential extension of N in $\hat{M}(M)$. According to 6.8, $\mathfrak{M}^v(\hat{M})\cong\mathfrak{M}^t(M)$ under the correspondence $N\rightarrow N\cap M$, $N\in\mathfrak{M}^v(\hat{M})$. For each $N\in\mathfrak{M}(\hat{M})$, N^v is also the minimal injective extension of N.

7.2. LEMMA. *If $N_i\in\mathfrak{M}^v(\hat{M})$, then $N_1\vee N_2=N_1+N_2$.*

Proof. The lemma is obvious if $N_1\subset N_2$, so let us assume $N_1\not\subset N_2$. Select $K\in\mathfrak{M}(\hat{M})$ maximal so that $K\subset N_1$, $K\cap N_2=0$. Clearly $K\in\mathfrak{M}^v(\hat{M})$ and $N_1=(N_1\cap N_2)\oplus K$; for N_1-K is an essential extension of $N_1\cap N_2$, and therefore $N_1-K\cong N_1\cap N_2$. Since N_2 and K are injective, so is $N_2\oplus K$. Thus $N_1+N_2=N_2\oplus K$, and evidently $N_1\vee N_2=N_1+N_2$.

We shall designate by $\mathfrak{C}(\hat{M}, R)$ the centralizer of R over $\hat{M}$, and we shall consider $\hat{M}$ as a $(\mathfrak{C}, R)$-module.

7.3. THEOREM. *For each $\alpha\in\mathfrak{C}(\hat{M}, R)$, $\alpha^{-1}v=v\alpha^{-1}$.*

Proof. If $N\in\mathfrak{M}(\hat{M})$ and $x\in(\alpha^{-1}N)^v$, then $x^{-1}(\alpha^{-1}N)=(\alpha x)^{-1}N\in\mathfrak{L}^{\blacktriangle}(R)$,

$\alpha x \in N^v$, and finally $x \in \alpha^{-1}N^v$. Thus $(\alpha^{-1}N)^v \subset \alpha^{-1}N^v$. Similarly, one proves that $\alpha^{-1}N^v \subset (\alpha^{-1}N)^v$, and 7.3 follows.

If $N \in \mathfrak{M}^{\blacktriangle}(M)$ and α is a homomorphism of the R-module N into M, then there exists a unique element $\beta \in \mathfrak{C}(\hat{M}, R)$ such that $\beta x = \alpha x$ for every $x \in N$. Thus, if there were two elements of $\mathfrak{C}$ agreeing with α on N, their difference γ would annihilate N, $\gamma^{-1}0 \supset N$. Since $\gamma^{-1}0 \in \mathfrak{M}^v(\hat{M})$ by 7.3, $\gamma^{-1}0 = \hat{M}$ and $\gamma = 0$. From these remarks, it is evident that $\mathfrak{C}(\hat{M}, R)$ is isomorphic to the extended centralizer of R over M as defined in [6]. Thus the following results on $\mathfrak{C}(\hat{M}, R)$ are generalizations of some of our previous results on the extended centralizer [6; 7].

For each $N \in \mathfrak{M}(\hat{M})$, let

$$\mathfrak{C}(N) = \{\alpha; \alpha \in \mathfrak{C}(\hat{M}, R), \alpha\hat{M} \subset N\}.$$

As in [7, 5.5], the principal right ideals of the regular ring $\mathfrak{C}(\hat{M}, R)$ may be identified as follows.

7.4. Lemma. *If $A \in \mathfrak{L}(\mathfrak{C})$, A is principal if and only if $A = \mathfrak{C}(N)$ for some $N \in \mathfrak{M}^v(\hat{M})$.*

Proof. If $N \in \mathfrak{M}^v(\hat{M})$ and N' is a complement of N, then $N + N' = \hat{M}$. Let $\epsilon \in \mathfrak{C}$ be defined as follows: $\epsilon x = x$, $x \in N$; $\epsilon N' = 0$. Then $(\epsilon\mathfrak{C})\hat{M} = N$. If $\alpha\hat{M} \subset N$ for $\alpha \in \mathfrak{C}$, then $\epsilon\alpha = \alpha$ and $\alpha \in \epsilon\mathfrak{C}$. Therefore $\mathfrak{C}(N) = \epsilon\mathfrak{C}$.

Conversely, if $A = \alpha\mathfrak{C}$ for some $\alpha \in \mathfrak{C}$ and $N = \alpha\hat{M}$, then $N \in \mathfrak{M}^v(\hat{M})$ by 7.3. Clearly $A = \mathfrak{C}(N)$, and the lemma is proved.

7.5. Theorem. *The lattice $\mathfrak{M}^t(M)$ is isomorphic to the lattice $\mathcal{P}(\mathfrak{C})$ of principal right ideals of $\mathfrak{C}(\hat{M}, R)$ under the correspondence $N \rightarrow \mathfrak{C}(N^v)$, $N \in \mathfrak{M}^t(M)$.*

Proof. We shall prove that $\mathfrak{M}^v(\hat{M}) \cong \mathcal{P}(\mathfrak{C})$, which will prove our theorem in view of 6.8. Evidently if K, $N \in \mathfrak{M}^v$ with $K \subset N$, then $\mathfrak{C}(K) \subset \mathfrak{C}(N)$. If $K \neq N$ and $\mathfrak{C}(N) = \alpha\mathfrak{C}$, then $\alpha\hat{M} = N$ and $\alpha \notin \mathfrak{C}(K)$; thus $\mathfrak{C}(K) \neq \mathfrak{C}(N)$. On the other hand, if α, $\beta \in \mathfrak{C}$ and $\alpha\mathfrak{C} \subset \beta\mathfrak{C}$, then $\alpha = \beta\gamma$ for some $\gamma \in \mathfrak{C}$ and $\alpha\hat{M} \subset \beta\hat{M}$; also, $\alpha\hat{M} \neq \beta\hat{M}$ if $\alpha\mathfrak{C} \neq \beta\mathfrak{C}$. This proves 7.5.

The center of $\mathfrak{M}^v(\hat{M})$ may be identified as follows:

7.6. Theorem. *If $N \in \mathfrak{M}^v(\hat{M})$, N is in the center of $\mathfrak{M}^v$ if and only if $\mathfrak{C}N \subset N$.*

Proof. Assume that $\alpha \in \mathfrak{C}$ and $x \in N$ such that $\alpha x \notin N$. Since $(\alpha x)^{-1}N \notin \mathfrak{L}^{\blacktriangle}(R)$, there exists $A \in \mathfrak{L}(R)$ such that $(\alpha x)A \neq 0$, $(\alpha x)A \cap N = 0$. Let N' be a complement of N containing $(\alpha x)A$. Clearly $(x - \alpha x)A \cap N = 0$ and $(x - \alpha x)A \cap N' = 0$. Thus, since $(x - \alpha x)A \neq 0$, N has another complement N'' containing $(x - \alpha x)A$. Consequently N is not in the center of $\mathfrak{M}^v(\hat{M})$.

Conversely, if N is not in the center of $\mathfrak{M}^v(\hat{M})$ and N' is some complement of N, then N' will in turn have a complement $N'' \neq N$. Let $\epsilon \in \mathfrak{C}$ be defined by: $\epsilon x = x$, $x \in N''$; $\epsilon x = 0$, $x \in N'$. Now each $x \in N$ has the form $x = x' + x''$, $x' \in N'$,

$x'' \in N''$, and for some $x \in N$, $x'' \notin N$. Hence $\epsilon N \not\subset N$. This proves **7.6.**

7.7. THEOREM. *If the closure operation s is homogeneous and if* $\{N_i\}$ *is the set of atoms in the center of* $\mathfrak{M}^v(\hat{M})$, *then*

$$\mathcal{C}(\hat{M}, R) \cong \sum_i{}^* \mathcal{C}(N_i, R).$$

Proof. If $N = \bigcup_i N_i$, then $N \subset' \hat{M}$ so that each R-homomorphism of N into $\hat{M}$ has a unique extension in $\mathcal{C}(\hat{M}, R)$. Since $\mathcal{C} N_i \subset N_i$ for each i, $\mathcal{C} N \subset N$ and $\mathcal{C}(\hat{M}, R) \cong \mathcal{C}(N, R)$. Each $\alpha \in \mathcal{C}(N, R)$ has the form $\alpha = \sum_i^* \alpha_i$ where $\alpha_i \in \mathcal{C}(N_i, R)$. Thus $\mathcal{C}(N, R) \cong \sum_i^* \mathcal{C}(N_i, R)$ and the theorem is proved.

It is easily shown that each $\mathcal{C}(N_i, R)$ of the theorem above is a primitive ring with minimal right ideals. In case $M = R^+$, $\mathcal{C}(\hat{M}, R)$ is known to be the maximal right quotient ring of R [6, p. 895]. Thus, if s is homogeneous, R has as maximal right quotient ring a full direct sum of primitive rings with minimal right ideals.

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