SEMI-DIRECT PRODUCTS WITH AMPLE HOMOMORPHISMS(1)

BY

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1. **Introduction.** If $G = HK$ where H is a normal subgroup of the group G and where K is a subgroup of G with the trivial intersection with H, then G is said to be a *semi-direct product* of H and K or a *splitting extension* of H by K. In a recent paper, D. G. Higman [5](2) has considered the problem of the existence of H when finite G and K are given and has extended the classical results. Following Cartan [2], Malcev [7] has restated the definition of this product in terms of automorphisms of H. Specifically, one considers the semi-direct product as an ordered triple $G = (H, K; \phi)$ where ϕ is a homomorphism from K into the automorphism group of H; $\phi: k \rightarrow \phi_k$. Special types of semi-direct products are the holomorphs and the dihedral groups. The former, for the case H abelian, are discussed in a paper of Mills [8]. The automorphism groups of the holomorphs of characteristically simple groups are treated in a paper of Gol'fand [3].

In this paper, some properties of semi-direct products are found and connections with the results in the literature are given. In §2, two special elements of Hom $((H, K; \phi), \phi(K))$ are constructed. It is proved that the extensions of a group K by a group of automorphisms of a group H determine extensions of K by the related relative holomorph, extensions which prove to be semi-direct products of H by the corresponding extensions of K.

A necessary and sufficient condition is found, in §3, for a pair of homomorphisms, one on H into $\phi(K)$, the other on K into $\phi(K)$, to be compounded to a homomorphism of G into $\phi(K)$, where $G = (H, K; \phi)$. If H is of class 2 and if $\mathfrak{I}(H)$ is the group of inner automorphisms of H, then the natural map θ on H onto $\mathfrak{I}(H)$ and each of the power maps $\pi_n: \alpha \rightarrow \alpha^n$ of the abelian group $\mathfrak{I}(H)$ can be compounded to a homomorphism of the relative holomorph [9] of $\mathfrak{I}(H)$ over H to $\mathfrak{I}(H)$. Continuing in §4, we show that for the holomorph if the natural map θ on H onto $\mathfrak{I}(H)$ can be compounded with an inner automorphism of the automorphism group of H generated by an inner automorphism of H, then H must be nilpotent of class 3.

In §5, two products are constructed from group inclusions. The ascending central series is determined for these, in one case in terms of repeated commutator quotients [1]. It is shown that each endomorphism of the auto-

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morphism group of H leads to an extension W of H with normal subgroup H_0 anti-isomorphic to H such that $W/H_0 \cong W/H$. Automorphisms of H are extended, in §6, to inner automorphisms of $(H, K; \phi)$, and a fundamental homomorphism of the latter is constructed onto a semi-direct product of the group of inner automorphisms of $(H, K; \phi)$ generated by the elements of H and of the group of such automorphisms generated by the elements of K.

The automorphisms of K which can be extended to inner automorphisms of $(H, K; \phi)$ turn out (§7) to be the inner automorphisms of K. If the ascending central series ends with the center and if $\phi(K) = \mathfrak{Z}(H)$, then $\mathfrak{Z}(K)$ can be injected into $\mathfrak{Z}(H, K; \phi)$.

Those automorphisms of G which induce (§8) a pair of automorphisms, one on H, the other on K, form a subgroup $\mathfrak{G}$. If H is of class 2 and if $\phi \in \text{Hom}\ (K, \mathfrak{Z}(H))$, then $\mathfrak{Z}(H)$ has an isomorphic image which is a direct summand of a normal subgroup of $\mathfrak{G}$.

If $\mathfrak{B}$ is a group of automorphisms of H which includes the inner automorphisms and is in the centralizer of the group of normal automorphisms, then the normal automorphism group of the relative holomorph of $\mathfrak{B}$ over H turns out to be a semi-direct product. For a class 2 group H, construct the centrally normal automorphisms, all those automorphisms which induce the identity on the center and on the inner automorphism group. Then construct the relative holomorph over H of this group of automorphisms, and form the centrally normal automorphism group of this relative holomorph. By modifying the proof of the result on the holomorph of $\mathfrak{B}$ above, we can show that our new group of centrally normal automorphisms splits into a direct sum of two groups, one of which lies in the $\mathfrak{G}$ of the relative holomorph.

The ascending central series is determined, in §9, for the case where ϕ is *ample* in the sense that $\phi(K) \subset \mathfrak{Z}(H)$. The members of this series and the related quotient groups turn out to be semi-direct products related to the ascending central series of H and of K and to the repeated commutator quotients of kernels of appropriate mappings. Even if ϕ is not ample, one can readily construct the first two members of the ascending central series. G is of class 2 if and only if H and K are of class 2 and each automorphism in $\phi(K)$ is a special type of normal automorphism of H.

A short discussion of the derivative occurs in §10. The factor-commutator group of G is the direct product of H modulo a subgroup of modified commutators and of the factor-commutator group of K.

Normalizers and centralizers of H and K are found in §11. The largest normal subgroup of G included in K is the kernel of ϕ. The normalizer tower of K in G is constructed by using a modification of the normalizer tower of the subgroup of universal fixed points (under the ϕ_k) of H in H. Again the commutator quotients will be indispensable in the consideration of this tower. The normalizer modulo the centralizer of K in G is isomorphic to $\mathfrak{Z}(K)$, also a property of a direct summand.

As for notation, $H \Delta G$ shall mean that H is a normal subgroup of the group G. The element e is used for the unity of every group, and (e) is the one-element subgroup which it generates. If A is a subset of G, then $\{A\}$ is to be the subgroup of G generated by the elements of A. $\mathfrak{A}(H)$, $\mathfrak{J}(H)$ and $\mathfrak{E}(H)$ are to be, respectively, the group of automorphisms, the group of inner automorphisms and the set of endomorphisms of the group H. Hom (A, B) is to be the set of homomorphisms on the group A into the group B. For a homomorphism α, we shall denote the kernel by kern α. ι will be reserved for the trivial automorphism or for an injection of a group A into a group B (isomorphism into). For an abelian group H, the mappings π_n mentioned above are endomorphisms. We write $\pi_{-1}=\omega$. For α, $\beta \in$ Hom (A, B), $\alpha+\beta$ shall denote the mapping (not necessarily a homomorphism) given by $(\alpha+\beta)(x)=\alpha(x)\beta(x)$. As usual, $\otimes$ is to denote cartesian product, while $\oplus$ will indicate direct sum (product). $\cong$ means that there is an isomorphism from, say, left to right. The members of $\mathfrak{J}(H)$ are the θ_h, where $\theta_h(x)=x^h =hxh^{-1}$. ν will be reserved for the trivial map: $\nu(a)=e$ for every $a\in A$. $\nu\in$ Hom (A, B). The $Z_i(G)$ are to be the members of the ascending central series [9] of G. $Z_1(G)$ is the center, and $Z_1(G/Z_n(G))\cong Z_{n+1}(G)/Z_n(G)$. G is said to be (*nilpotent*) *of class n* (or to be *n-nilpotent*) if $G=Z_n(G)$. $\mathfrak{T}_1(G)$, the set of normal automorphisms [9] of G, is the centralizer of $\mathfrak{J}(G)$ in $\mathfrak{A}(G)$. Let $\mathfrak{A}(A;\ A/B)$ be the subset of all those automorphisms of A which induce ι on A/B. Then $\mathfrak{T}_1(G)=\mathfrak{A}(G;\ G/Z_1(G))$. Let $\mathfrak{T}_n(G)=\mathfrak{A}(G;\ G/Z_n(G))$ [4]. Occasionally, if B is a non-normal subgroup of A, we shall write $\mathfrak{A}(A;A \bmod B)$ for the set of automorphisms α of $A\supset B$ where $x^{-1}\alpha(x)\in B$ for every $x\in A$. $\mathfrak{A}(H;B)$ is to be the set of automorphisms of H which induce the identity on the subgroup B of H. H' is to be the commutator subgroup $D^{(1)}(H)=D(H)$ of H with generators all $[h_1, h_2]=h_1h_2h_1^{-1}h_2^{-1}$ with h_1, $h_2\in H$. $H''=D^{(2)}(H) =D(D(H))$, $\cdots$, $D^{(n)}(H)=D(D^{(n-1)}(H))$, $\cdots$, are the members of the derived series of H [9]. $\mathfrak{C}(A, B)$ is to be the centralizer of A in B where A is a subgroup of the group B. $\mathfrak{C}^{(n)}(A, B)=\mathfrak{C}(\mathfrak{C}^{(n-1)}(A, B), B)$ where $\mathfrak{C}^{(1)}(A, B) =\mathfrak{C}(A, B)$. Replace $\mathfrak{C}$ by $\mathfrak{N}$ and the word *centralizer* by the word *normalizer* in the last two sentences. For a normal subgroup A and for any subgroup B of G, $A\div B$ $(=A\div_G B)$ is to be the set of all $g\in G$ such that $[g, b]\in A$ for every $b\in B$. $A\div B$ is called [1] the *commutator quotient* of A by B in G. It is a subgroup of G, normal if B is normal. The formally defined $K\div_G K$ is the subgroup $\mathfrak{N}(K, G)$ even though K need not be normal. A subgroup A of B is said to be α-admissible for $\alpha\in\mathfrak{E}(B)$ if $\alpha(a)\in A$ for every $a\in A$. The numbering of items starts anew with each section.

2. **Mappings into $\phi(K)$.** Let G^* be a group, and let H and K be subgroups of G^* such that (1) $G^*=\{H, K\}$, (2) $H\cap K=(e)$ and (3) $H\Delta G$. Then each element of G^* has a unique representation in the form hk, $h\in H$, $k\in K$, and $(hk)(h'k')=(h(kh'k^{-1}))(kk')$, where $h'\in H$, $k'\in K$ [3]. Alternately, let H and K be groups, and let ϕ be in Hom $(K, \mathfrak{A}(H))$. We write ϕ_u instead of

$\phi(u)$ for $u \in K$. $\phi_{uv} = \phi_u \phi_v$ where mappings are written to the left. $\phi(K)$ is to be the range of ϕ in $\mathfrak{A}(H)$ and is a subgroup thereof. Let us also write kern $\phi = \mathfrak{k}(\phi) = \mathfrak{k}$. For ordered pairs from the cartesian product $H \otimes K$, define multiplication by $(h, k)(h', k') = (h\phi_k(h'), kk')$ [7]. The element (e, e) is the obvious multiplicative identity, and $(\phi_{k-1}(h^{-1}), k^{-1})(h, k) = (e, e)$. The associative law for multiplication is quickly verifiable so that $G^\#$, $H \otimes K$ with the given multiplication, is a group. We write $G = (H, K; \phi)$ and call this group the *semi-direct product of H with respect to K via* ϕ [7]. A short argument shows that $G^\#$ is a splitting extension, i.e., an extension of H by K with a retractable factor system [9]; and conversely, every such extension is a semi-direct product. This formulation of the semi-direct product goes back to Cartan [2]. If K is a subgroup of $\mathfrak{A}(H)$, and if $\phi = \iota$, the injection of K into $\mathfrak{A}(H)$, then $(H, K; \iota)$ is the *relative holomorph of K over H* [9]. In particular, $(H, \mathfrak{A}(H); \iota)$ is the holomorph $\mathfrak{H}(H)$ of H.

$G^\#$ has a subgroup $H^* \cong H$ and a subgroup $K^* \cong K$, where H^* consists of all (h, e), and K^* is the set of all (e, k). It is immediate that (1) $G^\# = \{H^*, K^*\}$, that (2) $H^* \cap K^* = (e)$ and that (3) $H^* \Delta G^\#$. Conversely, if a group G^* has subgroups H and K satisfying (1), (2) and (3), we can define

$$\phi \in \mathrm{Hom}\,(K, \mathfrak{A}(H))$$

by letting ϕ_k be that automorphism of the normal subgroup H which is induced by the inner automorphism $\theta_k \in \mathfrak{A}(G)$. Then $G^\# = (H, K; \phi) \cong G^* = H^*K^*$ under the mapping given by $(h, k) \rightarrow hk$. For this reason we shall write $G = (H, K; \phi)$ in what follows. Thus, in a semi-direct product, ϕ_k is the relativization of $\theta_{(e,k)}$ to H. It is well known [9] that every automorphism of H can be extended to an inner automorphism of $\mathfrak{H}(H)$, so that we have verified the obvious generalization that *every element of* $\phi(K)$ *can be extended to an element of* $\mathfrak{J}(H, K; \phi)$. Since $\theta_{(h,k)}(x, e) = (\theta_h \phi_k(x), e)$ for every $x \in H$, $\theta_{(h,k)}$ relativizes to some ϕ_k, if and only if $\theta_h \in \phi(K)$. Let us say that ϕ is *ample* if $\phi(K) \supset \mathfrak{J}(H)$ and define $B_1 = B_1(G)$ as the set of all (h, k) with $\theta_{h^{-1}} = \phi_k$. Note that *if* ϕ *is ample, then the mapping* γ_1 *on* $G = (H, K; \phi)$ *onto* $\phi(K)$ *given by* $\gamma_1(h, k) = \theta_h \phi_k$ *is a homomorphism, and* kern $\gamma_1 = B_1(G)$.

Let $F_1 = F_1(H)$ be the set of all $x \in H$ for which $\phi_y(x) = x$ for every $y \in K$. $F_1(H)$ is a subgroup of H. If[3] $\phi(K) \subset \mathfrak{T}_1(H)$, then $F_1 \Delta H$, and the latter normal inclusion is equivalent, in any case, to $\phi(K) \subset \mathfrak{A}(H; H/\mathfrak{C}(F_1(H), H))$; for if $h \in H$, $f \in F_1(H)$ and if $y \in K$, then $F_1(H) \Delta H$ implies that $\phi_y(hfh^{-1}) = \phi_y(h) f \phi_y(h^{-1}) = hfh^{-1}$, or $h^{-1}\phi_y(h) \in \mathfrak{C}(F_1(H), H)$, and conversely. Now $(h, k) \in Z_1(G)$ if and only if $(h, k)(x, y) = (x, y)(h, k)$ for every $(x, y) \in G$. Equivalently, $h\phi_k(x) = x\phi_y(h)$ and $k \in Z_1(K)$. Taking $x = e$, we find $h \in F_1$ so that $\phi_k(x) = \theta_{h^{-1}}(x)$. Conversely, if $h \in F_1$, $k \in Z_1(K)$ and if $(h, k) \in B_1$, then $(h, k) \in Z_1(G)$. We have proved that $Z_1(H, K; \phi) = B_1(H, K; \phi) \cap (F_1(H)$

(3) See §1 for notation.

$\oplus Z_1(K))$. "By abuse of language," here $F_1(H)\oplus Z_1(K)$ stands for the isomorphic group consisting of all (h, k) with $h\in F_1(H)$, $k\in Z_1(K)$.

Let θ be the natural homomorphism of a group A onto $\mathfrak{J}(A)$ given by $\theta(a)=\theta_a$. Then *for ϕ ample and $G=(H, K; \phi)$, γ_2 defined by $\gamma_2(\theta_{(h,k)})=\theta_h\phi_k$ is an onto mapping in* Hom $(\mathfrak{J}(G), \phi(K))$ *such that $\gamma_1=\gamma_2\theta$ and* kern $\gamma_2=\theta(B_1)$, *where $G=(H, K; \phi)$.*

If G is the relative holomorph of a group K over a group H such that $K\supset\mathfrak{J}(H)$, then γ_1 is onto K and has the form $\gamma_1(h, k)=\theta_h k$. In particular, there exists a homomorphism γ_1 of $\mathfrak{H}(H)$ onto $\mathfrak{A}(H)$. Let γ_3 on $G=(H, K; \phi)$ into $\mathfrak{H}(H)$ be given by $\gamma_3(h, k)=(h, \phi_k)$. $\gamma_3\in$Hom $(G, (H, \phi(K); \iota))$. But there exists $\gamma_1'\in$Hom $((H, \phi(K); \iota), \phi(K))$ which is onto $\phi(K)$, where $\gamma_1'(h, \phi_k)=\theta_h\phi_k$, as we saw just above, provided that ϕ is ample. Clearly, $\gamma_1=\gamma_1'\gamma_3=\gamma_2\theta$. In what follows, let it be understood that ϕ is ample. Let γ_4 be the natural map on G onto K given by $\gamma_4(h, k)=k$. γ_4 induces an isomorphism: $G/H\cong K$. Then $\gamma_5=\phi\gamma_4$ is on G onto $\phi(K)$. $\gamma_5=\gamma_4'\gamma_3$ where γ_4' is the γ_4 of $(H, \phi(K); \iota)$. Now both γ_1 and γ_5 are on G onto $\phi(K)$. $\gamma_1(h, k)=\theta_h\phi_k$, and $\gamma_5(h, k)=\phi_k$. Hence $\gamma_1=\gamma_5$ if and only if H is abelian. Equivalently, $\gamma_1'=\gamma_4'$. Let γ_6 be the natural map of $\phi(K)$ onto $\phi(K)/\mathfrak{J}(H)$, where ϕ is ample. Then $\gamma_6\gamma_1=\gamma_6\gamma_5$. Let $B_5=$kern γ_5. $B_5\cong H\oplus\mathfrak{k}$. It is easy to show that $B_1\subset B_5$, $B_5\subset B_1$ and $B_1=B_5$ are equivalent conditions, and these, in turn, are equivalent to H being abelian. The only case where $\gamma_1=\alpha\gamma_5$ (or $\gamma_5=\alpha\gamma_1$) with $\alpha\in\mathfrak{A}(\phi(K))$ is that with $\alpha=\iota$, whence H is abelian. $B_1\cap B_5\cong Z_1(H)\oplus\mathfrak{k}$ so that $B_1\cap B_5=(e)$ if and only if G is a relative holomorph over a centerless group H of an extension of the inner automorphism group of H. In particular, if H is a complete group [9], then in $\mathfrak{H}(H)$, $B_1\cap B_5=(e)$.

The mapping γ_3 can be used to prove

LEMMA 1. *Let H and K_0 be groups, and let $\mathfrak{A}_0$ be a subgroup of $\mathfrak{A}(H)$. Then each extension of K_0 by $\mathfrak{A}_0$ determines an extension of K_0 by the relative holomorph of $\mathfrak{A}_0$ over H to a semi-direct product of H by the extension of K_0.*

Proof. Let $K/K_0\cong\mathfrak{A}_0$. Define $\phi\in$ Hom$(K, \mathfrak{A}_0)$ by $\phi(k)=\phi_k=$"kK_0" where "kK_0" is that element of $\mathfrak{A}_0$ which corresponds to $kK_0\in K/K_0$ under the given isomorphism. Let $G=(H, K; \phi)$. Let $\mathfrak{k}^*$ be the set of all $(e, k)\in G$, $k\in$kern ϕ. $\mathfrak{k}^*\cong K_0$. For γ_3 on G onto $(H, \phi(K); \iota)$ defined by $\gamma_3(h, k)=(h, \phi_k)$, kern $\gamma_3=\mathfrak{k}^*$. Since $\phi(K)=\mathfrak{A}_0$ and since $\mathfrak{k}^*\cong K_0$, there exists a group $G_0\cong G$ such that $K_0\Delta G_0$, and $G_0/K_0\cong(H, \mathfrak{A}_0; \iota)$.

3. **Further mappings on** $\phi(K)$. Let W be a group, and let $G=(H, K; \phi)$. For $\alpha\in$Hom (H, W), $\beta\in$Hom (K, W), define a mapping γ on G into W by $\gamma(h, k)=\alpha(h)\beta(k)$. It is easy to verify that $\gamma\in$Hom (G, W) if and only if

$$\beta(k)\alpha(h) = \alpha(\phi_k(h))\beta(k) \tag{1}$$

for every $h\in H$ and for every $k\in K$. If such a pair of mappings α and β obeys (1), we write $\gamma=\alpha\wedge\beta\in$Hom (G, W). Conversely, if $\gamma\in$Hom (G, W), if

$\alpha(h)=\gamma(h, e)$, $\beta(k)=\gamma(e, k)$, then $\alpha\in\text{Hom}(H, W)$, $\beta\in\text{Hom}(K, W)$ and $\gamma=\alpha\wedge\beta$ where α and β satisfy (1). Recall[3] that $\mathfrak{A}(A; A/B)$ is a subgroup of $\mathfrak{A}(A)$. We have

THEOREM 1. *Let $G=(H, K; \phi)$, and suppose that $\alpha\in\text{Hom}(H, W)$ where* kern α *is a characteristic subgroup of H, that $\beta\in\text{Hom}(K, W)$ where $\alpha\wedge\beta\in\text{Hom}(G, W)$, that $\sigma\in\mathfrak{A}(H)$ and that $\tau\in\mathfrak{A}(K; K/\mathfrak{k})$. Then $\alpha\sigma\wedge\beta\tau\in\text{Hom}(G, W)$ if and only if $\sigma\in\mathfrak{A}(H; H/\text{kern}\ \alpha)\div\phi(K)$.*

Proof. Since $\alpha\wedge\beta\in\text{Hom}(G, W)$, $\beta(k)\alpha(h)=\alpha(\phi_k(h))\beta(k)$ for all $h\in H$ and for all $k\in K$. For $\sigma\in\mathfrak{S}(H)$, $\tau\in\mathfrak{S}(K)$, $\alpha\sigma\wedge\beta\tau\in\text{Hom}(G, W)$ if and only if $\beta\tau(k)\alpha\sigma(h)=\alpha\sigma(\phi_k(h))\beta\tau(k)$. Since σ and τ are automorphisms, in the first of these identities we can replace k by $\tau(k)$ and h by $\sigma(h)$. Then $\beta\tau(k)\alpha\sigma(h)=\alpha(\phi_{\tau(k)}\sigma(h))\beta\tau(k)$. Thus $\alpha\sigma\wedge\beta\tau\in\text{Hom}(G, W)$ if and only if $\alpha\phi_{\tau(k)}(\sigma(h))=\alpha\sigma\phi_k(h)$ for every $h\in H$ and for every $k\in K$. Equivalently, $\phi_{\tau(k)}\sigma(h)\equiv\sigma\phi_k(h)$ mod kern α. By hypothesis, $\phi_{\tau(k)}=\phi_k$, and σ has an inverse σ^{-1}. Then the condition that $\alpha\sigma\wedge\beta\tau\in\text{Hom}(G, W)$ reduces to $\phi_k\sigma(h)\equiv\sigma\phi_k(h)$ mod kern α. Since kern α is characteristic, this latter congruence is the same as $\phi_k^{-1}\sigma^{-1}\phi_k\sigma(h)\equiv h$ mod kern α. Since kern α is characteristic, $\mathfrak{A}(H; H/\text{kern}\ \alpha)\Delta\mathfrak{A}(H)$ so that $\mathfrak{A}(H; H/\text{kern}\ \alpha)\div\phi(K)$ is defined. The theorem follows at once.

Now let $W=\mathfrak{A}(H)$. We saw above that $\gamma_1=\theta\wedge\phi$ is in $\text{Hom}(G, \mathfrak{A}(H))$; specifically, since ϕ is assumed to be ample, $\gamma_1\in\text{Hom}(G, \phi(K))$. Write $\beta(k)=\beta_k$. We omit the proof of

THEOREM 2. *Let $G=(H, K; \phi)$ with ample ϕ. For $\beta\in\text{Hom}(K, \phi(K))$, the following are equivalent*: (a) $\theta\wedge\beta\in\text{Hom}(G, \phi(K))$; (b) $\beta_k\equiv\phi_k$ mod $(\mathfrak{T}_1(H)\cap\phi(K))$ *for every $k\in K$*; (c) *for each $k\in K$ there exists $\zeta_k\in\text{Hom}(H, Z_1(H))$ with $\beta_k=\phi_k+\zeta_k$.*

Suppose that $G=(H, K; \phi)$ with ample ϕ. For $\alpha\in\text{Hom}(H, \phi(K))$, $\alpha\wedge\phi\in\text{Hom}(G, \phi(K))$ if and only if $\phi_k\alpha_h=\alpha_{\phi_k(h)}\phi_k$ for every $h\in H$ and for every $k\in K$, where we write $\alpha(h)=\alpha_h$. If $\alpha\wedge\phi\in\text{Hom}(G, \phi(K))$, then this mapping is a homomorphism onto $\phi(K)$. If $h\in$ kern α, $\phi_k=\alpha_{\phi_k(h)}\phi_k$ so that $\phi_k(h)\in$ kern α, and kern α is ϕ_k-admissible for every $k\in K$. Corresponding to Theorem 2 is

THEOREM 3. *Let $G=(H, K; \phi)$. For $\alpha\in\text{Hom}(H, \phi(K))$ any two of the following imply the third*: (a) $\alpha\wedge\phi\in\text{Hom}(G, \phi(K))$. (b) $\phi(K)\subset\mathfrak{A}(H; H/\text{kern}\ \alpha)$. (c) $\alpha(H)\subset Z_1(\phi(K))$.

COROLLARY. (a) *For $G=(H, K; \phi)$ with ample ϕ, $\phi(K)\subset\mathfrak{T}_1(H)$ if and only if $\mathfrak{J}(H)\subset Z_1(\phi(K))$.* (b) *$\mathfrak{J}(H)\subset Z_1(\phi(K))$ implies that $F_1\Delta H$.*

We now examine modifications of γ_1. Let $G=(H, K; \phi)$ with ample ϕ. For $\sigma\in\mathfrak{S}(H)$, $\tau\in\mathfrak{S}(K)$, $\gamma=\theta\sigma\wedge\phi\tau\in\text{Hom}(G, \phi(K))$ if and only if

$$\phi_{\tau(k)}(\sigma(h))\equiv\sigma(\phi_k(h))\ \text{mod}\ Z_1(H) \tag{2}$$

for every $h \in H$ and for every $k \in K$. Suppose, now, that

$$\gamma = \theta\sigma \wedge \phi\tau \in \text{Hom}\,(G, \phi(K)).$$

Then ψ on kern γ into G given by $\psi(h, k) = (\sigma(h), \tau(k))$ for every $(h, k) \in$ kern γ is readily seen to be into B_1. Conversely, if $(\sigma(h), \tau(k)) \in B_1$, then $(h, k) \in$ kern γ. This shows that if ψ is extended to all of G, then $\psi(\text{kern}\,\gamma) \subset B_1$, and that the complete inverse image in G of B_1 under ψ is kern γ.

LEMMA 2. *Let $G = (H, K; \phi)$ with ϕ ample, and let $\sigma \in \mathfrak{E}(H)$, $\tau \in \mathfrak{E}(K)$ generate $\psi \in \mathfrak{E}(G)$ by $\psi(h, k) = (\sigma(h), \tau(k))$. Then $\theta\sigma \wedge \phi\tau \in$* Hom $(G, \phi(K))$.

Proof. Necessary and sufficient for ψ to be an endomorphism when σ and τ are endomorphisms is that $\psi(h\phi_k(x), ky) = \psi(h, k)\psi(x, y)$ for every $h, x \in H$ and for every $k, y \in K$. The left side is $(\sigma(h)\sigma\phi_k(x), \tau(ky))$, while the right is $(\sigma(h)\phi_{\tau(k)}\sigma(x), \tau(ky))$. Hence ψ is in $\mathfrak{E}(G)$ if and only if

$$\sigma\phi_k = \phi_{\tau(k)}\sigma \tag{3}$$

for every $k \in K$. But this equality surely implies the congruence (2).

LEMMA 3. (a) *If σ is an automorphism and if $\mathfrak{k}$ is τ-admissible, then* kern $(\theta\sigma \wedge \phi\tau) \cap B_5 = B_1 \cap B_5$. (b) *If σ is an automorphism then*

$$(h, k) \in \text{kern}\,(\theta\sigma \wedge \phi\tau) \cap B_5$$

implies that $h \in Z_2(H)$.

LEMMA 4. *Let $G = (H, K; \phi)$ with ample ϕ and let $\sigma \in \mathfrak{A}(H)$, $\tau \in \mathfrak{A}(K)$. For $\gamma = \theta\sigma \wedge \phi\tau \in$* Hom $(G, \phi(K))$, *$(h, k) \in$* kern γ *implies that $\sigma(h) \equiv \sigma(\phi_k(h))$* mod $Z_1(H)$ *and that $\phi_{\tau(k)}(\sigma(h)) \equiv \sigma(h)$* mod $Z_1(H)$.

It should be noted that the latter conclusion has the equivalent form $[\theta_{\sigma(h)}, \phi_{\tau(k)}] = \iota$.

If H is abelian, then $\theta = \nu$. For $\tau \in \mathfrak{E}(K)$ one can readily verify that $\gamma = \nu \wedge \phi\tau \in \text{Hom}\,(G, \phi(K))$. Here $\gamma(h, k) = \phi_{\tau(k)}$.

If ϕ is ample, and if $\sigma \in \mathfrak{A}(H)$, then $\gamma_\sigma = \theta\sigma \wedge \phi \in \text{Hom}\,(G, \phi(K))$ if and only if $\phi_k\sigma(h) \equiv \sigma\phi_k(h)$ mod $Z_1(H)$, from (2). Since $Z_1(H)$ is characteristic and since $\mathfrak{T}_1(H) \Delta \mathfrak{A}(H)$, this latter congruence is equivalent to[3] $\sigma \in \mathfrak{T}_1(H) \div \phi(K)$. (If, in Theorem 1, we take $W = \phi(K)$, the same conclusion results.) Let $B_\sigma =$ kern γ_σ. Now suppose that $\sigma, \tau \in \mathfrak{T}_1(H) \div \phi(K)$. Let μ on B_σ onto B_τ be defined by $\mu(h, k) = (\tau^{-1}\sigma(h), k)$ for $(h, k) \in B_\sigma$. One can show that μ will be an isomorphism on B_σ onto B_τ if $\phi_k[\tau^{-1}\sigma(h)] = \tau^{-1}\sigma\phi_k(h)$ for every $h \in H$ and for every $k \in K$. But this is equivalent[3] to $\sigma \equiv \tau$ mod $\mathfrak{C}(\phi(K), \mathfrak{A}(H))$. Now $\sigma \equiv \tau$ mod $\mathfrak{T}_1(H)$ if the above congruence modulo $\mathfrak{C}$ holds; for, $\phi(K) \supset \mathfrak{J}(H)$ so that $\mathfrak{C}(\phi(K), \mathfrak{A}(H)) \subset \mathfrak{C}(\mathfrak{J}(H), \mathfrak{A}(H)) = \mathfrak{T}_1(H)$. Then $\sigma(h) \equiv \tau(h)$ mod $Z_1(H)$ for every $h \in H$. From this it follows that $\theta_{\sigma(h)} = \theta_{\tau(h)}$ for every $h \in H$. Hence $\gamma_\sigma = \theta\sigma \wedge \phi = \theta\tau \wedge \phi = \gamma_\tau$. We have proved

THEOREM 4. *Let $G=(H, K; \phi)$ with ϕ ample. If $\sigma, \tau \in \mathfrak{A}(H)$ with $\sigma \equiv \tau$* mod $\mathfrak{C}(\phi(K), \mathfrak{A}(H))$ *and with $\sigma, \tau \in \mathfrak{T}_1(H) \div \phi(K)$, then $\theta\sigma \wedge \phi = \theta\tau \wedge \phi$.*

Note that if $\sigma \in \mathfrak{T}_1(H)$, then $\theta\sigma \wedge \phi = \gamma_1$.

By (2), $\gamma^{(\sigma)} = \theta \wedge \phi\sigma \in \mathrm{Hom}\ (G, \phi(K))$, where $\sigma \in \mathfrak{A}(K)$, if and only if $\phi_{\sigma(k)}(h) \equiv \phi_k(h)$ mod $Z_1(H)$ for every $h \in H$ and for every $k \in K$. That is, $\phi_{k^{-1}\sigma(k)} \in \mathfrak{T}_1(H)$ is equivalent to $\gamma^{(\sigma)} \in \mathrm{Hom}\ (G, \phi(K))$. Observe that if $\mathfrak{k}$ is σ-admissible then σ induces an automorphism on $\phi(K)$ by $\phi_k \to \phi_{\sigma(k)}$.

Let K be abelian so that(3) $\omega \in \mathfrak{A}(K)$. If ϕ is ample, $\mathfrak{I}(H)$ is consequently abelian so that H is of class 2. For $\sigma \in \mathfrak{E}(H)$, (2) shows that $\theta\sigma \wedge \phi\omega \in \mathrm{Hom}$ if and only if $\phi_{k^{-1}}\sigma(h) \equiv \sigma\phi_k(h)$ mod $Z_1(H)$ for every $h \in H$ and for every $k \in K$. For any W, it is easily seen that $\alpha \in \mathrm{Hom}\ (H, W)$, kern α ϕ_k-admissible for every $k \in K$, $\beta \in \mathrm{Hom}\ (K, W)$ and $\alpha \wedge \beta \in \mathrm{Hom}\ (G, W)$, where $G = (H, K; \phi)$ and K is abelian, imply that $\alpha \wedge \beta\omega \in \mathrm{Hom}\ (G, W)$ if and only if each $\phi_{k^2} \in \mathfrak{A}(H; H/\mathrm{kern}\ \alpha)$.

Let H be a group of class 2, and let $\mathfrak{K}$ be an abelian extension in $\mathfrak{A}(H)$ of $\mathfrak{I}(H)$. Consider $G = (H, \mathfrak{K}; \iota)$, the relative holomorph of $\mathfrak{K}$ over H. For a given integer(3) n, $\theta \wedge \pi_n \in \mathrm{Hom}\ (G, \mathfrak{K})$ if and only if, by (1), $\kappa^n\theta_h(x) = \theta_{\kappa(h)}\kappa^n(x)$ for every $x, h \in H$ and for every $\kappa \in \mathfrak{K}$. Upon expansion, this turns out to be the equivalent of $\kappa^n(h) \equiv \kappa(h)$ mod $Z_1(H)$, or equally, $\kappa^n \equiv \kappa$ mod $\mathfrak{T}_1(H)$ and $\kappa^{n-1} \in \mathfrak{T}_1(H)$. Since, however, $\mathfrak{K}$ is an abelian extension of the abelian group $\mathfrak{I}(H)$, $\mathfrak{K} \subset \mathfrak{T}_1(H) = \mathfrak{C}(\mathfrak{I}(H), \mathfrak{A}(H))$. Hence

LEMMA 5. *Let H be a group of class 2, and let $\mathfrak{K}$ be an abelian extension in $\mathfrak{A}(H)$ of $\mathfrak{I}(H)$. Then for each integer n, $\theta \wedge \pi_n \in \mathrm{Hom}\ ((H, \mathfrak{K}; \iota), \mathfrak{K})$.*

4. **Some properties of the holomorph.** The results of §3 apply to $\mathfrak{H}(H)$ since $\phi = \iota$ is ample. For $\alpha \in \mathrm{Hom}\ (H, W)$, $\Delta \in \mathrm{Hom}\ (\mathfrak{A}(H), W)$, we have $\alpha \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), W)$ if and only if

$$(1) \qquad \Delta(\kappa)\alpha(h) = \alpha(\kappa(h))\Delta(\kappa)$$

for every $h \in H$ and for every $\kappa \in \mathfrak{A}(H)$. Take $W = \mathfrak{A}(H)$ so that $\Delta \in \mathfrak{E}(\mathfrak{A}(H))$. With the notation $\alpha(h) = \alpha_h$, *if* $\alpha \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$, $\Delta \in \mathfrak{E}(\mathfrak{A}(H))$, *then* $\alpha \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ *implies that* $\Delta(\theta_u) \equiv \alpha_u$ mod $\mathfrak{C}(\alpha(H), \mathfrak{A}(H))$ *for every* $u \in H$. By Theorem 2, $\theta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ if and only if $\Delta(\kappa) \equiv \kappa$ mod $\mathfrak{T}_1(H)$ for every $\kappa \in \mathfrak{A}(H)$. *If* $\Delta \in \mathfrak{E}(\mathfrak{A}(H))$, *then*

$$\Delta\theta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H)),$$

and $B_1 \subset \mathrm{kern}\ (\Delta\theta \wedge \Delta)$. If $\Delta \in \mathfrak{A}(\mathfrak{A}(H)) = \mathfrak{A}^2(H)$, kern $(\Delta\theta \wedge \Delta) = B_1$.

LEMMA 6. *Let H be an abelian group. If $\delta \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ such that there exists $\Delta \in \mathfrak{A}^2(H)$ with $\Delta\delta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$, then $H/\mathrm{kern}\ \delta$ is abelian with exponent $\leqq 2$.*

Proof. Since $\Delta\delta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, (1) shows that

$$\Delta(\kappa)\Delta(\delta(h)) = \Delta(\delta(\kappa(h))\Delta(\kappa)$$

for every $h \in H$ and for every $\kappa \in \mathfrak{A}(H)$. Hence $\kappa\delta_h\kappa^{-1}\delta_{\kappa(h^{-1})} = \iota$. Now H is abelian so that we may choose(3) $\kappa = \omega$. Then, for $x \in H$, $\omega\delta_h\omega\delta_h(x) = x$, whence $\delta_{h^2}(x) = x$, $\delta_{h^2} = \iota$ and h^2 in kern δ. In particular, the lemma holds if $\delta \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$.

Corollary. *Let H be an abelian group in which $x^2 = h \in H$ always has a solution. Then for $\delta \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ there exists $\Delta \in \mathfrak{A}^2(H)$ such that $\Delta\delta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ if and only if $\delta = \nu$. If $\delta = \nu$, any $\Delta \in \mathfrak{A}^2(H)$ may be chosen.*

Observe that if $\delta \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ and if $\Delta_1, \Delta_2 \in \mathfrak{E}(\mathfrak{A}(H))$ where kern Δ_1 = kern Δ_2 then $\Delta_1\delta \wedge \Delta_1 \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ if and only if $\Delta_2\delta \wedge \Delta_2$ is in this same Hom. In particular, if $\Delta \in \mathfrak{A}^2(H)$, then $\Delta\delta \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ if and only if $\delta \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$; i.e., $\kappa\delta_h\kappa^{-1} = \delta_{\kappa(h)}$ for every $\kappa \in \mathfrak{A}(H)$. For instance, $\theta \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$. *If $\delta \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$, if $H/$kern δ is abelian and if $\delta \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$, then $\delta(H) \subset \mathfrak{T}_1(H)$.*

Theorem 5. *If $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ for every $h \in H$, then H is of class 3.*

Proof. Here θ_{θ_h} means that element of $\mathfrak{A}^2(H)$ which maps $\kappa \in \mathfrak{A}(H)$ onto $\theta_h\kappa\theta_{h-1}$. By the remark after the italicized statement following (1), $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ if and only if $\theta_h\kappa\theta_{h^{-1}} \equiv \kappa \bmod \mathfrak{T}_1(H)$; equivalently, $\theta_h \in \mathfrak{T}_1(H) \div \mathfrak{A}(H)$. But $\theta_h\kappa\theta_{h^{-1}}\kappa^{-1}(x) = [h\kappa(h^{-1})]x[h\kappa(h^{-1})]^{-1} \equiv x \bmod Z_1(H)$ so that $h\kappa(h^{-1}) \in Z_1(H) \div H = Z_2(H)$. Thus every $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ if and only if(3) $\mathfrak{A}(H) = \mathfrak{T}_2(H)$. Then $\theta_x(h) \equiv h \bmod Z_2(H)$ for every $x, h \in H$. It follows that(3) $H' \subset Z_2(H)$ so that H is of class 3.

We have also proved that $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ implies that $h \in Z_3(H)$. Theorem 2 shows that $\theta \wedge \theta_\lambda \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, where $\lambda \in \mathfrak{A}(H)$, if and only if $\lambda \in \mathfrak{T}_1(H) \div \mathfrak{A}(H)$. Since this commutator quotient is included in $\mathfrak{T}_2(H)$ [4], $\theta \wedge \theta_\lambda \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ implies that $\lambda \in \mathfrak{T}_2(H)$. If H is of class 2, then $\mathfrak{A}(H) = \mathfrak{T}_2(H)$ so that $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ for every $h \in H$. It is easy to show that if all $\theta \wedge \theta_{\theta_h} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, then $\bigcap_h$ kern $(\theta \wedge \theta_{\theta_h})$ is the set of all $(x, \theta_{x^{-1}})$ where $x \in Z_2(H)$. Hence if H is of class 2 this intersection of kernels reduces to B_1. Likewise, $\theta \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ as we saw above, and kern $(\theta \wedge \iota) = B_1$. Suppose that α is an isomorphism of H into $\mathfrak{A}(H)$ for which $\alpha \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$. It is easy to show that $\alpha = \theta$ so that H is centerless. In the general case, if $\alpha \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ and if $\alpha \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, then $\alpha(Z_1(H)) \subset \mathfrak{T}_1(H) \cap \mathfrak{A}(H; H/\text{kern}\ \alpha)$. If H is abelian and if $\alpha \wedge \nu \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, then $H/$kern α is abelian with exponent $\leqq 2$. For, given any group H, $\alpha \wedge \nu \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ if and only if, from (1), $\kappa(h) \equiv h \bmod \text{kern}\ \alpha$ for every $h \in H$ and for every $\kappa \in \mathfrak{A}(H)$. If H is abelian, take $\kappa = \omega$. Even if H is not abelian, we can choose $\kappa = \theta_x$ and allow x to run over all of H. Then $H' \subset$kern α so that $\alpha \wedge \nu$

$\in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ implies that $H/\mathrm{kern}\ \alpha$ is abelian.

It is also valid that if $\delta \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ and if $\delta \wedge \iota$ and $\delta \wedge \nu$ are both in $\mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$, then $\delta(H) \subset Z_1(\mathfrak{A}(H))$. (Cf. the italicized statement just before Theorem 5.) If H has no outer automorphisms, (say, if it is complete [9]), then $\alpha \wedge \nu \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$ if and only if $H/\mathrm{kern}\ \alpha$ is abelian. If H is abelian, if $\alpha \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$, if $\Delta \in \mathfrak{C}(\mathfrak{A}(H))$, if $H/\mathrm{kern}\ \alpha$ has exponent 2 and if $\alpha \wedge \Delta \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, then $\Delta(\omega) \in \mathfrak{C}(\alpha(H), \mathfrak{A}(H))$. For, from (1) with $\kappa = \omega$, $\Delta(\omega)\alpha_h = \alpha_{h^{-1}}\Delta(\omega)$. Since $H/\mathrm{kern}\ \alpha$ has exponent 2, $(\alpha_h)^2 = \iota$ so that $\alpha_h = \alpha_{h^{-1}}$. Hence $\Delta(\omega) \in \mathfrak{C}$.

Now $\alpha \wedge \theta_\lambda \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ if and only if

$$\lambda\kappa\lambda^{-1}\alpha_h = \alpha_{\kappa(h)}\lambda\kappa\lambda^{-1} \tag{2}$$

for every $\kappa \in \mathfrak{A}(H)$. Equivalently, replacing κ by $\lambda^{-1}\kappa\lambda$, we have

$$\kappa\alpha_h = \alpha_{\lambda^{-1}\kappa\lambda(h)}\kappa. \tag{3}$$

Replacing κ by $\alpha_{h^{-1}}$, $[\lambda^{-1}\alpha_{h^{-1}}\lambda(h)]h^{-1} \in \mathrm{kern}\ \alpha$. In (3) take $\kappa = \lambda$ and $h \in \mathrm{kern}\ \alpha$. Then $\lambda(h) \in \mathrm{kern}\ \alpha$ so that $\mathrm{kern}\ \alpha$ is λ-admissible. Hence $\alpha_{h^{-1}}\lambda(h) \equiv \lambda(h) \bmod \mathrm{kern}\ \alpha$. Thus, replacing h by h^{-1}, we have $\alpha_h\lambda(h) \equiv \lambda(h) \bmod \mathrm{kern}\ \alpha$. In (2), replace κ by θ_h and reduce. Combining, we have

$$\alpha_h\lambda(h) \equiv \lambda(h) \bmod (Z_1(H) \cap \mathrm{kern}\ \alpha) \tag{4}$$

for every $h \in H$ and for every α and λ such that $\alpha \wedge \theta_\lambda \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$.

In (2), take $\kappa = \lambda$ to obtain $\lambda\alpha_h = \alpha_{\lambda(h)}\lambda$. In (2), replace κ by α_h. Then, by the preceding formula, $\lambda\alpha_h\lambda^{-1}\alpha_h = \alpha_{\lambda(h)h}$ on the left of (2). On the right of (2), $\alpha_{\alpha_h(h)}\lambda\alpha_h\lambda^{-1} = \alpha_{\alpha_h(h)\lambda(h)}$, by (3). Thus $\alpha \wedge \theta_\lambda \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$ implies that

$$\lambda(h)h \equiv \alpha_h(h)\lambda(h) \bmod \mathrm{kern}\ \alpha. \tag{5}$$

LEMMA 7. (a) *If* $\alpha \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ *and if* $\alpha \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}(H), \mathfrak{A}(H))$, *then* $\alpha_h(h) \equiv h \bmod (Z_1(H) \cap \mathrm{kern}\ \alpha)$ *for every* $h \in H$. (b) *If* $h \in H$ *has the property that every conjugate of* h *lies in the centralizer of* h *and if there exist* $\alpha \in \mathrm{Hom}\ (H, \mathfrak{A}(H))$ *and* $u \in H$ *with* $\alpha \wedge \theta_{\theta_u} \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, *then*

$$\alpha_h(h) \equiv h \bmod \mathrm{kern}\ \alpha.$$

Proof. (a) Since $\alpha \wedge \iota \in \mathrm{Hom}\ (\mathfrak{H}, \mathfrak{A})$, let $\lambda = \iota$ in (4). (b) In (5) take $\lambda = \theta_u$. Then $uhu^{-1}h \equiv \alpha_h(h)uhu^{-1} \bmod \mathrm{kern}\ \alpha$. But $(uhu^{-1})h = h(uhu^{-1})$, whence the conclusion is immediate.

We can obtain some direct information about the first few members of the ascending central series of $\mathfrak{H}(H)$, the center consisting, for instance, of all (h, ι) where $h \in F_1(H)$. For $(h, \alpha) \in Z_2(\mathfrak{H}(H))$, $\eta(h) \equiv h \bmod F_1(H)$ for every $\eta \in \mathfrak{A}(H)$. From this, one can show that $\mathfrak{H}(H)$ *is of class* 2 *if and only if* $\mathfrak{A}(H) = \mathfrak{A}(H; H/F_1(H))$. In general, $(h, \alpha) \in Z_2(\mathfrak{H}(H))$ if and only if (a) $\eta(h) \equiv h \bmod F_1(H)$ for every $\eta \in \mathfrak{A}(H)$, (b) $\alpha \in Z_1(\mathfrak{A}(H))$ and (c) $\theta_h\alpha \in \mathfrak{A}(H;$

$H/F_1(H)$). By induction, $(h, \alpha)\in Z_n(\mathfrak{H}(H))$ implies that $\alpha\in Z_{n-1}(\mathfrak{A}(H))$. One can readily show that if $\mathfrak{H}(H)$ is of class 3 then $\alpha(h')\equiv h' \bmod F_1(H)$ for every $\alpha\in\mathfrak{A}(H)$ and for every(8) $h'\in H'$.

5. **Inclusion products.** For $(h, k)\in G=(H, K; \phi)$, let $\gamma_7(h, k)=(\theta_h, \phi_k)\in Q=(\mathfrak{J}(H), \phi(K); \theta)$. γ_7 is a homomorphism with kernel $B_1\cap B_5\cong Z_1(H)\oplus\mathfrak{k}$. Let $\gamma_8(\theta_h, \phi_k)=\theta_h\phi_k$ (if ϕ is ample). γ_8 is likewise a homomorphism, and kern $\gamma_8=\gamma_7(B_1)$ with $\gamma_1=\gamma_8\gamma_7$. Consider a pair of groups H and K where $H\Delta K$. Then one can always form the semi-direct product $(H, K; \theta)$ which we shall abbreviate by $[H\Delta K]$. We shall call this *an inclusion product of the first kind*. If ϕ is ample, Q above is such a product. Another is $(H, H; \theta)$, the *central square* of H. Denote this square by $[H\Delta H]$, an extension of H by H for which $Z_1[H\Delta H]\cong B_1\cap B_5\cong[Z_1(H)\Delta Z_1(H)]$. For K of class 2 and for fixed integers m and n, the mapping δ given by $\delta(h, k)=\theta_{h^mk^n}$ lies in Hom $([H\Delta K], \mathfrak{J}(K))$.

For α and $\beta\in\mathfrak{A}(K)$, a routine argument shows that $\theta\alpha\wedge\theta\beta\in$ Hom $([H\Delta K], \mathfrak{J}(K))$ if and only if $\alpha(k)\equiv\beta(k) \bmod (Z_1(K)\div\alpha(H))$ for every $k\in K$. In particular, $\theta\alpha\wedge\theta\beta\in$ Hom $([H\Delta H], \mathfrak{J}(H))$ if and only if $\alpha\equiv\beta \bmod \mathfrak{T}_2(H)$. Hence if K is of class 2, $\theta\alpha\wedge\theta\beta\in$ Hom $([H\Delta K], \mathfrak{J}(H))$ for every α and $\beta\in\mathfrak{A}(K)$; and if H is of class 2, $\theta\alpha\wedge\theta\beta\in$ Hom $([H\Delta H], \mathfrak{J}(H))$ for every $\alpha, \beta\in\mathfrak{A}(H)$. In any event, for $\alpha, \beta\in$ Hom $(K, \mathfrak{J}(H))$, $\alpha\wedge\beta\in$ Hom $([H\Delta K], \mathfrak{J}(H))$ if and only if $\alpha(k)\equiv\beta(k) \bmod \mathfrak{C}(\alpha(H), \mathfrak{J}(H))$.

If L is a subgroup of K and if $H\Delta K$, then $L\cap H\Delta L$. At once we have, by a simple induction, that $Z_n[H\Delta K]=[Z_n(K)\cap H\Delta Z_n(K)]$.

If $L\Delta H\Delta K$, $L\Delta K$, then $[H\Delta K]$ maps onto $[(H/L)\Delta(K/L)]$ via the obvious homorphism with a kernel which is isomorphic to the central square of L. We may take $[L\Delta L]$ as a normal subgroup of $[H\Delta K]$. Let $\mathfrak{M}(L)$ be the set of all $(x, x^{-1})\in[H\Delta K]$ where $x\in L$. $\mathfrak{M}(L)$ is readily seen to be a normal subgroup of $[H\Delta K]$. In fact, $\theta_{(h,k)}(x, x^{-1})=(kxk^{-1}, kx^{-1}k^{-1})$ where $x\in L$, $h\in H$, $k\in K$. Recall [9] that $[H, K]$ is the mutual commutator group of H and K. If $H\Delta K$, then $[H, K]\Delta H, K$. $\mathfrak{M}(H)\Delta B_1$. We have

LEMMA 8. $\mathfrak{M}[H, K]\Delta[H\Delta K]'$.

Proof. Observe that $[(h^{-1}, h), (x, k)]=([k, h], [h, k])$. On the right is a typical generator of $\mathfrak{M}[H, K]$ where $h\in H$, $k\in K$. On the left is an element of $[H\Delta K]'$ (where x may be chosen at will in H).

A trivial argument shows that $[H\Delta K]\Delta[L\Delta M]$ if and only if $H\Delta M$, $H\Delta K\Delta M$, $H\Delta L\Delta M$ and $[L, K]\subset H$.

Suppose that $H\supset K$. An *inclusion product of the second kind* is given by $(H, K; \theta)$ and is denoted by $[H\supset K]$. $[H\supset H]=[H\Delta H]$. The θ of a product of the first kind is always ample, but this is not generally true for a product of the second kind. If H is of class 2 and if m and n are integers, one can verify that the mapping δ given by $\delta(h, k)=\theta_{h^mk^n}$ lies in Hom $([H\supset K], \mathfrak{J}(H))$ as before for inclusion products of the first kind. If $\alpha\in\mathfrak{C}(H)$, $\beta\in$ Hom (K, H)

where $K \subset H$, then $\theta\alpha \wedge \theta\beta \in \text{Hom}\ ([H \supset K], \mathfrak{Z}(H))$ if and only if (cf. above) $\alpha(k) \equiv \beta(k) \bmod (Z_1(H) \div \alpha(H))$ for every $k \in K$. If $\alpha \in \mathfrak{A}(H)$, then the congruence reduces to $\alpha(k) \equiv \beta(k) \bmod Z_2(H)$. By direct verification, one can show that $[H \supset K] \Delta [L \supset M]$ if and only if $K \subset H \Delta L$ and $K \Delta M \subset L$.

LEMMA 9. *If $K \Delta H$, then a necessary and sufficient condition that the mapping γ: $(h, k) \rightarrow (k, h)$ be an isomorphism on $[H \supset K]$ onto $[K \Delta H]$ is that $K \subset Z_1(H)$. Then these inclusion products are both isomorphic to $H \oplus K$.*

By $A(\div)^n B$ we shall mean $(A(\div)^{n-1}B) \div B$ where the symbol $(\div)^{n-1}$ is already defined, and $A(\div)^1 B = A \div B$. $A(\div)^0 B$ shall mean A.

THEOREM 6. *If $H \supset K$, then $Z_n[H \supset K]$ is the set of all (h, k), $h \in H$, $k \in K$, where $h \in \mathfrak{C}(K, H)(\div)^{n-1}K$ and where $hk \in Z_n(H)$.*

Proof. $(h, k) \in Z_1[H \supset K]$ if and only if $(h, k)(x, y) = (x, y)(h, k)$ for every $(x, y) \in [H \supset K]$. Equivalently, $hkxk^{-1} = xyhy^{-1}$ and $ky = yk$. Taking $x = e$, we see that $h \in \mathfrak{C}(K, H)$ and that $hkxk^{-1} = xh$, or $(hk)x = x(hk)$, so that $hk \in Z_1(H)$. Conversely, if $hk \in Z_1(H)$ and if $h \in \mathfrak{C}(K, H)$, then $(hk)xk^{-1} = xk^{-1}(hk)$ $= xhk^{-1}k = xh$, while $xyhy^{-1} = xhyy^{-1} = xh$. Moreover, $h \in \mathfrak{C}(K, H)$ and $hk \in Z_1(H)$ imply $k \in Z_1(K)$ so that $ky = yk$. This establishes the theorem for the case $n = 1$.

Now suppose that the theorem holds for the case n. $(h, k) \in Z_{n+1}$ if and only if each $[(h, k), (x, y)] \in Z_n$. But this commutator reduces to $([hk, xy] \cdot [y, k], [k, y])$. The assumption $h'k' \in Z_n(H)$ leads to the equivalence of $h' \in \mathfrak{C}(K, H)(\div)^{n-1}K$ and of $k' \in \mathfrak{C}(K, H)(\div)^{n-1}K$, as a short separate argument shows. Consequently, the induction assumption $(h, k) \in Z_{n+1}$ implies that $[hk, xy][y, k][k, y] = [hk, xy] \in Z_n(H)$ and that both $[hk, xy][y, k]$ and $[k, y] \in \mathfrak{C}(K, H)(\div)^{n-1}K$. Since xy ranges over all of H, $hk \in Z_{n+1}(H)$. Since y ranges over all of K, $k \in (\mathfrak{C}(K, H)(\div)^{n-1}K) \div K = \mathfrak{C}(K, H)(\div)^n K$. Hence $h \in \mathfrak{C}(K, H)(\div)^n K$, and the theorem is established.

If $L \Delta H$, $L \subset K$, $[H \supset K]$ maps homomorphically onto $[(H/L) \supset (K/L)]$ with kernel $[L \supset L]$, so that the latter is a normal subgroup of $[H \supset K]$. We can define $\mathfrak{M}^*(L)$ as the set of all (x, x^{-1}), $x \in L$, in the inclusion product of the second kind $[H \supset K]$ where $L \subset K$. Suppose, further, that $L \Delta H$. Then $\mathfrak{M}^*(L) \Delta [H \supset K]$.

There exists an onto mapping $\lambda_1 \in \text{Hom}\ ([H \Delta K], K)$ given by $\lambda_1(h, k)$ $= hk$. kern $\lambda_1 = \mathfrak{M}(H)$. There exists $\lambda_2 \in \text{Hom}\ ([H \supset K], H)$, an onto mapping, given by $\lambda_2(h, k) = hk$. kern $\lambda_2 = \mathfrak{M}^*(K)$, so that $\mathfrak{M}^*(K) \Delta [H \supset K]$ even if K is not normal in H.

THEOREM 7. *To each $\phi \in \text{Hom}\ (K, \mathfrak{A}(H))$ there exists an extension W of H with a normal subgroup H_0 anti-isomorphic to H such that $W/H \cong W/H_0$ $\cong (H, K; \phi)$.*

Proof. Let $W = [H^* \Delta(H, K; \phi)]$. Let H_0 be the set of all $((h, e), (h^{-1}, e))$

$\in W$ where $h\in H$. That is, $H_0=\mathfrak{M}(H^*)\triangle W$, where we recall that H^* is the set of all (h, e). Since $((h, e), (h^{-1}, e))((t, e), (t^{-1}, e))=((th, e), (h^{-1}t^{-1}, e))$, H and H_0 are anti-isomorphic. By the remarks above on λ_1, $W/\mathfrak{M}(H^*)$ is isomorphic to $(H, K; \phi)$. But W/H^{**} (written, "by abuse of language," as W/H) is also isomorphic to $(H, K; \phi)$. In particular, each endomorphism of $\mathfrak{A}(H)$ leads to such an extension.

6. **Extensions from $\mathfrak{A}(H)$ to $\mathfrak{J}(G)$.** We saw above in §2 that each element of $\phi(K)$ can be extended to an element of $\mathfrak{J}(G)$. Suppose that $g=(a, u)$, $a\in H, u\in K$. For $x\in H$, $(x, e)^g=(\theta_a\phi_u(x), e)$, so that those automorphisms of H which extend to inner automorphisms of G are precisely the elements of $\mathfrak{J}(H)\phi(K)$. In particular, each element of $\mathfrak{J}(H)$ extends to an element of $\mathfrak{J}(G)$. $\theta_a\in\mathfrak{J}(H)$ extends to $\theta_g\in\mathfrak{J}(G)$ for all $g\in G$ such that $g=(b, u)$, $b\in H$, $u\in K$, with $\phi_u=\theta_{b^-a}$. If, for instance, $\phi(K)=\mathfrak{J}(H)$, then to each pair $a\in H$, $u\in K$, there exists $b\in H$ such that $\theta_{(b,u)}$ extends from θ_a. More generally, if ϕ is ample, then $\alpha\in\mathfrak{A}(H)$ extends to some element of $\mathfrak{J}(G)$ if and only if $\alpha\in\phi(K)$. Since $\theta_{(a,u)}(x, e)=(\theta_a\phi_u(x), e)$, $\theta_{(a,u)}$ induces an element of $\mathfrak{J}(H)$ if and only if $\phi_u\in\mathfrak{J}(H)$. But $\phi^{-1}(\mathfrak{J}(H))\triangle K$ since ϕ ample implies $\mathfrak{J}(H)\triangle\phi(K)$, whence it is easy to see that the set of all elements of $\mathfrak{J}(G)$, each of which induces an element of $\mathfrak{J}(H)$, is a normal subgroup of $\mathfrak{J}(G)$.

$\theta_{(a,u)}$ and $\theta_{(b,v)}$ induce the same automorphism of H if and only if $\phi_{uv^{-1}}=\theta_{a^{-1}b}$, so that $\theta_{(a,e)}$ and $\theta_{(b,e)}$ generate the same element of $\mathfrak{A}(H)$ if and only if $a\equiv b \mod Z_1(H)$, while $\theta_{(e,u)}$ and $\theta_{(e,v)}$ generate the same element of $\mathfrak{A}(H)$ if and only if $u\equiv v \mod \mathfrak{k}$. There exist mappings $\delta_1\in\text{Hom }(H, \mathfrak{J}(G))$ and $\delta_2\in\text{Hom }(K, \mathfrak{J}(G))$ given by $\delta_1(a)=\theta_{(a,e)}$, $\delta_2(u)=\theta_{(e,u)}$. kern $\delta_1=Z_1(H)\cap F_1(H)$ and kern $\delta_2=Z_1(K)\cap\mathfrak{k}$. For a subgroup U of G, let $\mathfrak{J}(U, G)$ be that subgroup of $\mathfrak{J}(G)$ consisting of all θ_u, $u\in U$.

THEOREM 8. *Let* $G=(H, K; \phi)$. (a) *There exists* $\phi^{(1)}\in\text{Hom }(\mathfrak{J}(K^*, G), \mathfrak{A}(\mathfrak{J}(H^*, G)))$ *with* kern $\phi^{(1)}$ *consisting of all* $\theta_{(e,k)}$ *where* $\phi_k\in\mathfrak{A}(H; H/(Z_1(H)\cap F_1(H)))$. (b) *There exists a homomorphism* ψ_1 *on* G *onto the semi-direct product* $G_1=(\mathfrak{J}(H^*, G), \mathfrak{J}(K^*, G); \phi^{(1)})$, *and* kern $\psi_1\cong(Z_1(H)\cap F_1(H))\oplus(Z_1(K)\cap\mathfrak{k})$.

Proof. (a) Let $\phi^{(1)}(\theta_{(e,k)})=\alpha_k$, that automorphism of $\mathfrak{J}(H^*, G)\cong H/(Z_1(H)\cap F_1(H))$ which is induced by $\phi_k\cdot\alpha_k$ exists since both $Z_1(H)$ and $F_1(H)$ are ϕ_k-and $\phi_{k^{-1}}$-admissible. If $k\equiv k' \mod (Z_1(K)\cap\mathfrak{k})$, then $\phi_k=\phi_{k'}$, and $\alpha_k=\alpha_{k'}$, whence $\phi^{(1)}$ is single-valued. It is immediate that $\phi^{(1)}$ is a homomorphism. kern $\phi^{(1)}$ consists of all $\theta_{(e,k)}$ for which $\phi_k(h)\equiv h \mod (Z_1(H)\cap F_1(H))$. (b) Define ψ_1 by $\psi_1(h, k)=(\delta_1(h), \delta_2(k))\in G_1\cdot\psi_1$ is onto G_1, and kern ψ_1 is the set of all (h, k) where $h\in$kern δ_1 and $k\in$kern δ_2. If $h_i\in$kern δ_1, $k_i\in$kern δ_2, $(i=1,2)$, then $(h_1, k_1)(h_2, k_2)=(h_1h_2, k_1k_2)$ so that kern $\psi_1\cong$kern $\delta_1\oplus$kern δ_2. If $g_1=(h, k)$, $g_2(x, y)\in G$, then $\psi_1(g_1g_2)=\psi_1(h\phi_k(x), ky)=(\delta_1(h)\delta_1(\phi_k(x)), \delta_2(k)\delta_2(y))$. Since $\phi^{(1)}\delta_2(k)(\delta_1(x))=\alpha_k\theta_{(x,e)}=\theta_{(\theta_k(x),e)}=\delta_1(\phi_k(x))$, it follows that $\psi_1(g_1g_2)=\psi_1(g_1)\psi_1(g_2)$ so that ψ_1 is a homomorphism.

COROLLARY. *If ϕ is ample, then* kern $\psi_1=Z_1(G)$, *and* $G_1\cong\mathfrak{J}(G)$.

Proof. ϕ ample implies that $\mathfrak{J}(H)\subset\phi(K)$ so that $Z_1(H)\supset F_1(H)$. $(h, k)\in$ kern ψ_1 if and only if $h\in Z_1(H)\cap F_1(H)=F_1(H)$ and $k\in Z_1(K)\cap\mathfrak{k}$. If $(h, k)\in$ kern ψ_1, $k\in\mathfrak{k}$ and $\phi_k=\iota$, while $h\in F_1(H)\cap Z_1(H)$ implies that $\theta_{h^{-1}}=\iota$. By our earlier determination, $(h, k)\in Z_1(G)$. Conversely, if $(h, k)\in Z_1(G)$, $h\in F_1(H)=Z_1(H)\cap F_1(H)$, and $k\in Z_1(K)$ with $\theta_{h^{-1}}=\phi_k$. Since ϕ is ample, $F_1(H)\subset Z_1(H)$, whence $\theta_{h^{-1}}=\iota$. Therefore $\phi_k=\iota$, and $k\in\mathfrak{k}$, $Z_1(K)\cap\mathfrak{k}$.

LEMMA 10. *If ϕ is ample, then $\phi^{(1)}$ is ample.*

We shall use this lemma later in determining the ascending central series of semi-direct products with ample homomorphisms.

Suppose that $Z_1(H)\subset F_1(H)$, and that $\phi(Z_1(K))\subset\mathfrak{T}_1(H)$. (For instance, if ϕ is ample, $\phi(Z_1(K))\subset\mathfrak{T}_1(H)$.) Then, as in Theorem 8, we can show that there exists $\phi^{(1)}\in \operatorname{Hom}(\mathfrak{J}(K), \mathfrak{A}(H/F_1(H)))$ given by $\phi^{(1)}_{kZ_1(K)}(hF_1(H))=\phi_k(h)F_1(H)$, and there exists a homomorphism $\psi_{(1)}$ on $(H, K; \phi)$ onto $(H/F_1(H), J(K); \phi^{(1)})$ given by $\psi_{(1)}(h, k)=(hF_1(H), kZ_1(K))$, and kern $\psi_{(1)}\cong F_1(H)\oplus Z_1(K)$.

7. **Extensions from $\mathfrak{A}(K)$ to $\mathfrak{J}(G)$.**

LEMMA 11. *$\mathfrak{J}(K)$ is precisely that subset of $\mathfrak{A}(K)$, the elements of which can be extended to elements of each $\mathfrak{J}(H, K; \phi)$.*

If $\mathfrak{k}=K$, then $\mathfrak{J}(K)$ can be injected into $\mathfrak{J}(G)$. A related result is

THEOREM 9. *Let H be a group for which the ascending central series breaks off with $Z_1(H)$. If $\phi(K)=\mathfrak{J}(H)$, then $\mathfrak{J}(K)$ can be injected into $\mathfrak{J}(H, K; \phi)$.*

Proof. The mapping $\gamma:\theta_u\to\theta_{(e,u)}$ is single valued if and only if $Z_1(K)\subset\mathfrak{k}$. If γ is single valued, it is clearly in Hom $(\mathfrak{J}(K), \mathfrak{J}(G))$. $\theta_{(e,u)}=\iota$ if and only if $\phi_u=\iota$ where $u\in Z_1(K)$, so that, if γ is single valued, it is an isomorphism. Now suppose that $\phi(K)=\mathfrak{J}(H)$. Then $\mathfrak{J}(H)\subset\phi(K)$ implies that each ϕ_k with $k\in Z_1(K)$ is a normal automorphism. Since $\phi(K)\subset\mathfrak{J}(H)$, to each $k\in Z_1(K)$ there exists $h\in H$ with $\phi_k=\theta_h$. For $x\in H$, $\phi_k(x)=hxh^{-1}xz$, $z\in Z_1(H)$. Thus $h\in Z_2(H)$. Since, however, $Z_2(H)=Z_1(H)$, $\phi_k(x)=x$ so that $Z_1(K)\subset\mathfrak{k}$, and γ is the required injection.

COROLLARY. *If $Z_1(K)\subset\mathfrak{k}$, then the image of $\mathfrak{J}(K)$ in $\mathfrak{J}(H, K; \phi)$ under the above injection γ is a normal subgroup of $\mathfrak{J}(H, K; \phi)$ if and only if $\phi(K)\subset\mathfrak{A}(H; H/(F_1(H)\cap Z_1(H)))$.*

In particular, note that $\mathfrak{J}(K)$ can be injected into $\mathfrak{J}[H\Delta K]$ and if $Z_1(K)\subset Z_1(H)$, also into $\mathfrak{J}[H\supset K]$. The image under injection is a normal subgroup of $\mathfrak{J}[H\Delta K]$ if and only if $H\subset Z_2(K)$; of $\mathfrak{J}[H\supset K]$ if and only if $K\subset Z_2(H)$ (where $Z_1(K)\subset Z_1(H)$).

8. **Pair extensions.** An automorphism Γ of $G=(H, K; \phi)$ is called a *pair extension of α and of β over G* if Γ induces an automorphism α on H and an automorphism β on K. Then $\Gamma(h, k)=\Gamma(h, e)\Gamma(e, k)=(\alpha(h), e)(e, \beta(k))$

$=(\alpha(h), \beta(k))$. (Cf. Lemma 2.) Write $\Gamma=\Gamma_{\alpha,\beta}=(\alpha, \beta)$. Let $\mathfrak{G}=\mathfrak{G}(G)$ be the set of all pair extensions over G.

LEMMA 12. *$\mathfrak{G}$ is a group under automorphism composition; and for $\alpha\in\mathfrak{A}(H)$, $\beta\in\mathfrak{A}(K)$, $(\alpha, \beta)\in\mathfrak{G}$ if and only if*

$$\alpha\phi_k=\phi_{\beta(k)}\alpha \tag{1}$$

for every $k\in K$.

LEMMA 13. (a) *$(\alpha, \iota)\in\mathfrak{G}$ if and only if $\alpha\in\mathfrak{C}(\phi(K), \mathfrak{A}(H))$.* (b) *$(\iota, \beta)\in\mathfrak{G}$ if and only if $\beta\in\mathfrak{A}(K; K/\mathfrak{k})$.* (c) *If $(\alpha, \beta)\in\mathfrak{G}$, then $\alpha\in\mathfrak{C}(\phi(K), \mathfrak{A}(H))$ if and only if $\beta\in\mathfrak{A}(K; K/\mathfrak{k})$.*

If $G=[H\Delta H]$, then $\mathfrak{C}(\phi(K), \mathfrak{A}(H))$ becomes $\mathfrak{C}(\mathfrak{J}(H), \mathfrak{A}(H))=\mathfrak{T}_1(H)$, while $\mathfrak{A}(K; K/\mathfrak{k})$ reduces to $\mathfrak{A}(H; H/Z_1(H))$, a group which is also equal to $\mathfrak{T}_1(H)$. Hence, by (c), *if $(\alpha, \beta)\in\mathfrak{G}[H\Delta H]$, α is normal if and only if β is normal, while α and β normal imply $(\alpha, \beta)\in\mathfrak{G}[H\Delta H]$.* It is easy to see that $(\alpha, \beta)\in\mathfrak{T}_1[H\Delta H]$ if $\alpha, \beta\in\mathfrak{T}_1(H)$. Thus if $\alpha\in\mathfrak{T}_1(H)$, $(\alpha, \alpha)\in\mathfrak{T}_1[H\Delta H]$ so that *$\mathfrak{T}_1(H)$ nontrivial implies that $\mathfrak{T}_1[H\Delta H]$ is nontrivial.* In particular, if H is of class 2, if $(\alpha, \beta)\in\mathfrak{G}[H\Delta H]$ and if one of α or β is inner, then the other is normal.

LEMMA 14. *If $G=(H, K; \phi)$, then $\mathfrak{J}(G)\cap\mathfrak{G}(G)$ consists of all $\theta_{(h,k)}$ such that $h\in F_1(H)$, so that $\mathfrak{J}(G)\cap\mathfrak{G}(G)\cong(F_1(H)\oplus K)/Z_1(G)$. If $\theta_{(h,k)}\in\mathfrak{G}(G)$, then $\theta_{(h,k)}=(\theta_h\phi_k, \theta_k)$.*

By Lemma 13a, each $(\theta_h, \iota)\in\mathfrak{G}(H, K; \phi)$ if and only if $\mathfrak{J}(H)\subset\mathfrak{C}(\phi(K), \mathfrak{A}(H))$, or equivalently $\phi(K)\subset\mathfrak{T}_1(H)$. If $(\theta_h, \beta)\in\mathfrak{G}$ under these conditions, then $(\iota, \beta)\in\mathfrak{G}$. Likewise $(\theta_a, \beta)\in\mathfrak{G}$ with $a\in F_1(H)$ implies $(\iota, \beta)\in\mathfrak{G}$. For then, $\phi_k\theta_a=\theta_a\phi_k$ whence $\theta_a\in\mathfrak{C}(\phi(K), \mathfrak{A}(H))$ so that Lemmas 13 c, b are applicable. $\theta_{(a,k)}$ induces inner automorphisms on both H and K if and only if $a\in F_1(H)$ and $k\in\phi^{-1}(\mathfrak{J}(H))$.

THEOREM 10. *For a group H of class 2 and for $\phi\in\text{Hom}\,(K, \mathfrak{J}(H))$, let $\mathfrak{n}$ be the set of all $(\theta_h, \beta)\in\mathfrak{G}(H, K; \phi)$. Then $\mathfrak{n}\,\Delta\mathfrak{G}(H, K; \phi)$, and $\mathfrak{n}\cong\mathfrak{J}(H)\oplus\mathfrak{A}(K; K/\mathfrak{k})$.*

Proof. Since H is of class 2, $\mathfrak{J}(H)\subset\mathfrak{T}_1(H)$. Since $\phi(K)\subset\mathfrak{J}(H)$, the remarks before the theorem show that $(\theta_h, \beta)\in\mathfrak{G}$ implies that $(\iota, \beta)\in\mathfrak{G}$, so that, by Lemma 13b, $\beta\in\mathfrak{A}(K; K/\mathfrak{k})$. Conversely, suppose that $\beta\in\mathfrak{A}(K; K/\mathfrak{k})$. If $h\in H$, $\theta_h\phi_k=\phi_k\theta_h$ for every $k\in K$, since $\phi(K)\subset\mathfrak{T}_1(H)$. But $\beta(k)\equiv k \bmod \mathfrak{k}$ so that $\theta_h\phi_k=\phi_{\beta(k)}\theta_h$. By (1) (of this section), $(\theta_h, \beta)\in\mathfrak{G}$. If we define $\mathfrak{n}$ as above, we have proved that $\mathfrak{n}\cong\mathfrak{J}(H)\oplus\mathfrak{A}(K; K/\mathfrak{k})$. To show that $\mathfrak{n}\,\Delta\mathfrak{G}$, suppose that $(\gamma, \delta)\in\mathfrak{G}$ where $\gamma\in\mathfrak{A}(H)$, $\delta\in\mathfrak{A}(K)$. Then, for $(\theta_h, \beta)\in\mathfrak{n}$, $(\theta_h, \beta)^{(\gamma,\delta)}=(\gamma\theta_h\gamma^{-1}, \delta\beta\delta^{-1})$. Since $\mathfrak{J}(H)\,\Delta\mathfrak{A}(H)$, $\gamma\theta_h\gamma^{-1}\in\mathfrak{J}(H)$. For $y\in K$, $\delta\beta\delta^{-1}(y)=\delta[\delta^{-1}(y)k]$ where $k\in\mathfrak{k}$. By (1), $\gamma\phi_k=\phi_{\delta(k)}\gamma$. Hence $k\in\mathfrak{k}$ implies that $\gamma=\phi_{\delta(k)}\gamma$ so that $\delta(k)\in\mathfrak{k}$, whence $\delta\beta\delta^{-1}\in\mathfrak{A}(K; K/\mathfrak{k})$, and we have proved that $\mathfrak{n}\,\Delta\mathfrak{G}$.

If one could find a class 2 group H without outer automorphisms (the existence of such is an open question) $\mathcal{G}(H, K; \phi)$ would be isomorphic to $\mathfrak{n}$ and therefore to $\mathfrak{J}(H) \oplus \mathfrak{A}(K; K/\mathfrak{k})$. In particular, for such an H, $\mathcal{G}[H \Delta H] \cong \mathfrak{J}(H) \oplus \mathfrak{T}_1(H)$.

THEOREM 11. *If $G=(H, K; \phi)$, then the set of all pair extensions of the form (α, ι) where α is a normal automorphism of H which induces the identity on $H/F_1(H)$, and of all pair extensions of the form (ι, β) where β is a normal automorphism of K, generates the group of pair extensions which are also normal automorphisms of G.*

Proof. $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{T}_1$ if and only if $\alpha\phi_y = \phi_{\beta(y)}\alpha$ for every $y \in K$ and $(\alpha, \beta)(h, k) = (hr, ks)$ where $r \in F_1(H)$, $s \in Z_1(K)$ and $\theta_r = \phi_{s^{-1}}$. Hence, if $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{T}_1$, $(\alpha, \beta)(h, e) = (hr, e)$ where $\theta_r = \iota$, (i.e, $r \in Z_1(H)$). Then $(\alpha, \beta)(h, k) = (hr, ks)$ implies $s \in \mathfrak{k}$. Since $\alpha\phi_y = \phi_{\beta(y)}\alpha$, where $\phi_{\beta(y)} = \phi_y$, $\alpha \in \mathfrak{C}(\phi(K), \mathfrak{A}(H))$ so that, by Lemma 13a, $(\alpha, \iota) \in \mathcal{G}$. It follows that $(\iota, \beta) \in \mathcal{G}$. Now $(\alpha, \iota) \in \mathfrak{T}_1 \cap \mathcal{G}$ if and only if $(\alpha, \iota) \in \mathcal{G}$ and $\alpha(h) = hr$ where $r \in F_1(H) \cap Z_1(H)$; and $(\iota, \beta) \in \mathfrak{T}_1 \cap \mathcal{G}$ if and only if $(\iota, \beta) \in \mathcal{G}$ and $\beta(k) = ks$ where $s \in Z_1(K) \cap \mathfrak{k}$. If $(\alpha, \beta) \in \mathcal{G} \cap \mathfrak{T}_1$, we saw above that $r \in F_1(H) \cap Z_1(H)$. Hence $(\alpha, \iota) \in \mathcal{G} \cap \mathfrak{T}_1$. At once, $(\iota, \beta) \in \mathcal{G} \cap \mathfrak{T}_1$ so that $\alpha \in \mathfrak{C}(\phi(K), \mathfrak{A}(H)) \cap \mathfrak{A}(H; H/(F_1(H) \cap Z_1(H)))$ and so that $\beta \in \mathfrak{A}(K; K/(Z_1(K) \cap \mathfrak{k}))$. Conversely, if α and β are in these sets of automorphisms, respectively, then both (α, ι) and $(\iota, \beta) \in \mathfrak{T}_1 \cap \mathcal{G}$ so that $(\alpha, \beta) \in \mathfrak{T}_1 \cap \mathcal{G}$.

Let K_* be that subgroup of a group K which is generated by all $x^{-1}\alpha(x)$ where α ranges over all of $\mathfrak{A}(K)$ and x ranges over all of K. $K' \subset K_*$, and if K is abelian, $x^2 \in K_*$ for every $x \in K$. We might call K_* *the generalized derived subgroup of* K.

THEOREM 12. $\mathcal{G}(H, K; \phi) \cong \mathfrak{A}(H) \oplus \mathfrak{A}(K)$ *(in the natural way) if and only if $\mathfrak{k} \supset K_*$ and $\phi(K) \subset Z_1(\mathfrak{A}(H))$.*

If $\alpha \in \mathfrak{A}(H)$, there is one and only one pair extension of α with α as the first component of the pair over $\mathfrak{H}(H)$. Unless, therefore, $\Delta \in \mathfrak{J}(\mathfrak{A}(H))$, there is no pair extension of $\Delta \in \mathfrak{A}^2(H)$ with Δ as second component over $\mathfrak{H}(H)$. If $\Delta = \theta_\alpha \in \mathfrak{J}(\mathfrak{A}(H))$, then each $(\alpha', \Delta) \in \mathcal{G}(H, K; \phi)$ where $\alpha' \equiv \alpha \bmod Z_1(\mathfrak{A}(H))$. Note that $(\alpha, \theta_\alpha) = \theta_{(e,\alpha)}$ on $\mathfrak{H}(H)$. Since each (α', θ_α) above can be written $(\alpha', \theta_{\alpha'})$, $\mathcal{G}(\mathfrak{H}(H))$ consists of all $(\alpha, \theta_\alpha) = \theta_{(e,\alpha)}$.

$\mathfrak{J}(F_1(H), H) \subset \mathfrak{C}(\phi(K), \mathfrak{A}(H))$ since $\phi_k\theta_h(x) = h\phi_k(x)h^{-1}$ if $h \in F_1(H)$. If ϕ is ample there is a homomorphism γ_9 on $F_1(H) \oplus K$ onto $\phi(K)$ given by $\gamma_9(h, k) = \theta_h\phi_k$ for every $h \in F_1(H)$ and for every $k \in K$, where γ_1 restricted to the same domain of definition is γ_9. If kern $\gamma_9 = B_9$, $Z_1(H, K; \phi) \subset B_9 \subset B_1$ for ample ϕ.

We should note carefully the following: by Lemma 14, $\theta_{(h,k)}$ is a pair extension, namely $(\theta_h\phi_k, \theta_k)$, if and only if $h \in F_1(H)$. But each product $\theta_h\phi_k$

can be extended, by §6, to the inner automorphism $\theta_{(h,k)}$ of $(H, K; \phi)$. What $\theta_{(h,k)}$ does *not* do, in general, is to induce an inner automorphism on K, unless $h\in F_1(H)$, so that $\theta_{(h,k)}$ is always an extension of an automorphism of H, even though it is only exceptionally a pair extension.

Since $\mathfrak{T}_1(H)=\mathfrak{C}(\mathfrak{J}(H), \mathfrak{A}(H))$, $\mathfrak{J}(H)\subset\mathfrak{C}(\mathfrak{T}_1(H), \mathfrak{A}(H))=\mathfrak{C}^{(2)}(\mathfrak{J}(H),\mathfrak{A}(H))$. Let $\mathfrak{J}\subset\mathfrak{B}\subset\mathfrak{C}^{(2)}(\mathfrak{J})$ where $\mathfrak{B}$ is a group of automorphisms which we shall call an *ample group of automorphisms*; and form $\mathfrak{H}(H; \mathfrak{B})$, the relative holomorph [9] of $\mathfrak{B}$ over H. Let $F_1(H)$ be computed for the semi-direct product $\mathfrak{H}(H; \mathfrak{B})$. Note that $F_1(H)\subset Z_1(H)$, so that Hom $(\mathfrak{B}, F_1(H))$ is a group under homomorphism addition. There exists a $\Theta\in\text{Hom}(\mathfrak{A}(H; H/F_1(H)), \mathfrak{A}(\text{Hom}\ (\mathfrak{B}, F_1(H)))$ given by $\Theta:\gamma\rightarrow\Theta_\gamma$ where $\Theta_\gamma\Gamma=\gamma\Gamma$ for every $\gamma\in\mathfrak{A}(H; H/F_1(H))$ and for every $\Gamma\in\text{Hom}\ (\mathfrak{B}, F_1(H))$. We can show that the group of normal automorphisms of $\mathfrak{H}(H; \mathfrak{B})$ has a faithful representation as a splitting extension of an abelian group by a group of normal automorphisms of H. We have

THEOREM 13. *Let $\mathfrak{B}$ be an ample group of automorphisms of a group H. If $F_1(H)$ is computed for the relative holomorph of $\mathfrak{B}$ over H, then the normal automorphism group of this relative holomorph is isomorphic to*

$$(\text{Hom}\ (\mathfrak{B}, F_1(H)), \mathfrak{A}(H; H/F_1(H)); \Theta)$$

where $\Theta_\gamma\Gamma=\gamma\Gamma$ for every $\gamma\in\mathfrak{A}(H; H/F_1(H))$ and for every

$$\Gamma\in\text{Hom}\ (\mathfrak{B}, F_1(H)).$$

Proof. For $\Psi\in\mathfrak{T}_1(\mathfrak{H}(H; \mathfrak{B}))$ and for $(x, \beta)\in\mathfrak{H}(H; \mathfrak{B})=\mathfrak{H}$, $\Psi(x, \beta)=\Psi(x, \iota)\Psi(e, \beta)$. Now $Z_1(\mathfrak{H})$ is the set of all (y^{-1}, θ_y) where $y\in F_1(H)$ and where $\theta_y\in Z_1(\mathfrak{B})$. Since $\mathfrak{B}\supset\mathfrak{J}(H)$, $F_1(H)\subset Z_1(H)$ so that $Z_1(\mathfrak{H})$ is the set of all (f, ι) where $f\in F_1(H)$. Hence $\Psi(x, \iota)=(xf(x), \iota)=(\gamma(x), \iota)$ where $f(x)\in F_1(H)$, and $\Psi(e, \beta)=(\Gamma(\beta), \beta)$ where $\Gamma(\beta)\in F_1(H)$. It is easy to show that $\gamma\in\mathfrak{A}(H; H/F_1(H))$ and that $\Gamma\in\text{Hom}\ (\mathfrak{B}, F_1(H))$. It is clear that the semi-direct product $\mathcal{P}=(\text{Hom}\ (\mathfrak{B}, F_1(H)), \mathfrak{A}(H; H/F_1(H)); \Theta)$ exists. Let $\Omega(\Psi)=(\Gamma, \gamma)\in\mathcal{P}$. One can verify that Ω is a homomorphism with trivial kernel. It remains to show that Ω is onto. For $\Gamma\in\text{Hom}\ (\mathfrak{B}, F_1(H))$ construct Ψ_1 by $\Psi_1(x, \beta)=(x\Gamma(\beta), \beta)$. A routine check which uses the fact that $F_1(H)\subset Z_1(H)$ shows that Ψ_1 is an automorphism. Since $(\beta^{-1}(x^{-1}), \beta^{-1})\Psi_1(x, \beta)=(\Gamma(\beta), \iota)\in Z_1(\mathfrak{H})$, Ψ_1 is normal. Likewise, if $\gamma\in\mathfrak{A}(H; H/F_1(H))$, construct Ψ_2 by $\Psi_2(x, \beta)=(\gamma(x), \beta)$. Then $\Psi_2(x_1\beta_1(x_2), \beta_1\beta_2)=(\gamma(x_1)\gamma\beta_1(x_2), \beta_1\beta_2)$. Since, however, $\mathfrak{B}\subset\mathfrak{C}^{(2)}(\mathfrak{J})=\mathfrak{C}^{(1)}(\mathfrak{C}^{(1)}(\mathfrak{J}))$, and since $\gamma\in\mathfrak{A}(H; H/F_1(H))\subset\mathfrak{T}_1(H)=\mathfrak{C}^{(1)}(\mathfrak{J})$, $\gamma\beta_1=\beta_1\gamma$ so that $(\gamma(x_1)\gamma\beta_1(x_2), \beta_1\beta_2)=\Psi_2(x_1, \beta_1)\Psi_2(x_2, \beta_2)$, and Ψ_2 is a homomorphism. That Ψ_2 is onto with a trivial kernel is immediate, and the fact $(\beta^{-1}(x^{-1}), \beta^{-1})(\gamma(x), \beta)=(\beta^{-1}(x^{-1}\gamma(x)), \iota)$ with $x^{-1}\gamma(x)\in F_1(H)$ shows that Ψ_2 is normal. It is clear that $\Omega(\Psi_1\Psi_2)=(\Gamma, \gamma)$, so that Ω is onto $\mathcal{P}$, and the theorem is established.

Let $\mathfrak{W}(H)=\mathfrak{A}(H;\ Z_1(H))\cap\mathfrak{T}_1(H)$, the set of all normal automorphisms of H which reduce to the identity on the center, the abelian group of *centrally normal automorphisms* of H. Let $\mathfrak{B}(H)=\mathfrak{H}(H;\ \mathfrak{W}(H))$, the *centrally normal holomorph* of H.

COROLLARY. *Let H be of class 2. Then the group of centrally normal automorphisms of the centrally normal holomorph of H splits into a direct sum of a group of automorphisms which extend the identity on H, isomorphic to the group of homomorphisms of the group of centrally normal automorphisms of H into the center of H, and of a group of inner automorphisms which are pair extensions over the centrally normal holomorph of H, isomorphic to the group of centrally normal automorphisms of H.*

Proof. Since H is of class 2, $\mathfrak{W}(H)\supset\mathfrak{I}(H)$. If we identify $\mathfrak{W}(H)$ with the $\mathfrak{B}$ of the theorem (ignoring the upper bound given there for $\mathfrak{B}$) and if we compute $F_1(H)$ for $\mathfrak{H}(H;\mathfrak{B})$, we find that $Z_1(\mathfrak{H})$ is the set of all $(f,\iota), f\in F_1(H)$. Since $\mathfrak{B}\subset\mathfrak{A}(H;\ Z_1(H))$, $F_1(H)\supset Z_1(H)$ so that $Z_1(\mathfrak{H})=Z_1(\mathfrak{B}(H))$ is the set of all $(f,\ \iota), f\in Z_1(H)$. If $\Psi\in\mathfrak{W}(\mathfrak{B}(H))$, $\Psi(x,\ \iota)=(\gamma(x),\ \iota)$, as in the proof of the theorem, whence $x^{-1}\gamma(x)\in Z_1(H)$. Since $\Psi\in\mathfrak{A}(\mathfrak{B}(H);\ Z_1(\mathfrak{B}(H)))$, one can show that $\gamma\in\mathfrak{A}(H;\ Z_1(H))\cap\mathfrak{T}_1(H)=\mathfrak{W}(H)$. Likewise $\Psi(e,\ \beta)=(\Gamma(\beta),\ \beta)$ where $\Gamma\in\mathrm{Hom}\ (\mathfrak{B},\ Z_1(H))$. Let $\mathcal{P}=\mathrm{Hom}\ (\mathfrak{W}(H),\ Z_1(H))\oplus\mathfrak{W}(H)$, and set $\Omega(\Psi)=(\Gamma,\ \gamma)\in\mathcal{P}$. That Ω is an isomorphism into is immediate. If $(\Gamma,\ \gamma)\in\mathcal{P}$, let Ψ be defined by $\Psi(x,\ \beta)=(\gamma(x)\Gamma(\beta),\ \beta)$. It is clear that Ψ is an automorphism of $\mathfrak{B}(H)$ and that $\Omega(\Psi)=(\Gamma,\ \gamma)$, so that Ω is onto $\mathcal{P}$. Observe that γ is the mapping of H induced by Ψ. For $\gamma\in\mathfrak{W}(H)$, define Δ by $\Delta(x,\ \beta)=(\gamma(x),\ \beta)$. Then(3) $\Omega(\Delta)=(\nu,\ \gamma)$ so that Δ is that pair extension over $\mathfrak{B}(H)$ which induces γ on H and ι on $\mathfrak{B}$ and which lies in $\mathfrak{W}(\mathfrak{B}(H))$. For

$$\Gamma\in\mathrm{Hom}\ (\mathfrak{W}(H),Z_1(H)),$$

define Ψ by $\Psi(x,\ \beta)=(x\Gamma(\beta),\ \beta)$, so that Ψ is a member of $\mathfrak{W}(\mathfrak{B}(H))$ which induces the identity on H. Note that H of class 2 implies that $\mathfrak{B}(H)$ is of class 2 since $(x,\ \beta)^{(y,\alpha)}=(y\alpha(x)\beta(y^{-1}),\ \beta)=(yz,\ \beta)$ where $z\in Z_1(H)$ for $(x,\ \beta)$, $(y,\ \alpha)\in\mathfrak{B}(H)$. Hence the inner automorphisms of $\mathfrak{B}(H)$ lie in $\mathfrak{W}(\mathfrak{B}(H))$, and $\Omega(\theta_{(y,\alpha)})=(\Gamma,\ \gamma)$ where $\Gamma(\beta)=y\beta(y^{-1})$ and $\gamma=\theta_y\alpha$. It follows that $\Omega(\theta_{(e,\gamma)})=(\nu,\ \gamma)$. The group of pair extension automorphisms over the centrally normal holomorph of H which occurs in the corollary is now seen to be a group of inner automorphisms.

9. **The ascending central series.** Returning to the situation of Theorem 8, we recall the existence of $\phi^{(1)}\in\mathrm{Hom}\ (K_1,\ \mathfrak{A}(H_1))$, where $K_1=K/(Z_1(K)\cap\mathfrak{k})$ and $H_1=H/(Z_1(H)\cap F_1(H))$, such that $k^\#=k(Z_1(K)\cap\mathfrak{k})$ implies $\phi_{k^\#}{}^{(1)}(h(Z_1(H)\cap F_1(H)))=\phi_k(h)(Z_1(H)\cap F_1(H))$ for every $h\in H$ and for every $k\in K$. Let $\mathfrak{k}_{11}=\mathrm{kern}\ \phi^{(1)}$, and let $\mathfrak{k}_{11}^*$ be all $(e,\ k)$ where $k\in\mathfrak{k}_{11}$. Define $\mathfrak{k}_1^*$ by $(\psi_1^{-1}(\mathfrak{k}_{11}^*))\cap K^*$ where ψ_1 is the homomorphism on $G=(H,\ K;\ \phi)$ onto $G_1=(H_1,\ K_1;\ \phi^{(1)})$ which was described in Theorem 8. We follow the practice that if $U\subset H$ (respec-

tively K) then U^* is the set of all (u, e) (respectively (e, u)). Then $\mathfrak{k}_1$ is defined as the set of all $k \in K$ such that $(e, k) \in \mathfrak{k}_1^*$. If $k \in \mathfrak{k} = \text{kern } \phi$, $\phi_k(h) = h$ for every $h \in H$ so that $\mathfrak{k} \subset \mathfrak{k}_1$. Also $\mathfrak{k}_1 \Delta K$. Let $[F_2(H)]^* = H^* \cap \psi_1^{-1}[F_1(H_1)]^*$. "By abuse of language," we make such identifications as of H_1 with its isomorphic image $\mathfrak{I}(H^*, G)$. $h(Z_1(H) \cap F_1(H)) \in F_1(H_1)$ if and only if

$$\phi_k(h) \equiv h \bmod (Z_1(H) \cap F_1(H))$$

for every $h \in H$. Hence if $h \in F_1(H)$, then $h \in F_2(H)$ so that $F_1(H) \subset F_2(H)$. From G_1, we can, by the same process which yielded $G_1 = \psi_1(G)$, construct $G_2 = (H_2, K_2; \phi^{(2)})$ where there exists a homomorphism ψ_{12} on G_1 such that $\psi_{12}(G_1) = G_2$. Then $\psi_2 = \psi_{12}\psi_1$ carries G onto G_2 homomorphically. Let kern $\phi^{(2)} = \mathfrak{k}_{21}$, and let $\mathfrak{k}_2^* = K^* \cap \psi_2^{-1}(\mathfrak{k}_{12}^*)$ where $\mathfrak{k}_{12}^*$ is the set of all $(e, k) \in G_1$ with $k \in \mathfrak{k}_{12}$. Since $K_1^* \cap \psi_{12}^{-1}(\mathfrak{k}_{12}^*) = \mathfrak{k}_{22}^* \supset \mathfrak{k}_{11}^*$, $\mathfrak{k}_*^2 = K^* \cap \psi_2^{-1}(\mathfrak{k}_{12}^*) = K^* \cap \psi_1^{-1}\psi_{12}^{-1}(\mathfrak{k}_{12}^*) = K^* \cap \psi_1^{-1}(\mathfrak{k}_{22}^*) \supset K^* \cap \psi_1^{-1}(\mathfrak{k}_{11}^*) = \mathfrak{k}_1^*$, so that $\mathfrak{k}_1 \subset \mathfrak{k}_2$. Continuing in this way, we can construct a sequence of groups $\{G_n\} = \{(H_n, K_n; \phi^{(n)})\}$ and an ascending chain of normal subgroups of $K: \mathfrak{k} = \mathfrak{k}_0 \subset \mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \cdots$. Likewise, since $[F_2(H_1)]^* = H_1^* \cap \psi_{12}^{-1}[F_1(H_2)]^* \supset [F_1(H_1)]^*$, $[F_3(H)]^* = H^* \cap \psi_2^{-1}[F_1(H_2)]^* \supset H^* \cap \psi_1^{-1}[F_1(H_1)]^* = [F_2(H)]^*$. Hence an ascending chain of subgroups of H is constructed: $F_1(H) \subset F_2(H) \subset F_3(H) \subset \cdots$.

Suppose that R is a subgroup of H which is ϕ_k-admissible for every k in a subgroup S of K. Then the set (h, k) of all $h \in R$ and all $k \in S$ is a subgroup T of $G = (H, K; \phi)$, and $T \cong (R, S; \phi^*)$ where ϕ^* is ϕ restricted to S with each ϕ_k, $k \in S$, restricted to R. We shall write, "by abuse of language," $T = (R, S; \phi) \subset G = (H, K; \phi)$.

THEOREM 14. *If ϕ is ample for $G = (H, K; \phi)$, then for $n = 1, 2, 3, \cdots$, $G/Z_n(G) \cong (H_n, K_n; \phi^{(n)})$ where $Z_n(G) = (F_n(H), Z_n(K) \cap \bigcap_{i+j=n-1} (\mathfrak{k}_i(\div)^j K); \phi)$ and where $H_n = H/F_n(H)$, $K_n = K/(Z_n(K) \cap \bigcap_{i+j=n-1} (\mathfrak{k}_i(\div)^j K))$.*

Proof. By the corollary to Theorem 8 and by Lemma 12, $G/Z_n(G) \cong (H_n, K_n; \phi^{(n)})$. For $n = 1$, $(h, k) \in Z_1(G)$ if and only if $h \in F_1(H)$, $k \in Z_1(K)$ and $\theta_{h^{-1}} = \phi_k$. Since ϕ is ample, $F_1(H) \subset Z_1(H)$, whence $\theta_{h^{-1}} = \iota$. This places $k \in \mathfrak{k}$ so that $Z_1(G) = (F_1(H), Z_1(K) \cap \mathfrak{k}; \phi)$. Since $\mathfrak{k} = \bigcap_{i+j=0} (\mathfrak{k}_i(\div)^j K)$, $Z_1(G)$ has the required form. $G_1 = (H/(Z_1(H) \cap F_1(H)), K/(Z_1(K) \cap \mathfrak{k}); \phi^{(1)}) = = (H/F_1(H), K/(Z_1(K) \cap \mathfrak{k}); \phi^{(1)}) \cong G/Z_1(G)$, so that the latter has the required form.

Now suppose that the theorem has been established through the index n. Then $(h, k) \in Z_{n+1}(G)$ if and only if (a) $hF_n(H) \in F_1(H_n)$ and (b) $k(Z_n(K) \cap L_{n-1}) \in Z_1(K_n) \cap \mathfrak{k}_{1n}$ where $L_{n-1} = \bigcap_{i+j=n-1} (\mathfrak{k}_i(\div)^j K)$ and $\mathfrak{k}_{1n} = \text{kern } \phi^{(n)}$. $\phi^{(n)}$ is ample, by Lemma 10. $F_1(H) \Delta H$ since $F_1(H) \subset Z_1(H)$. If $F_n(H) \Delta H$, then $F_1(H/F_n(H)) = F_{n+1}(H)/F_n(H) \Delta H/F_n(H)$ so that $F_{n+1}(H) \Delta H$. Now (a) is equivalent to (a′) $h \in F_{n+1}(H)$, and (b) is equivalent to (b′) that both $k \in \mathfrak{k}_n$ and that $[k, y] \in Z_n(K) \cap L_{n-1}$ for every $y \in K$. Equivalently, $k \in (Z_n(K) \div K) \cap (\bigcap_{i+j=n-1} (\mathfrak{k}_i(\div)^{j+1} K)) \cap \mathfrak{k}_n = Z_{n+1}(K) \cap L_n$, (as we see by replacing $j+1$ by

j in $(L_{n-1} \div K) \cap \mathfrak{k}_n)$. Thus Z_{n+1} has the required form. Finally, a short induction establishes the given isomorphisms for H_n and K_n. One can also show that *if H is abelian, and if $F_n(H)$ is computed for $\mathfrak{H}(H)$, then 2^n is an exponent for $F_n(H)$.*

We saw that $Z_n[H \Delta H] = [Z_n(H) \Delta Z_n(H)]$. If $Z_n(H, K; \phi) = (Z_n(H), Z_n(K); \phi)$ for $n = 1, 2, \cdots, m$, we say that the ascending central series of $(H, K; \phi)$ is *regular through m*. If this series is regular for every m, the series is called *regular*. By using Theorems 14 and 8a, we can prove

THEOREM 15. *If ϕ is ample for $G = (H, K; \phi)$, then the ascending central series of G is regular (regular through m) if and only if*
(a) $\phi(K) \subset \mathfrak{A}(H; Z_j(H)/Z_{j-1}(H))$, $j = 1, 2, 3, \cdots$, $(j = 1, 2, 3, \cdots, m)$, *and*
(b) $\phi(Z_j(K)) \subset \mathfrak{T}_{j-1}(H)$, $j = 1, 2, 3, \cdots$, $(j = 1, 2, 3, \cdots, m)$.

COROLLARY 1. *If H is abelian, then $G = (H, K; \phi)$ has a regular ascending central series if and only if $G = H \oplus K$.*

COROLLARY 2. *If $\phi(K) = \mathfrak{J}(H)$ and if $\phi(Z_j(K)) \subset \theta(Z_j(H))$, $j = 1, 2, \cdots$, then $G = (H, K; \phi)$ has a regular ascending central series.* (*E.g.*, $G = [H \Delta H]$.)

Despite Theorems 6 and 14, it seems difficult to determine the ascending central series of $(H, K; \phi)$ for nonample ϕ. However, we have

THEOREM 16. $(h, k) \in Z_2(H, K; \phi)$ *if and only if* (a) $\phi_y(h) = g(y)h = hg'(y)$ *where $g(y), g'(y) \in F_1(H)$ for every $y \in K$,* (b) $k \in Z_2(K)$ *and* (c) $\phi_k(x) = h^{-1}xhf_1(x)$ *where $f_1(x) \in Z_1(H) \cap F_1(H)$ and $f_1(\phi_y(x)) = f_1(x)$ for all x, y.*

Proof. $(h, k) \in Z_2$ if and only if (a′) $s = h\phi_k(x)\phi_{[k,y]}(\phi_y(h^{-1})x^{-1}) \in F_1(H)$, (b′) $[k, y] \in Z_1(K)$ and (c′) $\phi_{[k,y]}\theta_s = \iota$, all for all (x, y). Now assume that (a′), (b′) and (c′) hold. From (a′), with $x = e$, $h\phi_{y^k}(h^{-1}) \in F_1(H)$. Since y^k ranges over all of K, $h\phi_y(h^{-1}) \in F_1(H)$, so that ($a_1$) $\phi_y(h) = g(y)h$, $g(y) \in F_1(H)$. From (a′), with $y = e$, $[\phi_k(x)]^h x^{-1} \in F_1(H)$. But (c′), with $y = e$, reduces to $[\phi_k(x)]^h x^{-1} t = t[\phi_k(x)]^h x^{-1}$ for every $t \in H$. Thus, $[\phi_k(x)]^h x^{-1} = f_1(x) \in Z_1(H) \cap F_1(H)$, and we have ($c_1$) $\phi_k(x) = h^{-1}xhf_1(x)$. In (a′), let $x = \phi_{k^{-1}}(h^{-1})$. Then s reduces to $\phi_{[k,y]}(\phi_y(h^{-1})\phi_{k^{-1}}(h)) \in F_1(H)$ so that $\phi_y(h^{-1})\phi_{k^{-1}}(h) \in F_1(H)$. From ($c_1$), $\phi_k(h) = hf_1(h)$ whence $\phi_{k^{-1}}(h) = h(f_1(h))^{-1}$. Hence $\phi_y(h^{-1})h(f_1(h))^{-1} \in F_1(H)$ and (a_2)$\phi_y(h) = hg'(y)$, $g'(y) \in F_1(H)$. Thus $(h, k) \in Z_2$ leads to (a), (c_1) and (b). (c_1) implies that $x = \phi_{k^{-1}}(h^{-1})\phi_{k^{-1}}(x)\phi_k{}^{-1}(h)f_1(x) = f_1(h)h^{-1}\phi_{k^{-1}}(x)h(f_1(h))^{-1}f_1(x)$ $= h^{-1}\phi_{k^{-1}}(x)hf_1(x)$ so that $\phi_{k^{-1}}(x) = hxh^{-1}(f_1(x))^{-1}$. Hence $\phi_{yk^{-1}y^{-1}}(t)$ $= \phi_y(h)t\phi_y(h^{-1})(f_1(\phi_{y^{-1}}(t)))^{-1}$, and $\phi_{[k,y]}(t) = h^{-1}\phi_y(h)t\phi_y(h^{-1})hf_1(t)(f_1(\phi_{y^{-1}}(t)))^{-1}$, since f_1 is an endomorphism of H into $Z_1(H)$. It follows that $s = xhf_1(x)h^{-1}\phi_y(h)$ $\cdot\phi_y(h^{-1})x^{-1}\phi_y(h^{-1})hf_1(\phi_y(h^{-1}))f_1(x^{-1})f_1(\phi_{y^{-1}}(x))f_1(h) = \phi_y(h^{-1})hf_1(\phi_y(h^{-1}))$ $\cdot f_1(\phi_{y^{-1}}(x))f_1(h)$, whence $t^s = \phi_y(h^{-1})hth^{-1}\phi_y(h)$. Thus $\phi_{[k,y]}(t^s) = h^{-1}\phi_y(h)$ $\cdot\phi_y(h^{-1})hth^{-1}\phi_y(h)\phi_y(h^{-1})hf_1(t)(f_1\phi_{y^{-1}}(t))^{-1} = tf_1(t)f_1(\phi_{y^{-1}}(t^{-1})) = t$, by (c′), whence ($c_2$) follows.

Conversely, if (a), (b) and (c) hold, then (b′) is implied by (b). s reduces, by (c), to $\phi_y(h^{-1})h$ mod $(Z_1(H)\cap F_1(H))$, as we saw above. By (a), $\phi_y(h^{-1})h \in F_1(H)$ so that (a′) $s\in F_1(H)$. As for (c′), $\phi_{[k,y]}(t^s)=tf_1(t)f_1(\phi_{y^{-1}}(t^{-1}))=t$, by (c), so that (c′) $\phi_{[k,y]}\theta_s=\iota$.

Corollary 1. *$(H, K;\phi)$ is nilpotent of class 2 if and only if H and K are of class 2 and $\phi(K)\subset\mathfrak{A}(H; H/(F_1(H)\cap Z_1(H)))$.*

Proof. By the theorem, $G=(H, K;\phi)=Z_2(G)$ if and only if (a″) $h^{-1}\phi_k(h)$, $\phi_k(h)h^{-1}\in F_1(H)$ for every (h, k), (b″) every $k\in Z_2(K)$ and (c″) $\phi_k(x) =h^{-1}xhf_1(x)$ where $f_1(\phi_y(x))=f_1(x)\in Z_1(H)\cap F_1(H)$ for every y, $k\in K$. Suppose that G is of class 2. From (c″), with $h=e$, $\phi_k(x)\equiv x$ mod $(Z_1(H)\cap F_1(H))$, and the second conclusion follows. From (b″), K is of class 2. In (c″), take $k=e$ so that all $[x, h]\in Z_1(H)$, and H is of class 2. Conversely, if the conclusions hold, then $\phi_k(x)=xf=fx$ where $f\in F_1(H)$ so that (a″) holds. Since $\phi_k(x)\equiv x$ mod $Z_1(H)$, $\phi(K)\subset\mathfrak{T}_1(H)$ so that $H'\subset F_1(H)$. From the second conclusion, there exists $f(x)\in F_1(H)\cap Z_1(H)$ such that $\phi_k(x)=xf(x) =h^{-1}xh(x^{-1}x^h)f(x)$, since $H'\subset Z_1(H)$. But $x^{-1}x^h\in F_1(H)\cap Z_1(H)$ since $H'\subset F_1(H)$. Thus $\phi_k(x)=h^{-1}xhf_1(x)$ where $f_1(x)\in F_1(H)\cap Z_1(H)$. Now $h^{-1}x^{-1}h\phi_k(x)=f_1(x)$ for every $x\in H$. Replace x by $\phi_y(x)$ and apply ϕ_y^{-1} to both sides. Then $\phi_{y^{-1}}(h^{-1})x^{-1}\phi_{y^{-1}}(h)\phi_{y^{-1}ky}(x)=f_1(\phi_y(x))$. Since $\phi_y\in\mathfrak{T}_1(H)$, one can reduce the above to $h^{-1}x^{-1}h\phi_{y^{-1}ky}(x)=f_1(\phi_y(x))$. By what has already been proved, $\phi_y(x)=xa$ and $\phi_k(x)=xb$ where a, $b\in F_1(H)\cap Z_1(H)$. Hence $\phi_{y^{-1}ky}(x)=xb=\phi_k(x)$ so that $f_1(x)=h^{-1}x^{-1}h\phi_k(x)=f_1(\phi_y(x))$. This completes the proof.

We should note that $(H, K;\phi)$ of class 2 implies that $H'\subset F_1(H)$ and that $\phi(K)$ is abelian.

Corollary 2. *Necessary and sufficient that $G=(H, K;\phi)$ be nilpotent of class 2 is that H and K be of class 2 and that $\psi_1(G)$ be direct ($\phi^{(1)}=\nu$).*

10. **The derivative.** $[(a, b), (c, d)]=(a\phi_b(c)\phi_{[b,d]}[\phi_d(a^{-1})c^{-1}], [b, d]) =(\rho(a, b, c, d), [b, d])$. $\rho(a, b, c, d)$ may be called a *skew commutator*. Now $[(e, b), (e, d)]=(e, [b, d])$, so that the generators of the derivative G' are all (ρ, e) and all $(e, [b, d])$. Hence every element of G' can be expressed as (s, t) with $t\in K'$, and where s is a product of skew commutators, their inverses, and the transforms of these skew commutators and of their inverses under the automorphisms from $\phi(K')$. A trivial verification shows that the inverse of a skew commutator is a skew commutator: $[\rho(a, b, c, d)]^{-1}=\rho(a', b', c', d')$ where $a'=\phi_{[b,d]}(c)$, $b'=d^{[b,d]}$, $c'=\phi_{[b,d]}(a)$ and $d'=b^{[b,d]}$. Let us denote by H^S the subgroup of H generated by the skew commutators. $H'\subset H^S\,\Delta H$.

Theorem 17. *If $G=(H, K;\phi)$, then $G'=(H^S, K';\phi)$, and $G/G'\cong(H/H^S)\oplus(K/K')$.*

Proof. It is readily established that $\phi_k(\rho(a, b, c, d))=\rho(\phi_k(a), kb, c, d)$

$\cdot\rho(\phi_{[kb,d]}(c), d^{kb}, e, k)$ so that $\phi_k(H^S)\subset H^S$ for every $k\in K$, and the first conclusion follows from the preliminary material. Since $\phi_b(c)c^{-1}=\rho(e, b, c, e)$, $\phi(K)\subset\mathfrak{A}(H; H/H^S)$. Let X be the direct sum $(H/H^S)\oplus(K/K')$. Define a map ζ on G into X by $\zeta(h, k)=(hH^S, kK')$. It is clear that ζ is onto X. Since $\phi(K)\subset\mathfrak{A}(H; H/H^S)$, $\zeta(h_1\phi_{k_1}(h_2), k_1k_2) = (h_1\phi_{k_1}(h_2)H^S, k_1k_2K') = (h_1h_2H^S, k_1k_2K')=(h_1H^S, k_1K')(h_2H^S, k_2K')$, so that $\zeta\in\text{Hom }(G, X)$. Finally, $(h, k)\in\text{kern }\zeta$ if and only if $h\in H^S$ and $k\in K'$; that is, if and only if $(h, k)\in G'$.

Observe that for $[H\Delta H]$ we have $H^S=H'$, so that $D^{(j)}[H\Delta H] =[D^{(j)}(H)\Delta D^{(j)}(H)]$, $j=1, 2, \cdots$.

11. **Normalizers and centralizers.** Recall that $B_1(G)$ is the set of all (h, k) with $\theta_h\phi_k=\iota$. Let $B_{11}=B_{11}(G)$ and $B_{12}=B_{12}(G)$ be the respective projections on H and on K of B_1 via the mappings $(h, k)\rightarrow h$ and $(h, k)\rightarrow k$. The sets B_{1i} are subgroups. We shall make such abbreviations as(3) $\mathfrak{C}(H^*, G)=\mathfrak{C}(H, G)$. Direct calculations establish that $\mathfrak{C}(H, G)=B_1$, that $\mathfrak{C}^{(2)}(H, G)$ is the set of all (h, k) with $\phi_k\in\mathfrak{A}(H; B_{11})$ and $k\in\mathfrak{C}(B_{12}, K)$, that $\mathfrak{C}(K, G)\cong F_1(H)\oplus Z_1(K)$, and that $\mathfrak{C}^{(2)}(K, G)$ is the set of all $(h, k)\in G$ for which $\phi_y(h)=h$ for every $y\in Z_1(K)$ and $h\in\mathfrak{C}(F_1(H), H)$. We have, at once, the result of Jordan [9, p. 51] that $\mathfrak{C}(H, \mathfrak{H}(H))$ is the set of all $(h, \theta_{h^{-1}})\in\mathfrak{H}(H)$. $G=\mathfrak{C}(H, G)$ if and only if G reduces to $H\oplus K$ with H abelian. $G=\mathfrak{C}^{(2)}(H, G)$ if and only if $B_{11}\subset F_1(H)$ and $B_{12}\subset Z_1(K)$. $G=\mathfrak{C}(K, G)$ if and only if G reduces to $H\oplus K$ with K abelian. $G=\mathfrak{C}^{(2)}(K, G)$ if and only if $F_1(H)\subset Z_1(H)$ and $Z_1(K)\subset\mathfrak{k}$. If H is abelian, then $\mathfrak{C}(H, G)\cong H\oplus\mathfrak{k}$ so that $\mathfrak{C}$ and $\mathfrak{k}$ are each nilpotent of the same class or both are non-nilpotent. If H is abelian, $\mathfrak{C}^{(2)}(H, G)=H\oplus Z_1(\mathfrak{k})$, so that $\mathfrak{C}^{(2)}=Z_1(\mathfrak{C}^{(1)})$. If H is abelian, and if H is maximal with respect to the property $H\Delta G$, then G is direct, or ϕ is an isomorphism and $G=(H, \phi(K); \iota)$, a relative holomorph. One can show that $G=\mathfrak{C}^{(2)}(H, G)$ *if and only if* $\mathfrak{C}(H, G)\subset\mathfrak{C}(K, G)$, *while* $G=\mathfrak{C}^{(2)}(K, G)$ *if and only if* $\mathfrak{C}(K, G)\subset Z_1(H)\oplus\mathfrak{k}(=\mathfrak{C}(H, G)\cap(H\oplus\mathfrak{k}))$.

Let us define a sequence of sets $\{E_n(H)\}$ by $E_1(H)=F_1(H), \cdots, E_{n+1}(H)$ $=$all h such that (a) $h\in\mathfrak{N}(E_n(H), H)$ and (b) $h\phi_y(h^{-1})\in E_n(H)$ for every $y\in K$. An inductive proof shows that the $E_n(H)$ form, under set inclusion, an ascending sequence of subgroups of H, each of which is ϕ_k-admissible for every $k\in K$. $E_n(H)\Delta E_{n+1}(H)$, $n=1, 2, \cdots$. Let $E_0(H)=(e)$. $E_2(H)$ *is the set of all* $h\in H$ *such that* $h^{-1}\phi_y(h)\in Z_1(F_1(H))$ *for every* $y\in K$.

Lemma 15. $\mathfrak{N}^{(j)}(K, G)=(E_j(H), K; \phi)$ *so that* (a) $G=\mathfrak{N}^{(j)}(K, G)$ *if and only if* $H=E_j(H)$; (b) $G=\mathfrak{N}^{(2)}(K, G)$ *if and only if* $\phi(K)\subset\mathfrak{A}(H; H \bmod Z_1\cdot(F_1(H)))$; (c) *if* $(H, K; \phi)$ *is of class c, then there exists* $j\leqq c$ *with* $H=E_j(H)$; (d) *if* ϕ *is ample and if* $G=\mathfrak{N}^{(2)}(K, G)$, *then* H *is of class* 2.

Proof. (d). By (b), $h^{-1}\phi_y(h)\in Z_1(F_1(H))=F_1(H)$. But each $\theta_x=$some ϕ_y, whence $h^{-1}h^x\in F_1(H)$ for every x and $h\in H$. Thus $H'\subset F_1(H)\subset Z_1(H)$.

Note that ϕ ample or $\phi(K)\subset\mathfrak{T}_1(H)$ implies that $E_n(H)\Delta H$ for all natural

n. Under such normality, the subgroups R of H such that $E_{n+1} \supset R \supset E_n$ are characterized by the property $\phi(K) \subset \mathfrak{A}(H; R/E_n)$.

If $A, B \Delta H$ with $A \subset B$, then(3) $(e) \div_B A = \mathfrak{C}(A, B)$. Let $\mathfrak{C}_1(A, B) = \mathfrak{C}(A, B)$. If $\mathfrak{C}_n(A, B)$ is defined as a subgroup of B, let $\mathfrak{C}_{n+1}(A, B) = \mathfrak{C}_n(A, B) \div_B A$. $\mathfrak{C}_n(A, B)$ is a subgroup of B normal in H. If $B \Delta H$, $B \subset V \Delta H$, then $\mathfrak{C}_n(A, B) \subset \mathfrak{C}_n(A, V)$. In particular, $Z_n(A) = \mathfrak{C}_n(A, A) \subset \mathfrak{C}_n(A, B)$ for every $A, B \Delta H$, $A \subset B$. $\mathfrak{C}_n(A, B) = (e)(\div)^n_B A$.

THEOREM 18. *If $G = (H, K; \phi)$ where each $E_n(H) \Delta H$, then for each n, $E_{n+1}(H)$ is the set of all $h \in H$ for which $h^{-1}\phi_y(h) \in \mathfrak{C}_n(F_1(H), E_n(H))$ for every $y \in K$.*

Proof. We saw above that $E_2(H)$ is the set of all h such that all $t = h^{-1}\phi_y(h) \in Z_1(F_1(H)) = \mathfrak{C}_1(F_1(H), E_1(H))$. Now suppose that the theorem is valid for the index n. Since $E_n(H) \Delta H$, $y \in K$, $h \in H$ and $x \in E_n(H)$ imply the existence of $z_1, z_2 \in \mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H))$ such that $\phi_y(x^h) = x^h z_1 = \phi_y(h) x z_2 \phi_y(h^{-1})$. Hence (A) $x^{-1}(h^{-1}\phi_y(h)) x z_2 (h^{-1}\phi_y(h))^{-1} = h^{-1} z_1 h$. $\mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H)) \Delta H$ so that (A) implies that $[x^{-1}, t] \in \mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H))$. That is, $t \in \mathfrak{C}_{n-1}(F_1(H), E_{n-1}(H)) \div_H E_n(H)$ where $t = h^{-1}\phi_y(h)$, $h \in E_{n+1}(H)$. $t \in E_n(H)$ so that $\div_H$ can be replaced by $\div_{E_n(H)}$. Since $E_{n-1}(H)$ and $F_1(H)$ are both included in $E_n(H)$, and since $f(A, B) = A \div B$ is monotonic up in A and down in B, $h \in E_{n+1}(H)$ implies that $t \in \mathfrak{C}_{n-1}(F_1(H), E_n(H)) \div_{E_n(H)} F_1(H)$. By the definition of the $\mathfrak{C}_n$ it follows that each $t \in \mathfrak{C}_n(F_1(H), E_n(H))$. The converse is immediate.

LEMMA 16. $\mathfrak{N}(K, G)/\mathfrak{C}(K, G) \cong \mathfrak{J}(K)$.

The specific homomorphism on $\mathfrak{N}$ to $\mathfrak{J}$ is $\gamma(f, k) = \theta_k$ for $f \in F_1(H)$. This quotient group is independent of H. Cf. [9, p. 47]. Note that the normalizer and centralizer of K coincide if and only if K is abelian.

Let R be a ϕ_k-admissible subgroup of H for every $k \in K$. $W = (R, K; \phi)$ is a subgroup of $G = (H, K; \phi)$. $(a, b) \in \mathfrak{C}(W, G)$ where $a \in H$, $b \in K$, if and only if $a\phi_b(r) = r\phi_k(a)$ and $b \in Z_1(K)$ for every $r \in R$ and for every $k \in K$. Choosing $r = e$, we have $a = \phi_k(a)$, so that $a \in F_1(H)$. Then $a\phi_b(r) = ra$ so that on R, $\phi_b\theta_a = \iota$. Conversely, $a \in F_1(H)$, $b \in Z_1(K)$ and $\phi_b\theta_a = \iota$ on R imply $(a, b) \in \mathfrak{C}(W, G)$. Likewise, $(a, b) \in \mathfrak{C}(W, G)$ implies that $a \in \mathfrak{N}(R, H)$. $\mathfrak{C}(W, G)$ has the "least possible value" $Z_1(G)$ if and only if R has the property (P) if $\alpha \in \mathfrak{A}(R)$ can be extended to $\phi_b \in \phi(K)$, where $b \in Z_1(K)$, and if α can be extended to $\theta_{a^{-1}} \in \mathfrak{J}(H)$, where $a \in F_1(H)$, then $\phi_b\theta_a = \iota$ over all of H.

Let $\mathfrak{C}\mathfrak{N}(K, G)$ be an abbreviation for $\mathfrak{C}(\mathfrak{N}(K, G), G)$.

THEOREM 19. *If $G = (H, K; \phi)$, and if $\mathfrak{C}(F_1(H), H) \subset F_1(H)$, then $\mathfrak{N}^{(2)}(K, G)/\mathfrak{C}\mathfrak{N}(K, G) \cong (\mathfrak{N}(F_1(H), H)/\mathfrak{C}(F_1(H), H)) \oplus \mathfrak{J}(K)$.*

Proof. Let $R = F_1(H)$. Suppose that $\phi_b\theta_a = \iota$ on $F_1(H)$ where $b \in Z_1(K)$ and $a \in F_1(H)$. $\phi_b = \iota$ on $F_1(H)$. Hence $a \in Z_1(F_1(H))$. Conversely, if $a \in Z_1(F_1(H))$

and if $b \in Z_1(K)$, then $f \in F_1(H)$ and $k \in K$ imply that $(a, b)(f, k) = (af, bk) = (fa, kb) = (f, k)(a, b)$ so that $(a, b) \in \mathfrak{C}\mathfrak{N}(K, G)$. We have proved that $\mathfrak{C}\mathfrak{N}(K, G) = (Z_1(F_1(H)), Z_1(K); \phi)(\cong Z_1(F_1(H)) \oplus Z_1(K))$. Let us define a mapping γ on $\mathfrak{N}^{(2)}(K, G)$ into $(\mathfrak{N}(F_1(H), H)/\mathfrak{C}(F_1(H), H)) \oplus \mathfrak{Z}(K)$ by $\gamma(h, k) = (\theta^*, \theta_k)$ where θ_h^* is θ_h restricted to $F_1(H)$. Here $h \in E_2(H)$ and $k \in K$. For $h_1, h_2 \in E_2(H)$, $k_1, k_2 \in K$, $h_1\phi_{k_1}(h_2) = h_1h_2z$, where $z \in Z_1(F_1(H))$. Now $\theta_z^* = \iota$ on $F_1(H)$ so that $\theta_{h_1h_2z}^* = \theta_{h_1}^*\theta_{h_2}^*$, and γ is a homomorphism. Since $\mathfrak{C}(F_1(H), H) \subset F_1(H)$ by hypothesis, it is clear that kern $\gamma = (Z_1(F_1(H)), Z_1(K); \phi) = \mathfrak{C}\mathfrak{N}(K, G)$. Hence $\mathfrak{N}^{(2)}/\mathfrak{C}\mathfrak{N} \cong T \oplus \mathfrak{Z}(K)$ where $T \subset \mathfrak{N}(F_1(H), H)/\mathfrak{C}(F_1(H), H)$. $h \in \mathfrak{N}(F_1(H), H)$ if and only if $\phi_y(h) = hc$, $c \in \mathfrak{C}(F_1(H), H)$. Suppose that $h \in \mathfrak{N}(F_1(H), H)$ and that $k \in K$. Then $\phi_y(h) = hc(y)$, $c(y) \in \mathfrak{C}(F_1(H), H) \subset F_1(H)$. Hence $h \in E_2(H)$, and $\gamma(h, k) = (\theta_h^*, \theta_k)$ so that γ is onto the required direct sum.

For a subgroup K of a group G, let $\mathfrak{H}_i(K, G) = \mathfrak{H}_i(K)$, *the inner normal hull of* K *in* G, be the largest subgroup of K which is normal in G. Let $\mathfrak{H}_0(K, G) = \mathfrak{H}_0(K)$, *the outer normal hull of* K *in* G, be the smallest normal subgroup of G in which K is included. $\mathfrak{H}_i = \mathfrak{H}_0$ if and only if $K \Delta G$, whence $\mathfrak{H}_i = \mathfrak{H}_0 = K$. As generators, $\mathfrak{H}_0(K)$ has all $(e, k)^{(x,y)} = (x\phi_{k^y}(x^{-1}), k^y)$; that is, all $(x\phi_y(x^{-1}), y)$. In $[H \Delta K]$ or $[H \supset K]$, the generators of $\mathfrak{H}_0(K)$ are all $([h, k], k)$ where $h \in H$, $k \in K$. *The inner normal hull of* K *in* $(H, K; \phi)$ *is* $\mathfrak{k}$. If a group K can be represented faithfully as a group of automorphisms of a group H, then one can minimize the inner normal hull by forming the relative holomorph $(H, K; \iota)$ of K over H. Here $\mathfrak{H}_i(K) = \mathfrak{k} = (e)$. We write, "by abuse of language," $\mathfrak{C}(\mathfrak{k}, G)$ rather than $\mathfrak{C}(\mathfrak{k}^*, G)$. (x, y) is in the latter if and only if $y \in \mathfrak{C}(\mathfrak{k}, K)$ as a short argument shows, and $\mathfrak{C}(\mathfrak{k}, G) = (H, \mathfrak{C}(\mathfrak{k}, K); \phi)$. If, as in $[K \Delta K]$, $\mathfrak{k} \subset Z_1(K)$, then $\mathfrak{C}(\mathfrak{k}, G) = G$ so that $\mathfrak{k} \subset Z_1(G)$.

Let K now be a group in which the ascending central series breaks off for some index n such that $Z_n(K) \neq K$. Then $K/Z_n(K)$ is a centerless group and is thus representable faithfully as a subgroup of its group of automorphisms. Form the product $G = (K/Z_n(K), K; \phi)$ where ϕ is the natural mapping on K onto $K/Z_n(K)$. Let a', for $a \in K$, be the coset $aZ_n(K)$. For $a, b, c \in K$, $(e', c)^{(a',b)} = (a'b'c'b'^{-1}a'^{-1}(b'c'b'^{-1})^{-1}, c^b) = ([ac^ba^{-1}(c^b)^{-1}]', c^b) = ([a, c^b]', c^b)$ so that $\mathfrak{H}_0(K)$ has for a complete set of generators all $([h, k]', k)$. Since $K \subset \mathfrak{H}_0(K)$, all $([h, k]', e)$ are included among the generators. Recall that groups K for which $K = D(K)$ are called *perfect* [9]. If we take G as indicated, we have proved

THEOREM 20. *Let* K *be a group for which the ascending central series terminates with* $Z_n(K) \neq K$ *and for which* $K/Z_n(K)$ *is perfect. Then there exists a proper extension* G *of* K *such that the outer normal hull of* K *in* G *is* G.

Since any homomorphic image of a perfect group is perfect, we can have K rather than $K/Z_n(K)$ perfect in the hypothesis.

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