AN ASYMPTOTIC FORMULA IN THE THEORY OF NUMBERS

BY

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Introduction. Ramanujan(1) conjectured that if α and β are fixed positive numbers

$$\sum_{\nu=1}^{n-1} \sigma_\alpha(\nu)\sigma_\beta(n-\nu) - \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{\zeta(\alpha+1)\zeta(\beta+1)}{\zeta(\alpha+\beta+2)} \sigma_{\alpha+\beta+1} = o(\sigma_{\alpha+\beta+1}(n))$$

where $\sigma_\alpha(n)$ is the sum of the αth powers of the divisors of n.

In 1927 Ingham(2) published a similar result about the sum

$$\sum_{\nu=1}^{n} d(\nu)d(n-\nu)$$

where $d(n)$ is the number of divisors of n, and pointed out that his elementary method would serve also to confirm Ramanujan's conjecture. The proof of this was carried out in detail by the author of the present note(3), who proved, in addition, that if the left-hand side of Ramanujan's formula is denoted by $E(n)$,

$$E(n) = \begin{cases} O(n^\omega \sigma_{-1}(n) \log n) & \textit{when } \alpha = \beta < 1 \\ O(n^\omega \log^c n) & \textit{otherwise}; \end{cases}$$

here $\omega = \alpha+\beta+1-\min(\alpha, \beta, 1)$ *and* $c=0$ *when* $\min(\alpha, \beta) > 1$; *when* $\min(\alpha, \beta) \leqq 1$, $c=0$ *if the three numbers* α, β, 1 *are different*, $c=1$ *if two of these numbers, but not all three, are equal, and* $c=2$ *if* $\alpha=\beta=1$.

If $\min(\alpha, \beta) \geqq 1$, there is a saving of 1 in the exponent of the error term, and it appears difficult to improve this result. If, however, $\min(\alpha, \beta) < 1$, and the added restriction

$$\alpha + \beta < 1 \tag{1}$$

is imposed, it is possible to adapt an analytic argument which Estermann(4) used to sharpen the above-mentioned result of Ingham, to proving the following:

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(1) Collected papers, p. 137.

(2) J. London Math. Soc. vol. 12 (1927) pp. 202–208.

(3) See Halberstam [1].

(4) See Estermann [2].

THEOREM.

$$\sum_{1}^{n-1} \sigma_\alpha(\nu)\sigma_\beta(n-\nu) = A_1\sigma_{\alpha+\beta+1}(n) + A_2 n^\alpha \sigma_{-\alpha+\beta+1}(n) + A_3 n^\beta \sigma_{\alpha-\beta+1}(n)$$
$$+ A_4 n^{\alpha+\beta}\sigma_{-\alpha-\beta+1}(n) + O(n^{\omega_1}(\log n)^{3+\kappa})$$

where A_1, A_2, A_3 *and* A_4 *are as defined in* (61), *and* $\omega_1 = 3/4 + (\alpha+\beta)/2$ *if* $\alpha+\beta < 1/2$, $\omega_1 = 1/2 + \alpha + \beta$ *if* $\alpha+\beta \geqq 1/2$, *and* $\kappa = 1$ *if* $\alpha+\beta = 1/2$ *and* 0 *otherwise.*

In the case $\alpha+\beta \geqq 1/2$ it is possible that the error term may absorb part of the approximating function.

The author wishes to express his indebtedness to Dr. Estermann for having suggested this investigation.

Notation. k, l, m, r, u are positive integers. τ is a complex number. θ, ϑ are real numbers. λ is a positive number less than 1.

Wherever the O-notation is used, the relations are uniform with respect to all variables except λ, α and β.

ξ_ν stands for $e^{2\pi i/\nu}$.

Throughout a and q are positive integers such that $a < q$, $(a, q) = 1$; these restrictions on a are implied automatically whenever a is the variable of summation.

Preliminary results. We define, for $\mathfrak{I}(\tau) > 0$,

$$f_\lambda(\tau) = \sum_{m=1}^{\infty} \sigma_\lambda(m) e^{2\pi i m\tau}.$$

Then, provided that $\mathfrak{I}(\tau) > 0$,

$$g(\tau) = f_\alpha(\tau) f_\beta(\tau) = \sum_{m=1}^{\infty} s(m) e^{2\pi i m\tau}$$

where

$$s(m) = \sum_{k,l,r,u;\, kl+ru=m} k^\alpha r^\beta = \sum_{\nu=1}^{m-1} \sigma_\alpha(\nu)\sigma_\beta(m-\nu).$$

Hence

$$s(n) = \int g(\tau) e^{-2\pi i n\tau} d\tau \tag{2}$$

with the integration taken over the line segment $(i/n, i/n+1)$. Writing $\tau = \theta + i/n$, (2) becomes

$$s(n) = e^{2\pi} \int_0^1 g(\theta + i/n) e^{-2\pi i n\theta} d\theta. \tag{3}$$

We dissect (0, 1) into Farey arcs of order $n^{1/2}$, and denote the typical arc by $M_{a,q}$. It is well known that if θ is in $M_{a,q}$, we can write $\theta = a/q + \vartheta$, where

$$-B_1 q^{-1} n^{-1/2} \leqq \vartheta \leqq B_2 q^{-1} n^{-1/2}$$

with $1/2 \leqq B_1 \leqq 1$, $1/2 \leqq B_2 \leqq 1$. Then, by (3),

$$s(n) = e^{2\pi} \sum_{q \leqq n^{1/2}} \int_{-q^{-1}n^{-1/2}}^{q^{-1}n^{-1/2}} \left\{ \sum_a \phi \xi_q^{-na} g(a/q + \eta) \right\} e^{-2\pi i n \vartheta} d\vartheta$$

where

$$\eta = i/n + \vartheta, \tag{4}$$

and $\phi = \phi(n, a, q, \vartheta) = 1$ or 0 according as $a/q + \vartheta$ does, or does not, belong to $M_{a,q}$.

We define

$$h_{n,q}(\eta) = \sum_a \phi \xi_q^{-na} g(a/q + \eta), \tag{5}$$

$$H_{n,q} = \int_{-q^{-1}n^{-1/2}}^{q^{-1}n^{-1/2}} h_{n,q}(\eta) e^{-2\pi i n \vartheta} d\vartheta. \tag{6}$$

Then

$$s(n) = e^{2\pi} \sum_{q \leqq n^{1/2}} H_{n,q}. \tag{7}$$

Our first objective is to find an approximating function for $f_\lambda(a/q + \eta)$ and to express the resulting error term in suitable form (see (39)).

By Mellin's transformation formula

$$f_\lambda(a/q + \eta) = \sum_{m=1}^{\infty} \sigma_\lambda(m) \xi_q^{am} e^{2\pi i m \eta} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s) z^{-s} F_\lambda(s) ds \tag{8}$$

where

$$z = -2\pi i \eta \tag{9}$$

and

$$F_\lambda(s) = \sum_{m=1}^{\infty} \sigma_\lambda(m) m^{-s} \xi_q^{ma}. \tag{10}$$

The functional equation for $F^i(s)$. We have that

$$\begin{aligned} F_\lambda(s) = F_\lambda(s; \xi_q^a) &= \sum_{k,l} k^\lambda \xi_q^{akl} k^{-s} l^{-s} = \sum_{b=1}^{q} \sum_{k,l;\, l \equiv b \pmod q} \xi_q^{akl} k^{\lambda - s} l^{-s} \\ &= \sum_{b=1}^{q} \sum_{k=1}^{\infty} \xi_q^{abk} k^{\lambda - s} \sum_{l=1;\, l \equiv b \pmod q}^{\infty} l^{-s} = \sum_{b=1}^{q} \zeta(s - \lambda; \xi_q^{ab}) \zeta(s; b, q) \end{aligned} \tag{11}$$

where

$$\zeta(s;\, \xi_q^b) = \sum_{k=1}^{\infty} \xi_q^{bk} k^{-s} \tag{12}$$

and

$$\zeta(s;\, b,\, q) = \sum_{k=1;\, k\equiv b \pmod{q}}^{\infty} k^{-s}. \tag{13}$$

We put $G(s) = -i(2\pi)^{s-1}\Gamma(1-s)$. Then

$$\zeta(s;\, b,\, q) = G(s)q^{-s}(e^{i\pi s/2}\zeta(1-s;\, \xi_q^b) - e^{-i\pi s/2}\zeta(1-s;\, \xi_q^{-b})) \tag{14}$$

and

$$\zeta(s;\, \xi_q^b) = G(s)q^{1-s}(e^{i\pi s/2}\zeta(1-s;\, -b,\, q) - e^{-i\pi s/2}\zeta(1-s;\, b,\, q)). \tag{15}$$

Equation (14) is proved in the same way as the functional equation for Riemann's ζ-function, and (15) follows from (14). We apply (14) and (15) to (11) and obtain

$$\begin{aligned} F_\lambda(s) = {} & G(s)G(s-\lambda)q^{1+\lambda-2s}\sum_{b=1}^{q}(e^{i\pi(s-\lambda)/2}\zeta(1+\lambda-s;\, -ab,\, q) \\ & - e^{-i\pi(s-\lambda)/2}\zeta(1+\lambda-s;\, ab,\, q))(e^{i\pi s/2}\zeta(1-s;\, \xi_q^b) - e^{-i\pi s/2}\zeta(1-s;\, \xi_q^{-b})) \\ = {} & G(s)G(s-\lambda)q^{1+\lambda-2s}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4) \end{aligned}$$

where

$$\begin{aligned} \Sigma_1 &= e^{i\pi(2s-\lambda)/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, -ab,\, q)\zeta(1-s;\, \xi_q^b), \\ \Sigma_2 &= e^{-i\pi(2s-\lambda)/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, ab,\, q)\zeta(1-s;\, \xi_q^{-b}) \\ &= e^{-i\pi(2s-\lambda)/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, -ab,\, q)\zeta(1-s;\, \xi_q^b), \\ \Sigma_3 &= -\, e^{i\pi\lambda/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, ab,\, q)\zeta(1-s;\, \xi_q^b), \\ \Sigma_4 &= -\, e^{-i\pi\lambda/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, -ab,\, q)\zeta(1-s;\, \xi_q^{-b}) \\ &= -\, e^{-i\pi\lambda/2}\sum_{b=1}^{q}\zeta(1+\lambda-s;\, ab,\, q)\zeta(1-s;\, \xi_q^b). \end{aligned}$$

Thus

$$F_\lambda(s) = 2G(s)G(s-\lambda)q^{1+\lambda-2s} \tag{16}$$
$$\cdot\left\{\cos\left(s-\frac{1}{2}\lambda\right)\pi\sum_{b=1}^{q}\zeta(1+\lambda-s;\,-ab,\,q)\zeta(1-s;\,\xi_q^b)\right.$$
$$\left.-\cos\frac{1}{2}\lambda\pi\sum_{b=1}^{q}\zeta(1+\lambda-s;\,ab,\,q)\zeta(1-s;\,\xi_q^b)\right\}.$$

We now define a' by

$$aa' \equiv 1 \pmod q, \qquad 0 < a' \leqq q;$$

then if r is the least positive residue mod q of $-ab$, we have $ra' \equiv -baa' \equiv -b \pmod q$, and this together with (11) enables us to rewrite (16) as

$$F_\lambda(s;\,\xi_q^a) = 2G(s)G(s-\lambda)q^{1+\lambda-2s}\left\{\cos\left(s-\frac{1}{2}\lambda\right)\pi\cdot F_\lambda(1+\lambda-s;\,\xi_q^{-a'})\right. \tag{17}$$
$$\left.-\cos\frac{1}{2}\lambda\pi\cdot F_\lambda(1+\lambda-s;\,\xi_q^{a'})\right\}$$

the required functional equation for $F_\lambda(s)$.

The poles of $F_\lambda(s)$ at $s=1$ and $s=1+\lambda$. It is easily seen that

$$\zeta(s;\,q,\,q) = q^{-s}\zeta(s), \qquad \sum_{b=1}^{q}\zeta(s;\,\xi_q^{ab}) = q^{1-s}\zeta(s).$$

Further, by (11),

$$F_\lambda(s) - q^{-s}\zeta(s)\sum_{b=1}^{q}\zeta(s-\lambda;\,\xi_q^{ab}) = \sum_{b=1}^{q}\zeta(s-\lambda;\,\xi_q^{ab})\left\{\zeta(s;\,b,\,q) - q^{-s}\zeta(s)\right\},$$

and, by the above, since the right-hand side is a regular function of s in the neighborhood of $s=1$, the residues of $F_\lambda(s)$ and $q^{1+\lambda-2s}\zeta(s)\zeta(s-\lambda)$ at $s=1$ are the same, namely

$$q^{\lambda-1}\zeta(1-\lambda). \tag{18}$$

It is clear from (11) that we can also write

$$F_\lambda(s) = \sum_{b=1}^{q}\zeta(s-\lambda;\,b,\,q)\zeta(s;\,\xi_q^{ab}).$$

This, combined with the above method, enables us to prove in exactly the same way that the residue of $F_\lambda(s)$ at $s=1+\lambda$ is

$$q^{-\lambda-1}\zeta(1+\lambda). \tag{19}$$

Let R_λ be the sum of residues of $\Gamma(s)z^{-s}F_\lambda(s)$ at $s=1$, $s=1+\lambda$. It follows from (18) and (19) that

$$R_\lambda = \Gamma(1+\lambda)\,\zeta(1+\lambda)(qz)^{-1-\lambda} + \zeta(1-\lambda)q^\lambda(qz)^{-1} \tag{20}$$

and from (8) that

$$f_\lambda(a/q+\eta) - R_\lambda = F_\lambda(0) + \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s)z^{-s}F_\lambda(s)ds. \tag{21}$$

We proceed to evaluate the two expressions on the right of (21).

Calculation of $F_\lambda(0)$. We rewrite (17) in the form

$$\begin{aligned}\frac{F_\lambda(s;\overset{a}{\xi_q})}{2G(s)G(s-\lambda)q^{1+\lambda-2s}} &= \left(\cos\left(s-\frac{1}{2}\lambda\right)\pi - \cos\frac{1}{2}\lambda\pi\right)F_\lambda(1-s+\lambda;\overset{-a'}{\xi_q}) \\ &\quad + \cos\frac{1}{2}\lambda\pi\big[F_\lambda(1-s+\lambda;\overset{-a'}{\xi_q}) \\ &\quad - F_\lambda(1-s+\lambda;\overset{a'}{\xi_q})\big].\end{aligned} \tag{22}$$

We observe that $G(0)G(-\lambda) = -\Gamma(1+\lambda)/(2\pi)^{2+\lambda}$, and that, by (19),

$$\begin{aligned}&\lim_{s\to 0}\left\{\left(\cos\left(s-\frac{1}{2}\lambda\right)\pi - \cos\frac{1}{2}\lambda\pi\right)F_\lambda(1-s+\lambda;\overset{-a'}{\xi_q})\right\} \\ &\qquad = \lim_{s\to 0}\left(s\pi\sin\frac{1}{2}\lambda\pi + \cdots\right)F_\lambda(1-s+\lambda;\overset{-a'}{\xi_q}) \\ &\qquad = -\pi\sin\frac{1}{2}\lambda\pi\cdot q^{-1-\lambda}\zeta(1+\lambda).\end{aligned} \tag{23}$$

This disposes of the first term on the right of (22). As for the second, we have, by (11), that

$$\begin{aligned}&\lim_{s\to 0}\left(F_\lambda(1-s+\lambda;\overset{-a'}{\xi_q}) - F_\lambda(1-s+\lambda;\overset{a'}{\xi_q})\right) \\ &\qquad = \lim_{s\to 1}\left(F_\lambda(s+\lambda;\overset{-a'}{\xi_q}) - F_\lambda(s+\lambda;\overset{a'}{\xi_q})\right) \\ &\qquad = \lim_{s\to 1}\sum_{b=1}^{q}\zeta(s+\lambda;b,q)\left(\zeta(s;\overset{-a'b}{\xi_q}) - \zeta(s;\overset{a'b}{\xi_q})\right) \\ &\qquad = \sum_{b=1}^{q}\zeta(1+\lambda;b,q)\left(\zeta(1;\overset{-a'b}{\xi_q}) - \zeta(1;\overset{a'b}{\xi_q})\right) \\ &\qquad = \sum_{b=1}^{q-1}E_b\left(\zeta(1;\overset{-a'b}{\xi_q}) - \zeta(1;\overset{a'b}{\xi_q})\right)\end{aligned} \tag{24}$$

where, for $0<b<q$,

$$\begin{aligned}E_b = \zeta(1+\lambda;b,q) &= \sum_{r=0}^{\infty}(b+rq)^{-1-\lambda} < \sum_{r=0}^{\infty}b^{-1-\lambda}(r+1)^{-1-\lambda} \\ &= b^{-1-\lambda}\zeta(1+\lambda).\end{aligned} \tag{25}$$

Now if $r_q(m)$ is the least non-negative residue mod q of m,

$$\zeta(1;\xi_q^{-a'b}) - \zeta(1;\xi_q^{a'b}) = \log\frac{1-\xi_q^{a'b}}{1-\xi_q^{-a'b}} = \log(-\xi_q^{a'b}) = i\left(-\pi + \frac{2\pi r_q(a'b)}{q}\right)$$
$$= 2\pi i\left(r_q(a'b) - \frac{1}{2}q\right)\Big/ q,$$

so that the right-hand side of (24) can be written

$$\frac{2\pi i}{q}\sum_{b=1}^{q-1}\left(r_q(a'b) - \frac{1}{2}q\right)E_b.$$

Moreover, for any integer m

$$r_q(m) - \frac{1}{2}q = \sum_{\nu=1}^{q} c_\nu \xi_q^{\nu m}$$

where

$$\text{(26)} \qquad c_q = -\frac{1}{2}, \qquad c_\nu = (\xi_q^{-\nu} - 1)^{-1} \qquad (0 < \nu < q);$$

for

$$c_\nu = q^{-1}\sum_{l=1}^{q}\left\{r_q(l) - \frac{1}{2}q\right\}\xi_q^{-l\nu} = -\frac{1}{2} + q^{-1}\sum_{l=1}^{q-1}\left(l - \frac{1}{2}q\right)\xi_q^{-l\nu}$$
$$= -\frac{1}{2} + q^{-1}\sum_{l=1}^{q-1} l\xi_q^{-l\nu} - \frac{1}{2}\sum_{l=1}^{q-1}\xi_q^{-l\nu}$$
$$= -\frac{1}{2} + q^{-1}\frac{(q-1)q}{2} - \frac{1}{2}(q-1) = -\frac{1}{2} \qquad (\nu = q),$$
$$= -\frac{1}{2} + (\xi_q^{-\nu} - 1)^{-1} + \frac{1}{2} = (\xi_q^{-\nu} - 1)^{-1} \qquad (0 < \nu < q).$$

Thus the right-hand side of (24) becomes

$$\text{(27)} \quad \frac{2\pi i}{q}\sum_{1\leqq b\leqq q-1;1\leqq\nu\leqq q} c_\nu E_b \xi_q^{a'b\nu} = \frac{2\pi i}{q}\sum_{l=1}^{q}\xi_q^{a'l}\left(\sum_{1\leqq b\leqq q-1;1\leqq\nu\leqq q;\, b\nu\equiv l \pmod q} c_\nu E_b\right).$$

Finally,

$$\sum_{\nu=1}^{q}|c_\nu| \leqq \frac{1}{2} + \sum_{1\leqq\nu\leqq q/2}\frac{1}{\sin \pi\nu/q} < \frac{1}{2} + \frac{1}{2}q\sum_{1\leqq\nu\leqq q/2}\frac{1}{\nu} < q\log(2q),$$

so that, by (25),

$$\left| \sum_{1 \leqq b \leqq q-1; 1 \leqq \nu \leqq q} c_\nu E_b \right| \leqq \left(\sum_{\nu=1}^{q} |c_\nu| \right)\left(\sum_{b=1}^{q-1} E_b \right) < \zeta^2(1+\lambda) q \log(2q). \tag{28}$$

If now we combine equations (22), (23), (24) and (27), we obtain

$$\begin{aligned} F_\lambda(0) = & - \Gamma(1+\lambda)(2\pi)^{-2-\lambda} \left\{ -2\pi \sin \frac{1}{2} \pi\lambda \cdot \zeta(1+\lambda) \right. \\ & \left. + 4\pi i q^\lambda \cos \frac{1}{2} \lambda\pi \sum_{l=1}^{\lambda} \left(\sum_{1 \leqq b \leqq q-1; 1 \leqq \nu \leqq q; b\nu \equiv l \pmod q} c_\nu E_b \right) \xi_q^{a'l} \right\} \\ = & \sum_{l=1}^{q} d_l \xi_q^{a'l}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} d_l = & \frac{-2i\Gamma(1+\lambda)\cos \lambda\pi/2}{(2\pi)^{1+\lambda}} q^\lambda \sum_{1 \leqq b \leqq q-1; 1 \leqq \nu \leqq q; b\nu \equiv l \pmod q} c_\nu E_b \quad (1 \leqq l \leqq q-1) \\ = & \frac{\Gamma(1+\lambda)\sin \lambda\pi/2}{(2\pi)^{1+\lambda}} - \frac{2i\Gamma(1+\lambda)\cos \lambda\pi/2}{(2\pi)^{1+\lambda}} \sum_{1 \leqq b \leqq q-1; 1 \leqq \nu \leqq q; b\nu \equiv l \pmod q} c_\nu E_b \\ & \qquad (l = q), \end{aligned} \tag{30}$$

so that, by (28),

$$\sum_{l=1}^{q} |d_l| = O(q^{1+\lambda} \log (2q)). \tag{31}$$

Calculation of the integral in (21). We denote the integral under consideration by J_λ. By (17), (10) and the definition of $G(s)$,

$$\begin{aligned} J_\lambda = & \int_{-1/2-i\infty}^{-1/2+i\infty} 2G(s)G(s-\lambda) q^{1+\lambda-2s}\Gamma(s) z^{-s} \left\{ \cos\left(s - \frac{1}{2}\lambda\right)\pi \right. \\ & \left. \cdot F_\lambda(1-s+\lambda; \xi_q^{-a'}) - \cos \frac{1}{2} \lambda\pi F_\lambda(1-s+\lambda; \xi_q^{a'}) \right\} ds \\ = & \sum_{m=1}^{\infty} \sigma_\lambda(m) \xi_q^{-a'm} e_m - \sum_{m=1}^{\infty} \sigma_\lambda(m) \xi_q^{a'm} \epsilon_m, \end{aligned} \tag{32}$$

where

$$e_m = -\left(\frac{q}{2\pi m}\right)^{1+\lambda} \int_{-1/2-i\infty}^{-1/2+i\infty} \left(\frac{4\pi^2 m}{q^2 z}\right)^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} \cos\left(s - \frac{1}{2}\lambda\right)\pi \, ds \tag{33}$$

and

$$\epsilon_m = -\left(\frac{q}{2\pi m}\right)^{1+\lambda} \cos \frac{1}{2} \lambda\pi \int_{-1/2-i\infty}^{-1/2+i\infty} \left(\frac{4\pi^2 m}{q^2 z}\right)^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} ds. \tag{34}$$

LEMMA 1. *If* $\Re(\delta)>0$, *then*

$$\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} ds = O(|\delta|^{-1/2}),$$

and

$$\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} \cos\left(s-\frac{1}{2}\lambda\right)\pi \cdot ds = O(|\delta|^{-1/2}+|\delta|^{1+\lambda}e^{-\Re\delta}).$$

Proof. Of these two results the first is trivial. To prove the second, we assume first of all that $\mathfrak{I}(\delta)\leqq 0$. By expanding $\cos(s-\lambda/2)\pi$ it is easy to show that the second integral is equal to

$$\cos\frac{1}{2}\lambda\pi\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} e^{i\pi s}ds - ie^{i\lambda\pi/2}\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s\Gamma(1+\lambda-s)ds.$$

It is clear that

$$\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s \frac{\Gamma(1+\lambda-s)}{\sin \pi s} e^{i\pi s}ds = O(|\delta|^{-1/2}),$$

and

$$\int_{-1/2-i\infty}^{-1/2+i\infty} \delta^s\Gamma(1+\lambda-s)ds = \int_{3/2+\lambda-i\infty}^{3/2+\lambda+i\infty} \Gamma(w)\delta^{1+\lambda-w}dw = 2\pi i\delta^{1+\lambda}e^{-\lambda}.$$

Hence the result if $\mathfrak{I}(\delta)\leqq 0$; the proof for $\mathfrak{I}(\delta)>0$ is similar.

We apply Lemma 1 with $\delta=4\pi^2m/(q^2z)$ to e_m and ϵ_m. We note that by (4), (9) and because $|\vartheta|\leqq q^{-1}n^{-1/2}$, $q\leqq n^{1/2}$,

$$\Re(\delta) = \Re\left(\frac{4\pi^2m}{q^2z}\right) = \frac{8\pi^3m}{q^2|z|^2n} = \frac{2\pi m}{q^2n(\vartheta^2+n^{-2})} \geqq \pi m > 0,$$

whence

$$e^{-\Re\delta} = \exp\left(-\Re\frac{4\pi^2m}{q^2z}\right) \leqq 2\left(\Re\frac{4\pi^2m}{q^2z}\right)^{-2} < \frac{2}{\pi m}\left(\Re\frac{4\pi^2m}{q^2z}\right)^{-1} = \frac{q^2|z|^2n}{4\pi^4m^2}.$$

It follows from Lemma 1 together with these two inequalities, and equations (33) and (34), that

$$|e_m| + |\epsilon_m| = O(q^{2+\lambda}|z|^{1/2}m^{-(3/2+\lambda)} + nq^{1-\lambda}|z|^{1-\lambda}m^{-2}) = O(n(q|z|)^{1-\lambda}m^{-t})$$

where $t=\min(2, 3/2+\lambda)$. Hence

$$(35)\quad \sum_{m=1}^{\infty}\sigma_\lambda(m)\{|e_m|+|\epsilon_m|\} = O\left(n(q|z|)^{1-\lambda}\sum_{m=1}^{\infty}\sigma_\lambda(m)m^{-t}\right) = O(n(q|z|)^{1-\lambda}).$$

If now we put

(36) $$d_l^* = \sum_{m=1;m\equiv -l \pmod q}^{\infty} \sigma_\lambda(m) e_m - \sum_{m=1;m\equiv l \pmod q}^{\infty} \sigma_\lambda(m) \epsilon_m,$$

it follows from (32) that

(37) $$J_\lambda = \sum_{l=1}^{q} d_l^* \xi_q^{la'}$$

where, by (35) and (36),

(38) $$\sum_{l=1}^{q} |d_l^*| = O(n(q|z|)^{1-\lambda}).$$

In view of equations (29) and (37) we are now able to rewrite (21) as

(39) $$f_\lambda(a/q + \eta) - R_\lambda = \sum_{l=1}^{q} j_l \xi_q^{la'}$$

where

(40) $$j_l = d_l + \frac{1}{2\pi i} d_l^* \qquad (1 \leqq l \leqq q),$$

so that, by (31) and (38),

(41) $$\sum_{l=1}^{q} |j_l| = O(q^{1+\lambda} \log (2q) + n(q|z|)^{1-\lambda}) = O(n(q|z|)^{1-\lambda} \log (2q)).$$

Calculation of $g(a/q+\eta) - R_\alpha R_\beta$. We note that $g - R_\alpha R_\beta = f_\alpha f_\beta - R_\alpha R_\beta = (f_\alpha - R_\alpha)R_\beta + (f_\beta - R_\beta)f_\alpha$; hence, by (39) and (20),

(42) $$g(a|q + \eta) - R_\alpha R_\beta = \sum_{l=1}^{q} k_l \xi_q^{a'l}$$

where $k_1, k_2, \cdots, k_q$ are numbers, independent of a, such that, by (20) and (41),

(43) $$\begin{aligned}\sum_{l=1}^{q} |k_l| &= O\,(n^2(q|z|)^{2-\alpha-\beta} \log^2 (2q) + n(q|z|)^{-\alpha-\beta} \log (2q)) \\ &= O\,(n^{1+\alpha+\beta} q^{-(\alpha+\beta)} \log^2 n).\end{aligned}$$

The error term. We put

(44) $$T = R_\alpha R_\beta \sum_a \phi \xi_q^{-na}$$

and

(45) $$V = R_\alpha R_\beta \sum_a \xi_q^{-na} = c_q(n) R_\alpha R_\beta$$

where $c_q(n)$ is Ramanujan's function. We define

(46) $$t_{n,q} = \int_{-q^{-1}n^{-1/2}}^{q^{-1}n^{-1/2}} Te^{-2\pi i n\vartheta} d\vartheta$$

and

(47) $$v_{n,q} = \int_{-\infty}^{\infty} Ve^{-2\pi i n} \, d\vartheta.$$

We have by (5), (42) and (44) that

(48) $$\begin{aligned} h_{n,q}(\eta) - T &= \sum_a \phi\{g(a|q+\eta) - R_\alpha R_\beta\}\xi_q^{-na} = \sum_a \phi \sum_{l=1}^{q} k_l \xi_q^{la'-na} \\ &= \sum_{l=1}^{q} k_l \sum_a \phi \xi_q^{la'-na} \end{aligned}$$

It has been shown by Estermann(5) that an estimate for the inner sum on the right can be derived from an estimate of Kloosterman's sum. Moreover, a "best possible" estimate of $O(q^{1/2})$ for Kloosterman's sum was obtained by A. Weil(6) some years ago. If then we combine Estermann's method with Weil's result we find that

(49) $$\sum_a \phi\xi_q^{la'-na} = O(q^{1/2} \log (4q))$$

and

(50) $$\sum_a (\phi - 1)\xi_q^{-na} = O(q^{1/2} \log (4q)).$$

It follows from (43), (48) and (49) that

(51) $$h_{n,q}(\eta) - T = O(n^{1+\alpha+\beta} q^{1/2-\alpha-\beta} \log^3 n),$$

and from (20), (44), (45) and (50) that

$$T - V = R_\alpha R_\beta \sum_a (\phi - 1)\xi_q^{-na} = O((q|z|)^{-2-\alpha-\beta} q^{1/2} \log n).$$

Now it is clear from our definition of ϕ that $T = V$ for $|\vartheta| \leqq q^{-1}n^{-1/2}/2$ so that $T - V$ can be nonzero only for $|\vartheta| > q^{-1}n^{-1/2}/2$. Since, by (4) and (9), $|z| > 2\pi|\vartheta|$, we have that

(52) $$T - V = O((q|\vartheta|)^{-2-\alpha-\beta} q^{1/2} \log n), \qquad \left(|\vartheta| > \frac{1}{2} q^{-1} n^{-1/2}\right).$$

(5) See Estermann [3].
(6) See Weil [4]. The so-called Kloosterman sum is $\sum_a \xi_q^{la+ma'}$.

By (6), (46) and (51)

$$H_{n,q} - t_{n,q} = O(n^{1/2+\alpha+\beta} q^{-1/2-\alpha-\beta} \log^3 n). \tag{53}$$

Furthermore, our definition of $\phi(n, a, q, \vartheta)$ implies that

$$t_{n,q} = \int_{-\infty}^{\infty} T e^{-2\pi i n \vartheta} d\vartheta.$$

Hence, by (47) and (52),

$$\begin{aligned} t_{n,q} - v_{n,q} &= O\left(\int_{q^{-1}n^{-1/2}/2}^{\infty} (q\vartheta)^{-2-\alpha-\beta} q^{1/2} \log n \cdot d\vartheta \right) \\ &= O(n^{(1+\alpha+\beta)/2} q^{-1/2} \log n) = O(n^{1/2+\alpha+\beta} q^{-(\alpha+\beta+1/2)} \log n). \end{aligned} \tag{54}$$

It follows from (7), (53) and (54) that

$$\begin{aligned} s(n) - e^{2\pi} \sum_{q \leqq n^{1/2}} v_{n,q} &= O\left(n^{1/2+\alpha+\beta} \log^3 n \sum_{q \leqq n^{1/2}} q^{-1/2-\alpha-\beta} \right) \\ &= O(n^{\omega_1} (\log n)^{3+\kappa}) \end{aligned} \tag{55}$$

where $\omega_1 = 1/2 + \alpha + \beta$ if $\alpha + \beta \geqq 1/2$, $\omega_1 = 1/4 + (1 + \alpha + \beta)/2$ if $\alpha + \beta \leqq 1/2$, and $\kappa = 1$ if $\alpha + \beta = 1/2$ and 0 otherwise.

Finally, it is well known that

$$c_q(n) = \sum_{d \mid (q,n)} \mu\left(\frac{q}{d}\right) d,$$

so that by (47), (45), (20), (9) and (4)

$$\begin{aligned} \sum_{q > n^{1/2}} v_{n,q} &= \sum_{q > n^{1/2}} c_q(n) \int_{-\infty}^{\infty} R_\alpha R_\beta e^{-2\pi i n \vartheta} d\vartheta = O\left(n^{1+\alpha+\beta} \sum_{q > n^{1/2}} \mid c_q(n) \mid q^{-2-\alpha-\beta} \right) \\ &= O\left(n^{1+\alpha+\beta} \sum_{q > n^{1/2}} q^{-2-\alpha-\beta} \sum_{d \mid (q,n)} d \right) \\ &= O\left(n^{1+\alpha+\beta} \sum_{d \mid n} d \sum_{q > n^{1/2}, d \mid q} q^{-2-\alpha-\beta} \right) \\ &= O\left(n^{1+\alpha+\beta} \sum_{d \mid n} d^{-1-\alpha-\beta} \sum_{m > n^{1/2}/d} m^{-2-\alpha-\beta} \right) \\ &= O\left(n^{(1+\alpha+\beta)/2} \sum_{d \mid n} 1 \right) = O(n^{\omega_1}). \end{aligned}$$

Hence, by (55),

$$s(n) - e^{2\pi} \sum_{q=1}^{\infty} v_{n,q} = O(n^{\omega_1}(\log n)^{3+\kappa}). \tag{56}$$

The dominant term. By (20)

$$R_\alpha R_\beta = a_1(qz)^{-2-\alpha-\beta} + a_2 q^\alpha (qz)^{-2-\beta} + a_3 q^\beta (qz)^{-2-\alpha} + a_4 q^{\alpha+\beta}(qz)^{-2} \tag{57}$$

where

$$\begin{aligned} a_1 &= \Gamma(1+\alpha)\Gamma(1+\beta)\zeta(1+\alpha)\zeta(1+\beta), & a_2 &= \Gamma(1+\beta)\zeta(1+\beta)\zeta(1-\alpha) \\ a_3 &= \Gamma(1+\alpha)\zeta(1+\alpha)\zeta(1-\beta), & a_4 &= \zeta(1-\alpha)\zeta(1-\beta). \end{aligned} \tag{58}$$

LEMMA 2. *If z is as defined by* (9) *and* (4), *and if $k>1$, then*

$$\int_{-\infty}^{\infty} z^{-k} e^{-2\pi i n\vartheta} d\vartheta = e^{-2\pi} n^{k-1}/\Gamma(k).$$

Proof. The integral on the left is equal to

$$\int_{-\infty}^{\infty} \left(\frac{2\pi}{n} - 2\pi i\vartheta\right)^{-k} e^{-2\pi i n\vartheta} d\vartheta = n^k \int_{-\infty}^{\infty} (2\pi - 2\pi i n\vartheta)^{-k} e^{-2\pi} e^{2\pi - 2\pi i n\vartheta} d\vartheta$$
$$= \frac{e^{-2\pi} n^{k-1}}{2\pi i} \int_{2\pi - i\infty}^{2\pi + i\infty} e^w w^{-k} dw = \frac{e^{-2\pi} n^{k-1}}{\Gamma(k)}.$$

In view of (45), (47) and (57) $v_{n,q}$ involves four integrals of the type considered in Lemma 2, and an application of the lemma gives

$$e^{2\pi} v_{n,q} = c_q(n) \left\{ \frac{a_1}{\Gamma(2+\alpha+\beta)} \frac{n^{1+\alpha+\beta}}{q^{2+\alpha+\beta}} + \frac{a_2}{\Gamma(2+\beta)} \frac{n^{1+\beta}}{q^{2+\beta-\alpha}} + \frac{a_3}{\Gamma(2+\alpha)} \frac{n^{1+\alpha}}{q^{2+\alpha-\beta}} + a_4 \frac{n}{q^{2-\alpha-\beta}} \right\}. \tag{59}$$

LEMMA 3.

$$\sum_{q=1}^{\infty} c_q(n) q^{-k} = \frac{1}{\zeta(k)} \sigma_{1-k}(n), \qquad (k > 1).$$

Proof. The sum on the left is equal to

$$\sum_{q=1}^{\infty} \sum_{d|n, d|q} \mu\left(\frac{q}{d}\right) d q^{-k} = \sum_{d|n} d \sum_{q=1; d|q}^{\infty} \mu\left(\frac{q}{d}\right) q^{-k} = \sum_{d|n} d^{1-k} \sum_{m=1}^{\infty} \mu(m) m^{-k}$$
$$= \frac{\sigma_{1-k}(n)}{\zeta(k)}.$$

It follows from (56), (59) and Lemma 3 that

$$\begin{aligned} s(n) &= A_1 n^{1+\alpha+\beta}\sigma_{-1-\alpha-\beta}(n) + A_2 n^{1+\beta}\sigma_{-1-\beta+\alpha}(n) + A_3 n^{1+\alpha}\sigma_{-1-\alpha+\beta}(n) \\ &\quad + A_4 n\sigma_{-1+\alpha+\beta}(n) + O(n^{\omega_1}(\log n)^{3+\kappa}), \end{aligned} \tag{60}$$

where, by (58), (59) and Lemma 3,

$$\begin{aligned} A_1 &= \frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(2+\alpha+\beta)} \frac{\zeta(1+\alpha)\zeta(1+\beta)}{\zeta(2+\alpha+\beta)}, \quad & A_2 &= \frac{1}{1+\beta} \frac{\zeta(1+\beta)\zeta(1-\alpha)}{\zeta(2+\beta-\alpha)}, \\ A_3 &= \frac{1}{1+\alpha} \frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)}, & A_4 &= \frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)}. \end{aligned} \tag{61}$$

Since $n^k\sigma_{-k}(n) = \sigma_k(n)$, (60) implies our theorem.

References

1. H. Halberstam, *Four asymptotic formulae in the theory of numbers*, J. London Math. Soc. vol. 24 (1949) pp. 13–21.

2. T. Estermann, *On the representations of a number as a sum of two products*, Proc. London Math. Soc. 2 vol. 31 (1929) pp. 123–133.

3. ———, *Vereinfachter Beweis eines Satzes von Kloosterman*, Abh. Math. Sem. Hansischen Univ. vol. 7 (1929) pp. 83–98.

4. A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 204–207.

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