

ON SOME SIMPLE GROUPS DEFINED BY C. CHEVALLEY

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Introduction. For any semi-simple Lie algebra \mathfrak{g} over the field of complex numbers and an arbitrary field K , C. Chevalley [2] defined a group \mathfrak{G} by a uniform method. The group \mathfrak{G} turns out to be simple when \mathfrak{g} is simple (except for a few exceptional cases), and yields some new classes of simple groups not contained in the theory of classical groups, if, in particular, \mathfrak{g} is one of the exceptional simple Lie algebras.

In this paper first we consider the case where \mathfrak{g} is one of the classical algebras A_n, B_n, C_n, D_n , and identify the group \mathfrak{G} with classical groups, thus answering a question raised in the last section of [2]. Our results are as follows:

(a) if \mathfrak{g} is of the type A_n , then \mathfrak{G} is the special projective group $\text{PSL}(n+1, K)$;

(b) if \mathfrak{g} is of the type C_n , then \mathfrak{G} is the quotient group of the symplectic group $\text{Sp}(2n, K)$ over its center;

(c) if \mathfrak{g} is of the type D_n , then \mathfrak{G} is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=1}^n \xi_i \xi_{-i}$ ($3 \leq n$);

(d) if \mathfrak{g} is of the type B_n and if K is not of characteristic 2 then \mathfrak{G} is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=0}^n \xi_i \xi_{-i}$ ($2 \leq n$); if \mathfrak{g} is of type B_n and if K is of characteristic 2 then \mathfrak{G} is a subgroup of $\text{Sp}(2n, K)$; if, moreover, K is perfect⁽¹⁾ then $G = \text{Sp}(2n, K)$. The author has been unable to identify \mathfrak{G} in case K is not perfect and of characteristic 2.

Secondly, we consider the case where \mathfrak{g} is the exceptional algebra G_2 , and identify the group \mathfrak{G} with the groups defined by L. E. Dickson [4; 5].

1. The group defined by Chevalley. Let \mathfrak{g} be a semi-simple Lie algebra over the field of complex numbers and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} . Denote by (X, Y) the Killing form of \mathfrak{g} . For a root⁽²⁾ r of \mathfrak{g} , we set $H_r = 2(r, r)^{-1}r$, which is called the co-weight (co-poids) attached to r . It is known that the additive group \mathfrak{H} generated by all co-weights H_r is a free abelian group of rank n , where n is the dimension of \mathfrak{h} . Chevalley proves the existence of a system $\{X_r\}$ of root vectors satisfying the following conditions (1.1)–(1.2):

$$(1.1) \quad [X_r, X_{-r}] = H_r, \quad \text{for all roots } r;$$

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(1) A field K of characteristic $p > 0$ is called *perfect* if every element in K is a p th power of an element in K .

(2) Roots of a semi-simple algebra over the field of complex numbers can be identified with elements in the Cartan subalgebra by which the roots are defined. See [7].

(1.2) if r, s , and $r+s$ are roots then $[X_r, X_s] = N_{r,s}X_{r+s}$ with $N_{r,s} = \pm(p+1)$, where p is the greatest integer $i \geq 0$ such that $s - ir$ is a root.

If a system $\{X_r\}$ of root vectors satisfies (1.1) and (1.3) below then it also satisfies (1.2) (see [2, pp. 22–23]).

(1.3) $N_{-r,-s} = N_{r,s}$ for any two roots r, s such that $r+s$ is a root.

Let now $\{\tilde{H}_1, \dots, \tilde{H}_n\}$ be a basis of the group \mathcal{H} and $\{X_r\}$ a system of root vectors satisfying (1.1)–(1.2). Then the set $\{\tilde{H}_1, \dots, \tilde{H}_n\}$ together with $\{X_r\}$ forms a basis $\{\tilde{H}_i, X_r\}$ of \mathfrak{g} . It is easily seen that the coefficients appearing in the multiplication table of the basis $\{\tilde{H}_i, X_r\}$ are all integers. Chevalley proves that the coefficients of the matrix $A_r(t)$ representing the automorphism

$$x_r(t) = \exp t(\text{ad } X_r),$$

where t is a complex variable, of \mathfrak{g} with respect to the basis $\{\tilde{H}_i, X_r\}$ are all polynomials in t with integral coefficients. Therefore for any field K we can define a Lie algebra \mathfrak{g}_K over K by using the basis $\{\tilde{H}_i, X_r\}$, and also automorphisms $x_r^*(t)$, where $t \in K$, of \mathfrak{g}_K by using the matrix $A_r(t)$. Then the group considered by Chevalley is the group $\mathfrak{G}(\mathfrak{g}; K)$ (denoted by the symbol \mathfrak{G}' in [2]) generated by all $x_r^*(t)$, where r runs over all roots of \mathfrak{g} while t takes on all elements in K . This group was proved to be simple except for a few exceptional cases.

Since every \tilde{H}_i is a linear combination of the co-weights H_r with integral coefficients, it follows that an element in $\mathfrak{G}(\mathfrak{g}; K)$ is uniquely determined by its effect on H_r^* and X_r^* , where H_r^* and X_r^* are elements in \mathfrak{g}_K corresponding to H_r and X_r in \mathfrak{g} , respectively. In particular an element in \mathfrak{G} is the identity if and only if it leaves every H_r^* and every X_r^* invariant.

2. The classical algebras. For the classical algebras A_n, B_n, C_n, D_n the co-weights H_r and the system $\{X_r\}$ of root vectors satisfying (1.1) and (1.3) can be computed easily (cf. [7]).

The algebra A_n is the algebra of all $(n+1) \times (n+1)$ matrices of trace zero. Denote by E_{ij} the $(n+1) \times (n+1)$ matrix whose (i, j) -entry is 1 and whose other entries are all zero. Then

$$(2.1) \quad \{H_r\} = \{E_{ii} - E_{jj} \mid i \neq j\}, \quad \{X_r\} = \{E_{ij} \mid i \neq j\},$$

and if $H_r = E_{ii} - E_{jj}$ then $X_r = E_{ij}$, $X_{-r} = E_{ji}$. The mapping ϕ which maps a matrix T to $-T'$, where T' is the transpose of T , is an automorphism of A_n . We have $\phi(E_{ij}) = -E_{ji}$, and hence it is easily seen that the system $\{X_r\}$ satisfies the condition (1.3).

The algebra D_n is the algebra of all $2n \times 2n$ matrices X which satisfy $f_D(X\xi, \eta) + f_D(\xi, X\eta) = 0$ identically, where

$$f_D(\xi, \eta) = \xi_{-n}\eta_n + \dots + \xi_{-1}\eta_1 + \xi_1\eta_{-1} + \dots + \xi_n\eta_{-n}.$$

Denote by E_{ij} the $2n \times 2n$ matrix such that $E_{ij}\xi = \xi'_i$, where $\xi'_i = \xi_j$, $\xi'_k = 0$ for $k \neq i$, and set

$$F_{ij} = E_{-ij} - E_{-ji}, \quad H_{ij} = F_{i,-i} + F_{j,-j}.$$

Then we have

$$(2.2) \quad \begin{aligned} \{H_r\} &= \{H_{ij} \mid i < j, i \neq -j\}, \\ \{X_r\} &= \{F_{ij} \mid i < j, i \neq -j\}, \end{aligned}$$

where if $H_r = H_{ij}$ then $X_r = F_{ij}$, $X_{-r} = F_{-j,-i}$. The mapping $\phi(T) = -T'$ is also an automorphism of D_n , and hence the system $\{X_r\}$ satisfies (1.3).

The algebra B_n may be regarded as the algebra of all $(2n+1) \times (2n+1)$ matrices X satisfying $f_B(X\xi, \eta) + f_B(\xi, X\eta) = 0$ identically, where

$$f_B(\xi, \eta) = 2\xi_0\eta_0 + \sum_{i=1}^n (\xi_{-i}\eta_i + \xi_i\eta_{-i}).$$

It is readily seen that a matrix $X = (x_{ij})$ belongs to B_n if and only if

$$x_{-ij} + x_{-ji} = 0, \quad x_{-i0} + 2x_{0i} = 0, \quad x_{00} = 0$$

for all $i \neq 0$ and $j \neq 0$. Define the matrices E_{ij} as in the case of algebra D_n , and set

$$\begin{aligned} F_{ij} &= E_{-ij} - E_{-ji}, & F_{i0} &= 2E_{-i0} - E_{0i}, & F_{0j} &= -F_{j0}, \\ H_{ij} &= F_{i,-i} + F_{j,-j}, & H_{i0} &= 2F_{i,-i}, & H_{0j} &= 2F_{j,-j} \end{aligned}$$

for $i \neq 0, j \neq 0$. Then we see easily that

$$(2.3) \quad \{H_r\} = \{H_{ij} \mid i < j, i \neq -j\}$$

is the set of all co-weights (with respect to a certain Cartan subalgebra) and that

$$(2.4) \quad \{X_r\} = \{F_{ij} \mid i < j, i \neq -j\}$$

is a system of root vectors satisfying (1.1), where, $X_r = F_{ij}$ and $X_{-r} = F_{-j,-i}$ if $H_r = H_{ij}$. This system also satisfies (1.3). This is seen as follows: Let $P = 2E_{00} + \sum_{i=1}^n (E_{ii} + E_{-i,-i})$. Then the mapping $\phi(X) = -P^{-1}X'P$ is an automorphism of B_n such that $\phi(F_{ij}) = -F_{-j,-i}$, that is, $\phi(X_r) = -X_{-r}$ for all roots r , and hence (1.3) follows immediately.

The algebra C_n is the algebra of all $2n \times 2n$ matrices X which satisfy $f_C(X\xi, \eta) + f_C(\xi, X\eta) = 0$ identically, where

$$f_C(\xi, \eta) = \xi_{-n}\eta_n + \cdots + \xi_{-1}\eta_1 - \xi_1\eta_{-1} - \cdots - \xi_n\eta_{-n}.$$

Define E_{ij} as in the case of the algebra D_n and set

$$(2.5) \quad F_{ij} = \begin{cases} E_{-ij} + \text{sign}(ij)E_{-ji}, & \text{if } i \neq j, \\ E_{-ii} & \text{if } i = j, \end{cases}$$

where $\text{sign } k$ denotes $+1$ if $k > 0$, and -1 if $k < 0$, and

$$H_{ij} = \begin{cases} F_{i,-i} + F_{j,-j}, & \text{if } i \neq j, \\ F_{i,-i} & \text{if } i = j. \end{cases}$$

Then we have

$$(2.6) \quad \begin{aligned} \{H_r\} &= \{H_{ij} \mid i \leq j, i + j \neq 0\}, \\ \{X_r\} &= \{F_{ij} \mid i \leq j, i + j \neq 0\}, \end{aligned}$$

and if $H_r = H_{ij}$ then $X_r = F_{ij}$, $X_{-r} = F_{-j,-i}$. The mapping $\phi: X \rightarrow -X'$ is an automorphism of C_n such that $\phi(X_r) = -X_{-r}$ for all roots. Thus (1.3) is also satisfied by the system $\{X_r\}$.

3. The group \bar{G} (\mathfrak{g} , K). For any two matrices X , Y with complex coefficients we have the formula

$$(3.1) \quad \exp(\text{ad } X) \cdot Y = (\exp X)Y(\exp(-X)),$$

where $(\text{ad } X) \cdot Y = XY - YX$. This can be verified easily if we use the identity

$$\frac{(\text{ad } X)^k}{k!} \cdot Y = \sum_{i+j=k} \frac{X^i}{i!} Y \frac{(-X)^j}{j!}.$$

For the matrix algebras $\mathfrak{g} = A_n, C_n, D_n$, every root vector $X_r = F_{ij}$ (we set $F_{ij} = E_{ij}$, $i \neq j$, for the algebra A_n) given in the preceding section satisfies $X_r^2 = 0$, and hence

$$(3.2) \quad \exp(tX_r) = I + tF_{ij},$$

where I is the unit matrix. For the algebra $\mathfrak{g} = B_n$, every root vector $X_r = F_{ij}$ such that $i \neq 0, j \neq 0$ satisfies $X_r^2 = 0$, and hence we have (3.2) for this X_r also. For the root vector $X_r = F_{i0}$, of the algebra B_n we have $X_r^2 = -2E_{-ii}$, $X_r^3 = 0$. Therefore

$$(3.3) \quad \exp(tX_r) = I + t(2E_{-i0} - E_{0i}) - t^2E_{-ii},$$

for $i = \pm 1, \pm 2, \dots, \pm n$. From (3.2) and (3.3) we see that $\exp(tX_r)$ can always be represented as a matrix whose entries are polynomials in t with integral coefficients. Now, the matrix $A_r(t)$ representing the automorphism $x_r(t)$ of \mathfrak{g} with respect to the basis $\{\tilde{H}_i, X_r\}$ can be obtained from the formula

$$(3.4) \quad x_r(t) \cdot Y = (\exp(tX_r))Y(\exp(-tX_r)), \quad Y \in \mathfrak{g},$$

which follows from (3.1), and the expression for $\exp(tX_r)$ given in (3.2)–(3.3).

Let K be an arbitrary field and \mathfrak{g} one of the algebras A_n, B_n, C_n, D_n . In the preceding section all co-weights H_r and root vectors X_r of \mathfrak{g} are represented by matrices with integral coefficients, and consequently, every element in the basis $\{\tilde{H}_i, X_r\}$ is also represented as a matrix with integral coefficients.

Therefore we see that \mathfrak{g}_K is represented in this way as a matrix algebra over K . For $t \in K$, define the matrix $\exp(tX_r^*)$, by the right hand side of (3.3) if X_r is one of the root vectors F_{i0}, F_{0j} (in case $\mathfrak{g} = B_n$), and by that of (3.2) if X_r is not, and consider the automorphism $x_r^*(t)$ defined by $A_r(t)$. Then (3.4) yields

$$(3.5) \quad x_r^*(t) \cdot Y^* = (\exp(tX_r^*))Y^*(\exp(-tX_r^*))$$

for any Y^* in \mathfrak{g}_K . Denote by $\overline{\mathfrak{U}}(\mathfrak{g}, K)$ the multiplicative group of matrices generated by all $\exp(tX_r^*)$, where r runs over all roots of \mathfrak{g} while t runs over all elements in K . For any $S \in \overline{\mathfrak{U}}$ define the mapping S^σ of \mathfrak{g}_K into itself by $S^\sigma \cdot Y^* = SY^*S^{-1}$, where $Y^* \in \mathfrak{g}_K$. Then (3.5) shows that S^σ is a product of certain automorphisms of \mathfrak{g}_K of the form $x_r^*(t)$, and hence belongs to the group $\mathfrak{U}(\mathfrak{g}, K)$ defined in §1. The mapping $\sigma: S \rightarrow S^\sigma$ is clearly a homomorphism of $\overline{\mathfrak{U}}$ onto \mathfrak{U} . Denote by \mathfrak{Z} the kernel of the homomorphism σ . By the remark given at the end of §1, we see that an element S in $\overline{\mathfrak{U}}$ belongs to \mathfrak{Z} if and only if S commutes with all X_r^* and H_r^* . If $S \in \mathfrak{Z}$ commutes with all X_r^* then S belongs to the center of $\overline{\mathfrak{U}}$, for $\overline{\mathfrak{U}}$ is generated by elements of the form $\exp(tX_r^*)$. Conversely, suppose that S belongs to the center of $\overline{\mathfrak{U}}$. Then S commutes with all $\exp(tX_r^*)$. If \mathfrak{g} is one of the algebras A_n, C_n, D_n , then $\exp(tX_r^*) = I + tX_r^*$. Hence S commutes with all X_r^* . Since $[X_r^*, X_{-r}^*] = H_r^*$, we see that S also commutes with all H_r^* . Therefore S belongs to \mathfrak{Z} . Thus we have proved that \mathfrak{Z} coincides with the center of $\overline{\mathfrak{U}}(\mathfrak{g}; K)$ if \mathfrak{g} is one of the algebras A_n, C_n, D_n . Consider now the case $\mathfrak{g} = B_n$. If S belongs to the center of $\overline{\mathfrak{U}}(B_n, K)$ then S commutes with $\exp X_r^*$, and hence with $(\exp X_r^* - I)^2 = (X_r^*)^2$. Therefore, if K is of characteristic $\neq 2$, then S commutes with all X_r^* and hence belongs to \mathfrak{Z} , as before. If K is of characteristic 2, then from (3.3) we see that S commutes with all $E_{0i} + E_{-ii}, i = \pm 1, \pm 2, \dots, \pm n$. Then it follows easily that S is of the form aI , with $a \in K$. Therefore S commutes with all X_r^* and H_r^* , and consequently it belongs to \mathfrak{Z} . Thus we have proved

$$(3.6) \quad \mathfrak{U}(\mathfrak{g}, K) \cong \overline{\mathfrak{U}}(\mathfrak{g}, K)/\mathfrak{Z},$$

where \mathfrak{Z} is the center of $\overline{\mathfrak{U}}(\mathfrak{g}, K)$.

4. The group $\mathfrak{U}(A_n, K)$. It is well known that the matrices of the form $I + tE_{ij}$, where $t \in K, i \neq j$, generate the special linear group $\text{SL}(n+1, K)$. Therefore $\overline{\mathfrak{U}}(A_n, K) = \text{SL}(n+1, K)$. By (3.6) it follows that $\mathfrak{U}(A_n, K)$ is isomorphic to the special projective group $\text{PSL}(n+1, K)$.

5. The group $\mathfrak{U}(C_n, K)$. We show that the group $\mathfrak{U}(C_n, K)$ coincides with the symplectic group $\text{Sp}(2n, K)$ (in its matrix representation). It is known [6] that the group $\text{Sp}(2n, K)$ is generated by the matrices

$$I + tE_{-ii}, I + t(E_{-ij} - E_{-ji}),$$

and

$$S_i = I - E_{ii} - E_{-i, -i} + E_{-ii} - E_{i, -i},$$

where $t \in K$, $i, j = 1, 2, \dots, n$, except when $n = 1$ and K is of order 2 or 3. In view of (2.5)–(2.6), the first two sets of matrices belong to $\overline{\mathfrak{G}}(C_n, K)$. In order to obtain the equality $\overline{\mathfrak{G}} = \text{Sp}(2n, K)$, it thus suffices to show that the matrices S_i also belong to $\overline{\mathfrak{G}}$. As is mentioned in the preceding section, however, the group $\text{SL}(2, K)$ is generated by matrices of the form $I + tE_{ij}$, where $t \in K$, $i \neq j$. Therefore S_i can be written as a product of matrices of the forms $I + tE_{-ii}$ and $I + tE_{i,-i}$, which belong to $\overline{\mathfrak{G}}$ by definition. Except when $n = 1$ and K is of order 2 or 3, therefore we obtain $\overline{\mathfrak{G}} = \text{Sp}(2n, K)$. Hence, by (3.6), $\mathfrak{G}(C_n, K)$ is isomorphic to the quotient of the symplectic group $\text{Sp}(2n, K)$ over its center. It should be noted that $\text{Sp}(2n, K)$ is the group of all $2n \times 2n$ matrices X which satisfy

$$f_c(X\xi, X\eta) = f_c(\xi, \eta) \text{ identically.}$$

6. The group $\mathfrak{G}(D_n, K)$. First we shall show that the group $\overline{\mathfrak{G}}(D_n, K)$ contains the commutator subgroup \mathfrak{D}' of the orthogonal group \mathfrak{D} consisting of all $2n \times 2n$ matrices which leave the quadratic form $\sum_{i=1}^n \xi_i \xi_{-i}$ invariant. By (2.2), the group $\overline{\mathfrak{G}}$ is generated by the matrices

$$W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i}),$$

where $t \in K$ and $i, j = \pm 1, \dots, \pm n$; $i \neq \pm j$. Denote by P_{ij} , Q_i and $R_{i,t}$ the matrices corresponding to the permutations $(\xi_i \xi_j)(\xi_{-i} \xi_{-j})$, $(\xi_i \xi_{-i})$ and the transformations $\xi'_i = t\xi_i$, $\xi'_{-i} = t^{-1}\xi_{-i}$, $\xi'_k = \xi_k$ ($k \neq i, -i$), respectively. It is known [3] that the group \mathfrak{D} is generated by the matrices $W_{i,j,t}$, P_{ij} , Q_i , and $R_{i,t}$. Therefore, $\mathfrak{D}' \leq \overline{\mathfrak{G}}$ is proved if we can show that for any distinct $|i|, |j|$, $|k|$ the matrices $(P_{ik}P_{ij})^2$, Q_iQ_j , $R_{i,t}R_{j,t}^{-1}$, and $R_{i,t}^2$ are in $\overline{\mathfrak{G}}$. By elementary computations we have

$$\begin{aligned} (P_{ik}P_{ij})^2 &= W_{j,-k,1}W_{-j,k,1}W_{j,-k,1}W_{i,-j,1}W_{-i,j,1}W_{i,-j,1}, \\ (6.1) \quad Q_iQ_j &= W_{j,-i,1}W_{-j,i,1}W_{j,-i,1}W_{i,j,1}W_{-i,-j,1}W_{i,j,1}, \\ R_{i,t}R_{j,t}^{-1} &= W_{-i,j,t-1}W_{-j,i,1}W_{-i,j,s}W_{-j,i,-t}, \\ R_{i,t}R_{j,t} &= W_{-i,-j,t-1}W_{j,i,1}W_{-i,-j,s}W_{j,i,-t}, \end{aligned}$$

where $s = t^{-1} - 1$. Thus the matrices $(P_{ik}P_{ij})^2$, Q_iQ_j , $R_{i,t}R_{j,t}^{-1}$ are shown to be in $\overline{\mathfrak{G}}$. Since $R_{i,t}$ and $R_{j,t}$ commute, we have $R_{i,t}^2 = (R_{i,t}R_{j,t}^{-1})(R_{i,t}R_{j,t})$. Hence $R_{i,t}^2$ is also in $\overline{\mathfrak{G}}$. Therefore $\mathfrak{D}' \leq \overline{\mathfrak{G}}$ is proved.

Now we shall show $\overline{\mathfrak{G}} \leq \mathfrak{D}'$, assuming that $3 \leq n$. For any $W_{i,j,t}$ take k such that $|k| \neq |i|, |j|$. Then we have

$$W_{i,j,t} = W_{-k,j,-1}^{-1}W_{i,k,t}^{-1}W_{-k,j,-1}W_{i,k,t},$$

which shows that $\overline{\mathfrak{G}} \leq \mathfrak{D}'$. Therefore we have $\overline{\mathfrak{G}} = \mathfrak{D}'$ provided that $3 \leq n$, and hence $\mathfrak{G}(D_n, K)$ is the commutator group of projective orthogonal group defined by the quadratic form $\sum_{i=1}^n \xi_i \xi_{-i}$.

7. **The group $\mathfrak{G}(B_n, K)$.** First we consider the case where K is not of characteristic 2. We shall show that the group $\overline{\mathfrak{G}}(B_n, K)$ is isomorphic to the commutator group \mathfrak{D}' of the orthogonal group \mathfrak{D} defined by the quadratic form $\sum_{i=0}^n \xi_i \xi_{-i}$, provided that $2 \leq n$. By (2.4), the group $\overline{\mathfrak{G}}$ is generated by the matrices

$$\begin{aligned} W_{i,j,t} &= I + t(E_{i,-j} - E_{j,-i}), \\ V_{i,t} &= I + t(2E_{-i0} - E_{0i}) - t^2 E_{-ii}, \end{aligned}$$

where $t \in K$, $i, j = \pm 1, \dots, \pm n$; $i \neq \pm j$. Define the matrices P_{ij} , Q_i , $R_{i,t}$ as in §6 for $i, j \neq 0$. It is known [4] that the group \mathfrak{D} is generated by the matrices $W_{i,j,t}$, $V_{i,t}$, P_{ij} , Q_i , $R_{i,t}$. The formula (6.1) shows $\mathfrak{D}' \leq \overline{\mathfrak{G}}$.

We shall show $\overline{\mathfrak{G}} \leq \mathfrak{D}'$ under the assumption that $2 \leq n$. By an elementary computation we have

$$W_{i,j,2t} = V_{i,t} V_{j,1} V_{i,t}^{-1} V_{j,1}^{-1},$$

which shows that every $W_{i,j,t}$ is in \mathfrak{D}' , since K is not of characteristic 2. Also we have

$$V_{i,t} = W_{ji,t}(W_{-ji,1}^{-1} V_{j,-t}^{-1} W_{-ji,1} V_{j,-t}).$$

Hence we see that $V_{i,t}$ is also in \mathfrak{D}' . Thus $\overline{\mathfrak{G}} \leq \mathfrak{D}'$, and hence $\overline{\mathfrak{G}} = \mathfrak{D}'$ is proved. Therefore if K is not of characteristic 2 then $\mathfrak{G}(B_n, K)$ is the commutator group of the projective orthogonal group defined by the quadratic form $\sum_{i=0}^n \xi_i \xi_{-i}$.

Consider now the case where K is of characteristic 2. Denote by \mathfrak{M} a vector space over K spanned by the indeterminates ξ_i , $i=0, \pm 1, \dots, \pm n$, and by \mathfrak{M}' the subspace of \mathfrak{M} spanned by ξ_i , $i=\pm 1, \dots, \pm n$. The matrix $W_{i,j,t}$ is the matrix of the linear transformation of \mathfrak{M} :

$$(7.1) \quad \xi'_i = \xi_i + t\xi_{-j}, \quad \xi'_j = \xi_j + t\xi_{-i}, \quad \xi'_k = \xi_k \quad (k \neq i, j),$$

while $V_{i,t} = I + tE_{0i} + t^2 E_{-ii}$ is the matrix of the linear transformation:

$$(7.2) \quad \xi'_0 = \xi_0 + t\xi_i, \quad \xi'_{-i} = \xi_{-i} + t^2 \xi_i, \quad \xi'_k = \xi_k \quad (k \neq 0, -i).$$

Therefore the subspace \mathfrak{M}' reduces the group $\overline{\mathfrak{G}}$. Moreover, (7.1) and (7.2) shows that the linear transformations in $\overline{\mathfrak{G}}$ leave the bilinear form $f_c(\xi, \eta) = \sum_{i=1}^n (\xi_i \eta_{-i} - \xi_{-i} \eta_i)$ invariant. It was noted in §3 that the center of $\overline{\mathfrak{G}}$ consists of elements of the form aI with $a \in K$. Since aI leaves $f_c(\xi, \eta)$ invariant, we have $a^2 = 1$ and hence $a = 1$. Then $\mathfrak{G} = \overline{\mathfrak{G}}$ and hence $\overline{\mathfrak{G}}$ is simple. (We exclude the exceptional cases. See [2, p. 65]). Therefore the representation of $\overline{\mathfrak{G}}$ induced in the space \mathfrak{M}' is faithful. Thus we may regard \mathfrak{G} as a multiplicative group of $2n \times 2n$ matrices generated by $W_{i,j,t} = I + t(E_{i,-j} - E_{j,-i})$ and $V_{i,t} = I + t^2 E_{-ii}$, where $t \in K$ and $i, j = \pm 1, \pm 2, \dots, \pm n$. We have mentioned in §5 that the symplectic group $\text{Sp}(2n, K)$ is generated by $W_{i,j,t}$ and $I + tE_{-ii}$. Thus the group $\mathfrak{G}(B_n, K)$, in the case when K is of characteristic 2,

is a subgroup of $\text{Sp}(2n, K)$. If, however, the field K is perfect then $\mathfrak{G} = \text{Sp}(2n, K)$.

8. The algebra G_2 as a subalgebra of B_3 . E. Cartan [1, p. 146] gave a representation of G_2 as a subalgebra of the algebra B_3 , which we shall derive here in a form convenient for our purposes. The algebra B_3 is defined as the algebra of all 7×7 matrices X satisfying $f_B(X\xi, \eta) + f_B(\xi, X\eta) = 0$ identically, where

$$f_B(\xi, \eta) = 2\xi_0\eta_0 + \sum_{i=1}^3 (\xi_{-i}\eta_i + \xi_i\eta_{-i}).$$

Denote by E_{ij} the 7×7 matrix such that $\xi'_i = E_{ij}\xi$ where $\xi'_i = \xi_j$, $\xi'_k = 0$ for $k \neq i$, and set

$$\begin{aligned} F_{ij} &= E_{-ij} - E_{-ji}, & F_{i0} &= 2E_{-i0} - E_{0i}, & F_{0i} &= -F_{i0}, \\ H_{ij} &= F_{i,-i} + F_{j,-j}, & H_{i0} &= 2F_{i,-i}, & H_{0i} &= 2F_{i,-i} \end{aligned}$$

for $i, j = 1, 2, 3$. Then $\{H_r\} = \{H_{ij} | i < j, i \neq -j\}$ is the set of all co-weights of B_3 with respect to the Cartan subalgebra \mathfrak{h} spanned by all H_r , and $\{X_r\} = \{F_{ij} | i < j, i \neq -j\}$ is a system of root vectors such that if $X_r = F_{ij}$ then $H_r = H_{ij}$ and $X_{-r} = F_{-j,-i}$. It was shown in [5] that there is an automorphism ϕ of B_3 such that $\phi(X_r) = -X_{-r}$ for all roots r of B_3 . Cartan showed that the six elements

$$U_{ij} = F_{i,-j}, \quad (0 < i, 0 < j, i \neq j),$$

together with the six elements

$$U_{0i} = F_{i0} + F_{i' i''}, \quad U_{i0} = F_{0,-i} + F_{-i' -i''},$$

where $i = 1, 2, 3$ and where $(i' i'')$ is an even permutation of $1, 2, 3$, form a system of root vectors of a subalgebra \mathfrak{g} of B_3 isomorphic to G_2 . The Cartan subalgebra of \mathfrak{g} is given by $\mathfrak{g} \cap \mathfrak{h}$. We have

$$\begin{aligned} (8.1) \quad [U_{ij}, U_{ji}] &= H_{i,-j}, \\ [U_{0i}, U_{i0}] &= H_{i0} + H_{i' i''}, \end{aligned}$$

where $i \neq 0, j \neq 0$, since $[F_{i0}, F_{-i' -i''}] = 0$, $[F_{0,-i}, F_{i' i''}] = 0$. We have also the identities

$$[H_{i,-j}, U_{ij}] = 2U_{ij}, \quad [H_{i0} + H_{i' i''}, U_{0i}] = 2U_{0i}.$$

Hence we see that the elements on the right hand side of (8.1) are co-weights of the algebra \mathfrak{g} . Moreover, it is easily seen that the automorphism ϕ of B_3 induces an automorphism of the subalgebra \mathfrak{g} such that $\phi(U_{ij}) = -U_{ji}$, $\phi(U_{0i}) = -U_{i0}$. Therefore the system $\{U_{ij}, U_{0i}, U_{i0}\}$ of root vectors can be used to define the group $\mathfrak{G}(G_2, K)$.

9. The group $\mathfrak{G}(G_2, K)$. Let K be an arbitrary field. By the same argument as in §3 we see that $\mathfrak{G}(G_2, K)$ is isomorphic to the quotient group over

its center of the multiplicative group $\overline{\mathfrak{G}}(G_2, K)$ generated by the matrices

$$(9.1) \quad \begin{cases} \exp(tU_{ij}) = I + tU_{ij}, \\ \exp(tU_{0i}) = I + tU_{0i} - t^2E_{-ii}, \\ \exp(tU_{i0}) = I + tU_{i0} - t^2E_{-ii}, \end{cases}$$

where $t \in K$ and $i, j = 1, 2, 3$.

Dickson [4, §9; 5], guided by the above mentioned representation of the algebra G_2 given by Cartan, considered the multiplicative group \mathfrak{G} consisting of all 7×7 matrices X with entries in K which satisfy the conditions (9.2)–(9.4):

$$(9.2) \quad \det X = 1;$$

$$(9.3) \quad X \text{ leaves the quadratic form}$$

$$q(\xi) = \xi_0^2 + \xi_1\xi_{-1} + \xi_2\xi_{-2} + \xi_3\xi_{-3}$$

invariant, that is, $q(X\xi) = q(\xi)$ identically;

$$(9.4) \quad X \text{ leaves invariant the system of equations}$$

$$\xi_0\eta_i - \xi_i\eta_0 + \xi_{-i'}\eta_{-i''} - \xi_{-i''}\eta_{-i'} = 0$$

where $(i' i'')$ are cyclic even permutations of 1, 2, 3, or of $-1, -2, -3$, when X operates cogrediently upon the two sets of variables $(\xi_i), (\eta_i)$, $-3 \leq i \leq 3$.

Dickson proved, assuming that K has more than two elements, that the group \mathfrak{G} is simple and generated by the matrices in (9.1) and the matrices $R_{i,t}R_{j,t}^{-1}$, $i \neq j$, $i \neq 0$, $j \neq 0$, $t \in K$, where $R_{i,t}$ is the matrix of the transformation

$$\xi'_i = t\xi_i, \quad \xi'_{-i} = t^{-1}\xi_{-i}, \quad \xi'_k = \xi_k \quad (k \neq i, -i).$$

In view of the formulas (6.1), the matrix $R_{i,t}R_{j,t}^{-1}$ can be represented as a product of matrices in (9.1), and hence it follows that the group \mathfrak{G} is generated by the matrices in (9.1). Therefore we have $\mathfrak{G} = \overline{\mathfrak{G}}(G_2, K)$. Since \mathfrak{G} is simple, we have also $\mathfrak{G}(G_2, K) = \overline{\mathfrak{G}}(G_2, K) = \mathfrak{G}$.

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