MULTIPLICATION ON SPHERES (II)

BY

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1. **Introduction.** Let A be a space with a basepoint $e \in A$. A *multiplication* is a continuous function of $A \times A$ into A, written $(x, y) \to x \cdot y$ where $x, y \in A$, such that $x \cdot e = x$ and $e \cdot y = y$. The multiplication in which $(x, y) \to y \cdot x$ is called the *inversion* of the original. We say that two multiplications are *homotopic* if they are given by homotopic maps $A \times A \to A$. In the first of these two notes[1], [6] we studied *homotopy-commutative* multiplications, i.e. those which are homotopic to their inversions. In the present note, we study homotopy-associative multiplications, according to the following definition. Consider the two maps g, h: $A \times A \times A \to A$ such that

$$g(x, y, z) = (x \cdot y) \cdot z, \qquad h(x, y, z) = x \cdot (y \cdot z) \qquad\qquad (x, y, z \in A).$$

An associative multiplication is one in which $g = h$. I define a *homotopy-associative* multiplication to be one such that g and h are homotopic. If a multiplication is homotopy-associative then so is its inversion, and so is any homotopic multiplication.

We prove theorems which determine all the classes of homotopy-associative multiplications on spheres[2]. Let S^n denote a topological n-sphere, $n \geqq 1$, such as the unit sphere in $(n+1)$-dimensional euclidean space. We shall prove

THEOREM (1.1). *If S^n admits a multiplication, then the homotopy classes of multiplication on S^n are in* (1, 1) *correspondence with the elements of the homotopy group $\pi_{2n}(S^n)$.*

If $n = 1$, we can represent the points of S^1 by complex numbers of unit modulus, which multiply according to the associative law. By (1.1), every multiplication on S^1 is homotopic to the complex multiplication, since $\pi_2(S^1) = 0$. Therefore we have

THEOREM (1.2). *Every multiplication on S^1 is homotopic to the complex multiplication, and hence is both homotopy-commutative and homotopy-associative.*

If $n = 3$, we can represent the points of S^3 by quaternions of unit modulus, which multiply associatively. The inversion of quaternionic multiplication is

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(1) Numbers in square brackets refer to the list at the end of this paper.

(2) The sphere of dimension zero admits two multiplications, which are not homotopic. Both of them are commutative and associative. This case is henceforth excluded from the discussion.

also associative(3), and the two structures are related by an orientation-reversing isomorphism. By Corollary 2 on p. 48 of [12], an associative multiplication on a sphere is a group multiplication, and therefore a Lie group multiplication, by Satz I of [11]. In the case of S^3 the group is semi-simple, by (16.1) of [1], and hence is isomorphic to the quaternionic group-structure, by the Cartan classification. Therefore the multiplication is homotopic to quaternionic multiplication or to its inversion, according as the isomorphism is orientation-preserving or not.

We shall prove:

THEOREM (1.3). *There are twelve homotopy classes of multiplication on S^3, which can be divided into six pairs, such that either of the classes of a pair consists of the inversions of the multiplications in the other class. Four pairs consist of multiplications which are homotopy-associative, and two pairs consist of multiplications which are not.*

Notice that by our previous remarks, six of the eight classes of homotopy-associative multiplications do not contain an associative multiplication.

To complete the investigation we prove

THEOREM (1.4). *There exists no homotopy-associative multiplication on S^n unless $n=1$ or 3.*

For example, the Cayley multiplication on S^7 is not homotopy-associative.

The results in this introduction are applications of our two main theorems. We state these theorems in the next section, and show how Theorems (1.1), $\cdots$, (1.4) are deduced from them. Then we go on to prove the main theorems.

The case of S^3 has previously been analyzed by G. W. Whitehead in unpublished work. I wish to thank him for encouragement to proceed with my account, and to thank P. J. Hilton for informing me of (4.1) below.

2. **The main theorems.** Let M denote the cartesian product of p spheres of various dimensions, say

$$M = S_1 \times \cdots \times S_r \times \cdots \times S_p \qquad (p \geqq 1).$$

Let z_r be a basepoint in S_r, and let Σ denote the complement in M of the product cell

$$(S_1 - z_1) \times \cdots \times (S_r - z_r) \times \cdots \times (S_p - z_p).$$

If $p=3$, for example, then

$$\Sigma = (z_1 \times S_2 \times S_3) \cup (S_1 \times z_2 \times S_3) \cup (S_1 \times S_2 \times z_3).$$

Let A be a space, and let u, v: $M \to A$ be maps which agree on Σ. Then a

(3) Of course, the inversion of any associative multiplication is associative.

separation element(4) $d(u, v) \in \pi_m(A)$ is defined, where m is the dimension of M, and we have

PROPOSITION (2.1). *The maps u, v are homotopic, relative*(5) *to Σ, if, and only if, $d(u, v)=0$.*

The first of our two main theorems is:

THEOREM (2.2). *Let A be a sphere which admits a multiplication. Let $u, v\colon M \to A$ be maps which agree on Σ. If $u \simeq v$, then $u \simeq v$, rel. Σ. Hence, by* (2.1), *u and v are homotopic if, and only if, $d(u, v)=0$.*

We apply (2.2) first to prove (1.1). Suppose that S^n admits a multiplication $f\colon S^n \times S^n \to S^n$. Let e denote the identity, so that

$$f(x, e) = f(e, x) = x \qquad (x \in S^n).$$

If f' is another multiplication, then f and f' agree on the set $\Sigma = e \times S^n \cup S^n \times e$, and so the separation element $d(f, f') \in \pi_{2n}(S^n)$ is defined. By (2.2), with $p=2$, if two multiplications are homotopic then they are homotopic relative to Σ, so that at each stage of the deformation we have a multiplication. It follows that the transformation $f' \to d(f, f')$ is independent of the homotopy class of f'. On the other hand, given an element $\delta \in \pi_{2n}(S^n)$ there exists a multiplication f' such that $d(f, f')=\delta$, and any two such multiplications are homotopic, by the standard theory of the separation element. Therefore f determines a (1, 1)-correspondence between classes of multiplication and elements of $\pi_{2n}(S^n)$. This proves (1.1). The proof of (1.2) has already been given, and we now state the theorem on which (1.3) and (1.4) are based.

The Hopf construction applied to f determines an element $c(f)$ in $\pi_{2n+1}(S^{n+1})$ with Hopf invariant unity. If f' is another multiplication then

$$c(f') = c(f) + Ed(f, f'), \tag{2.3}$$

by (9.2) of [8], where $Ed(f, f')$ denotes the suspension of the separation element. Consider the homomorphisms

$$\pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \xleftarrow{F} \pi_{r+1}(S^{2n+1}),$$

where E is the Freudenthal suspension, and F is induced by $c(f)$. By Corollary 1 on p. 282 of [9] we have

PROPOSITION (2.4). *The homomorphisms E and F are both isomorphisms into, and $\pi_{r+1}(S^{n+1})$ is expressed as the direct sum of their images, so that*

$$\pi_r(S^n) + \pi_{r+1}(S^{2n+1}) \approx \pi_{r+1}(S^{n+1}).$$

Consider now the two maps $g, h\colon S^n \times S^n \times S^n \to S^n$ which are given by

(4) The reader is referred to [8] for a summary of the theory of the separation element.

(5) A homotopy relative to a subspace is one throughout which the images of points in the subspace stay fixed.

$$g(x, y, z) = f(f(x, y), z), \qquad h(x, y\ z) = f(x, f(y, z)),$$

where $x, y, z \in S^n$. Since g and h agree on the set

$$\Sigma = (e \times S^n \times S^n) \cup (S^n \times e \times S^n) \cup (S^n \times S^n \times e),$$

a separation element $d(g, h) \in \pi_{3n}(S^n)$ is defined. We write $d(f) = d(g, h)$, and refer to $d(f)$ as the obstruction to homotopy-associativity, because (2.2) gives us

COROLLARY (2.5). *The multiplication f is homotopy-associative if, and only if, $d(f) = 0$.*

The second of our main theorems effectively expresses $d(f)$ in terms of $c(f)$, as follows.

THEOREM (2.6). *The element $c(f) \in \pi_{2n+1}(S^{n+1})$ and the element $d(f) \in \pi_{3n}(S^n)$ are related by the congruence*

$$Ed(f) \equiv Pc(f), \bmod F\pi_{3n+1}(S^{2n+1}),$$

where $Pc(f)$ denotes the Whitehead product of $c(f)$ and a certain generator of $\pi_{n+1}(S^{n+1})$.

Since S^n admits a multiplication, n is odd, by V of [3]. By (1.9) of [7], $Pc(f)$ is not contained in $F\pi_{3n+1}(S^{2n+1})$ if $n > 3$. Hence, by (2.6), $Ed(f) \neq 0$ unless $n = 1$ or 3. Hence, by (2.5), f is not homotopy-associative unless $n = 1$ or 3. This proves (1.4). There remains (1.3), which we deal with in the next section.

3. **Multiplications on S^3.** Let γ denote the Hopf class in $\pi_7(S^4)$, i.e. the element which is obtained by the Hopf construction from q, the quaternionic multiplication. We recall (from (4.3) and (7.2) of [10], for example) that $\pi_6(S^3)$ is a cyclic group of order twelve, generated by an element ω such that

$$(3.1) \qquad E\omega = 2\gamma + [\iota, \iota],$$

where ι generates $\pi_4(S^4)$. We prove

THEOREM (3.2). *Let f be a multiplication on S^3 such that $d(f, q) = m\omega$, where m is an integer. Then f is homotopy-associative if, and only if, $m \equiv 0$ or 1,* mod 3.

By §3 and (4.3) of [2] we have

PROPOSITION (3.3). *The group $\pi_9(S^3)$ is cyclic of order three, generated by an element δ such that $E\delta = [[\iota, \iota], \iota]$.*

Let $\bar{\gamma}$ denote the element of $\pi_7(S^4)$ which is obtained by the Hopf construction from $\bar{q}$, the inversion of quaternionic multiplication. Then

$$(3.4) \qquad \gamma + \bar{\gamma} + [\iota, \iota] = 0,$$

by (2.3) of [6], and so it follows from (3.1) that

$$\gamma = \bar{\gamma} + E\omega. \tag{3.5}$$

By (2.5) we have $d(q)=d(\bar{q})=0$, since q and $\bar{q}$ are associative. Therefore

$$\begin{aligned} &\text{(a)} \\ &\text{(b)} \end{aligned} \begin{cases} [\gamma, \iota] = \gamma \circ \beta, \\ [\bar{\gamma}, \iota] = \bar{\gamma} \circ \bar{\beta}, \end{cases} \tag{3.6}$$

by (2.6), where $\beta, \bar{\beta} \in \pi_{10}(S^7)$. Since $\pi_{10}(S^7)=E\pi_9(S^6)$, composition with β on the left is a linear operation. Therefore

$$\bar{\gamma} \circ \beta + E\omega \circ \beta + [\bar{\gamma}, \iota] + [[\iota, \iota], \iota] = 0,$$

by substituting in (3.6a) from (3.4) and (3.5). Hence, by (3.3) and (3.6b),

$$(\bar{\gamma} \circ \beta + \bar{\gamma} \circ \bar{\beta}) + (E\delta + E\omega \circ \beta) = 0.$$

But $E\omega \circ \beta \in E\pi_9(S^3)$, since $\beta \in E\pi_9(S^6)$, and so it follows that $\bar{\gamma} \circ (\beta + \bar{\beta}) \in E\pi_9(S^3)$. Hence we obtain from (2.4) that

$$\begin{aligned} &\text{(a)} \\ &\text{(b)} \end{aligned} \begin{cases} \beta + \bar{\beta} = 0, \\ E\delta + E\omega \circ \beta = 0. \end{cases} \tag{3.7}$$

Now let f be the multiplication in (3.2), such that $d(f, q) = m\omega$. Let α denote the element of $\pi_7(S^4)$ which is obtained from f by the Hopf construction. Then $\gamma - \alpha = mE\omega$, by (2.3). Therefore

$$\alpha \circ \beta - \gamma \circ \beta = -mE\omega \circ \beta = mE\delta, \tag{3.8}$$

by (3.7b). Also $\gamma - \alpha = m(\gamma - \bar{\gamma})$, by (3.5), and so

$$\begin{aligned} [\alpha, \iota] &= (1-m)[\gamma, \iota] + m[\bar{\gamma}, \iota] \\ &= (1-m)\gamma \circ \beta + m\bar{\gamma} \circ \bar{\beta}, \text{ by (3.6)}, \\ &= (1-m)\gamma \circ \beta + m(E\omega - \gamma) \circ \beta, \text{ by (3.5), (3.7a)}, \\ &= (1-2m)\gamma \circ \beta - mE\delta, \text{ by (3.7b)}. \end{aligned}$$

Substituting for $\gamma \circ \beta$ from (3.8) we conclude that

$$[\alpha, \iota] = (1-2m)\alpha \circ \beta + 2m(m-1)E\delta.$$

Hence and from (2.4), (2.6) it follows that

$$d(f) = 2m(m-1)\delta. \tag{3.9}$$

Since δ has order three, it follows from (3.9) that $d(f)=0$ if, and only if, $m \equiv 0$ or 1, mod 3. Hence, by (2.5), f is homotopy-associative if, and only if, $m \equiv 0$ or 1, mod 3. This proves (3.2).

We now prove (1.3). By (1.1), there are twelve classes of multiplication on S^3, since $\pi_6(S^3)$ has twelve elements. By (3.2), eight of these classes consist of multiplications which are homotopy-associative, and four consist of multiplications which are not. If two multiplications are homotopic then so are their inversions. By (1.1) of [6], no multiplication is homotopic to its own inversion. Hence the twelve classes fall into six pairs, such that either of the classes of a given pair consists of the inversions of the multiplications in the other class. If a multiplication is homotopy-associative then so is its inversion. Therefore either all the multiplications of a pair are homotopy-associative or none of them are. This completes the proof of (1.3).

It remains for us to prove the two theorems, (2.2) and (2.6), from which all these other results follow.

4. **Products of spheres.** In this section and the next we prove (2.2), which is concerned with the classification of mappings of products of spheres. We first of all prove[6]

THEOREM (4.1). *Let M be a manifold which is the cartesian product of spheres of various dimensions. Let b_r denote the Betti number of M in dimension r, $r=1, 2, \cdots$. Then the suspension of M has the same homotopy type as a bunch of spheres containing b_r spheres of dimension $r+1$, $r=1, 2, \cdots$.*

Let M be the product of p spheres, ($p\geqq 1$), say

$$M = S_1 \times \cdots \times S_r \times \cdots \times S_p.$$

Let z_r be a basepoint in S_r, and let $e_r = S_r - z_r$. We regard S_r as the complex composed of z_r and e_r, and we regard M as the product complex. Consider the q distinct subcomplexes

$$M_i = T_1 \times \cdots \times T_r \times \cdots \times T_p \qquad (1 \leqq i \leqq q),$$

where $T_1 = z_1$ or $S_1, \cdots, T_r = z_r$ or $S_r, \cdots$. There are 2^p such alternatives, so that $q = 2^p$. We arrange that $M_1 = M$, and that M_q is the basepoint

$$z = z_1 \times \cdots \times z_r \times \cdots \times z_p.$$

Let f_i: $M \to M_i$ denote the canonical retraction. Let Σ_i be a sphere of the same dimension as M_i $(1 \leqq i < q)$, such that $\Sigma_i \cap \Sigma_j = z$ if $i \neq j$, and let g_i: $M_i \to \Sigma_i$ be a map of degree one. By the Künneth formula, there are b_r spheres Σ_i of dimension r, $r=1, 2, \cdots$, where b_r is the rth Betti number of M. Let S denote the union of the spheres Σ_i, and let

$$h_i\colon M \to S \qquad (1 \leqq i \leqq q)$$

denote the map determined by $g_i f_i$ if $i < q$, the constant map to z if $i = q$.

Let I denote the interval $0 \leqq t \leqq 1$, regarded as the complex which consists of the open interval and the endpoints. Then $\hat{M}$, the suspension of M, is an identification complex of the product complex $M \times I$. Hence we represent

(6) This elegant theorem was proved by P. J. Hilton in 1954 (unpublished). The proof given here is my own. I understand that the theorem and its generalizations were discovered independently by various others, including K. Hardie and D. Puppe, who are preparing detailed accounts.

points of M by pairs (x, t), where $x \in M$, $t \in I$, such that $(x, 0) = (x, 1) = \hat{z}$, the suspension of z. Let $\hat{S}$ denote the suspension of S, which is a bunch of spheres containing b_r spheres of dimension $r+1$, $r = 1, 2, \cdots$. Let $h: \hat{M} \to \hat{S}$ be the map which is defined by

$$\begin{aligned} h(x, t) &= (h_1(x), qt) && (0 \leqq t \leqq 1/q), \\ &= (h_2(x), qt - 1) && (1/q \leqq t \leqq 2/q), \end{aligned}$$

and so forth. Then h maps each of the cells of $\hat{M}$ with degree one onto one of the spheres which make up $\hat{S}$. Hence h induces an isomorphism of the homology groups of $\hat{M}$ and $\hat{S}$. Since $\hat{M}$ and $\hat{S}$ are simply-connected, it therefore follows from Theorem 3 of [13] that h is a homotopy-equivalence. This proves (4.1).

5. **Proof of** (2.2). Let K be a CW-complex (see [13]) with a subcomplex L such that $K - L = e^m$, an m-cell of K. Let $\alpha \in \pi_{m-1}(L)$ be the homotopy class of the attaching map of e^m. Let z be a 0-cell of L. We prove

LEMMA (5.1). *Suppose that the r-fold suspension of α is zero for some $r \geqq 0$. Let A be a space and let u, v: $K \to A$ be maps which agree on L, so that the separation element $d(u, v) \in \pi_m(A)$ is defined. If $u \simeq v$, rel. z, then the r-fold suspension of $d(u, v)$ is equal to zero.*

Let $\tilde{u}, \tilde{v}: \tilde{K} \to \tilde{A}$ denote the r-fold suspensions of u, v, respectively. $\tilde{K}$ contains $\tilde{L}$, the r-fold suspension of L, as a subcomplex so that $\tilde{K} - \tilde{L} = e^{m+r}$, the (open) r-fold suspension of e^m. Then $\tilde{u}$ agrees with $\tilde{v}$ on $\tilde{L}$, since u agrees with v on L, and so the separation element $d(\tilde{u}, \tilde{v}) \in \pi_{m+r}(\tilde{A})$ is defined, and

$$\tilde{E}d(u, v) = d(\tilde{u}, \tilde{v}), \tag{5.2}$$

where $\tilde{E}$ denotes r-fold suspension. Since $\tilde{E}(\alpha) = 0$, by hypothesis, e^{m+r} is attached inessentially to $\tilde{L}$. Hence there is an element $\beta \in \pi_{m+r}(\tilde{K})$ the injection of which generates $\pi_{m+r}(\tilde{K}, \tilde{L})$. Therefore

$$d(\tilde{u}, \tilde{v}) = \tilde{u}_*(\beta) - \tilde{v}_*(\beta), \tag{5.3}$$

by (B1b) on p. 108 of [14], where

$$\tilde{u}_*, \tilde{v}_*: \pi_{m+r}(\tilde{K}) \to \pi_{m+r}(\tilde{A})$$

are the homomorphisms induced by $\tilde{u}, \tilde{v}$, respectively. If $u \simeq v$, rel. z, then $\tilde{u} \simeq \tilde{v}$, and so $\tilde{u}_* = \tilde{v}_*$. Hence $d(\tilde{u}, \tilde{v}) = 0$, by (5.3), and so (5.1) follows from (5.2).

As in §4, consider a product of spheres

$$M = S_1 \times \cdots \times S_r \times \cdots \times S_p.$$

Let m be the dimension of M, and consider the m-cell

$$e^m = e_1 \times \cdots \times e_r \times \cdots \times e_p.$$

Let $\alpha \in \pi_{m-1}(\Sigma)$ be the homotopy class of the map by which e^m is attached.

Then it follows from (4.1) that $E(\alpha)=0$, and so by applying (5.1) we obtain

THEOREM (5.4). *Let u, v: $M\to A$ be maps which agree on Σ such that $u\simeq v$, rel. z. Then $Ed(u, v)=0$.*

Hence, by (2.1), we obtain

COROLLARY (5.5). *Suppose that E: $\pi_m(A)\to\pi_{m+1}(\tilde{A})$ is an isomorphism into. Then $u\simeq v$, rel. z, if, and only if, $u\simeq v$, rel. Σ.*

We use (5.5) to prove (2.2). Let A be a sphere with multiplication, say $A=S^n$. Let u, v: $M\to A$ be homotopic maps which agree on Σ. We have to prove that u and v are homotopic, relative to Σ. If $m=1$ this follows at once from the homotopy-extension theorem, since Σ is a single point. Let $m>1$. If $n=1$, then $\pi_m(A)=0$ and so $u\simeq v$, rel. Σ, by (2.1). Let $n>1$. Then $\pi_1(A)=0$, and so $u\simeq v$, rel. z, by the homotopy-extension theorem. Also E is an isomorphism into, by (2.4). Hence $u\simeq v$, rel. Σ, by (5.5). This completes the proof of (2.2).

We now commence the proof of (2.6), which requires more elaborate notation.

6. **Notations.** Let V^{r+1} $(r\geqq 0)$ denote the $(r+1)$-element which consists of points $(x_0, x_1, \cdots, x_i, \cdots)$ in Hilbert space, such that $x_i=0$ if $i>r$ and such that

$$x_0^2+x_1^2+\cdots+x_r^2\leqq 1.$$

Let S^r denote the boundary of V^{r+1}, where

$$x_0^2+x_1^2+\cdots+x_r^2=1,$$

and let e be the point where $x_0=-1$. Let E^r_+, E^r_- denote the hemispheres where $x_r\geqq 0$, $x_r\leqq 0$, respectively.

If $r\geqq 1$ let p^r, q^r: $V^r\to S^r$ denote the projections, parallel to the axis of x_r, onto E^r_+, E^r_-, respectively. Thus, p^r and q^r map V^r homeomorphically, leaving S^{r-1} fixed. Let (s, x) denote the point of V^r which divides e and x in the ratio s: $1-s$, where $x\in S^{r-1}$ and $0\leqq s\leqq 1$. Thus $(0, x)=e$, and $(1, x)=x$. We represent S^r $(r\geqq 1)$ as the suspension of S^{r-1}, so that if $x\in S^{r-1}$ then

$$\begin{cases}(x, t) = q^r(2t, x) \text{ if } 0\leqq t\leqq 1/2,\\ (x, t) = p^r(2-2t, x) \text{ if } 1/2\leqq t\leqq 1.\end{cases}\tag{6.1}$$

7. **The key lemma.** We prove a lemma which is the key to the proof of (2.6), the second of our main theorems. Let A be a space and let

$$u, v: S^m\times S^n\to A \qquad (m, n\geqq 1)$$

be maps. Suppose that

$$u|\Sigma = v|\Sigma,\tag{7.1}$$

where $\Sigma = e \times S^n \cup S^m \times e$, so that the separation element $d(u, v) \in \pi_{m+n}(A)$ is defined. Let $p^m, q^m: V^m \to S^m$ be the projections defined in the previous section, and let

$$p, q: V^m \times S^n \to S^m \times S^n$$

denote the products of p^m, q^m, respectively, with the identity map of S^n. Suppose that v is invariant under reflection in the plane $x_m = 0$, i.e. suppose that

$$vp = vq. \tag{7.2}$$

Since p^m and q^m agree on S^{m-1}, and since $p^m V^m \cup q^m V^m = S^m$, it follows from (7.1) and (7.2) that the compositions

$$up, uq: V^m \times S^n \to A$$

agree on the set $S^{m-1} \times S^n \cup V^m \times e$, and so the separation element $d(up, uq) \in \pi_{m+n}(A)$ is defined. The purpose of this section is to prove:

LEMMA (7.3). *Let u, v be a pair of maps which satisfy* (7.1) *and* (7.2). *Then*(7) $d(u, v) \pm \mathrm{M}d(up, uq)$.

Consider the map $v': S^m \times S^n \to A$ which is defined by

$$v'(x, y) = v(e, y) \qquad (x \in S^m,\ y \in S^n). \tag{7.4}$$

Then trivially $v'p = v'q$. As a first step towards proving (7.3) I assert that there exists a map $u': S^m \times S^n \to A$, which agrees with v' on Σ, and such that

$$\begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix} \begin{cases} d(u, v) = d(u', v'), \\ d(up, uq) = d(u'p, u'q). \end{cases} \tag{7.5}$$

For there is a homotopy $v_t: S^m \times S^n \to A$, such that $v_0 = v'$ and $v_1 = v$, by (7.2), which is defined by

$$\begin{aligned} v_t((x, s), y) &= v((x, st), y) && (0 \leqq s \leqq 1/2), \\ &= v((x, t - st), y) && (1/2 \leqq s \leqq 1), \end{aligned}$$

where $x \in S^{m-1}$ and $y \in S^n$. Let $w_t = v_t | \Sigma$. Then $w_1 = v | \Sigma = u | \Sigma$, by (7.1). Hence, by the homotopy extension theorem, there exists a homotopy $u_t: S^m \times S^n \to A$ such that $u_1 = u$, and such that

$$u_t | \Sigma = w_t = v_t | \Sigma. \tag{7.6}$$

The separation element $d(u_t, v_t) \in \pi_{m+n}(A)$ is defined for each value of t, and so $d(u_0, v_0) = d(u_1, v_1)$. This proves (7.5a), with $u' = u_0$. Let $z \in V^m \times e$. Then $p(z), q(z) \in \Sigma$, and since $v_t p = v_t q$, by the definition of v_t, it follows from (7.6) that

$$u_t p(z) = v_t p(z) = v_t q(z) = u_t q(z).$$

(7) The ambiguity of sign which occurs here and elsewhere below depends only on orientations, not on the maps u, v.

Hence, and since p and q agree on $S^{m-1}\times S^n$, it follows that the homotopies u_tp, u_tq: $V^m\times S^n\to A$ coincide on $S^{m-1}\times S^n\cup V^m\times e$. Therefore $d(u_tp, u_tq)$ is defined for each value of t, and so $d(u_0p, u_0q)=d(u_1p, u_1q)$. This completes the proof of (7.5), with $u'=u_0$.

The proof of (7.3) continues as follows. Consider the homotopies p_t, q_t: $V^m\times S^n\to S^m\times S^n$ which are defined by

$$(7.7)\qquad \begin{cases} \text{(a)}\quad p_t((s, x), y) = \left(\left(x, 1-\frac{s}{2}(1-t)\right), y\right), \\ \text{(b)}\quad q_t((s, x), y) = \left(\left(x, \frac{s}{2}(1+t)\right), y\right), \end{cases}$$

where $x\in S^{m-1}$, $y\in S^n$, and $0\leqq s\leqq 1$. We have $p_0=p$, $q_0=q$, by (6.1). Notice that p_t and q_t agree on $S^{m-1}\times S^n$, and that

$$p_t(V^m\times y)\cup q_t(V^m\times y) = S^m\times y,$$

for each value of t. But u' is constant on $S^m\times e$, by (7.4) and (7.6). Therefore the homotopies $u'p_t$, $u'q_t$: $V^m\times S^n\to A$ agree on $S^{m-1}\times S^n\cup V^m\times e$. Therefore $d(u'p_t, u'q_t)$ is defined for each value of t, and hence is independent of t. Write $p'=p_1$, $q'=q_1$. We have proved that

$$(7.8)\qquad d(u'p, u'q) = d(u'p', u'q').$$

The final stage is to prove that

$$(7.9)\qquad d(u', v') = \pm d(u'p', u'q').$$

By (7.7b), q' maps $V^m\times S^n$ onto $S^m\times S^n$ with degree ± 1, and maps $S^{m-1}\times S^n\cup V^m\times e$ into $e\times S^n\cup S^m\times e$. Therefore

$$(7.10)\qquad d(u', v') = \pm d(u'q', v'q'),$$

depending on the degree of q'. However, if $x\in S^{m-1}$, $y\in S^n$ and $0\leqq s\leqq 1$ then

$$\begin{aligned} v'q'((s, x), y) &= v'((x, s), y), \text{ by (7.7b)}, \\ &= v'(e, y), \text{ by (7.4)}, \\ &= u'(e, y), \text{ by (7.6)}, \\ &= u'p'((s, x), y), \text{ by (7.7a)}. \end{aligned}$$

Therefore $v'q'=u'p'$, and (7.9) follows from (7.10).

To sum up, we have that

$$\begin{aligned} d(u, v) &= d(u', v'), \text{ by (7.5a)}, \\ &= \pm d(u'p', u'q'), \text{ by (7.9)}, \\ &= \pm d(u'p, u'q), \text{ by (7.8)}, \\ &= \pm d(up, uq), \text{ by (7.5b)}. \end{aligned}$$

This completes the proof of Lemma (7.3).

8. **The reduced product complex.** We regard S^n as the complex which is composed of e and its complementary n-cell, e^n. Let K denote the reduced product complex of S^n, as defined in [4]. Thus, K contains S^n as a subcomplex, and has cell-structure

$$S^n \cup e^{2n} \cup \cdots \cup e^{mn} \cup \cdots,$$

where e^r is an r-cell, $r=2n, 3n, \cdots$. The mn-section of K is the image of the product complex

$$S^n \times \cdots \times S^n \times \cdots \times S^n \qquad (m \text{ factors}),$$

under the identifications which are specified in [4]. For example, the $2n$-section of K is the image of the map w: $S^n \times S^n \to K$ which identifies

$$(8.1) \qquad w(x, e) = w(e, x) = x \qquad (x \in S^n).$$

In general, we denote the image of the point

$$(x_1, \cdots, x_i, \cdots, x_m) \qquad (x_i \in S^n)$$

under the identification map by

$$[x_1, \cdots, x_i, \cdots, x_m].$$

An associative multiplication, T, with identity e, is defined on K by the formula:

$$(8.2) \qquad T([x_1, \cdots, x_p], [y_1, \cdots, y_q]) = [x_1, \cdots, x_p, y_1, \cdots, y_q],$$

where all the coordinates lie in S^n.

Now suppose that S^n admits a multiplication f: $S^n \times S^n \to S^n$, with identity e. Then f determines a retraction r of K onto S^n such that

$$(8.3) \qquad rT(x, y) = f(r(x), y) \qquad (x \in K, y \in S^n).$$

For example, if $x, y, z \in S^n$ then

$$(8.4) \qquad \begin{cases} \text{(a)} & rw(x, y) = f(x, y), \\ \text{(b)} & r[x, y, z] = f(f(x, y), z). \end{cases}$$

Let i: $S^n \to K$ denote the inclusion map, and define $f' = if$. Then $ri = 1$, since r is a retraction, and so

$$(8.5) \qquad rf' = rif = f = rw,$$

by (8.4a). Let p^{2n}, q^{2n}: $V^{2n} \to S^{2n}$ mean the same as in §6, and let ψ^{2n}: $V^{2n} \to S^n \times S^n$ mean the same as in [8]. Thus, $\psi^{2n}(S^{2n-1}) = e \times S^n \cup S^n \times e$, and ψ^{2n} maps $V^{2n} - S^{2n-1}$ homeomorphically. We define a map k: $S^{2n} \to K$ such that

$$(8.6) \qquad \begin{cases} \text{(a)} & kp^{2n} = w\psi^{2n}, \\ \text{(b)} & kq^{2n} = f'\psi^{2n}. \end{cases}$$

These equations specify the values of k on the upper and lower hemispheres of S^n, respectively. Next we define two maps $u, v: S^{2n} \times S^n \to S^n$ such that

$$(8.7) \qquad \begin{cases} \text{(a)} & u(x, y) = rT(y, k(x)), \\ \text{(b)} & v(x, y) = rT(k(x), y), \end{cases}$$

where $x \in S^{2n}$, $y \in S^n$. Since T is a multiplication, u and v agree on the set $e \times S^n \cup S^{2n} \times e$. Let $d(f)$ denote the obstruction to homotopy-associativity which is defined in §2. We prove

LEMMA (8.8). *In* $\pi_{3n}(S^n)$ *we have* $d(u, v) = \pm d(f)$.

By (8.3) and (8.7b), if $x \in S^{2n}$ and $y \in S^n$ then $v(x, y) = f(rk(x), y)$. By (8.6a), however, we have

$$\begin{aligned} rkp^{2n} = rw\psi^{2n} &= rf'\psi^{2n}, \text{ by (8.5)}, \\ &= rkq^{2n}, \text{ by (8.6b)}. \end{aligned}$$

Therefore v satisfies (7.2), and hence

$$(8.9) \qquad d(u, v) = \pm\, d(up, uq),$$

by (7.3). We recall from §2 that $d(f)$ is the separation element of the maps $g, h: S^n \times S^n \times S^n \to S^n$ which are defined by

$$g(x, y, z) = f(f(x, y), z), \qquad h(x, y, z) = f(x, f(y, z)),$$

where $x, y, z \in S^n$. Let ξ be a point of V^{2n} such that $\psi^{2n}(\xi) = (y, z)$. I assert that

$$(8.10) \qquad \begin{cases} \text{(a)} & g(x, y, z) = u(p^{2n}(\xi), x), \\ \text{(b)} & h(x, y, z) = u(q^{2n}(\xi), x). \end{cases}$$

In the case of (a) we have

$$\begin{aligned} u(p^{2n}(\xi), x) &= rT(x, kp^{2n}(\xi)), \text{ by (8.7a)}, \\ &= rT(x, w\psi^{2n}(\xi)), \text{ by (8.6a)}, \\ &= rT(x, [y, z]), \text{ by definition}, \\ &= r[x, y, z], \text{ by (8.2)}, \\ &= f(f(x, y), z), \text{ by (8.4b)}. \end{aligned}$$

This verifies (a). In the case of (b) we have

$$\begin{aligned} u(q^{2n}(\xi), x) &= rT(x, kq^{2n}(\xi)), \text{ by (8.7a)}, \\ &= rT(x, f'\psi^{2n}(\xi)), \text{ by (8.6b)}, \\ &= rT(x, f(y, z)), \text{ by definition}, \\ &= f(x, f(y, z)), \text{ by (8.3)}. \end{aligned}$$

This completes the proof of (8.10). We rewrite (8.10) in the form

$$g\psi = up, \qquad h\psi = uq,$$

where ψ: $V^{2n}\times S^n \rightarrow S^n\times S^n\times S^n$ is the map defined by

$$\psi(\xi, x) = (x, \psi^{2n}(\xi)) \qquad\qquad (\xi \in V^{2n}, x \in S^n).$$

Since ψ is a map of degree ± 1, such that

$$\psi(S^{2n-1}\times S^n \cup V^{2n}\times e) \subset (e\times S^n\times S^n)\cup(S^n\times e\times S^n)\cup(S^n\times S^n\times e),$$

we have $d(g, h) = \pm d(g\psi, h\psi)$, i.e. $d(f) = \pm d(up, uq)$. But $d(u, v) = \pm d(up, uq)$, by (8.9). Hence $d(f) = \pm d(u, v)$, which proves (8.8).

9. **Proof of** (2.6). The identification of $d(f)$, the obstruction to homotopy-associativity, is completed in this section. We continue straight on from §8. Thus, f is a multiplication on S^n, and r: $K\rightarrow S^n$ is the retraction which f defines. Consider the homomorphism induced by r, and the canonical isomorphism ϕ as defined in §10 of [5]:

$$\pi_{m+1}(S^{n+1}) \overset{\phi}{\leftarrow} \pi_m(K) \overset{r_*}{\rightarrow} \pi_m(S^n).$$

Let $\alpha\in\pi_{2n}(K)$ denote the homotopy class of k. Then $\alpha = d(w, f')$, by (8.6), and so it follows from (5.7) of [8] that

$$\phi(\alpha) = c(f), \tag{9.1}$$

where $c(f)$ is the element of $\pi_{2n+1}(S^{n+1})$ which is obtained from f by the Hopf construction. Notice also that

$$r_*(\alpha) = r_*d(w, f') = d(rw, rf') = 0, \tag{9.2}$$

by (8.5). By (6.1) of [8] there is a generator $\iota\in\pi_{n+1}(S^{n+1})$ such that

$$[\phi(\alpha), \iota] = \phi d(u', v'), \tag{9.3}$$

where u', v': $S^{2n}\times S^n\rightarrow K$ are the maps defined by

$$u'(x, y) = T(y, k(x)), \qquad v'(x, y) = T(k(x), y),$$

for $x\in S^{2n}$, $y\in S^n$. Moreover

$$r_*d(u', v') = d(ru', rv') = d(u, v),$$

by (8.7), and so we obtain from (8.8) that

$$r_*d(u', v') = \pm\, d(f). \tag{9.4}$$

We proceed to prove (2.6). By (2.4) there exist elements $\beta\in\pi_{3n}(S^n)$, $\gamma\in\pi_{3n+1}(S^{2n+1})$ such that

$$[\phi(\alpha), \iota] = E(\beta) + F(\gamma); \tag{9.5}$$

where $F(\gamma) = c(f)\circ\gamma$. Now E is equal to the composition

$$\pi_{3n}(S^n) \xrightarrow{i_*} \pi_{3n}(K) \xrightarrow{\phi} \pi_{3n+1}(S^{n+1}),$$

by (10.2a) of [5]. Also $\gamma = E\gamma'$, where $\gamma' \in \pi_{3n}(S^{2n})$, by the Freudenthal suspension theorems. Therefore

$$\begin{aligned} F(\gamma) &= \phi(\alpha) \circ E(\gamma'), \text{ by (9.1)}, \\ &= \phi(\alpha \circ \gamma'), \end{aligned}$$

by (10.4) of [5]. Hence and by (9.3) we can rewrite (9.5) in the form

$$\phi d(u', v') = \phi i_*(\beta) + \phi(\alpha \circ \gamma').$$

Therefore $d(u', v') = i_*(\beta) + \alpha \circ \gamma'$, since ϕ is an isomorphism. But $r_*(\alpha \circ \gamma') = r_*(\alpha) \circ \gamma' = 0$, by (9.2). Hence $r_* d(u', v') = r_* i_*(\beta) = \beta$, since r is a retraction. Therefore $\beta = \pm d(f)$, by (9.4), and since $\phi(\alpha) = c(f)$, by (9.1), we finally obtain from (9.5) that

$$Ed(f) = \pm \lfloor c(f), \iota \rfloor, \text{ mod } F\pi_{3n+1}(S^{2n+1}).$$

This completes the proof of (2.6), our second main theorem.

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