ON PARTIALLY STABLE ALGEBRAS[1]

BY

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1. **Introduction.** Consider a simple commutative power-associative algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p \neq 2, 3, 5$. Then $\mathfrak{A}$ has a unity element $1 = u + v$, where u and v are orthogonal nontrivial idempotents of $\mathfrak{A}$. There is then a consequent decomposition of $\mathfrak{A}$ as the vector space direct sum

$$\mathfrak{A} = \mathfrak{A}_u(1) + \mathfrak{A}_u(1/2) + \mathfrak{A}_u(0),$$

where $\mathfrak{A}_u(\lambda)$ consists of all elements x_λ of $\mathfrak{A}$ such that $ux_\lambda = \lambda x_\lambda$. If $\mathfrak{A}_u(\lambda)\mathfrak{A}_u(1/2) \subseteq \mathfrak{A}_u(1/2)$ for $\lambda = 0, 1$ we call u a *stable* idempotent of $\mathfrak{A}$ and say that $\mathfrak{A}$ is *u-stable*. We shall say that $\mathfrak{A}$ is *partially stable* if $\mathfrak{A}$ contains a stable idempotent.

The most interesting remaining problem in the theory of commutative power-associative algebras is the determination of all simple algebras of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p > 5$. In this paper we shall essentially solve the problem for the partially stable case.

2. **Known results.** Our assumption on the characteristic of $\mathfrak{F}$ implies that a commutative algebra $\mathfrak{A}$ is power-associative if and only if $P(x, y, s, t) = 0$ for every x, y, s and t of $\mathfrak{A}$, where

$$\begin{aligned} P(x, y, s, t) = {} & 4(xy)(st) + 4(xs)(yt) + 4(xt)(ys) - x[y(st) + s(ty) + t(ys)] \\ & - y[x(st) + s(tx) + t(xs)] - s[x(yt) + y(tx) + t(xy)] \\ & - t[x(ys) + y(sx) + s(xy)]. \end{aligned} \tag{1}$$

We shall assume throughout this paper that $\mathfrak{A}$ is a simple commutative power-associative algebra of degree two over an algebraically closed field $\mathfrak{F}$ of characteristic $p > 5$ and that u is a stable idempotent of $\mathfrak{A}$. We designate the unity element of $\mathfrak{A}$ by 1 and write $v = 1 - u$, so that

$$z = u - v = 2u - 1, \qquad z^2 = 1. \tag{2}$$

Then the spaces $\mathfrak{A}_u(\lambda)$ are characterized by the properties

$$x_1 z = x_1, \qquad x_0 z = -x_0, \qquad x_{1/2} z = 0, \tag{3}$$

for every x_λ of $\mathfrak{A}_u(\lambda)$. The following properties of $\mathfrak{A}$ are known[2].

Presented to the Society, April 14, 1956; received by the editors May 17, 1956.

(1) This paper was sponsored in part by the Office of Ordnance Research, United States Army, under Contract DA-11-022-ORD-1571.

(2) These results were obtained in the author's *on commutative power-associative algebras of degree two,* Trans. Amer. Math. Soc. vol. 74 (1953) pp. 323–343.

LEMMA 1. *There exists an element* w *in* $\mathfrak{A}_u(1/2)$ *such that* $w^2=1$.

LEMMA 2. *Let* $\mathfrak{C}=\mathfrak{A}_u(1)+\mathfrak{A}_u(0)$, *and* $\mathfrak{B}$ *be the set of all elements* b *of* $\mathfrak{C}$ *such that* $(bw)w=b$. *Then* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{C}$, *both* $\mathfrak{A}_u(1)$ *and* $\mathfrak{A}_u(0)$ *are isomorphic to* $\mathfrak{B}$, $\mathfrak{A}_u(1)=u\mathfrak{B}$, $\mathfrak{A}_u(0)=(1-u)\mathfrak{B}$, $\mathfrak{C}=\mathfrak{B}+\mathfrak{B}z$.

LEMMA 3. *Let* a *and* b *be in* $\mathfrak{B}$. *Then* $(wa)b=(wb)a=w(ab)$, $(wa)(wb)=w(ab)$, $w(\mathfrak{B}z)=0$, $w(au)=w(av)=1/2wa$.

We may always write

$$\mathfrak{A}_u(1/2) = w\mathfrak{B} + \mathfrak{G}, \tag{4}$$

where $\mathfrak{G}$ consists of all elements g of $\mathfrak{A}_u(1/2)$ such that $wg=0$. Indeed $\mathfrak{A}$ contains the idempotents $e=1/2(1+w)$ and $f=1/2(1-w)$ such that

$$\mathfrak{A}_e(1) = \mathfrak{A}_f(0) = e\mathfrak{B}, \quad \mathfrak{A}_e(0) = \mathfrak{A}_f(1) = f\mathfrak{B}, \quad A_e(1/2) = \mathfrak{A}_f(1/2) = \mathfrak{G} + \mathfrak{B}z. \tag{5}$$

Since $[\mathfrak{A}_u(1/2)]^2\subseteq\mathfrak{C}$ we may write

$$(wa)g = f_g(a) + z\phi_g(a), \tag{6}$$

where $f_g(a)$ and $\phi_g(a)$ are clearly linear functions on $\mathfrak{B}$ to $\mathfrak{B}$ for every g of $\mathfrak{G}$. Also ga is in $\mathfrak{A}_u(1/2)$ for every a of $\mathfrak{B}$, and so we may write

$$ga = gS_a + wa', \tag{7}$$

where a' is in $\mathfrak{B}$ and S_a is a linear transformation of $\mathfrak{G}$ which is also a linear function of a. Then $w(ga)=a'$. However, it is known that

$$w(ga) + (wa)g = z\phi_g(a). \tag{8}$$

Thus we have the first part of the following lemma, where the last part is also known.

LEMMA 4. *Let* a *be in* $\mathfrak{B}$ *and* g *be in* $\mathfrak{G}$. *Then*

$$ga = gS_a - wf_g(a), \qquad w(ga) = -f_g(a), \qquad g(az) = -w\phi_g(a). \tag{9}$$

Our last known result may be stated as follows.

LEMMA 5. *The product* gh *of any two elements* g *and* h *of* $\mathfrak{G}$ *is in* $\mathfrak{B}$. *If* a *and* b *are in* $\mathfrak{B}$ *the product* $g=(wa)(bz)$ *is in* $\mathfrak{G}$ *and*

$$(wa)(bz) = -(wb)(az). \tag{10}$$

3. **Some properties of** $f_g(a)$, $\phi_g(a)$ **and** S_a. We shall first compute $P(w, az, bz, g)$ for a, b in $\mathfrak{B}$ and g in $\mathfrak{G}$. Since $w(az)=w(bz)=wg=0$ we obtain $w[(ab)g+w\phi_g(a)\cdot(bz)+w\phi_g(b)\cdot az]+g[w(ab)]-(az)\phi_g(b)-(bz)\phi_g(a)=w[(ab)g]+g[(ab)w]-z[a\phi_g(b)+b\phi_g(a)]=0$. By (8) we have the following result.

LEMMA 6. *The function* $\phi_g(a)$ *is a derivation of* $\mathfrak{B}$ *for every* g *of* $\mathfrak{G}$, *that is,*

$$\phi_g(ab) = a\phi_g(b) + b\phi_g(a). \tag{11}$$

We next compute $P(w, w, wa, gb)$ for g in $\mathfrak{G}$ and a in $\mathfrak{B}$, to see that $4(wa)(gb)+8a[w(gb)]=4(wa)(gb)-8af_g(b)=2w[a(gb)-waf_g(b)+w\cdot(wa)(gb)]+(wa)[gb-2wf_g(b)]+3(gb)(wa)-4af_g(b)$. It follows that $-4af_g(b)=2w[(gb)a]+2w[w\cdot(wa)(gb)]$, that is

$$w[(gb)a] + w[w\cdot(wa)(gb)] = -2af_g(b). \tag{12}$$

We also compute $P(w, g, a, b)$ and obtain $4(wb)(ga)+4(wa)(gb)=w[(ga)b+(gb)a+g(ab)]+3g[w(ab)]+a[w(bg)+(wb)g]+b[w(ag)+(wa)g]=w[(ga)b+(gb)a]-f_g(ab)+3[f_g(ab)+z\phi_g(ab)]+z[a\phi_g(b)+b\phi_g(a)]$. By Lemma 6 this relation reduces to

$$4[(wb)(ga) + (wa)(gb)] - w[(ga)b + (gb)a] = 2f_g(ab) + 4z\phi_g(a). \tag{13}$$

The last computation needed for the basic result we shall obtain is that of $P(wa, g, w, b)$, which yields $4(gb)a+4(wb)[(wa)g]=4(gb)a+4(wb)f_g(a)-4(wb)[z\phi_g(a)]=(wa)[w(gb)+(wb)g]+w[(wa)(gb)+g\cdot w(ab)+b\cdot(wa)g]+3g(ab)+b[w\cdot(wa)g+ag]=-(wa)[z\phi_g(b)]+w[(wa)(gb)]+wf_g(ab)+wbf_g(a)+3g(ab)+b(ag)+wbf_g(a)$. Then

$$\begin{aligned} w[2bf_g(a) - f_g(ab)] + (wa)[z\phi_g(b)] - 4(wb)[z\phi_g(a)] \\ = w[(wa)(gb)] + [3g(ab) + b(ag) - 4(gb)a]. \end{aligned} \tag{14}$$

Multiply (14) by w to yield

$$2bf_g(a) - f_g(ab) = w[w\cdot(wa)(gb)] - 3f_g(ab) + w[b(ag) - 4(gb)a], \tag{15}$$

that is,

$$2[bf_g(a) + f_g(ab)] = w[w\cdot(wa)(gb)] + w[b(ag) - 4(gb)a]. \tag{16}$$

Substitute the value of $w[w\cdot(wa)(gb)]$ from (12) in (16) and obtain $-2af_g(b)-w[(gb)a]+w[b(ag)-4(gb)a]=2[bf_g(a)+f_g(ab)]$, that is,

$$w[b(ag) - 5(gb)a] = 2[bf_g(a) + af_g(b) + f_g(ab)]. \tag{17}$$

Interchange a and b in (17), and obtain

$$w[b(ag) - 5(bg)a] = w[a(gb) - 5(ga)b]. \tag{18}$$

Then $6w[b(ag)-a(bg)]=0$, that is,

$$w[b(ag)] = w[a(bg)], \tag{19}$$

and so (17) becomes

$$f_g(ab) + af_g(b) + bf_g(a) = -2w[(bg)a]. \tag{20}$$

Substitute this expression for $w[(gb)a]$ back in (12) and obtain $4af_g(b)=f_g(ab)+bf_g(a)+af_g(b)-2w[w\cdot(wa)(gb)]$ and thus

$$2w[w\cdot(wa)(gb)] = f_g(ab) + bf_g(a) - 3af_g(b). \tag{21}$$

We now multiply (13) by w twice and can substitute the results of (20) and (21) to obtain $2[f_g(ab)+bf_g(a)-3af_g(b)]+2[f_g(ab)+af_g(b)-3bf_g(a)]+f_g(ab)+bf_g(a)+af_g(b)=2f_g(ab)$ from which $3f_g(ab)-3bf_g(a)-3af_g(b)=0$. We have derived the following property.

LEMMA 7. *The function $f_g(a)$ is a derivation of $\mathfrak{B}$ for every g of $\mathfrak{G}$, that is,*

$$f_g(ab) = af_g(b) + bf_g(a). \tag{22}$$

Let us now return to the original identities (12), (13), (14). We shall substitute the values from (6), (9) and use (11). We obtain no new result from (12). However, (13) becomes $4(wb)[gS_a-wf_g(a)]+4(wa)[gS_b-wf_g(b)]-w[(gS_a)b-wbf_g(a)+(gS_b)a-waf_g(b)]=4[f_{gS_a}(b)+z\phi_{gS_a}(b)-bf_g(a)]+4[f_{gS_b}(a)+z\phi_{gS_b}(a)-af_g(b)]-w[gS_aS_b-wf_{gS_a}(b)+gS_aS_b-wf_{gS_b}(a)-wf_g(ab)]=5[f_{gS_a}(b)+f_{gS_b}(a)]-3f_g(ab)+4z[\phi_{gS_a}(b)+\phi_{gS_b}(a)]=2f_g(ab)+4z\phi_g(ab)$. Hence we have derived the property

$$f_{gS_a}(b) + f_{gS_b}(a) = f_g(ab), \tag{23}$$

and the property

$$\phi_{gS_a}(b) + \phi_{gS_b}(a) = \phi_g(ab) \tag{24}$$

of our derivations f_g and ϕ_g.

We also substitute in (14) to see that $w[2bf_g(a)-f_g(ab)]+(wa)[z\phi_g(b)]-4(wb)[z\phi_g(a)]=w[(wa)(gS_b)]-w[af_g(b)]+3[gS_{ab}-wf_g(ab)]+gS_aS_b-wf_{gS_a}(b)-wbf_g(a)-4gS_bS_a+4wf_{gS_b}(a)+4waf_g(b)$. Since $w[(wa)gS_b]=w[f_{gS_b}(a)+z\phi_{gS_b}(a)]=wf_{gS_b}(a)$, we see that the component in $\mathfrak{G}$ yields

$$(wa)[z\phi_g(b)] - 4(wb)[z\phi_g(a)] = 3gS_{ab} + gS_aS_b - 4gS_bS_a, \tag{25}$$

while the component in $w\mathfrak{B}$ yields

$$\begin{aligned} 2bf_g(a) - f_g(ab) = f_{gS_b}(a) &- af_g(b) - 3f_g(ab) - f_{gS_a}(b) \\ &- bf_g(a) + 4f_{gS_b}(a) + 4af_g(b). \end{aligned} \tag{26}$$

Then (26) is equivalent to

$$\begin{aligned} 5f_{gS_b}(a) - f_{gS_a}(b) &= 3bf_g(a) - 3af_g(b) + 2f_g(ab) \\ &= 5bf_g(a) - af_g(b). \end{aligned} \tag{27}$$

By interchanging a and b we derive $5f_{gS_a}(b)-f_{gS_b}(a)=5af_g(b)-bf_g(a)$, $25f_{gS_a}(b)-5f_{gS_a}(a)=25af_g(b)-5bf_g(a)$, from which $24f_{gS_a}(b)=24af_g(b)$. Our assumption on the characteristic of $\mathfrak{F}$ yields the following result.

LEMMA 8. *The function $f_g(b)$ satisfies the property*

$$f_{gS_a}(b) = af_g(b) \tag{28}$$

for every a of $\mathfrak{B}$.

We similarly interchange a and b in (25) to see that $(wb)[z\phi_g(a)] - 4(wa)[z\phi_g(b)] = 3gS_{ab} + gS_bS_a - 4gS_aS_b$ and so $4(wb)[z\phi_g(a)] - 16(wa)[z\phi_g(b)] = 12gS_{ab} + 4gS_bS_a - 16gS_aS_b$. Add this result to (25) and obtain $-15(wa) \cdot [z\phi_g(b)] = 15gS_{ab} - 15gS_aS_b$, from which

$$gS_{ab} = gS_aS_b - (wa)[z\phi_g(b)] = gS_bS_a - (wb)[z\phi_g(a)], \tag{29}$$

for every g of $\mathfrak{G}$ and every a and b of $\mathfrak{B}$.

The next step in our derivation is a computation of $P(z, g, az, b)$ for a, b in $\mathfrak{B}$ and g in $\mathfrak{G}$. Then $4(gb)a + 4[g(bz)](az) = 4[gS_b - wf_g(b)]a - 4[w\phi_g(b)](az) = 4gS_bS_a - 4wbf_g(a) - 4waf_g(b) + 4(wa)[z\phi_g(b)] = 4gS_bS_a - 4wf_g(ab) + 4(wa)[z\phi_g(b)] = 3g(ab) + (az)[g(bz)] + b(ga) = 3gS_{ab} - 3wf_g(ab) - (az)[w\phi_g(b)] + gS_aS_b - wbf_g(a) - waf_g(b) = 3gS_{ab} - 4wf_g(ab) + gS_aS_b + (wa)[z\phi_g(b)]$.

Thus

$$3gS_{ab} + gS_aS_b - 4gS_bS_a = 3(wa)[z\phi_g(b)]. \tag{30}$$

By (29) we see that $3gS_{ab} = 3gS_aS_b - 3(wa)[z\phi_g(b)]$ and $-4gS_bS_a = -4gS_aS_b + 4(wa)[z\phi_g(b)] - 4(wb)[z\phi_g(a)]$. Thus (30) implies that $(wa)[z\phi_g(b)] - 4(wb)[z\phi_g(a)] = 3(wa)[z\phi_g(b)]$, $[wa][z\phi_g(b)] = -2(wb)[z\phi_g(a)]$. Interchanging a and b yields $(wb)[z\phi_g(a)] = -2(wa)[z\phi_g(b)] = 4(wb)[z\phi_g(a)]$, $3(wb)[z\phi_g(a)] = 0$ and $(wb)[z\phi_g(a)] = 0$. Using (29) we have the following result.

LEMMA 9. *The linear transformations S_a have the property*

$$S_{ab} = S_aS_b = S_bS_a, \tag{31}$$

for every a and b of $\mathfrak{B}$. Thus the mapping $a \to S_a$ is a homomorphism of $\mathfrak{B}$ onto the associative algebra $\mathfrak{B}^*$ of all S_a. Moreover

$$(wa)[z\phi_g(b)] = 0, \tag{32}$$

for every a and b of $\mathfrak{B}$ and g of $\mathfrak{G}$.

4. **The structure of $\mathfrak{B}$.** Let $\mathfrak{M}$ be the set of all finite sums of products of a finite number of factors in $\mathfrak{B}$ at least one of which is an associator

$$q = a(bc) - (ab)c, \tag{33}$$

where a, b, c are in $\mathfrak{B}$. Clearly $\mathfrak{M}$ is an ideal of $\mathfrak{B}$ and is zero when $\mathfrak{B}$ is associative. Then (31) implies the following result.

LEMMA 10. *Let m be an element of $\mathfrak{M}$. Then $S_m = 0$.*

For (31) implies that $S_q = S_aS_{bc} - S_{ab}S_c = S_aS_bS_c - S_aS_bS_c = 0$. If $d = d_1 \cdots d_t$ with d_i in $\mathfrak{B}$ then $S_d = S_{d_t} \cdots S_{d_t}$. If one factor d_i is an associator q then $S_q = 0$, $S_d = 0$. Hence $S_m = 0$ for every m of $\mathfrak{M}$.

We now apply (24) to obtain

$$\phi_{gS_{ab}}(c) + \phi_{gS_c}(ab) = \phi_g(ab \cdot c). \tag{34}$$

By (24) and $S_{ab}=S_aS_b$ we have

$$\phi_{gS_{ab}}(c) + \phi_{gS_{ac}}(b) = \phi_{gS_a}(bc). \tag{35}$$

Use (34) twice to obtain

$$\phi_g(ab\cdot c) - \phi_{gS_c}(ab) + \phi_g(ac\cdot b) - \phi_{gS_b}(ac) = \phi_{gS_a}(bc), \tag{36}$$

that is

$$\phi_g(ab\cdot c + ac\cdot b) = \phi_{gS_a}(bc) + \phi_{gS_b}(ac) + \phi_{gS_c}(ab). \tag{37}$$

The right member of (37) is unaltered by permutations of the three letters a, b, c. Hence $\phi_g(ab\cdot c+ac\cdot b)=\phi_g(a\cdot bc+ac\cdot b)$, that is

$$\phi_g(q) = 0, \tag{38}$$

for all associators q. Thus $\phi_g(d)$ is independent of the association of the factors of d in $\mathfrak{B}$. We use the fact that $\mathfrak{B}$ is commutative to see that if q is any factor of d we may write $\phi_g(d)=\phi_g(qr)$, where r is a product of all of the remaining factors of d. Then $\phi_g(d)=\phi_g(qr)=\phi_{gS_q}(r)+\phi_{gS_r}(q)$. If q is an associator we have $S_q=0$, $\phi_h(q)=0$ for any h of $\mathfrak{G}$, $\phi_g(d)=0$. By (23) we have $f_g(d)=0$ and have derived the following result.

LEMMA 11. *If m is in $\mathfrak{M}$ and g is in $\mathfrak{G}$ the relations $f_g(m)=\phi_g(m)=0$ hold. Thus*

$$\mathfrak{G}(\mathfrak{M} + \mathfrak{M}z + w\mathfrak{M}) = 0. \tag{39}$$

We next derive the following result.

LEMMA 12. The relation $\mathfrak{B}[(w\mathfrak{M})(\mathfrak{B}z)]\subseteq(w\mathfrak{M})(\mathfrak{B}z)$ holds.

For let a, b be in $\mathfrak{B}$ and m be in $\mathfrak{M}$. Then the computation of $P(wa, b, m, z)$ yields

$$\begin{aligned}4[w(ab)](mz) + 4[w(am)](bz) + 3[w(bm)](az)&\\ = -b[(wm)(az)] + m[(wa)(bz)].&\end{aligned} \tag{40}$$

The element $g=(wa)(bz)$ is in $\mathfrak{G}$ and so $m[(wa)(bz)]=0$ by (39). Also $[w(ab)](mz)=-(wm)(abz)$ is in $(w\mathfrak{M})(\mathfrak{B}z)$, am and bm are in $\mathfrak{M}$ and so $b[(wm)(az)]$ is in $(w\mathfrak{M})(\mathfrak{B}z)$ as desired.

LEMMA 13. *The relation $[(w\mathfrak{M})(\mathfrak{B}z)](\mathfrak{B}z)\subseteq wM$ holds.*

For proof we compute $P(wm)$, bz, a, where a and b are in $\mathfrak{B}$ and m is in $\mathfrak{M}$ to obtain $4[(wm)(bz)](az)+4w[(am)b]=3w[m(ab)]+(bz)[(wm)(az)]+w[(bm)a]$. Interchange a and b to obtain $4[(wm)(az)](bz)+4w[(bm)a] = 3w[m(ab)] + (az)[(wm)(bz)] + w[(am)b]$. Then $16[(wm)(bz)](az) =4(bz)[(wm)(az)]+4w[(bm)a]+12w[m(ab)]-16w[(am)b]=[(wm)(bz)](az) +w[(am)b]+3w[m(ab)]-4w[(bm)a]+4w[(bm)a]+12w[m(ab)]-16w[(am)b]$ that is, $15[(wm)(bz)](az)=15w[m(ab)]-15w[(am)b]$. Hence

$$(41) \qquad [(wm)(bz)](az) = w[m(ab) - (ma)b].$$

If we now use the fact that m is in $\mathfrak{M}$ our lemma follows.

Our next result is obtained from a computation of $P(wa, g, b, z)$, where a and b are in $\mathfrak{B}$ and g is in $\mathfrak{G}$. Then $4[(wa)g](bz) = (wa)[g(bz)] + g[(wa)(bz)] + b[(wa)g \cdot z] + z[(wa)(gb) + (wa)g \cdot b + w(ab) \cdot g] = -a\phi_g(b) + g[(wa)(bz)] + 2b[\phi_g(a) + zf_g(a)] + \phi_g(ab) + zf_g(ab) + z\{(wa)[gS_b - wf_g(b)]\} = 4[\phi_g(a) + zf_g(a)]b$. Since $(wa)(bz)$ is in $\mathfrak{G}$ the product $g[(wa)(bz)]$ is in $\mathfrak{B}$, $g[(wa)(bz)] = 4b\phi_g(a) + a\phi_g(b) - 2b\phi_g(a) - a\phi_g(b) - b\phi_g(a) - \phi_{gS_b}(a)$. This yields the following result.

LEMMA 14. *Let a and b be in $\mathfrak{B}$ and g be in $\mathfrak{G}$. Then*

$$(42) \qquad g[(wa)(bz)] = b\phi_g(a) - \phi_{gS_b}(a).$$

Let us now apply (42) with a in $\mathfrak{M}$. Then $\phi_g(a) = \phi_{gS_b}(a) = 0$ and so $g[(wa)(bz)] = 0$, that is, the following result holds.

LEMMA 15. *The relation $[(w\mathfrak{M})(\mathfrak{B}z)]\mathfrak{G} = 0$ holds.*

We are now ready to define what we shall show is an ideal of $\mathfrak{A}$. Let

$$(43) \qquad \mathfrak{H} = \mathfrak{M} + \mathfrak{M}z + w\mathfrak{M} + (w\mathfrak{M})(\mathfrak{B}z).$$

By Lemmas 11 and 15 we have

$$(44) \qquad \mathfrak{H}\mathfrak{G} = 0.$$

Now $\mathfrak{M}\mathfrak{A} = \mathfrak{M}\mathfrak{B} + \mathfrak{M}(\mathfrak{B}z) + \mathfrak{M}(w\mathfrak{B}) + \mathfrak{M}\mathfrak{G} \subseteq \mathfrak{M} + \mathfrak{M}z + w\mathfrak{M} \subseteq \mathfrak{A}$. Also $(\mathfrak{M}z)\mathfrak{A} = (\mathfrak{M}z)\mathfrak{B} + (\mathfrak{M}z)(\mathfrak{B}z) + (\mathfrak{M}z)(w\mathfrak{B}) \subseteq \mathfrak{M}z + \mathfrak{M} + (w\mathfrak{M})(\mathfrak{B}z) \subseteq \mathfrak{H}$ by (10). We next show that $(w\mathfrak{M})\mathfrak{A} = (w\mathfrak{M})\mathfrak{B} + (w\mathfrak{M})(\mathfrak{B}z) + (w\mathfrak{M})(w\mathfrak{B}) + (w\mathfrak{M})\mathfrak{G} \subseteq w\mathfrak{M} + (w\mathfrak{M})(\mathfrak{B}z) + \mathfrak{M} \subseteq \mathfrak{H}$. Finally $(w\mathfrak{M})(\mathfrak{B}z)\mathfrak{A} = [(w\mathfrak{M})(\mathfrak{B}z)][\mathfrak{B} + \mathfrak{B}z + w\mathfrak{B} + \mathfrak{G}] \subseteq (w\mathfrak{M})(\mathfrak{B}z) + w\mathfrak{M} + [(w\mathfrak{M})(\mathfrak{B}z)](w\mathfrak{B})$ by Lemmas 12 and 13. Compute $P[wa, bz, w, c]$ to obtain $4[(wa)(bz)](wc) + 4[a(bc)]z = (wa)[(bz)(wc)] + (bz)(3ac) + w[(wa)(bz) \cdot c + (wa)(bcz) + w(ac) \cdot (bz)] = (wa)[(bz)(wc)] + 3(ac)(bz) + w[(wa)(bz) \cdot c]$ since $(wa)(bcz)$ and $[w(ac)](bz)$ are in $\mathfrak{G}$. If a is in $\mathfrak{M}$ then $[(wa)(bz)]c$ is in $[(w\mathfrak{M})(\mathfrak{B}z)]\mathfrak{B} \subseteq (w\mathfrak{M})(\mathfrak{B}z)$ by Lemma 12 and w annihilates the product. Hence $4[(wa)(bz)]wc = -4[a(bc)]z + 3[(ac)b]z$ is in $\mathfrak{M}z$ and we have completed our proof of the fact that $\mathfrak{H}$ is an ideal of $\mathfrak{A}$.

If $\mathfrak{H} = \mathfrak{A}$ then $\mathfrak{M} = \mathfrak{B}$ and $(w\mathfrak{M})(\mathfrak{B}z) = \mathfrak{G}$. Then $\mathfrak{G}\mathfrak{A} = 0$ implies that $\mathfrak{G}$ is a proper ideal of $\mathfrak{A}$, $\mathfrak{G} = 0$, $\mathfrak{H} = \mathfrak{M} + \mathfrak{M}z + w\mathfrak{M} = \mathfrak{B} + \mathfrak{B}z + w\mathfrak{B} = \mathfrak{A}$. Then $\mathfrak{M} = \mathfrak{B}$, $(w\mathfrak{B})(\mathfrak{B}z) = 0$, and (41) implies that $w\{w[m(ab) - (ma)b]\} = m(ab) - (ma)b = 0$ for every a, b, m of $\mathfrak{B}$. Hence $\mathfrak{B}$ is associative. If $\mathfrak{H} \neq \mathfrak{A}$ then $\mathfrak{H} = \mathfrak{M} = 0$ and $\mathfrak{B}$ is again associative. We state this result as follows.

THEOREM 1. *Let $\mathfrak{A}$ be a simple u-stable commutative power-associative algebra of degree two over an algebraically closed field of characteristic $p > 5$. Then $\mathfrak{A}_u(1)$ is associative.*

5. Additional properties of u-stable algebras. We now use the fact that $\mathfrak{B}$ is always associative. Since $\mathfrak{B}$ is commutative and has a unity element we may write $\mathfrak{B}=e\mathfrak{F}+\mathfrak{N}$, where $e=1$ is the unity element of $\mathfrak{A}$ and $\mathfrak{N}$ is the radical of $\mathfrak{B}$. If $\mathfrak{N}=0$ it is easy to see that $\mathfrak{A}$ is a Jordan algebra of degree two. Assume henceforth that $\mathfrak{N}\neq 0$. We begin by deriving the following result.

LEMMA 16. *Let g be in $\mathfrak{G}$ and b be a nonsingular element of $\mathfrak{B}$. Then $gb=0$ only if $g=0$.*

For $b=\alpha e+c$ where c is in $\mathfrak{N}$ and $\alpha\neq 0$. Then $gb=gS_b-wf_g(b)=0$ only if $gS_b=\alpha g+gS_c=0$, $g=gS_a$ where $a=-\alpha^{-1}c$ is in $\mathfrak{N}$. Then $g\neq 0$ implies that $a\neq 0$, $S_a\neq 0$, $S_a^k\neq 0$ for some integer k such that $S_a^{k+1}=0$. However $g=gS_a=gS_a^2=\cdots=gS_a^k=gS_a^{k+1}$, a contradiction.

We next compute $P(z, za, zb, g)$ where a and b are in $\mathfrak{B}$ and g is in $\mathfrak{G}$. Thus $4a[g(zb)]+4b[g(za)]=3g(zab)+(zb)(ga)+(za)(gb)$ and so $-[4a\phi_g(b)+4b\phi_g(a)]w=-4w\phi_g(ab)=-3w\phi_g(ab)+(zb)[gS_a-wf_g(a)]+(za)[gS_b-wf_g(b)]=-3w\phi_g(ab)-w[\phi_{gS_a}(b)+\phi_{gS_b}(a)]-(zb)[wf_g(a)]-(za)[wf_g(b)]$. By (24) we have

$$[wf_g(a)](zb)+[wf_g(b)](za)=0. \tag{45}$$

Let us proceed to derive the following result.

LEMMA 17. *Let $g=(wa)(bz)$ where a and b are in $\mathfrak{B}$. Then $f_g(c)=\phi_g(c)=0$ for every c of $\mathfrak{B}$.*

Put $h=(wc)(bz)$, $k=(wa)(bcz)$, $t=[w(ac)]\cdot bz$. Then $wh=wk=wt=0$. Also $gc=gS_c-wf_g(c)$. Form $P(wa, bz, c, w)$ to obtain $4g(wc)+4a(bc)z=(wa)h+(bz)(3ac)+c(abz)+w(gc+k+t)=f_h(a)+z\phi_h(a)+4(abc)z-f_g(c)$. Hence $4[f_g(c)+z\phi_g(c)]=f_h(a)+z\phi_h(a)-f_g(c)$. It follows that $5f_g(c)=f_h(a)$ and $4\phi_g(c)=\phi_h(a)$. If we interchange a and c the result is an interchange of g and h. Then $5f_h(a)=f_g(c)$ and $25f_g(c)=f_g(c)$, $f_g(c)=0$. Similarly $4\phi_h(a)=\phi_g(c)$ and $16\phi_g(c)=\phi_g(c)$, $\phi_g(c)=0$.

The following tool formula will be derived as the next stage of our derivation.

LEMMA 18. *Let a, b, c be in $\mathfrak{B}$. Then*

$$[w(bc)](az)=b[(wc)(az)]+c[(wb)(az)]. \tag{46}$$

Form $P(wa, b, c, z)$ to get $4[w(ab)](cz)+4[w(ac)](bz)=3(wa)[(bc)z]+b[(wa)(cz)]+c[(wa)(bz)]$. But then $b[(wa)(cz)]+c[(wa)(bz)]=4[w(ab)](cz)+4[w(ac)](bz)+3[w(bc)](az)$. Form $P(w, az, bz, cz)$ to get $0=(az)[w(bc)]+(bz)[w(ac)]+(cz)[w(ab)]$. Then $[w(bc)](az)=-b[(wa)(cz)]-c[(wa)(bz)]$ and we have (46).

Formula (46) may be used as follows. From (45) and (46) we have $[wf_g(b)][z(ac)]+[wf_g(ac)](zb)=0=-[w(ac)][f_g(b)z]+w[af_g(c)+cf_g(a)]$

$\cdot(zb) = a[wf_g(c)\cdot(zb)] + f_g(c)[(wa)(bz)] + c[wf_g(a)\cdot(zb)] + f_g(a)[(wc)(bz)] + a[wf_g(b)\cdot(zc)] + c[wf_g(b)\cdot(az)]$ from which we have the following result.

LEMMA 19. *Let a, b, c be in $\mathfrak{B}$ and g be in $\mathfrak{G}$. Then*

$$f_g(c)[(wa)(bz)] + f_g(a)[(wc)(bz)] = 0. \tag{47}$$

As a consequence of Lemma 19 we have the following result.

LEMMA 20. *Let $f_g(a)$ be a nonsingular element of $\mathfrak{B}$ for some g of $\mathfrak{G}$ and some a of $\mathfrak{B}$. Then $(wc)(bz)=0$ for every b and c of $\mathfrak{B}$.*

For (47) implies that $f_g(a)[(wa)(bz)]=0$. By Lemma 16 we have $(wa)(bz)=0$. If c is any element of $\mathfrak{B}$ such that $f_g(c)$ is nonsingular then $(wc)(bz)=0$. By (47) we have $(wa)(bz)=0$ for every a.

We may also combine (32) with Lemma 16 and (46). By (46) we have $[w\phi_g(bc)](az) = w[c\phi_g(b) + b\phi_g(c)]\cdot(az) = \phi_g(b)[(wc)(az)] + \phi_g(c)[(wb)(az)] + c[w\phi_g(b)\cdot(az)] + b[w\phi_g(c)\cdot(az)]$. The last two terms vanish by (32). Hence

$$\phi_g(b)[(wc)(az)] + \phi_g(c)[(wb)(az)] = 0. \tag{48}$$

Consequently $\phi_g(b)[(wb)(az)]=0$. By an argument like that used to prove Lemma 20 we have

LEMMA 21. *Let $\phi_g(b)$ be nonsingular for some b in $\mathfrak{B}$ and some g in $\mathfrak{G}$. Then $(wa)(cz)=0$ for every a and c in $\mathfrak{B}$.*

Let a be in $\mathfrak{B}$ and g be in $\mathfrak{G}$. We compute $P(wa, z, g, g)$ to obtain $0=2g[g(az)\cdot w]+(aw)(g^2z) = -2g\phi_g(a)+(aw)(g^2z)$. Thus

$$gc = (aw)(g^2z), \qquad c = \phi_g(a), \qquad f_g[\phi_g(a)] = 0. \tag{49}$$

We are now ready to derive the following fundamental structure theorem.

THEOREM 2. *Let $\mathfrak{A}$ be an algebra for which there exists an element b of $\mathfrak{B}$ and an element g of $\mathfrak{G}$ such that either $f_g(b)$ or $\phi_g(b)$ is nonsingular. Then $[\mathfrak{A}_u(1/2)](Bz)=0$ so that $\phi_g(a)=0$ for every a of $\mathfrak{B}$. Also $(xa)b=x(ab)$ for every x of $\mathfrak{A}_u(1/2)$ and every a and b of $\mathfrak{B}$, $[\mathfrak{A}_u(1/2)]^2\subseteq\mathfrak{B}$.*

For Lemmas 20 and 21 imply that $(w\mathfrak{B})(\mathfrak{B}z)=0$. By (49) we have $gc=0$ for every g of $\mathfrak{G}$ and a of $\mathfrak{B}$ where $c=\phi_g(a)$. By Lemma 16 we see that $\phi_g(a)$ is singular for every a of $\mathfrak{B}$. Hence our hypothesis reduces to the assumption that $f_g(b)$ is nonsingular for some b of $\mathfrak{B}$ and g of $\mathfrak{G}$. We also have $gc=gS_c=0$ by (49) and $f_{gS_c}(b)=cf_g(b)=\phi_g(a)f_g(b)=0$ and the hypothesis that $f_g(b)$ is nonsingular implies that $\phi_g(a)=0$ for every a of $\mathfrak{B}$. If $f_h(b)$ is singular for an h in $\mathfrak{G}$ then $f_k(b)=f_g(b)+f_h(b)$ is nonsingular for $k=g+h$, $\phi_k(a)=\phi_h(a)=0$ for every a of $\mathfrak{B}$. Thus $\phi_h(a)=0$ for every h of $\mathfrak{G}$ and a of $\mathfrak{B}$. By (9) we have $\mathfrak{G}(\mathfrak{B}z)=0$, $[\mathfrak{A}_u(1/2)](\mathfrak{B}z)=0$. By (6), Lemma 3, and Lemma 5 we have $[\mathfrak{A}_u(1/2)]^2\subseteq\mathfrak{B}$. By Lemma 3 we know that $[(wc)a]b=w[c(ab)]=(wc)(ab)$ for every c, a, b in $\mathfrak{B}$. Also $(ga)b=[gS_a-wf_g(a)]b=gS_aS_b-wf_{gS_a}(b)-w[f_g(a)b]$

$=gS_{ab}-w[af_g(b)+bf_g(a)]=gS_{ab}-wf_g(ab)=g(ab)$ and our theorem is proved.

6. **Reduction to the nonsingular case.** We may write

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{L}, \qquad \mathfrak{C} = \mathfrak{A}_u(1) + \mathfrak{A}_u(0), \qquad \mathfrak{L} = \mathfrak{A}_u(1/2).$$

The element w of $\mathfrak{L}$ is not unique. Each determination of an element w_1 of $\mathfrak{L}$, such that $w_1^2=1$, results in a further decomposition

$$\mathfrak{C} = \mathfrak{B}_1 + z\mathfrak{B}_1, \qquad \mathfrak{L} = w_1\mathfrak{B}_1 + \mathfrak{G}_1,$$

where $\mathfrak{B}_1$ and $\mathfrak{G}_1$ depend upon w_1. The derivations defined by

$$(w_1b_1)g_1 = f_{g_1}(b_1) + z\phi_{g_1}(b_1)$$

also depend upon w_1. Assume then that $f_{g_1}(b_1)$ is singular for every g_1 in $\mathfrak{G}_1$, every b_1 in $\mathfrak{B}_1$, *and every choice of* w_1. By Theorem 2 it follows that $\phi_{g_1}(b_1)$ is singular for every choice of w_1.

Let the element w be selected, and define, for this choice,

$$\mathfrak{M} = \mathfrak{N} + \mathfrak{N}z, \tag{50}$$

where $\mathfrak{N}$ is the radical of the corresponding associative algebra $\mathfrak{B}$. Then $\mathfrak{M}$ is the radical of the associative algebra

$$\mathfrak{C} = e\mathfrak{F} + z\mathfrak{F} + \mathfrak{M}, \tag{51}$$

and our hypothesis implies that

$$(w\mathfrak{B})\mathfrak{G} \subseteq \mathfrak{M}, \qquad \mathfrak{G}(\mathfrak{B}z) \subseteq w\mathfrak{N}, \tag{52}$$

$$\mathfrak{G}\mathfrak{N} \subseteq \mathfrak{G}\mathfrak{N}^* + w\mathfrak{N}, \tag{53}$$

where $\mathfrak{N}^*$ is the *algebra* of all linear transformations S_a for a in $\mathfrak{N}$. We proceed to derive the following critical result.

LEMMA 22. *The inclusion relations* $(\mathfrak{L}\mathfrak{M})\mathfrak{C}\subseteq\mathfrak{L}\mathfrak{M}$, $(\mathfrak{L}\mathfrak{M})\mathfrak{L}\subseteq\mathfrak{M}$ *hold.*

For $\mathfrak{L}\mathfrak{M} = (w\mathfrak{B})\mathfrak{M} + \mathfrak{G}\mathfrak{M} = (w\mathfrak{B})\mathfrak{N} + (w\mathfrak{B})(\mathfrak{N}z) + \mathfrak{G}\mathfrak{N} + \mathfrak{G}(\mathfrak{N}z) = w\mathfrak{N} + (w\mathfrak{B})(\mathfrak{N}z) + \mathfrak{G}\mathfrak{N}^*$. Then $(\mathfrak{L}\mathfrak{M})\mathfrak{C} \subseteq (w\mathfrak{N})\mathfrak{B} + [(w\mathfrak{N})(\mathfrak{B}z)]\mathfrak{B} + (\mathfrak{G}\mathfrak{N}^*)\mathfrak{B} + (w\mathfrak{N})(\mathfrak{B}z) + [(w\mathfrak{N})(\mathfrak{B}z)](\mathfrak{B}z) + (\mathfrak{G}\mathfrak{N}^*)(\mathfrak{B}z)$. By Lemma 17 we know that $[(w\mathfrak{N})(\mathfrak{B}z)](\mathfrak{B}z)=0$. Evidently $(w\mathfrak{N})\mathfrak{B}\subseteq w\mathfrak{N}$, $(\mathfrak{G}\mathfrak{N}^*)(\mathfrak{B}z)\subseteq w\mathfrak{N}$. If a is in $\mathfrak{N}$ then $(gS_a)b=gS_aS_b-waf_g(b)$ and so $(\mathfrak{G}\mathfrak{N}^*)\mathfrak{B}\subseteq\mathfrak{G}\mathfrak{N}^*+w\mathfrak{N}$. Also if a is in $\mathfrak{N}$ and b and c are in $\mathfrak{B}$ we have $[(wb)(az)]c=[(wb)(az)]S_c$. Then $c=\alpha e+d$ where d is in $\mathfrak{N}$ and $[(wb)(az)]S_c=\alpha(wb)(az)+[(wb)(az)]S_d$ is in $(w\mathfrak{B})(\mathfrak{N}z)+\mathfrak{G}\mathfrak{N}^*$. Hence $[(w\mathfrak{B})(\mathfrak{N}z)]\mathfrak{B}\subseteq\mathfrak{L}\mathfrak{M}$. Also $(w\mathfrak{N})(\mathfrak{B}z)=(w\mathfrak{B})(\mathfrak{N}z)$ so we have completed our proof of the relation $(\mathfrak{L}\mathfrak{M})\mathfrak{C}\subseteq\mathfrak{L}\mathfrak{M}$.

We next compute $(\mathfrak{L}\mathfrak{M})\mathfrak{L}=(w\mathfrak{N})(w\mathfrak{B})+[(w\mathfrak{B})(\mathfrak{N}z)](w\mathfrak{B})+(\mathfrak{G}\mathfrak{N}^*)(w\mathfrak{B})+(w\mathfrak{N})\mathfrak{G}+[(w\mathfrak{B})(\mathfrak{N}z)]\mathfrak{G}+(\mathfrak{G}\mathfrak{N}^*)\mathfrak{G}$. The first component is $(w\mathfrak{N})(w\mathfrak{B})=\mathfrak{N}\mathfrak{B}\subseteq\mathfrak{N}$. The second component is zero by Lemma 17 and our hypothesis implies that $(\mathfrak{G}\mathfrak{N}^*)(w\mathfrak{B})+(w\mathfrak{N})\mathfrak{G}\subseteq\mathfrak{G}(w\mathfrak{B})\subseteq\mathfrak{M}$. By our hypotheses and Lemma 14 we have $\mathfrak{G}[(w\mathfrak{B})(\mathfrak{B}z)]\subseteq\mathfrak{N}$. It therefore remains only to show that

$$(\mathfrak{G}\mathfrak{N}^*)\mathfrak{G} \subseteq \mathfrak{G}, \tag{54}$$

that is, that $(gS_a)h$ is in $\mathfrak{M}$, and hence in $\mathfrak{N}$, for every a of $\mathfrak{N}$ and every g and h of $\mathfrak{G}$.

Up to this point in our argument we have used the assumption of the singularity of $f_g(a)$ and $\phi_g(a)$ only for the initially selected element w of $\mathfrak{L}$. We now compute $P(g, h, a, w)$ for g and h in $\mathfrak{G}$ and a in $\mathfrak{B}$ to obtain $4(gh)(wa) = g[h(wa) + (hw)a] + h[(wa)g + w(ag)] + w[(gh)a + (ga)h + (ha)g] + a[(gh)w] = g[z\phi_h(a)] + h[z\phi_g(a)] + w[g(hS_a) - wf_h(a)\cdot g + h(gS_a) - wf_g(a)\cdot h] + 2(gh)(wa)$. It follows that $2(gh)a + \phi_g[\phi_h(a)] + \phi_h[\phi_g(a)] + f_g[f_h(a)] + f_h[f_g(a)] = g(hS_a) + h(gS_a)$.

Write $a \equiv b$ for a and b in $\mathfrak{B}$ if $a-b$ is in $\mathfrak{N}$. In the singular case the result just derived implies that

$$2(gh)a \equiv g(hS_a) + h(gS_a). \tag{55}$$

When a is in $\mathfrak{N}$ we see that gh in $\mathfrak{B}$ implies that

$$g(hS_a) + h(gS_a) \equiv 0. \tag{56}$$

Let b be in $\mathfrak{N}$ and replace g by gS_b in (56) to obtain

$$(gS_b)(hS_a) + h(gS_{ab}) \equiv 0. \tag{57}$$

Also replace h by hS_b in (56) and then interchange a and b to obtain

$$(gS_b)(hS_a) + g(hS_{ab}) \equiv 0. \tag{58}$$

Subtract (58) from (57) to see that $g(hS_{ab}) \equiv h(gS_{ab})$. But ab is in $\mathfrak{N}$ and so (56) implies that $g(hS_{ab}) \equiv -h(gS_{ab})$. Then (58) implies that

$$g(hS_{ab}) \equiv (gS_a)(hS_b) \equiv 0, \tag{59}$$

that is,

$$\mathfrak{G}(\mathfrak{G}\mathfrak{N}^* \cdot \mathfrak{N}^*) \subseteq \mathfrak{N}, \qquad (\mathfrak{G}\mathfrak{N}^*)(\mathfrak{G}\mathfrak{N}^*) \subseteq \mathfrak{N}. \tag{60}$$

We are now ready to use our full singularity hypothesis.

If g is any element of $\mathfrak{G}$ and λ is in $\mathfrak{F}$ we find that $s = (\lambda w + g)^2 = \lambda^2 + g$ is in $\mathfrak{B}$. Then s is nonsingular for a value of $\lambda \neq 0$ in $\mathfrak{F}$. It follows that there exists a nonsingular polynomial $t = \psi(s)$ in $\mathfrak{F}[s] \subseteq \mathfrak{B}$ such that $w_1 = (\lambda w + g)t$ has the property that $w_1^2 = 1$, $w_1 = \lambda wt - wf_g(t) + gS_t = wa + gS_b + \alpha g$ where $t = \alpha + b$, $\alpha \neq 0$ is in $\mathfrak{F}$, b is in $\mathfrak{N}$, $a = \lambda t - f_g(t)$ is in $\mathfrak{B}$. We have already shown that $(\mathfrak{L}\mathfrak{M})(w_1\mathfrak{B}_1) \subseteq \mathfrak{M}$ for every w_1 of $\mathfrak{L}$ where $\mathfrak{B}_1 = \mathfrak{B}_{w_1}$ for the singular case. Clearly $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}z$ is unchanged when w is replaced by w_1, and thus $(\mathfrak{L}\mathfrak{M})w_1 \subseteq \mathfrak{M}$. If c is in $\mathfrak{N}$ and h is in $\mathfrak{G}$ we know that hS_c is in $\mathfrak{L}\mathfrak{M}$, and $w_1(hS_c) = (wa + gS_b + \alpha g)(hS_c) = \alpha g(hS_c) + (wa)(hS_c) + (gS_b)(hS_c)$ is in $\mathfrak{M}$. By the singularity hypothesis $(wa)(hS_c)$ is in $\mathfrak{M}$ and by (60), $(gS_b)(hS_c)$ is in $\mathfrak{M}$. Hence $g(hS_c)$ is in $\mathfrak{M}$ for every g and h of $\mathfrak{G}$ and c of $\mathfrak{N}$, $\mathfrak{G}(\mathfrak{G}\mathfrak{N}^*) \subseteq \mathfrak{N}$ and our proof of Lemma 22 is complete.

Lemma 22 now implies that $(\mathfrak{M}+\mathfrak{L}\mathfrak{M})\mathfrak{A}=(\mathfrak{M}+\mathfrak{L}\mathfrak{M})(\mathfrak{E}+\mathfrak{L})=\mathfrak{E}\mathfrak{M}+\mathfrak{E}(\mathfrak{L}\mathfrak{M})+\mathfrak{L}\mathfrak{M}+(\mathfrak{L}\mathfrak{M})\mathfrak{L}\subseteq\mathfrak{M}+\mathfrak{L}\mathfrak{M}$. Thus $\mathfrak{M}+\mathfrak{L}\mathfrak{M}$ is an ideal of $\mathfrak{A}$ which does not contain the unity quantity of $\mathfrak{A}$. The hypothesis that $\mathfrak{A}$ is simple implies that $\mathfrak{M}=0$ contrary to hypothesis. Thus the singular case cannot exist and hence all simple stable algebras are given by Theorems 1 and 2.

7. **The final structure theorems.** Let $\mathfrak{B}$ be a commutative associative algebra with a unity element e over a field $\mathfrak{F}$ of characteristic p and $\mathfrak{B}$ have degree one so that $\mathfrak{B}=e\mathfrak{F}+\mathfrak{N}$, where $\mathfrak{N}$ is the radical of $\mathfrak{B}$. We form a vector space

$$\mathfrak{L} = (y_0\mathfrak{B}, \cdots, y_m\mathfrak{B}) \tag{61}$$

which is the sum (but not necessarily the vector space direct sum) of $m+1$ homomorphic images $y_i\mathfrak{B}$ of $\mathfrak{B}$. We then form the vector space direct sum

$$\mathfrak{T} = \mathfrak{B} + \mathfrak{L}, \tag{62}$$

and propose to define a product on $\mathfrak{T}\mathfrak{T}$ to $\mathfrak{T}$ preserving the commutative associative product in $\mathfrak{B}$.

Let $\mathfrak{D}$ be a set of $(m+1)^2$ derivations D_{ij} of $\mathfrak{B}$ subject to the conditions

$$D_{ij} = -D_{ji}, \qquad D_{ii} = 0 \qquad (i, j = 0, \cdots, m). \tag{63}$$

Put

$$y_0^2 = e, \qquad y_0y_j = 0, \qquad y_iy_j = b_{ij} = b_{ji} \qquad (i, j = 1, \cdots, m) \tag{64}$$

where the b_{ij} are arbitrary elements of $\mathfrak{B}$ and define

$$(y_ia)(y_jb) = f_{ij}(a, b) = f_{ji}(b, a) = (b_{ij})(ab) + \phi_{ij}(a, b), \tag{65}$$

$$\phi_{ij}(a, b) = (aD_{ij})b - a(bD_{ij}). \tag{66}$$

Assume also that

$$(y_ia)b = b(y_ia) = y_i(ab) \tag{67}$$

for every a and b of $\mathfrak{B}$. We have now completed our definition of an algebra $\mathfrak{T}$ over $\mathfrak{F}$.

The algebra $\mathfrak{T}$ is power-associative. Indeed the relation

$$f_{ij}(b, ac) + f_{ij}(ab, c) = 2af_{ij}(b, c) \tag{68}$$

is equivalent to $\phi_{ij}(b, ac)+\phi_{ij}(ab, c)+(y_iy_j)[b(ac)+(ab)c]=2a[\phi_{ij}(b, c)+(y_iy_j)(bc)]$. But $\mathfrak{B}$ is associative and so (67) is equivalent to $\phi_{ij}(b, ac)+\phi_{ij}(ab, c)=2a\phi_{ij}(b, c)$. This relation is an easy consequence of (66). Then (67) is known to be equivalent to

$$(xa)(xb) = x^2(ab) \tag{69}$$

for every x of $\mathfrak{L}$ and a and b of $\mathfrak{B}$. But (66) implies that

$$(xa)b = x(ab) \tag{70}$$

for every x of $\mathfrak{L}$ and a and b of $\mathfrak{B}$. It should now be clear that $\mathfrak{B}+x\mathfrak{B}$ is an associative algebra for every x of $\mathfrak{L}$, and it is then true that $\mathfrak{T}$ is power-associative.

The algebra $\mathfrak{T}$ may be imbedded in a vector space direct sum

$$\mathfrak{A} = \mathfrak{T} + z\mathfrak{B} \tag{71}$$

where $(za)(zb)=ab$ for every a and b of $\mathfrak{B}$, $\mathfrak{C}=\mathfrak{B}+z\mathfrak{B}$ is associative and $(z\mathfrak{B})\mathfrak{L}=0$. It is then easy to see that $\mathfrak{A}$ is power associative. Clearly $\mathfrak{A}$ is partially stable of degree two where $2u=1+z$.

An ideal $\mathfrak{R}$ of $\mathfrak{B}$ will be called a $\mathfrak{D}$-ideal if aD is in $\mathfrak{R}$ for every a of $\mathfrak{R}$ and D of $\mathfrak{D}$. We say that $\mathfrak{B}$ is $\mathfrak{D}$-simple if $\mathfrak{B}$ has no proper $\mathfrak{D}$-ideals. We now have the following simplicity condition:

THEOREM 3. *The algebra $\mathfrak{A}$ of* (71) *is simple if and only if $\mathfrak{B}$ is $\mathfrak{D}$-simple and there exists no element g in $\mathfrak{L}$ such that $gx=0$ for every x of $\mathfrak{L}$.*

For let $\mathfrak{W}$ be an ideal of $\mathfrak{A}$ and $\mathfrak{W}\neq\mathfrak{A}$. The intersection $\mathfrak{B}_0$ of $\mathfrak{W}$ and $\mathfrak{B}$ is an ideal of $\mathfrak{B}$ and $\mathfrak{B}_0\neq\mathfrak{B}$ since otherwise $\mathfrak{B}_0$ would contain the unity element e of $\mathfrak{A}$, e would be in $\mathfrak{W}$, $\mathfrak{W}=\mathfrak{A}$. If c is in $\mathfrak{B}_0$ then $(y_ic)y_j=(y_iy_j)c+cD_{ij}$ is in $\mathfrak{W}$ and hence in $\mathfrak{B}_0$, $(y_iy_j)c$ is in $\mathfrak{B}_0$, cD_{ij} is in $\mathfrak{B}_0$ for every c of $\mathfrak{B}_0$ and $\mathfrak{B}_0$ is a $\mathfrak{D}$-ideal contrary to hypothesis. Hence $\mathfrak{B}_0=0$. If cz is in $\mathfrak{W}$ with c in $\mathfrak{B}$, then $(cz)z=c$ is in $\mathfrak{B}_0$ and so $cz=0$. If y is any nonzero element of $\mathfrak{W}$ we may write $y=b+cz+g$ where b and c are in $\mathfrak{B}$ and g is in $\mathfrak{L}$. Then $y_1=z(zy)=b+cz$ is in $\mathfrak{W}$ and so are $(y_1w)w=b$ and $cz=y_1-b$. Hence $b=c=0$ and $g\neq 0$. But xg is in $\mathfrak{B}_0$ for every x of $\mathfrak{L}$ and so $xg=0$ for every x of $\mathfrak{L}$ contrary to hypothesis. Hence $\mathfrak{A}$ is simple.

We are now in a position to derive our final structure theorem.

THEOREM 4. *Every simple commutative power associative algebra is an algebra of the kind described in Theorem* 3.

For we have already seen that $\mathfrak{A}=\mathfrak{B}+\mathfrak{B}z+\mathfrak{L}$ where the conditions above hold for products $\mathfrak{B}\mathfrak{L}$ and $(\mathfrak{B}z)\mathfrak{L}$. We may clearly write $\mathfrak{G}=\mathfrak{H}+\mathfrak{H}\mathfrak{N}^*$ where $\mathfrak{H}$ is a subspace of $\mathfrak{G}$. Indeed if $\mathfrak{G}\mathfrak{N}^*$ has dimension μ and $\mathfrak{G}$ has dimension $m+\mu$ then there exist m linearly independent elements $y_1, \cdots, y_m$ in $\mathfrak{G}$ such that if $\mathfrak{H}=y_1\mathfrak{F}+\cdots+y_m\mathfrak{F}$ then $\mathfrak{G}=\mathfrak{H}+\mathfrak{G}\mathfrak{N}^*$. Thus if h is in $\mathfrak{H}$ and g is in $\mathfrak{G}$ we have $h+gS_a=0$ for a in $\mathfrak{B}$ only if $h=gS_a=0$. Then $\mathfrak{G}\mathfrak{N}^*=(\mathfrak{H}\mathfrak{N}^*, \mathfrak{H}\mathfrak{N}^{*2}, \cdots, \mathfrak{H}\mathfrak{N}^{*k})\subseteq\mathfrak{H}\mathfrak{N}^*\subseteq\mathfrak{G}\mathfrak{N}^*$, $\mathfrak{G}=\mathfrak{H}+\mathfrak{H}\mathfrak{N}^*$. Put $y_0=w$, define $y_iy_j=b_{ij}$ and we have (64). It then remains only to show that (65), (66) hold for derivations D_{ij} of $\mathfrak{B}$.

We already know that $(wa)(gb)=(y_0a)[gS_b-wf_g(b)]=f_{gS_b}(a)-af_g(b)$ $=bf_g(a)-af_g(b)$. This yields the result

$$(y_0a)(y_jb) = \phi_{0j}(a, b) \tag{72}$$

of (65) where $D_{0j}(a)=f_g(a)$ for $g=y_j$. There remain the formulas for $(y_ia)(y_jb)$ where $i, j=1, \cdots, m$.

If h is in $\mathfrak{H}$ we have already derived the property that there exist elements c in $\mathfrak{B}$ and $\lambda\neq 0$ in $\mathfrak{F}$ such that $w_1=\lambda wc+hc$, $w_1^2=1$. Also xy is in $\mathfrak{B}$ for every x and y of $\mathfrak{L}$ by Theorem 2. But w_1 defines a subalgebra $\mathfrak{B}_1$ of $\mathfrak{C}$ such that $(w_1b_1)w_1=b_1$ for every b_1 of $\mathfrak{B}_1$. Hence $\mathfrak{B}_1\subseteq\mathfrak{B}$. But $\mathfrak{B}_1$ and $\mathfrak{B}$ are both isomorphic to $\mathfrak{A}_u(1)$ and so $\mathfrak{B}_1=\mathfrak{B}$. If k is in $\mathfrak{H}$ there exists an element q in $\mathfrak{G}_1$ and an element d in $\mathfrak{B}$ such that $k=w_1d+q$. Here $\mathfrak{G}_1$ is the set of all elements g_1 of $\mathfrak{L}$ such that $g_1w_1=0$. Then $(w_1a)(kb)=(w_1a)[(w_1d+q)b]=(ab)d+bf_q^{(1)}(a)-af_q^{(1)}(b)$. However $(w_1a)(kb)=[\lambda w(ca)+h(ca)](kb)=[h(ca)](kb)+\lambda bf_k(ca)-(\lambda ca)f_k(a)$. Replace a by ac^{-1} to obtain

$$(ha)(kb) = (ab)(dc^{-1}) + bf_{h,k}(a) - af_{h,k}(b). \tag{73}$$

Here $f_{h,k}(a)=f_q^{(1)}(a)-\lambda f_k(a)$ is clearly a derivation of $\mathfrak{B}$ for every h and k of $\mathfrak{H}$. Take $a=b=1$ to get $hk=dc^{-1}$. Thus $(ha)(kb)=(ab)(hk)+bf_{h,k}(a)-af_{h,k}(b)$ and so $(kb)(ha)=(ab)(hk)+af_{k,h}(b)-bf_{k,h}(a)=(ha)(kb)$ so we have

$$f_{h,k}(a) = -f_{k,h}(a), \qquad f_{h,h}(a) = 0. \tag{74}$$

Put $b_{ij}=y_iy_j$, $h=y_i$, $k=y_j$ to obtain (65), (66) where $aD_{ij}=f_{h,k}(a)$. This completes our proof.

The condition that $\mathfrak{B}$ be $\mathfrak{D}$-simple is a restriction on $\mathfrak{B}$ which leads to an unsolved question on associative algebras which is being studied. The restrictions on $\mathfrak{L}$ implied by the condition $gx=0$ for every x of $\mathfrak{L}$ only if $g=0$ is another simplicity condition which requires further study.

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