A SCALAR TRANSPORT EQUATION

BY
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1. Introduction and physical background. This paper is concerned with the existence and uniqueness of the solutions of the following equation:

$$
\begin{aligned}
\frac{\partial f(x,t)}{\partial t} = {} & \frac{1}{2}\int_0^x f(y,t)f(x-y,t)\phi(y,x-y)dy \\
& - f(x,t)\int_0^\infty f(y,t)\phi(x,y)dy \\
& + \int_x^\infty f(y,t)\psi(y,x)dy - \frac{f(x,t)}{x}\int_0^x y\psi(x,y)dy.
\end{aligned}
\tag{1}
$$

Here the variables x, y, t are non-negative, and the functions $f(x, 0)$, $\phi(x, y)$ and $\psi(x, y)$ are assumed to be known. The main result (Theorem 1) is that under certain hypotheses on $f(x, 0)$, $\phi(x, y)$ and $\psi(x, y)$ there exists a continuous solution $f(x, t)$, valid for $x, t \geqq 0$, which is non-negative, analytic in t for each x, and integrable in x for each t. Another hypothesis guarantees uniqueness.

A special form of equation (1), with $\psi \equiv 0$, was treated from a practical point of view in [3]. More recently an existence theorem has been proved by Morgenstern [4], which applies to a general class of equations including the case $\psi \equiv 0$ of equation (1). The method of proof used in the present paper applies not only to equation (1) but also to certain other equations of the form

$$\frac{\partial f(x,t)}{\partial t} = A[f(x,t), f(x,t), t] + B[f(x,t), t]$$

where A and B are suitable bounded operators and A is bilinear and symmetric, while B is linear. In particular, the method yields some results of [4]. However, this subject will not be treated here.

Equation (1) arises in a number of problems in physics, meteorology and colloid chemistry. Perhaps it might also be used to treat certain situations in sociology and in astrophysics. Consider a non-negative scalar quantity, say mass, and assume that some fixed volume of space contains a number of randomly moving mass-particles. The particle-mass distribution varies as a result of two processes, coalescence and breakdown. It is assumed that the

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total number of particles is large enough to justify the use of the density function $f(x, t)$; $f(x, t)dx$ is then the average number of particles of mass x to $x+dx$. This average and all other averages are referred to a unit volume. The coalescence function $\phi(x, y)$ is introduced by assuming that the average number of coalescences between particles of mass x to $x+dx$ and those of mass y to $y+dy$, is $f(x, t)f(y, t)\phi(x, y)dxdydt$ during the time-interval $t-t+dt$. Likewise, the breakdown function $\psi(x, y)$ enters through the assumption that $f(x, t)\psi(x, y)dxdydt$ is the average number of particles of mass y to $y+dy$ created from the breakdown of particles of mass x to $x+dx$, during the time-interval $t-t+dt$. Conservation of mass implies that $\partial f(x, t)/\partial t$ is equal to a sum of four terms which express, respectively, the rates of:

(a) production of particles x by coalescence of particles y and $x-y$ $(0 \leqq y \leqq x)$,

(b) disappearance of particles x, due to their coalescence with particles y $(0 \leqq y < \infty)$,

(c) production of particles x in the process of breakdown of particles y $(x \leqq y < \infty)$, and

(d) disappearance of particles x, due to their breakdown into particles y $(0 \leqq y \leqq x)$.

When the various rates are expressed in terms of $f(x, t)$, $\phi(x, y)$ and $\psi(x, y)$, one obtains equation (1). The following relations hold in virtue of their obvious physical significance:

$$f(x, 0) \geqq 0,$$

$$0 \leqq \phi(x, y) = \phi(y, x),$$

$$0 \leqq \psi(x, y), \qquad \psi(x, y) = 0 \text{ if } y > x,$$

$$\int_0^x y\psi(x, y)dy \leqq x.$$

The last inequality expresses the fact that a mass x cannot break down into fragments of combined mass exceeding x.

Two generalizations of the above formulation suggest themselves: (1) time-dependent coalescence and breakdown, $\phi(x, y, t)$ and $\psi(x, y, t)$, and (2) presence of several kinds of particles. The latter case might be of importance in the treatment of nucleation in meteorology and in colloid chemistry. These alternative formulations will be treated elsewhere.

The following example illustrates the applicability of equation (1). Chandrasekhar [1] considered the problem of colloidal coagulation. He arrived at an infinite system of differential equations

$$\frac{dx_n(t)}{dt} = \frac{1}{2}\sum_{k=1}^{n-1} x_k(t)x_{n-k}(t) - x_n(t)\sum_{k=1}^{\infty} x_k(t), \qquad n = 1, 2, \cdots,$$

which he solved by an inductive procedure. His solution may be obtained by solving equation (1) under the assumptions $\phi(x, y) \equiv 1$, $\psi(x, y) \equiv 0$, $f(x, 0) = \delta(x-1)$; one obtains thus

$$f(x, t) = \sum_{k=1}^{\infty} x_k(t)\delta(x - k).$$

In this connection, see [3]. The use of the Dirac delta-function may be dispensed with by introducing Stieltjes integrals into equation (1). Another application of equation (1) will be found in [3].

The existence proof presented in this paper is constructive, and the solution $f(x, t)$ of equation (1) is exhibited in a form allowing computation and estimation of errors due to approximation.

2. **The operators $[f, g]$ and Lf.** In the remainder of this paper $\phi(x, y)$ and $\psi(x, y)$ will be two functions defined for $x, y \geqq 0$ and satisfying the following hypotheses:

(H_1) $\phi(x, y)$ is continuous,

$$0 \leqq \phi(x, y) = \phi(y, x) \leqq \text{l.u.b. } \phi(x, y) = A < \infty,$$

(H_2) $\psi(x, y)$ is continuous,

$$0 \leqq \psi(x, y) \leqq \text{l.u.b. } \psi(x, y) = C < \infty,$$

$$\int_0^x y\psi(x, y)dy \leqq x,$$

$$\int_0^x \psi(x, y)dy \leqq E - 1 < \infty.$$

Let $f(x)$ and $g(x)$ be two functions in the class $L'(0, \infty)$. Define the operators $[f, g]$ and Lf as follows:

$$[f(x), g(x)] = \frac{1}{2}\left\{\int_0^x f(y)g(x - y)\phi(y, x - y)dy - f(x)\int_0^\infty g(y)\phi(x, y)dy - g(x)\int_0^\infty f(y)\phi(x, y)dy\right\}, \tag{2}$$

$$Lf(x) = \int_x^\infty f(y)\psi(y, x)dy - \frac{f(x)}{x}\int_0^x y\psi(x, y)dy. \tag{3}$$

It follows from these definitions that $[f, g]$ is a symmetric, bilinear operator and Lf is a linear operator. If f and g are continuous, so are $[f, g]$ and Lf. Certain estimates, essential in the later work, will now be developed.

By elementary integral inequalities

$$| [f, g] | \leqq \frac{A}{2} \left\{ \int_0^x | f(y) | \, | g(x-y) | \, dy + | f(y) | \int_0^\infty | g(y) | \, dy + | g(x) | \int_0^\infty | f(y) | \, dy \right\},$$

and therefore

$$\text{l.u.b.} \, | [f, g] | \leqq \frac{A}{2} \left\{ \min \left(\text{l.u.b.} \, | f(x) | \int_0^\infty | g(y) | \, dy, \ \text{l.u.b.} \, | g(x) | \int_0^\infty | f(y) | \, dy \right) + \text{l.u.b.} \, | f(x) | \int_0^\infty | g(y) | \, dy + \text{l.u.b.} \, | g(x) | \int_0^\infty | f(y) | \, dy \right\}.$$

Since

$$\min(a, b) \leqq (a+b)/2,$$

it follows that

$$\text{l.u.b.} \, | [f, g] | \leqq \frac{3A}{4} \left\{ \text{l.u.b.} \, | f(x) | \int_0^\infty | g(y) | \, dy + \text{l.u.b.} \, | g(x) | \int_0^\infty | f(y) | \, dy \right\}. \tag{4}$$

In particular,

$$\text{l.u.b.} \, | [f, f] | \leqq \frac{3A}{2} \, \text{l.u.b.} \, | f(x) | \int_0^\infty | f(y) | \, dy. \tag{5}$$

Similarly one obtains

$$\int_0^\infty | [f, g] | \, dx \leqq \frac{3A}{2} \int_0^\infty | f(y) | \, dy \int_0^\infty | g(y) | \, dy. \tag{6}$$

In particular,

$$\int_0^\infty | [f, f] | \, dx \leqq \frac{3A}{2} \left(\int_0^\infty | f(y) | \, dy \right)^2. \tag{7}$$

Similar estimates are obtained for Lf:

$$\text{l.u.b.} \, | Lf | \leqq C \int_0^\infty | f(y) | \, dy + \text{l.u.b.} \, | f(x) |, \tag{8}$$

and by an elementary transformation of a double integral,

$$
(9)\qquad
\begin{aligned}
\int_0^\infty |Lf|\,dx &\leqq \int_0^\infty \int_x^\infty |f(y)|\,\psi(y, x)dydx + \int_0^\infty |f(y)|\,dy \\
&= \int_0^\infty \int_0^x |f(x)|\,\psi(x, y)dydx + \int_0^\infty |f(y)dy \\
&\leqq (E - 1 + 1)\int_0^\infty |f(y)|\,dy \\
&= E\int_0^\infty |f(y)|\,dy.
\end{aligned}
$$

It may be remarked here that $[f, g]$ and Lf satisfy identically the equations

$$
\int_0^\infty x[f, g]dx = 0, \qquad \int_0^\infty xLfdx = 0.
$$

From the physical formulation of the problem leading to equation (1), it will be obvious that the above equations express the conservation of mass.

3. **Local existence of solutions.**

LEMMA 1. *Let $f(x, 0)$ be a continuous, non-negative, bounded and integrable function. Let $0 < l.u.b.\ f(x, 0) = B < \infty$ and $\int_0^\infty f(x, 0)dx = N < \infty$. Let $\phi(x, y)$ and $\psi(x, y)$ satisfy hypotheses* (H_1) *and* (H_2). *Then there exists a number $m = m(A, B, C, N, E)$, $0 < m < \infty$, such that equation* (1) *possesses a solution $f(x, t)$ valid on the interval $0 \leqq t < 1/m$. This solution is continuous, analytic in t for each x, and integrable in x for each t.*

Proof. Equation (1) may be written as

$$
\frac{\partial f(x, t)}{\partial t} = [f(x, t), f(x, t)] + Lf(x, t).
$$

Consider the integrated form of the latter equation:

$$
(10)\qquad f(x, t) = f(x, 0) + \int_0^t \{[f(x, t), f(x, t)] + Lf(x, t)\}dt
$$

and assume that

$$
(11)\qquad f(x, t) = \sum_{k=0}^\infty a_k(x)t^k.
$$

If this power-series is substituted for $f(x, t)$ in (10) one obtains, on equating the coefficients of like powers of t,

$$
(12)\qquad a_0(x) = f(x, 0), \qquad a_{k+1}(x) = \frac{1}{k+1}\left\{\sum_{i+j=k} [a_i, a_j] + La_k\right\}.
$$

Let

$$A_n = \int_0^\infty |a_n(x)|\,dx, \qquad B_n = \text{l.u.b.}\ |a_n(x)|,$$

so that $A_0 = N$ and $B_0 = B$. The following estimates follow from the recursion-formula (12) and from the estimates (4)–(9):

$$B_{n+1} \leqq \frac{1}{n+1}\frac{3A}{2}\left\{\sum_{i+j=n} A_iB_j + B_n + CA_n\right\}, \tag{13}$$

$$A_{n+1} \leqq \frac{1}{n+1}\frac{3A}{2}\left\{\sum_{i+j=n} A_iA_j + EA_n\right\}. \tag{14}$$

Assume that

$$A_k \leqq Nm^k, \qquad B_k \leqq Bm^k, \qquad k = 0, 1, \cdots, n. \tag{15}$$

Then it follows, on estimating the right-hand sides of (13) and (14), that the inequalities in (15) hold also for $k = n+1$, provided that

$$m = \frac{3AN}{2} + \max\left(E, \frac{CN}{B}\right). \tag{16}$$

Since (15) is valid for $k=0$, this completes the induction, and it follows that (15) is valid for all k. The coefficients $a_k(x)$ are continuous functions of x for all k; this is easily shown by induction on k in the recursion-formula (12). Therefore, with m defined by equation (16), the power-series in (11) converges uniformly on the interval $0 \leqq t < 1/m$, and it represents on that interval a function $f(x, t)$ satisfying all the conditions of the lemma. On differentiating both sides of equation (10) w.r.t. t, the proof of the lemma is completed.

The following estimates, valid on the interval $0 \leqq t < 1/m$, will be needed later on.

$$\begin{aligned}
&|f(x,t)| \leqq \frac{B}{1-mt}, && \int_0^\infty |f(x,t)dx \leqq \frac{N}{1-mt},\\
&\left|\frac{\partial f(x,t)}{\partial t}\right| \leqq \frac{Bm}{(1-mt)^2}, && \int_0^\infty \left|\frac{\partial f(x,t)}{\partial t}\right| dx \leqq \frac{Nm}{(1-mt)^2},\\
&\left|\frac{\partial^2 f(x,t)}{\partial t^2}\right| \leqq \frac{2Bm^2}{(1-mt)^3}, && \int_0^\infty \left|\frac{\partial^2 f(x,t)}{\partial t^2}\right| dx \leqq \frac{2Nm^2}{(1-mt)^3}.
\end{aligned} \tag{17}$$

4. Non-negativity of $f(x, t)$.

LEMMA 2. *Under the hypotheses of Lemma* 1 *the solution $f(x, t)$ of equation* (1) *obtained in the last section, is non-negative on the interval* $0 \leqq t < 1/m$.

Proof. Consider a fixed interval $0 \leqq t \leqq \tau$, where $0 < \tau < 1/m$, and a fixed positive integer n. Define

$$(18)\quad f_{0n}(x) = f(x, 0), \quad f_{k+1n}(x) = f_{kn}(x) + \frac{\tau}{n}\{[f_{kn}(x), f_{kn}(x)] + Lf_{kn}(x)\}, \qquad k = 0, 1, \cdots, n-1.$$

Let

$$(19)\qquad T_{kn} = \int_0^\infty |f_{kn}(x)|\,dx, \qquad L_{kn} = \text{l.u.b. } |f_{kn}(x)|,$$

so that $T_{0n} = N$ and $L_{0n} = B$. From the recursion-formula in (18) and from the estimates (4)–(9) one obtains

$$(20)\qquad T_{k+1n} \leqq T_{kn}\left(1 + \frac{\tau E}{n}\right) + \frac{3AN\tau}{2n} T_{kn}^2,$$

$$(21)\qquad L_{k+1n} \leqq L_{kn}\left(1 + \frac{\tau}{n}\right) + \frac{C\tau}{n} T_{kn} + \frac{3AN\tau}{2n} T_{kn}L_{kn}.$$

Consider first the inequality (20); in order to obtain an upper bound on T_{kn}, one may replace the inequality sign in (20) by an equality sign. Let

$$a = 1 + \frac{\tau E}{n},$$

$$t_{kn} = \frac{3AN\tau}{2n\alpha} T_{kn}, \; h(x) = a(x + x)^2.$$

Then the equation, obtained from (20) by replacing the inequality sign by an equality sign, becomes

$$t_{k+1n} = h(t_{kn}).$$

On the interval $0 < x < 1$ the function $h(x)$ satisfies the inequality

$$0 < h(x) < \frac{ax}{1 - x};$$

on taking kth iterates throughout, it follows that

$$(22)\qquad 0 < h^{(k)}(x) < a^k x \bigg/ \left(1 - \frac{a^k - 1}{a - 1}\, x\right),$$

where $h^{(k)}(x)$ denotes the kth iterate of $h(x)$ and the denominator of the fraction on the right is positive. Now

$$a^k < a^n = \left(1 + \frac{\tau E}{n}\right)^n < e^{\tau E}.$$

Returning to the estimating of T_{kn}, one obtains by means of (22)

$$T_{kn} \leqq T_{nn} \leqq \frac{3ANe^{\tau E}}{3A - 2E(e^{\tau E} - 1)} = C_1 < \infty, \tag{23}$$

provided only that

$$\tau < \frac{1}{E} \log \left(1 + \frac{3A}{2E}\right).$$

Now one obtains an estimate for L_{kn} in the same way,

$$\begin{aligned} L_{kn} \leqq L_{nn} \leqq B \exp\left(\tau\left(1 + \frac{3AC_1N^2}{2}\right)\right) \\ + \frac{2CC_1N}{2 + 3AC_1N^2}\left[\exp\left(\tau\left(1 + \frac{3AC_1N^2}{2}\right)\right) - 1\right] = C_2 < \infty. \end{aligned} \tag{24}$$

The estimates (23) and (24) imply that the functions $f_{kn}(x)$ are bounded and integrable. Induction on k in the recursion-formula in (18) shows that the functions $f_{kn}(x)$ are continuous. When the recursion-formula in (18) is written out in full, it becomes

$$\begin{aligned} f_{k+1n}(x) = \frac{\tau}{2n}\int_0^x f_{kn}(y)f_{kn}(x-y)\phi(y, x-y)dy + \frac{\tau}{n}\int_x^\infty f_{kn}(y)\psi(y, x)dy \\ + f_{kn}(x)\left\{1 - \frac{\tau}{n}\left[\int_0^\infty f_{kn}(y)(x, y)dy + \frac{1}{x}\int_0^x y(x, y)\right]\right\}. \end{aligned} \tag{25}$$

Assume now that

$$n > n_0 = \tau(AC_1 + 1).$$

Then the expression in the curly brackets in equation (25) is non-negative, and induction on k shows that $f_{kn}(x) \geqq 0$, $k = 0, 1, \cdots, n$. Define now

$$F_n(x, t) = f_{kn}(x), \qquad \frac{k\tau}{n} \leqq t < \frac{k+1}{n}\tau.$$

Let

$$\alpha_k = \underset{x, k\tau/n \leqq t < ((k+1)/n)\tau}{\text{l.u.b.}} \left| f(x, t) - F_n(x, t) \right| = \text{l.u.b.} \left| f(x, t) - f_{kn}(x) \right|,$$

$$\begin{aligned} \beta_k &= \underset{k\tau/n \leqq t < ((k+1)/n)\tau}{\text{l.u.b.}} \int_0^\infty \left| f(x, t) - F_n(x, t) \right| dx \\ &= \text{l.u.b.} \int_0^\infty \left| f(x, t) - f_{kn}(x) \right| dx. \end{aligned}$$

Recursion-formulae will now be obtained for α_k and β_k. Assume that $k \geqq 1$, then

$$(26)\qquad \begin{aligned} f(x, t) - f_{kn}(x) &= f(x, t) - f\left(x, t - \frac{\tau}{n}\right) + f\left(x, t - \frac{\tau}{n}\right) - f_{k-1n}(x) \\ &\quad - \frac{\tau}{n}\left\{[f_{k-1n}(x), f_{k-1n}(x)] + Lf_{k-1n}(x)\right\}. \end{aligned}$$

Since $f(x, t)$ is analytic in t, one may apply to it mean-value theorems. In this way one obtains

$$f(x, t) - f\left(x, t - \frac{\tau}{n}\right) = \frac{\tau}{n} \frac{\partial f(x, t - (\tau/n))}{\partial t} + \frac{\theta_1 \tau^2}{n^2} \frac{\partial^2 f(x, t - (\theta_2 \tau/n))}{\partial t^2} \qquad (0 \leqq \theta_i \leqq 1).$$

After some manipulation of equation (26) one finds that

$$(27)\qquad \begin{aligned} &| f(x, t) - f_{kn}(x) | \\ &\leqq \left| f\left(x, t - \frac{\tau}{n}\right) - f_{k-1n}(x) \right| + \frac{\tau}{n}\left\{ \left| \left[f\left(x, t - \frac{\tau}{n}\right) + f_{k-1n}(x), f\left(x, t - \frac{\tau}{n}\right) \right.\right.\right. \\ &\qquad \left.\left. - f_{k-1n}(x) \right] \right| + \left.\left| L\left[f\left(x, t - \frac{\tau}{n}\right) - f_{k-1n}(x) \right] \right| \right\} \\ &\qquad + \frac{\theta_1 \tau^2}{n^2} \left| \frac{\partial^2 f(x, t - (\theta_2 \tau/n))}{\partial t^2} \right|. \end{aligned}$$

The estimates (17), (23) and (24) imply that

$$\text{l.u.b.} \left| f\left(x, t - \frac{\tau}{n}\right) + f_{k-1n}(x) \right| \leqq C_3,$$

$$\text{l.u.b.} \int_0^\infty \left| f\left(x, t - \frac{\tau}{n}\right) + f_{k-1n}(x) \right| dx \leqq C_4,$$

$$\text{l.u.b.} \left| \frac{\partial^2 f(x, t - (\theta_2 \tau/n))}{\partial t^2} \right| \leqq C_5,$$

where C_3, C_4, C_5 are finite, positive constants. On taking the l.u.b. throughout (27) and using the estimates (4)–(9) and (17), one obtains

$$(28)\qquad \alpha_k \leqq \alpha_{k-1} + \frac{3A\tau}{4n}(C_3\beta_{k-1} + C_4\alpha_{k-1}) + \frac{\tau}{n}(C\beta_{k-1} + \alpha_{k-1}) + \frac{C_5\tau^2}{n^2}.$$

A similar inequality is obtained for the β_k by integrating (27) w.r.t. x from 0 to ∞:

$$(29)\qquad \beta_k \leqq \beta_{k-1} + \frac{3A\tau}{2n} C_3\beta_{k-1} + \frac{\tau}{n} E\beta_{k-1} + \frac{C_6\tau^2}{n^2}.$$

Here the constant C_6 enters through the estimate in (17),

$$\int_0^\infty \left|\frac{\partial^2 f(x, t)}{\partial t^2}\right| dx \leqq C_6 < \infty.$$

Applying the mean-value theorem, one obtains

$$\alpha_0 \leqq C_7 \frac{\tau}{n}, \qquad \beta_0 \leqq C_8 \frac{n}{\tau}, \tag{30}$$

where

$$\text{l.u.b.} \left|\frac{\partial f(x, t)}{\partial t}\right| \leqq C_7,$$

$$\text{l.u.b.} \int_0^\infty \left|\frac{\partial f(x, t)}{\partial t}\right| dx \leqq C_8.$$

The constants C_7 and C_8 are finite by the estimates in (17). For the purpose of bounding α_k and β_k from above, the inequality signs in (28) and (29) are replaced by equality signs. Then one obtains, in the same way as in deriving (23) and (24), the estimates

$$\beta_k \leqq \beta_n \leqq \frac{C_9}{n}, \tag{31}$$

$$\alpha_k \leqq \alpha_n \leqq \frac{C_{10}}{n}, \tag{32}$$

where

$$C_9 \leqq \tau \exp\left(\tau\left(E + \frac{3AC_3}{2}\right)\right)\cdot\left(C_8 + \frac{2C_6}{2E + 3AC_3}\right),$$

$$C_{10} \leqq \tau \exp\left(\tau\left(1 + \frac{3AC_4}{4}\right)\right)\cdot\left(C_7 + \frac{4\tau^2 C_5 + C_9(4C + 3AC_3)}{4 + 3AC_4}\right).$$

It follows that $\lim_{n\to\infty} |f(x, t) - F_n(x, t)| = 0$. Since $F_n(x, t) \geqq 0$, it follows that $f(x, t) \geqq 0$. Lemma 2 is thus proved on the interval $0 \leqq t \leqq \tau$, where $\tau < (1/E) \cdot \log(1 + 3A/2E)$. However, this bound on t depends on absolute constants only, and the whole construction may now be repeated for the interval $\tau \leqq t \leqq 2\tau$, $2\tau \leqq t \leqq 3\tau$ etc. In this way one shows that $f(x, t) \geqq 0$ on the interval $0 \leqq t \leqq 1/m$. It will be observed that new values of N and B must be used on each partial interval. The estimates in (17) guarantee that these values are finite.

5. Global existence of solutions.

LEMMA 3. *Under the hypotheses of Lemma* 1 *equation* (1) *possesses a solution* $f(x, t)$, *valid on the interval* $0 \leqq t < \infty$. *This solution is continuous, bounded, non-negative, analytic in* t *for each* x, *and integrable in* x *for each* t.

Proof. The local solution $f(x, t)$, valid on the interval $0 \leqq t < 1/m$, was constructed in §3 on the hypotheses of continuity, boundedness, non-negativity and integrability for $f(x, 0)$. According to Lemmas 1 and 2, all these hypotheses hold also for $f(x, 1/2m)$. Therefore one may construct a new local-existence interval starting at $t = 1/2m$, and the whole process may be continued. In this way one obtains a sequence of local-existence intervals: $0 \leqq t \leqq \Delta_1$, $\Delta_1 \leqq t \leqq \Delta_2, \cdots$ where

$$\Delta_k \leqq \Delta_{k+1}, \qquad \Delta_0 = 0,$$

$$\Delta_{k+1} - \Delta_k = \frac{1}{3AN(\Delta_k) + 2 \max \{E, CN(\Delta_k)/B(\Delta_k)\}}. \tag{33}$$

The functions $N(t)$ and $B(t)$ are defined by

$$N(t) = \int_0^\infty f[x, t) dt, \qquad B(t) = \operatorname*{l.u.b.}_x f(x, t).$$

Since the sequence $\{\Delta_k\}$ is monotone nondecreasing, it follows that either $\Delta_k \to \infty$ or $\Delta_k \to \Delta$, $0 < \Delta < \infty$. If $\Delta_k \to \infty$, the lemma is proved. Assume therefore that $\Delta_k \to \Delta$, $0 < \Delta < \infty$. By the recursion-formula in (33)

$$\lim_{\Delta_k \to \Delta} \left[3AN(\Delta_k) + 2 \max \left\{ E, \frac{CN(\Delta_k)}{B(\Delta_k)} \right\} \right] = \infty. \tag{34}$$

It will be shown that this leads to a contradiction. Integrate both sides of equation (10) w.r.t. t from 0 to ∞. All the integrals are absolutely convergent on the interval $0 \leqq t < \Delta$, and the result of integrating (10) may be rearranged to yield the equation

$$\begin{aligned} N(t) = N &- \frac{1}{2} \int_0^t \int_0^\infty \int_x^\infty f(x, t) f(y, t) \phi(x, y) dy dx dt \\ &- \int_0^t \int_0^\infty \int_0^x \frac{f(x, t)}{x} y \psi(x, y) dy dx dt \\ &+ \int_0^t \int_0^\infty \int_x^\infty f(y, t) \psi(y, x) dy dx dt. \end{aligned}$$

The contribution of the first two integrals (signs included) is nonpositive by Lemma 2. Therefore

$$N(t) \leqq N + \int_0^t \int_0^\infty \int_x^\infty f(y, t)\psi(y, x)dydxdt$$

$$(35) \qquad = N + \int_0^t \int_0^\infty \int_0^x f(x, t)\psi(x, y)dydxdt$$

$$\leqq N + (E-1) \int_0^t \int_0^\infty f(x, t)dxdt = N + (E-1)\int_0^t N(t)dt.$$

By a well-known inequality [2, Chapter 1], this implies

$$(36) \qquad N(t) \leqq N\left\{1 + \frac{e^{(E-1)t} - 1}{E-1}\right\}.$$

In equation (1) the first and the third integral are non-negative, by Lemma 2. Therefore

$$\frac{\partial f(x, t)}{\partial t} \geqq -f(x, t)\int_0^\infty f(y, t)\phi(x, y)dy - \frac{f(x, t)}{x}\int_0^x y\psi(x, y)dy.$$

By the hypotheses (H_1) and (H_2), this implies

$$(37) \qquad \frac{\partial f(x, t)}{\partial t} \geqq -f(x, t)[1 + AN(t)],$$

and therefore

$$(38) \qquad f(x, t) \geqq f(x, 0)\exp\left(-t - A\int_0^t N(t)dt\right),$$

and finally,

$$(39) \qquad B(t) \geqq B\exp\left(-t - A\int_0^t N(t)dt\right).$$

The estimates (36) and (39) show that on the interval $0 \leqq t < \Delta$ the function $N(t)$ is uniformly bounded from above and $B(t)$ is uniformly bounded away from 0. This clearly contradicts (34). Therefore, the local-existence intervals cover the whole interval $0 \leqq t < \infty$, and the proof ot Lemma 3 is completed.

6. **Uniqueness.** Let $g(x, t)$ be a function satisfying the following hypothesis

$g(x, t)$ is continuous for $x, t \geqq 0$,

(H_3) $g(x, t)$ is integrable in x for each t,

$$\int_0^\infty |g(x, t)|dx \leqq K < \infty \text{ on any interval } 0 \leqq t \leqq T,\ T < \infty.$$

By Lemma 3 the solution $f(x, t)$ of equation (1), constructed in §5, satisfies the hypothesis (H_3). Moreover, the following result holds:

LEMMA 4. *$f(x, t)$ is the only solution of equation* (1) *which satisfies the hypothesis* (H_3) *and which assumes the initial value* $f(x, 0)$.

Proof. Assume that there are two such solutions, $f(x, t)$ and $g(x, t)$. Then

$$f(x, t) = f(x, 0) + \int_0^t \{ [f(x, t), f(x, t)] + Lf(x, t) \} dt$$

and

$$g(x, t) = f(x, 0) + \int_0^t \{ [g(x, t), g(x, t)] + Lg(x, t) \} dt.$$

Therefore

$$\begin{aligned} &| f(x, t) - g(x, t) | \\ &\leqq \int_0^t \{ | [f(x, t) + g(x, t), f(x, t) - g(x, t)] | + | L[f(x, t) - g(x, t)] | \} dt. \end{aligned}$$

Integration w.r.t. x from 0 to ∞, yields

$$\begin{aligned} &\int_0^\infty | f(x, t) - g(x, t) | dx \\ &\leqq \int_0^\infty \int_0^t \{ | [f(x, t) + g(x, t), f(x, t) - g(x, t)] | \\ &\qquad + | L[f(x, t) - g(x, t)] | \} dt dx \\ &= \int_0^t \int_0^\infty \{ | [f(x, t) + g(x, t), f(x, t) - g(x, t)] | \\ &\qquad + | L[f(x, t) - g(x, t]] | \} dx dt \qquad (40) \\ &\leqq \int_0^t \left\{ \frac{3A}{2} \int_0^\infty | f(x, t) + g(x, t) | dx \int_0^\infty | f(x, t) - g(x, t) dx \right. \\ &\qquad \left. + E \int_0^\infty | f(x, t) - g(x, t) | dx \right\} dt. \end{aligned}$$

The interchange of the order of integration is justified by Fubini's Theorem, and the other steps follow from the estimates (4)–(9). Let

$$h(t) = \int_0^\infty | f(x, t) - g(x, t) | dt.$$

By the hypotheses of the lemma there exists a constant C_{11} such that

$$\int_0^\infty |f(x, t) + g(x, t)| \, dx \leqq C_{11} < \infty$$

on any fixed interval $0 \leqq t \leqq T$ $(0 < T < \infty)$. Then (40) implies that

$$h(t) \leqq \left(\frac{3AC_{11}}{2} + E\right) \int_0^t h(t) dt \qquad (0 \leqq t \leqq T).$$

Also, by definition, $h(0) = 0$. Therefore $h(t) = 0$ on the interval $0 \leqq t \leqq T$. From the continuity of $f(x, t)$ and $g(x, t)$ it follows that $f(x, t) \equiv g(x, t)$ for $0 \leqq t \leqq T$. Since T is arbitrary, the proof is complete.

THEOREM 1. *Under the hypotheses of Lemma* 1, *equation* (1) *possesses a solution* $f(x, t)$, *valid for* $x, t \geqq 0$, *which is continuous, bounded, non-negative, integrable in* x *for each* t, *and analytic in* t *for each* x. *Under the hypotheses of Lemma* 4, *this solution is unique.*

The proof follows immediately from Lemmas 1 to 4.

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