LOCALIZATION ON SPHERES[1]

BY
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1. **Introduction.** Let Ω be the unit $(k-1)$-dimensional sphere in Euclidean k-space, $k\geqq 3$, and let $S=\sum_{n=1}^{\infty} Y_n(x)$ be a series of surface spherical harmonics defined on Ω with $Y_n(x)=O(n^\beta)$ uniformly for x on Ω, $\beta\geqq 0$. To S can be associated a Riemann function, $F(x)=\sum_{n=1}^{\infty}(-1)^w Y_n(x)[n(n+p)]^{-w}$, where w is the smallest integer greater than $(\beta+1)/2$ and $p=k-2$. We then show in this paper that if $F(x)$ is of class C^{2w} on a domain D of Ω, S is uniformly (C, α) summable, $\alpha>p+2w$, to $\Delta^w F(x)$ in any domain D_1 whose closure is contained in the interior of D, where $\Delta^w=\Delta(\Delta^{w-1})$ and Δ is the Laplace operator on Ω.

This result extends to spheres the localization theorems obtained by Riemann and Zygmund for the circle [3, p. 286; 4].

2. **Definitions and notation.** λ will always designate the value $(k-2)/2 = p/2$ and P_n^λ will designate the Gegenbauer (ultra-spherical) polynomials defined by the equation

$$(1-2r\cos\theta+r^2)^{-\lambda}=\sum_{n=0}^{\infty} P_n^\lambda(\cos\theta)r^n.$$

With the help of these functions, we can associate to every function $f(x)$ integrable on Ω a sequence of surface spherical harmonics by means of the equation

$$Y_n(x)=\Gamma(\lambda)(2\pi^{\lambda+1})^{-1}(n+\lambda)\int_\Omega P_n^\lambda[(x, y)]f(y)d\Omega(y)$$

where $d\Omega(y)$ is the natural $(k-1)$-dimensional volume element on Ω and where (x, y) is the inner product, that is if $x=(\xi_1, \cdots, \xi_k)$ and $y=(\eta_1, \cdots, \eta_k)$ then $(x, y)=\xi_1\eta_1+\cdots+\xi_k\eta_k$.

As shown in [2, Chapter 11], $r^n Y_n(x)$ gives rise to an homogeneous harmonic polynomial of degree n in Euclidean k-space, E_k. We define $S[f] =\sum_{n=0}^{\infty} Y_n(x)$ and call this the Fourier series of surface harmonics associated to f.

We shall denote the spherical cap with center x_0 and curvilinear radius h, $0\leqq h\leqq\pi$, by $D(x_0, h)$, that is

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$$D(x_0, h) = \{x, (x, x_0) \geqq \cos h\}.$$

$F(x)$ will be said to be of class C^{2w} in the interior of $D(x_0, h)$, $0<h<\pi/2$, if the following holds: Select a rectangular coordinate system in E_k so that every point ξ in E_k is described by the k-tuple $\xi=(\xi_1, \cdots, \xi_k)$ and so that in particular $x_0=(1, 0, \cdots, 0)$. Then with $|\xi|^2=\xi_1^2+\cdots+\xi_k^2$, we have the interior of $D(x_0, h)=D'(x_0, h)$ described by $\{\xi, |\xi|^2=1 \text{ and } \xi_1>\cos h\}$, and $F(x)=F(\xi_1, \cdots, \xi_k)$ for $x=\xi$ in $D'(x_0, h)$. $F(x)$ is then said to be in class C^{2w} in $D'(x_0, h)$ if $G(\xi_2, \cdots, \xi_k)=F[(1-\xi_2^2-\cdots-\xi_k^2)^{1/2}, \xi_2, \cdots, \xi_k]$ is in class C^{2w} in the interior of the $(k-2)$-dimensional sphere $\xi_2^2+\cdots+\xi_k^2<\sin^2 h$.

If $F(x)$ is in class C^{2w} in $D'(x_0, h)$, then it is easily seen that the function $f(\xi)=F(\xi_1|\xi|^{-1}, \cdots, \xi_k|\xi|^{-1})$ is in class C^{2w} in the interior of the upper half-cone defined by $\{\xi, \xi_1>|\xi| \cos h\}$. If $w\geqq 1$, we define $\Delta F(x)=|\xi|^2\nabla^2 f(\xi)$ where ∇^2 is the Laplace operator in the space coordintes and $x=\xi|\xi|^{-1}$ is in $D'(x_0, h)$. It is to be noticed that since ∇^2 is invariant with respect to any rectangular coordinate system in E_k so is Δ when the origin is fixed. It is clear that for any ray in the interior of the above upper half-cone $|\xi|^2\nabla^2 f(\xi)$ is independent of the distance from the origin, but the exact motivation for using this expression as the definition of $\Delta F(x)$ can be seen from the form of ∇^2 when expressed in terms of a spherical coordinate system [2, p. 235]. In particular, using the spherical coordinate system described in [2, p. 233], if x_0 is the first pole of the spherical coordinate system and $F(x)$ is a function of (x_0, x) which is in class C^2 in $D'(x_0, h)$ then for x in $D'(x_0, h)-x_0$,

$$\Delta F(x) = F_{\theta_1\theta_1}(\cos\theta_1) + p\cot\theta_1 F_{\theta_1}(\cos\theta_1).$$

$F(x)$ will be said to be of class C^{2w} on a domain D of Ω if $F(x)$ is of class C^{2w} in the interior of every $D(x_0, h)$ contained in D with $0<h<\pi/2$. In particular if $F_1(x)$ and $F_2(x)$ are two functions which are in class C^2 on Ω, then it is not difficult to show that Green's second identity holds, i.e.

$$\int_\Omega F_1(x)\Delta F_2(x)d\Omega(x) = \int_\Omega F_2(x)\Delta F_1(x)d\Omega(x).$$

When we write $\sum_{n=0}^\infty Y_n(x)$ is a series of surface spherical harmonics, it will be understood that each $Y_n(x)$ is either a surface spherical harmonic of order n or a function identically zero on Ω.

By the terminology $\sum_{n=0}^\infty Y_n(x)$ is (C, α) summable, we shall mean the usual Cesàro summability defined for example in [3, Chapter 3]. The symbol A_n^α, $\alpha\geqq 0$, will be defined by the relation $\sum_{n=0}^\infty A_n^\alpha x^n=(1-x)^{-\alpha-1}$.

Δ^0 will be interpreted as the identity operator.

3. **Statement of main result.** If $Y_n(x)$ is a surface spherical harmonic, it is well known that $\Delta Y_n(x)=-n(n+p)Y_n(x)$. As mentioned in the introduction if $S=\sum_{n=1}^\infty Y_n(x)$ with $Y_n(x)=O(n^\beta)$, $\beta\geqq 0$, uniformly for x on Ω, the function $F(x)=\sum_{n=1}^\infty(-1)^w Y_n(x)[n(n+p)]^{-w}$ is continuous on the $(k-1)$-sphere,

where w is the smallest integer greater than $(\beta+1)/2$, since the series converges uniformly on Ω. Furthermore it is clear that $F(x)$ is the formal wth anti-Laplacian of S. We shall consequently call F the Riemann function associated with S.

We shall prove the following theorem:

THEOREM. *Let $S=\sum_{n=1}^{\infty} Y_n(x)$ be a series of surface spherical harmonics with $Y_n(x)=O(n^{\beta})$, $\beta\geqq 0$, uniformly for x on Ω. Let F be the Riemann function associated with S. Then if F is of class C^{2w} on a domain D contained on Ω, S is uniformly (C, α) summable, $\alpha>p+2w$, to $\Delta^w F$ in every closed domain contained in D.*

4. **Fundamental lemmas.** Before proving the main result we shall first prove some lemmas, the first several dealing with one-dimensional trigonometric series.

LEMMA 1. *Let $Q_1(\xi)$ be an odd polynomial and $Q_2(\xi)$ be an even polynomial without a constant term. Suppose that Q_1 is of degree j and Q_2 is of degree $j+1$ Then for $h>0$ and $\alpha>j$,*

(a) *$\sum_{m=0}^{\infty} Q_1(2m+1)\sin(2m+1)\theta$ is uniformly (C, α) summable to zero for $h\leqq\theta\leqq\pi-h$.*

(b) *$\sum_{m=0}^{\infty} Q_2(m)\cos m\theta$ is uniformly $(C, \alpha+1)$ summable to zero for $h\leqq\theta\leqq 2\pi-h$.*

To prove (a), by induction it is clearly sufficient to show that $\sum_{m=0}^{\infty}(2m+1)^j\sin(2m+1)\theta$ is uniformly (C, α) summable to zero for θ in $[h, \pi-h]$. With no loss of generality, we assume that $j<\alpha<j+1/2$. Let $\lambda(\theta)$ be the localizing function for the intervals $[h/2, \pi-h/2]$ and $[h, \pi-h]$, that is the function which is one in the latter interval and zero outside the former interval mod 2π and is in class C^{∞}. Then setting $F(\theta)=-\pi/8[(\theta-\pi/2)^2-(\pi/2)^2]$ for $0\leqq\theta\leqq\pi$, making it odd in the interval $[-\pi, \pi]$, and continuing it periodically of period 2π over the real line, we have that the Fourier series of F is given by $S[F]=\sum_{m=0}^{\infty}(2m+1)^{-3}\sin(2m+1)\theta$. Then by [4, p. 101]

$$\sum_{m=0}^{n}(2m+1)^j\sin(2m+1)\theta \tag{1}$$
$$-(-1)^{(j+3)/2}(2\pi)^{-1}\int_{h/2}^{\pi-h/2}F(t)\lambda(t)\frac{d^{j+3}}{dt^{j+3}}\left[\frac{\sin(2n+1)(\theta-t)/2}{\sin(\theta-t)/2}\right]dt$$

is uniformly (C, α) summable to zero for θ in $[h/2, \pi-h/2]$.

But the integral in (1) represents a constant multiple of the nth partial sum of the Fourier series of the function $d^{j+3}[F(\theta)\lambda(\theta)]/d\theta^{j+3}$ and consequently converges to zero uniformly for θ in $[h, \pi-h]$. (a) of the lemma then follows from (1) and this last fact.

(b) follows in a manner similar to (a) except for $F(x)$, we use the continuous function whose Fourier series is given by $\sum_{m=1}^{\infty} m^{-2}\cos mx$.

LEMMA 2. *With $2\lambda = p$, j a non-negative integer and $h>0$, the trigonometric series*

$$\text{(a)} \qquad \sum_{n=0}^{\infty} (2n + p)[n(n + p)]^j P_n^\lambda(1) \sin (2n + p)\theta$$

is uniformly (C, α), $\alpha > p+2j$, summable to zero for θ in $[h, \pi-h]$ with p odd.

$$\text{(b)} \qquad \sum_{n=0}^{\infty} (2n + p)[n(n + p)]^j P_n^\lambda(1) \cos (n + \lambda)\theta$$

is uniformly (C, α), $\alpha > p+2j$, summable to zero for θ in $[h, 2\pi-h]$ with p even.

We prove (a) by means of Lemma 1 part (a); (b) follows in a similar manner from Lemma 1 part (b).

Let us suppose first that p is odd and $\geqq 3$. Then since $P_n^\lambda(1) = K(n+p-1) \cdots (n+1)$ where K is a constant independent of n, to prove (a) it is sufficient to show that the series

$$\sum_{n=-(p-1)/2}^{\infty} (2n + p)[(2n + p)^2 - p^2]^j (n + p - 1) \cdots (n + 1) \sin (2n + p)\theta$$

is (C, α) uniformly summable to zero for θ in $[h, \pi-h]$, or what is the same thing that the series with $q = (p-1)/2$.

$$\text{(2)} \qquad \sum_{n=0}^{\infty} (2n + 1)[(2n + 1)^2 - p^2]^j \pi_{i=-q+1}^{q} (n + i) \sin (2n + 1)\theta$$

is (C, α) uniformly summable to zero for θ in $[h, \pi-h]$.

But by Lemma 1 part (a), to show that (2) holds it is sufficient to show that there is an odd polynomial Q_1 of degree $2j+p$ such that

$$Q_1(2n + 1) = (2n + 1)[(2n + 1)^2 - p^2]^j \pi_{i=-q+1}^{q} (n + i).$$

This last fact follows on observing that

$$\pi_{i=-q+1}^{q} (n + i) = 2^{-2q} \pi_{i=1}^{q} [(2n + 1)^2 - (2i - 1)^2].$$

So part (a) is proved when $p \geqq 3$.

If $p=1$, $P_n^\lambda(1) = 1$ and the series involved in part (a) is

$$4^{-j} \sum_{n=0}^{\infty} (2n + 1)[(2n + 1)^2 - 1]^j \sin (2n + 1)\theta$$

and the proof follows immediately from Lemma 1, part (a).

As mentioned earlier, similar considerations involving Lemma 1, part (b), will prove part (b) of the present lemma, which proof we now regard as complete.

LEMMA 3. *Given $h>0$, the integral $I(\theta)=\int_{\theta/2}^{\pi/2}(\cos\theta-\cos 2t)^{-1/2}dt$ is uniformly bounded for θ in the interval $h\leqq\theta<\pi$.*

Let us suppose from the start that $h<\pi/4$. Then using the trigonometric identity for the difference of two cosines, transforming the integral to one over the interval $[\theta, \pi]$, and using the fact that $\sin t\geqq 2t/\pi$ for $0\leqq t\leqq\pi/2$, we obtain

$$I(\theta) \leqq \pi^{1/2}2^{-3/2}\int_{\theta}^{\pi}(t-\theta)^{-1/2}[\sin(\theta+t)/2]^{-1/2}dt. \tag{3}$$

Suppose $h\leqq\theta\leqq\pi/2$. Then it follows from the concavity of the sine curve that

$$I(\theta) \leqq \pi^{1/2}(2^3\sin h)^{-1/2}\int_{\theta}^{\pi}(t-\theta)^{-1/2}dt \leqq M_h. \tag{4}$$

Suppose $\pi/2\leqq\theta<\pi$. Then for $\theta\leqq t<\pi$, $\sin(\theta+t)/2\geqq\pi^{-1}(\pi-\theta)$. We, therefore, obtain from (3) that

$$I(\theta) \leqq 2^{-1}\pi(\pi-\theta)^{-1/2}\int_{\theta}^{\pi}(t-\theta)^{-1/2}dt. \tag{5}$$

The conclusion to the lemma then follows from (4) and (5).

If $S=\sum_{n=0}^{\infty}Y_n(x)$ is a series of surface spherical harmonics, by $\Delta^w S$ with w a non-negative integer, we shall mean the series that one obtains from S by formally applying the operator Δ^w, that is

$$\Delta^w S = \sum_{n=0}^{\infty}\Delta^w Y_n(x) = \sum_{n=0}^{\infty}(-1)^w[n(n+p)^w]Y_n(x).$$

We next introduce the important series of surface spherical harmonics,

$$T_{x_0}^{\lambda} = \sum_{n=0}^{\infty}(n+\lambda)P_n^{\lambda}[(x_0, x)] \tag{6}$$

and prove the following key lemma:

LEMMA 4. *For $0<h<\pi$, the series $[1-(x_0, x)^2]^{(p-1)/2}\Delta^w T_{x_0}^{\lambda}$ is uniformly (C, α) summable to zero, $\alpha>p+2w$, for all x such that $(x_0, x)\leqq\cos h$.*

From (6), using x_0 as the first pole of a spherical coordinate system on Ω, we see that the lemma will be proved if we can show that

$$\begin{aligned} f_n^{w,\alpha,\lambda}(\cos\theta) &\equiv \sum_{j=0}^{n}(\sin\theta)^{p-1}[j(j+p)]^{w}(2j+p)P_j^{\lambda}(\cos\theta)A_{n-j}^{\alpha} \\ &= o(n^{\alpha}) \text{ uniformly for } \theta \text{ in } [h, \pi]. \end{aligned} \tag{7}$$

To do this, let us first suppose that p is odd and $h\leqq\theta<\pi$. Then using Mehler's integral representation [2, p. 177], we have that

$$P_n^\lambda(\cos\theta) = K_\lambda P_n^\lambda(1)(\sin\theta)^{1-p}\int_{\theta/2}^{\pi/2}\sin(2n+p)t(\cos\theta-\cos 2t)^{\lambda-1}dt$$

where K_λ is a constant depending on λ. We then see from (7) that

$$f_n^{w,\alpha,\lambda}(\cos\theta) = K_\lambda\int_{\theta/2}^{\pi/2} V_n^{w,\alpha,\lambda}(t)(\cos\theta-\cos 2t)^{\lambda-1}dt \tag{8}$$

where $V_n^{w,\alpha,\lambda}(t) = \sum_{j=0}^n P_j^\lambda(1)[j(j+p)]^w(2j+p)A_{n-j}^\alpha \sin(2n+p)t$. But for $h\leqq\theta<\pi$ the absolute value of the expression on the right side of (8) is majorized by

$$|K_\lambda| \max_{\theta \text{ in } [h/2,\pi/2]} \left| V_n^{w,\alpha,\lambda}(\theta)\right| \int_{\theta/2}^{\pi/2}(\cos\theta-\cos 2t)^{\lambda-1}dt. \tag{9}$$

By Lemma 3, the integral in (9) is uniformly bounded for $h\leqq\theta<\pi$. We conclude from (7), (8), and (9) and the continuity of $f_n^{w,\alpha,\lambda}(\cos\theta)$ that there is a constant $K_{\lambda,h}$ such that for $h\leqq\theta\leqq\pi$

$$\left| f_n^{w,\alpha,\lambda}(\cos\theta)\right| \leqq K_{\lambda,h} \max_{\theta \text{ in } [h/2,\pi/2]} \left| V_n^{w,\alpha,\lambda}(\theta)\right| . \tag{10}$$

By Lemma 2, however, $V^{w,\alpha,\lambda}(\theta)=o(n^\alpha)$ uniformly in the interval $h/2\leqq\theta\leqq\pi/2$. Consequently we see from (10) that the desired result in (7) is established and the lemma is proved in all cases for which p is odd.

Let us now suppose that p is even and $h\leqq\theta<\pi$. Then using Mehler's integral representation we have that

$$P_n^\lambda(\cos\theta) = K_\lambda P_n^\lambda(1)(\sin\theta)^{1-p}\int_\theta^\pi \cos(n+\lambda)t(\cos\theta-\cos t)^{\lambda-1}dt$$

where K_λ is a constant depending on λ. Proceeding as in the case with p odd, we obtain that there is a positive constant $K_{\lambda,h}$ such that for $h\leqq\theta\leqq\pi$,

$$\left| f_n^{w,\alpha,\lambda}(\cos\theta)\right| \leqq K_{\lambda,h} \max_{\theta \text{ in } [h,\pi]} \left| U_n^{w,\lambda,\alpha}(\theta)\right| \tag{11}$$

where

$$U_n^{w,\lambda,\alpha}(\theta) = \sum_{j=0}^n P_j^\lambda(1)[j(j+p)]^w(2j+p)\cos(n+\lambda)\theta A_{n-j}^\alpha.$$

By Lemma 2, part (b), $U_n^{w,\alpha,\lambda}(\theta)=o(n^\alpha)$ uniformly for $h\leqq\theta\leqq\pi$ and we conclude from (11) that the desired result in (7) is established, which fact concludes the proof of the lemma.

LEMMA 5. *Let $0<h_1<h_2<\pi$. Let $F_1(x)$ be a bounded measurable function on Ω which is equal to zero in $D(x_0, h_2)$. Then for x in $D(x_0, h_1)$, $\Delta^w S[F_1]$ is uni-*

formly summable (C, α) *to zero,* $\alpha > p+2w$.

Setting

$$\Delta^{w} \sigma_n^{\alpha,\lambda}[(x, y)] = \sum_{j=0}^{n} (-1)^{w}(j+\lambda)[j(j+p)]^{w} A_{n-j}^{\alpha} P_n^{\lambda}[(x, y)]/A_n^{\alpha} \tag{12}$$

we have from Lemma 4 that for every $h>0$, there is a function $M_h(n)$ which is independent of x and which tends to zero as n tends to infinity such that for y in $\Omega - D(x, h)$

$$\left| [1-(x, y)^2]^{(p-1)/2} \Delta^{w} \sigma_n^{\alpha,\lambda}[(x, y)] \right| \leqq M_h(n). \tag{13}$$

To prove the lemma we observe that the (C, α) sum of rank n of $\Delta^{w}S[F]$ is given by

$$K_\lambda \int_{\Omega} F_1(y) \Delta^{w} \sigma_n^{\alpha,\lambda}[(x, y)] d\Omega(y) \tag{14}$$

where K_λ is a constant depending on λ. We also observe that for x in $D(x_0, h_1)$, $F_1(y) = 0$ for all y in $D(x, h_3)$ where $h_3 = 2 \arcsin [\sin h_2/2 - \sin h_1/2]$. Letting K' designate the bound of $|F_1(y)|$ on Ω, we consequently obtain that the absolute value of the expression in (14) is majorized by

$$K_\lambda K' \int_{\Omega - D(x, h_3)} \left| \Delta^{w} \sigma_n^{\alpha,\lambda}[(x, y)] \right| d\Omega(y) \tag{15}$$

for x in $D(x_0, h_1)$. But from (13) the integral in (15) in turn is majorized by

$$\begin{aligned} &M_{h_3}(n) \int_{\Omega - D(x, h_3)} [1-(x, y)^2]^{(1-p)/2} d\Omega(y) \\ &\quad = M_{h_3}(n) 2\pi^{\lambda+1/2} \left[\Gamma\left(\lambda + \frac{1}{2}\right) \right]^{-1} \int_{h_3}^{\pi} (\sin\theta)^{1-p} (\sin\theta)^{p} d\theta. \end{aligned} \tag{16}$$

(For the calculation in (16) see [2, pp. 233 and 234].)

Since $M_{h_3}(n)$ tends to zero as n tends to infinity, we obtain from (14), (15), and (16) that for x in $D(x_0, h_1)$, $\Delta^{w}S[F_1]$ is uniformly (C, α) summable to zero, which is the desired result.

Given two spherical caps, $D(x_0, h_2)$ and $D(x_0, h_3)$, with $0 < h_2 < h_3 < \pi$, we shall call $\psi(x)$ a localizing function for these caps if $\psi(x)$ is in class C^∞ on Ω and if $\psi(x)$ equals one for x in $D(x_0, h_2)$ and equals zero for x in $\Omega - D(x, h_3)$.

LEMMA 6. *Let* $S = \sum_{n=1}^{\infty} Y_n(x)$ *be a series of surface spherical harmonics with* $Y_n(x) = O(n^\beta)$, $\beta \geqq 0$. *Let* $F(x)$ *be the Riemann function associated with* S *and let* $\psi(x)$ *be a localizing function for the spherical caps* $D(x_0, h_2)$ *and* $D(x_0, h_3)$ *with* $0 < h_2 < h_3 < \pi$. *Then the series* $S - \Delta^{w}S[\psi F]$ *is uniformly* (C, α) *summable to zero in the cap* $D(x_0, h_1)$ *where* $0 < h_1 < h_2$ *and* $\alpha > p+2w$.

To prove this lemma set $F_1(x)=F(x)-\psi(x)F(x)$. Then by Lemma 5, the proof to the present lemma will be complete as soon as we show that $S-\Delta^w S[\psi F]=\Delta^w S[F_1]$. To accomplish this fact all we need show is that for $n\geqq 1$,

$$Y_n(x) - K_\lambda(n+\lambda)\Delta^w \int_\Omega P_n^\lambda[(x, y)]\psi(x)F(y)d\Omega(y)$$

$$= K_\lambda(n+\lambda)\Delta^w \int_\Omega P_n^\lambda[(x, y)][F(y) - \psi(y)F(y)]d\Omega(y)$$

where $K_\lambda=\Gamma(\lambda)[2\pi^{\lambda+1}]^{-1}$, or what is the same thing that $Y_n(x)=K_\lambda(n+\lambda)\Delta^w \cdot\int_\Omega P_n^\lambda[(x, y)]F(y)d\Omega(y)$. But this last fact follows immediately from the manner in which $F(x)$ has been constructed from S; so the proof to the lemma is complete.

5. **Proof of the theorem.** By the Heine-Borel theorem, to prove the theorem it is clearly sufficient to show that if $D(x_0, h_1)$ is contained in D, then S is uniformly (C, α) summable to $\Delta^w F$ for x in $D(x_0, h_1)$. In order to show this fact we proceed as follows: Given $D(x_0, h_1)$ contained in D, we introduce two more spherical caps $D(x_0, h_2)$ and $D(x_0, h_3)$ with $0<h_1<h_2<h_3<\pi$ and $D(x_0, h_3)$ contained in D. Letting $\psi(x)$ be a localizing function in class C^∞ on Ω for $D(x_0, h_2)$ and $D(x_0, h_3)$, we observe that $\psi(x)F(x)$ is a function in class C^{2w} on Ω. Furthermore applying Green's second identity w-times to $\int_\Omega P_n^\lambda[(x, y)][\psi(y)F(y)]d\Omega(y)$, we obtain that $\Delta^w S[\psi F]=S[\Delta^w(\psi F)]$. Now $\Delta^w(\psi F)$ is a continuous function on Ω. Therefore as is well-known **[1]**, $S[\Delta^w(\psi F]$ is (C, η), $\eta>p$, uniformly summable to $\Delta^w(\psi F)$ on Ω. Since $\psi=1$ on $D(x_0, h_2)$, we obtain in particular that, $S[\Delta^w(\psi F)]$ is (C, α) summable to $\Delta^w F$, uniformly for x in $D(x_0, h_1)$.

We next obtain from Lemma 6, that

$$S - \Delta^w S[\psi F] = S - S[\Delta^w(\psi F)]$$

is uniformly (C, α) summable to zero for x in $D(x_0, h_1)$ and therefore from the above discussion that S is uniformly (C, α) summable to $\Delta^w F$ for x in $D(x_0, h_1)$, which fact concludes the proof of the theorem.

Bibliography

1. K. K. Chen, *On the Cesàro-summability of the Laplace's series of hyperspherical functions*, The Science Reports of the Tôhoku Imperial University vol. 17 (1928) pp. 1073–1089.

2. A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, vol. 2, New York, 1953.

3. A. Zygmund, *Trigonometric series*, Warasaw, 1935.

4. ———, *Sur la théorie riemannienne des séries trigonométriques*, Math. Zeit. vol. 24 (1926) pp. 47–104.

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