

PEANO SPACES WHICH ARE EITHER STRONGLY CYCLIC OR TWO-CYCLIC⁽¹⁾

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1. Introduction. Let M be a Peano space⁽²⁾ $p \in M$, and $\{p_n\}$ any infinite sequence of distinct points of M converging to p . The space M is said to be *strongly arcwise connected* at p provided there is in M a simple arc containing infinitely many of the points $\{p_n\}$, *strongly cyclic* at p provided there is in M a simple closed curve containing infinitely many of the points $\{p_n\}$, and *two-cyclic* at p provided M contains two arcs Γ_1 and Γ_2 such that each contains infinitely many of the points $\{p_n\}$ and $\Gamma_1 \cap \Gamma_2 = p$. If M is strongly arcwise connected, strongly cyclic, or two-cyclic at each of its points, then M is said to be strongly arcwise connected, strongly cyclic, or two-cyclic respectively. Strongly arcwise connected Peano spaces have been studied by Hall and Puckett [1; 2]⁽³⁾.

It is clear that if M is either strongly cyclic or two-cyclic at p , then M is strongly arcwise connected at p ; and that if M is a cyclic Peano space which is two-cyclic at p , then M is strongly cyclic at p . Moreover, it is easily shown that if M is either strongly cyclic or two-cyclic at p , then p lies in at least one, and at most a finite number, of true cyclic elements of M . Thus only cyclic Peano spaces will be considered.

An example of a cyclic Peano space which is strongly arcwise connected at a point p , but which fails to be strongly cyclic at p follows. Let M be the point set in the Euclidean plane consisting of the closed interval from 0 to 1 on the x -axis, the closed line segment of $y=x$ from $(0, 0)$ to $(1, 1)$, and the closed line segments from $(1/n, 0)$ to $(1/n, 1/n)$ on the lines $x=1/n$, $n \in I$. Considering the sequence of points $\{p_n\}$ where $p_n = (1/n, 1/2n)$, $n \in I$, it is seen that M fails to be strongly cyclic at the point $(0, 0)$.

In this paper two characterizations of strongly cyclic Peano spaces will be obtained. The first will show that a cyclic Peano space which is strongly arcwise connected at p , but not strongly cyclic at p , is essentially like the example above. The second states that a cyclic Peano space M is strongly cyclic if and only if for any infinite collection $\{V_i\}$ of open sets in M , there exists a simple closed curve in M intersecting infinitely many of these sets.

For an example of a cyclic Peano space which is strongly cyclic at p , but

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⁽²⁾ All terms and symbols not explicitly defined in this paper may be found in [4] or [5].

⁽³⁾ Numbers in brackets refer to the bibliography at the end of the paper.

not two-cyclic at p , let M denote the square in the Euclidean plane with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$, either with or without its interior. Considering the sequence of points $(1/n, 0)$, $n \in I$, it is seen that M is not two-cyclic at the point $(0, 0)$. If M is considered without its interior, then M fails to be two-cyclic at $(0, 0)$ because this point lies on a free arc of M . If M is considered with its interior, define in M an arc Γ consisting of p and the union of the line segments of $x = 1/n$ from $(1/n, 0)$ to $(1/n, 1/n)$, the line segments from $(1/2n, 0)$ to $(1/2n-1, 0)$ on the x -axis, and the line segments from $(1/2n+1, 1/2n+1)$ to $(1/2n, 1/2n)$ on the line $y=x$, $n \in I$. Thus given $\epsilon > 0$, there exists an arc $\gamma \subset \Gamma$ such that γ has p as one endpoint, has diameter less than ϵ , and $M - \gamma$ has an infinite collection of components converging to p .

It will also be shown that a cyclic Peano space which is strongly cyclic at p , but not two-cyclic at p , is essentially like the above example.

2. Lemmas. In this section several lemmas are given which will be useful in obtaining the principal results.

LEMMA 2.1. *If a limit point p of an arcwise connected set R is not regularly accessible from R , then there exists a positive number η such that $R \cap S(p, \eta)$ has an infinite collection of distinct components $\{B_i\}$ such that $p \in \lim \inf \{B_i\}$. Moreover, $\overline{C} \cap F[S(p, \eta)] \neq \emptyset$ for every component C of $R \cap S(p, \eta)$.*

Proof. This is a consequence of a lemma of E. E. Betz [3, p. 128].

LEMMA 2.2. *Let $\alpha = qxp$ be an arc of a Peano space M and $\{p_n\}$ an infinite sequence of distinct points of $M - \alpha$ converging to p . If, for every $\epsilon > 0$, $S(p, \epsilon)$ contains an arc spanning $\alpha' = \alpha - (q \cup p)$ and containing a point of the sequence $\{p_n\}$, then M contains an arc α^* from q to p containing infinitely many of the points $\{p_n\}$.*

LEMMA 2.3. *Let M be a cyclic Peano space and $J = \alpha \cup \beta$ a simple closed curve in M where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$. Let $\{p_n\}$ be an infinite sequence of distinct points of $M - J$ converging to p , $\{x_n\}$ and $\{y_n\}$ infinite sequences of distinct points converging monotonically to p on α and β respectively. Assume that M contains a collection $\{\lambda_n\}$ of distinct arcs converging to p such that λ_n is an arc $x_n p_n y_n$ spanning J , and that there is a positive number η for which $(M - J) \cap S(p, \eta)$ has an infinite collection $\{B_i\}$ of components such that $\lambda_i \subset B_i$ and $\overline{B_i} \cap F[S(p, \eta)] \neq \emptyset$ for every i . Then there is a simple closed curve in M containing infinitely many of the points $\{p_n\}$.*

Proof. There is no loss of generality in assuming that the collection $\{B_i\}$ converges to a connected set L , and that L contains a nondegenerate subarc zp of α . Let δ be a positive number such that $S(p, \delta) \cap \alpha \subset zp$. There exists a point $r \in S(p, \delta) \cap \alpha$ such that $r \neq x_i$ for all i . There exists an integer n_1 , such that in the natural order from q to p on α , $r < x_n$ for $n \geq n_1$, and $x_{n_1-1} < r$. Let U be a neighborhood of r such that $\overline{U} \cap \beta = \emptyset$, $\overline{U} \subset S(p, \delta)$, and $\overline{U} \cap \alpha \subset x_{n_1-1} x_{n_1}$. There exists an integer $n_2 \geq n_1 + 2$ such that $B_{n_2} \cup s$ contains

an arc stw where $s \in \bar{U} \cap \alpha$ and $w = \lambda_{n_2} \cap stw$. Let Γ_1 be the union of the subarc qs of α , the arc stw , the subarc wy_{n_2} of λ_{n_2} , and the subarc $y_{n_2}p$ of β ; Γ_2 the union of the subarc qy_{n_1+1} of β , the arc λ_{n_1+1} , and the subarc $x_{n_1+1}p$ of α . Then $\Gamma_1 \cup \Gamma_2$ is a simple closed curve containing the point p_{n_1+1} .

Let δ_2 be a positive number such that $\bar{S}(p, \delta_2) \cap (\Gamma_1 \cup \Gamma_2)$ contains only points of the subarcs $x_{n_2}p$ and $y_{n_2}p$ of α and β respectively. The above construction may be repeated in $S(p, \delta_2)$, thereby yielding a simple closed curve containing a second point of the sequence $\{p_n\}$. The proof of the lemma follows by continuing this process inductively.

LEMMA 2.4. *Let M be a Peano space, $p \in M$, $\alpha = qxp$ an arc of M , and $\{p_n\}$ an infinite sequence of distinct points of M converging monotonically to p on α . Assume there is a positive number η and an infinite collection $\{D_i\}$ of components of $(M - \alpha) \cap S(p, \eta)$ such that (a) $p \in \liminf \{D_i\}$; (b) $\bar{D}_i \cap F[S(p, \eta)] \neq \emptyset$ for every i . Then there exist arcs Γ_1 and Γ_2 in M each of which contains infinitely many of the points $\{p_n\}$ and such that $\Gamma_1 \cap \Gamma_2 = p$.*

Proof. There is no loss of generality in assuming that the collection $\{D_i\}$ converges to a connected set L , and that L contains a nondegenerate subarc zp of α . Let σ_1 be a positive number such that $\bar{S}(p, \sigma_1) \cap \alpha \subset zp$. In the natural order from q to p on α let t_1 be the last point of $\alpha \cap [\bar{S}(p, \sigma_1) - S(p, \sigma_1)]$, and p_1^* the first point of the sequence $\{p_n\}$ such that $t_1 < p_1^*$. Let σ_2 be a positive number such that $\sigma_2 < \sigma_1/2$ and $\bar{S}(p, \sigma_2) \cap \alpha$ contains only points of α which follow p_1^* in the natural order from q to p on α . In the natural order from q to p on α let t_2 be the last point of $\alpha \cap [\bar{S}(p, \sigma_2) - S(p, \sigma_2)]$, and p_2^* the first point of the sequence $\{p_n\}$ such that $t_2 < p_2^*$. Continuing this process inductively yields sequences $\{\sigma_i\}$, $\{t_i\}$, and $\{p_i^*\}$ with the following properties:

(i) $\{\sigma_i\}$ is a null sequence of positive numbers such that $\sigma_{i+1} < \sigma_i/2$, and $\bar{S}(p, \sigma_{i+1}) \cap \alpha$ contains only points of α which follow p_i^* in the natural order from q to p on α .

(ii) $\{p_i^*\}$ is a subsequence of the $\{p_i\}$ such that $p_i^* \in S(p, \sigma_i) - \bar{S}(p, \sigma_{i+1})$.

(iii) t_i is the last point of $\alpha \cap [\bar{S}(p, \sigma_i) - S(p, \sigma_i)]$ in the natural order from q to p on α .

(iv) In the natural order from q to p on α , $t_i < p_i^* < t_{i+1}$, $i \in I$.

Let $\{V_i\}$ be a null sequence of regions such that $t_i \in V_i$ and $\bar{V}_i \cap \alpha$ contains only points interior to the subarc $p_{i-1}^*p_i^*$ of α . Then there exists a collection of distinct arcs $\{ef_{i+3}\}$ spanning α and such that ef_{i+3} lies in $V_i \cup [\bar{S}(p, \sigma_i) - S(p, \sigma_{i+3})] \cup V_{i+3}$. The proof of the lemma is completed by defining the arcs Γ_1 and Γ_2 as follows:

$$\Gamma_1 = p \cup \left[\bigcup_{n \in I} (e_{4n-3}f_{4n}) \right] \cup \left[\bigcup_{n \in I} (f_{4n}p_{4n}^*e_{4n+1}) \right],$$

$$\Gamma_2 = p \cup \left[\bigcup_{n \in I} (e_{4n-1}f_{4n+2}) \right] \cup \left[\bigcup_{n \in I} (f_{4n+2}p_{4n+2}^*e_{4n+3}) \right]$$

where the collections $\{f_{4n}p_{4n}^*e_{4n+1}\}$ and $\{f_{4n+2}p_{4n+2}^*e_{4n+3}\}$ are subarcs of α .

3. Separation theorem. In this section the following theorem is proved.

THEOREM 3.1. *Let M be a cyclic Peano space which is strongly arcwise connected at p but not strongly cyclic at p , and let J be any simple closed curve in M containing p . Let $J = \alpha \cup \beta$ where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$. Then there exist in M an infinite sequence of distinct points $\{p_n\}$, and a sequence of simple arcs $\{\lambda_n\}$ having the following properties:*

- (i) $p_n \in M - J$ for every n .
- (ii) $\{p_n\}$ converges to p .
- (iii) No simple closed curve in M contains infinitely many points of the sequence $\{p_n\}$.
- (iv) Every simple arc in M containing infinitely many points of the sequence $\{p_n\}$ intersects each of the arcs α and β in a set of points having p as a limit point.
- (v) Each λ_n is an arc $x_n y_n$ spanning J and containing the point p_n .
- (vi) $\lambda_m \cap \lambda_n = \emptyset$ for $m \neq n$.
- (vii) $\{\lambda_n\}$ converges to p .
- (viii) $\bigcup_{n \in I} x_n \subset \alpha' = \alpha - (q \cup p)$, $\bigcup_{n \in I} y_n \subset \beta' = \beta - (q \cup p)$.
- (ix) The sequences $\{x_n\}$ and $\{y_n\}$ converge monotonically to p on the arcs α and β respectively.
- (x) For $m \neq n$, the sets $\lambda'_m = \lambda_m - (x_m \cup y_m)$ and $\lambda'_n = \lambda_n - (x_n \cup y_n)$ lie in different components of $M - J$. Furthermore, these components converge to p . Thus for any simple closed curve J of M containing p , every arc of J having p as an interior point separates M into infinitely many distinct components.
- (xi) If $\{R_i\}$ is the collection of components of $M - J$ such that $\lambda'_i \subset R_i$, then for no value of i is there an arc of $\overline{R_i} - \beta$ spanning α and containing the point p_i , or an arc of $\overline{R_i} - \alpha$ spanning β and containing the point p_i .

Proof. Since M is not strongly cyclic at p , there exists in M an infinite sequence of distinct points $\{p_n\}$ satisfying (ii) and (iii). From (iii) it follows that J contains at most a finite number of the points $\{p_n\}$. Hence no generality is lost in assuming that (i) holds.

Since M is strongly arcwise connected at p , there exists in M an arc containing infinitely many of the points $\{p_n\}$. Let λ be an arbitrary arc satisfying this condition. No generality is lost in assuming that λ contains p_n for every n , that λ has p as one endpoint, and that the sequence $\{p_n\}$ converges monotonically to p on λ . Let $\lambda = bdp$ where b is the endpoint of λ distinct from p .

If one of the arcs α and β , say α , contains a subarc $\alpha_1 = stp$ such that $\alpha_1 \cap \lambda = p$, then M is strongly cyclic at p . This contradiction proves (iv).

There exists a positive number ϵ such that in $S(p, \epsilon)$ no subarc of λ containing a point of the sequence $\{p_n\}$ spans one of the arcs α and β without intersecting the other. If this is not true, then with the aid of Lemma 2.2 it is seen that M becomes strongly cyclic at p . Thus there exists a sequence

$\{\lambda_n\}$ of distinct arcs converging to p such that $\lambda_n = x_n y_n$ is a subarc of λ spanning J and containing a point of the sequence $\{p_n\}$, $x_n \in \alpha'$, $y_n \in \beta'$, and the sequences $\{x_n\}$ and $\{y_n\}$ converge monotonically to p on α and β respectively. Since there is no loss of generality in assuming that $p_n \in \lambda_n$ for every n , this proves (v), (vi), (vii), (viii), and (ix).

To prove (x), assume that R is a component of $M - J$ containing infinitely many of the sets $\{\lambda'_i\}$, and hence p is a limit point of R . If p is accessible from R , it is easily shown that M becomes strongly cyclic at p . Thus p is not regularly accessible from R . Thus by the lemma of E. E. Betz referred to for the proof of Lemma 2.1 and by Lemma 2.1 there exists a positive number $\epsilon > 0$ such that for every positive number η , $\eta < \epsilon$, $R \cap S(p, \eta)$ has an infinite number of components $\{B_i\}$ with $p \in \liminf B_i$, $\bar{B}_i \cap F[S(p, \eta)] \neq \emptyset$ for every i , and for every positive number δ such that $\delta < \eta$, $B_i \cap S(p, \delta) \neq \emptyset$ for all but a finite number of values of i . There exists a value η_0 of η such that no component of $R \cap S(p, \eta_0)$ contains more than one element of the collection $\{\lambda'_i\}$. If this is not the case then with the aid of Lemma 2.2 it is easily shown that M is strongly cyclic at p . Thus it may be assumed that $R \cap S(p, \eta_0)$ has an infinite collection of components $\{B_i\}$ such that $\lambda'_i \subset B_i$ for every i . But this makes M strongly cyclic at p since the hypothesis of Lemma 2.3 is satisfied. This contradiction shows that each component of $M - J$ can contain at most a finite number of the sets $\{\lambda'_i\}$. Hence it may be assumed that $M - J$ has an infinite collection of components $\{R_i\}$ such that $\lambda'_i \subset R_i$ for every i . The proof of (x) is complete if it is observed that if the sequence $\{R_i\}$ does not converge to p , then again the hypothesis of Lemma 2.3 is satisfied.

The proof of the theorem is complete if it is noted that if (xi) is not true, then M becomes strongly cyclic at p by an application of Lemma 2.2.

4. The properties $P(r, \alpha)$ and $P(r, \beta)$. In this section let M be a cyclic Peano space, $J = \alpha \cup \beta$ a simple closed curve in M where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$, and $r \in M - J$. Let $\alpha' = \alpha - (q \cup p)$, $\beta' = \beta - (q \cup p)$. The next two theorems are conveniently stated in terms of a set $R(J, r)$ and properties $P(r, \alpha)$ or $P(r, \beta)$ which are defined as follows.

DEFINITION 4.1. By the set $R(J, r)$ is meant the component of $M - J$ containing r .

DEFINITION 4.2. The arc α is said to possess the properties $P(r, \alpha)$ with respect to the set $R(J, r)$ provided it possesses both of the following properties:

$P_1(r, \alpha)$: No point of $R(J, r)$ separates r from $F[R(J, r)] \cap \alpha'$ in $\bar{R}(J, r) - \beta$.

$P_2(r, \alpha)$: $F[R(J, r)] \cap \alpha'$ is nondegenerate. The properties $P(r, \beta)$ are defined by an interchange of α and β .

THEOREM 4.3 (SPANNING THEOREM). *In order that there exists an arc in M which spans α , contains r , and is disjoint from β , it is necessary and sufficient that α possesses the properties $P(r, \alpha)$ with respect to $R(J, r)$.*

Proof. The necessity is immediate. To prove the sufficiency observe that the arc β and the points of $M - \beta$ give an upper semicontinuous decomposition of M . Let M' be the hyperspace of this decomposition and $f(M) = M'$ the associated monotone map. Let $f(\beta) = b$, and note that $f(M - \beta) = M' - b$ is one-to-one.

By property $P_2(r, \alpha)$ there exist distinct points z_1 and z_2 belonging to $\alpha' \cap F[R(J, r)]$. Thus z_i does not separate r from z_j ($i \neq j$) in $M - \beta$, hence no such separation occurs in $M' - b$. By property $P_1(r, \alpha)$ no point separates r and $z_1 \cup z_2$ in $M - \beta$, hence no such separation occurs in $M' - b$. Thus by a theorem of Hall and Puckett [2, Theorem 2.2, p. 555], $M' - b$ contains an arc from z_1 to z_2 having r as an interior point. Clearly the inverse under f of this arc is an arc which contains a subarc $a_1 r a_2$ spanning α' and such that $a_1 r a_2 - (a_1 \cup a_2) \subset R(J, r)$.

THEOREM 4.4. *Let each of the arcs α and β possess at most one of the properties $P_i(r, \alpha)$ and $P_i(r, \beta)$ respectively, $i = 1, 2$. If there exists an arc $\gamma = arb$ such that $\gamma \cap \alpha' = a$, $\gamma \cap \beta' = b$, and $\gamma - (a \cup b) \subset R(J, r)$, then there exists a region V in $R(J, r)$ and two distinct points c and d of the arc γ such that the following conditions hold:*

- (i) $r \in V$.
- (ii) $F(V) \supset c \cup d$ where c and d are distinct from p and q .
- (iii) $F(V) \subset c \cup d \cup q \cup p$.
- (iv) $\bar{V}$ is locally connected, hence a Peano space.

Proof. If there exists a region V of $R(J, r)$ satisfying (i), (ii), and (iii), then $\bar{V}$ is locally connected since $\bar{V} - V$ is a finite set.

Since α possesses at most one of the properties $P_i(r, \alpha)$, $i = 1, 2$, the subarc ar of γ contains a point c which either separates r from $F[R(J, r)] \cap \alpha'$ in $\bar{R}(J, r) - \beta$, or is the single point of $F[R(J, r)] \cap \alpha'$. Similarly the subarc br of γ contains a point d which either separates r from $F[R(J, r)] \cap \beta'$ in $\bar{R}(J, r) - \alpha$, or is the single point of $F[R(J, r)] \cap \beta'$. Thus the component of $R(J, r) - (c \cup d)$ which contains r is the region V .

5. Principal theorems. In this section the principal theorems on strongly cyclic Peano spaces are obtained.

THEOREM I. *In order that a cyclic Peano space M which is strongly arcwise connected at a point p fail to be strongly cyclic at p , it is necessary and sufficient that there exists in M a closed set D containing p and a separation $M - D = (\bigcup_{i \in I} V_i) \cup N$, where $\{V_i\}$ is a collection of distinct components of $M - D$ having the following properties:*

- (i) $V_i \rightarrow p$.
- (ii) For each i , $F(V_i) = c_i \cup d_i$ where $c_i \cup d_i \subset D - p$, $c_i \neq d_i$.
- (iii) No simple closed curve of M intersects infinitely many of the regions V_i .

Proof. The sufficiency is immediate. To prove the necessity let $J = \alpha \cup \beta$

be any simple closed curve in M containing p , where $\alpha = qxp$ and $\beta = qyp$ are arcs such that $\alpha \cap \beta = q \cup p$, $q \neq p$. Let $\{p_n\}$ and $\{\lambda_n\}$ respectively be the sequences of points and arcs of Theorem 3.1. By Theorem 3.1 there exists an infinite collection of distinct components $\{R_i\}$ of $M - J$ converging to p with $\lambda'_i \subset R_i$. Thus R_i is the set $R(J, p_i)$ of Definition 4.1. By Theorems 3.1 (xi) and 4.4 each component R_i contains a region V_i such that $p_i \in V_i$, $F(V_i) \subset c_i \cup d_i \cup q \cup p$, and $F(V_i) \supset c_i \cup d_i$ where $c_i \cup d_i \subset \lambda_i$, $c_i \neq d_i$. Since the sequence $\{R_i\}$ converges to p , it may be assumed $q \notin F(V_i)$ for any i . If $p \in F(V_k)$ for some integer k , then p is accessible from V_k by Theorem 4.4 (iv). But this makes M strongly cyclic at p contrary to assumption. Thus it may be assumed that $F(V_i) = c_i \cup d_i$ for every i . The necessity of (i) and (ii) is established by letting $D = p \cup [\bigcup_{i \in I} (c_i \cup d_i)]$.

To prove the necessity of (iii), let J_1 be a simple closed curve in M containing p . If $J_1 \cap V_k \neq \emptyset$ for some integer k , then J_1 contains the points c_k and d_k . If the subarc $c_k d_k$ of J_1 lying in $\bar{V}_k$ is replaced by the subarc $c_k p_k d_k$ of λ_k , then a simple closed curve containing the points p_k and p is obtained. If J_1 intersects infinitely many of the sets V_i , then it is seen that M is strongly cyclic at p . This contradiction completes the proof of the theorem.

THEOREM II. *In order that a cyclic Peano space M be strongly cyclic it is necessary and sufficient that for every infinite collection of open sets $\{V_i\}$ in M there exists in M a simple closed curve intersecting infinitely many of these sets.*

Proof. The necessity is immediate. To prove the sufficiency, observe that M is strongly arcwise connected. If M is not strongly cyclic, then by Theorem I there is an infinite collection of open sets in M such that no simple closed curve of M intersects infinitely many of these sets. Since this would contradict the hypothesis, it follows that M is strongly cyclic.

COROLLARY 5.1. *If a Peano space M is not separated by any two of its points, then M is strongly cyclic.*

COROLLARY 5.2. *If M is a cyclic Peano space which fails to be strongly cyclic at the point p , then p is an im kleinen cycle point of M .*

THEOREM 5.3. *If M is a cyclic Peano space which is strongly arcwise connected at p , but not strongly cyclic at p , then given $\epsilon > 0$, there exists in M an arc $\gamma = txp$ having diameter less than ϵ and an infinite collection of distinct components $\{K_i\}$ of $M - \gamma$ converging to p .*

Proof. Let $\{\alpha_i\}$ and $\{\beta_i\}$ be the collection of subarcs $\{x_i x_{i+1}\}$ and $\{y_i y_{i+1}\}$ of α and β respectively. Let $\alpha'_i = \alpha_i - (x_i \cup x_{i+1})$. Consider $M - \Gamma$ where $\Gamma = p \cup (\bigcup_{i \in I} \lambda_i) \cup (\bigcup_{i \in I} \alpha_{2i-1}) \cup (\bigcup_{i \in I} \beta_{2i})$. If a component K of $M - \Gamma$ should contain infinitely many of the sets α'_{2i} , then p is a limit point of K . Since M is not strongly cyclic at p , p is not regularly accessible from K . But

this makes M strongly cyclic at p by applications of Lemmas 2.1 and 2.4. Thus $M - \Gamma$ has an infinite collection $\{K_i\}$ of components each of which contains a set α'_{2i} . Now $K_i \rightarrow p$, for otherwise an application of Lemma 2.4 would make M strongly cyclic at p . Thus every subarc of Γ having p as one endpoint gives the desired separation of M , thereby proving the theorem.

6. Two-cyclic Peano spaces. It is known that every Peano space has a basis each element of which is an open connected set having property S [4, p. 219]. Thus the closure of an arbitrary element of such a basis is a Peano space. Throughout this section we denote by Σ such a basis for the Peano space M under consideration.

THEOREM III. *Let M be a cyclic Peano space and $p \in M$. Suppose that M is strongly cyclic at p , but not two-cyclic at p . Then either p belongs to a free arc of M or given $\epsilon > 0$, there exists in M an arc $\gamma = txp$ having diameter less than ϵ and an infinite collection of distinct components $\{K_i\}$ of $M - \gamma$ converging to p .*

The proof of Theorem III requires that some background material be developed.

THEOREM 6.1. *Let M be the Peano space of Theorem III and U an element of Σ such that $p \in U$. Then the following conditions hold:*

- (i) *The components of $\bar{U} - p$ may be classified as follows:*
 - (a) $\{X_i\}$: *those components of $\bar{U} - p$ such that p is an endpoint of $\bar{X}_i$; or if p is not an endpoint of $\bar{X}_i$, then $\bar{X}_i$ is not strongly cyclic at p .*
 - (b) $\{Y_j\}$: *those components of $\bar{U} - p$ such that $\bar{Y}_j$ is strongly cyclic at p .*
- (ii) *For each i and j , $\bar{X}_i \cap (\bar{U} - U) \neq \emptyset$, $\bar{Y}_j \cap (\bar{U} - U) \neq \emptyset$.*
- (iii) *Each of the collections $\{X_i\}$ and $\{Y_j\}$ is finite.*
- (iv) *If the collection $\{X_i\}$ is nonvacuous, then Theorem III is true.*

Proof. Statement (i) is just a classification of the components of $\bar{U} - p$. Statement (ii) follows from the fact that M is cyclic, while (iii) is a consequence of the local connectivity of M .

To prove (iv), let $X_1 \in \{X_i\}$. If p is an endpoint of $\bar{X}_1$, Theorem III is easily seen to hold. If p is not an endpoint of $\bar{X}_1$ but $\bar{X}_1$ is not strongly cyclic at p , then Theorem III follows from Theorem 5.3.

In view of Theorem 6.1(iv), it may be assumed that for the Peano space M of Theorem III the collection $\{X_i\}$ is empty for every element U of Σ such that $p \in U$. The space M then possesses the properties which follow:

PROPERTY 1. *There exists in M an infinite sequence of distinct points $\{p_n\}$ converging to p such that if Γ_1 and Γ_2 are two arcs of M having p as one endpoint and each containing infinitely many of the points $\{p_n\}$, then $\Gamma_1 \cap \Gamma_2$ consists of a set of points having p as a limit point. Moreover, there exists in M an arc $\alpha = qxp$ such that the points $\{p_n\}$ converge monotonically to p on α .*

PROPERTY 2. *If $U \in \Sigma$ and $p \in U$, then all but a finite number of the points*

$\{p_n\}$ of property 1 lie in a true cyclic element E of the set $\bar{Y}_1$ where $\bar{U}-p = \bigcup_{j=1}^h Y_j$.

Proof. Let $U \in \Sigma$ such that $p \in U$. Then $\bar{U}-p = \bigcup_{j=1}^h Y_j$. In view of Property 1 it may be assumed that all but a finite number of the points $\{p_n\}$ lie in Y_1 . Let E be the true cyclic element of $\bar{Y}_1$ containing p . Since M is cyclic, every component of $\bar{Y}_1-E$ must intersect $\bar{U}-U$, hence the components of $\bar{Y}_1-E$ cannot have limit points in E arbitrarily close to p . Thus all but a finite number of the points $\{p_n\}$ lie in E .

PROPERTY 3. $E-\alpha = H \cup N$, where H is the set of all points $x \in E$ for which there exists in E a nondegenerate arc xp with $xp \cap \alpha = p$, N is the set of all points $x \in E$ for which no such arc exists.

PROPERTY 4. If, for every component W of $E-\alpha$, $\bar{W} \cap F(M-E) \neq \emptyset$, then p is not a limit point of N and the collection $\{C_i\}$ of components of H is finite. Thus there exists a subarc vp of α which does not separate M . Moreover, in this case there exists at least one component of H which has a set of limit points in $\alpha-p$ having p as a limit point.

Proof. By the definition of N , p is not accessible from N . Thus if p is a limit point of N , the hypothesis of Lemma 2.4 is easily seen to be satisfied. Since this makes M two-cyclic at p , p is not a limit point of N . If H has an infinite collection of components, the hypothesis of Lemma 2.4 is again satisfied. Thus H has a finite number of components in this case. In the natural order from q to p on α , let d be the last point of $\bar{N} \cap \alpha$. Thus for any point v interior to the subarc dp , the subarc vp of α does not separate M . The proof of Property 4 is complete if it is observed that if H does not have a component which has a set of limit points in $\alpha-p$ having p as a limit point, then there is an element $V \in \Sigma$ containing p for which the collection $\{X_i\}$ is nonempty.

PROPERTY 5. If, for arbitrarily small elements U of Σ such that $p \in U$ there exists a component W of $E-\alpha$ such that $\bar{W} \cap F(M-E) = \emptyset$, then Theorem III is true.

The following restrictions may be imposed on all further discussion.

RESTRICTIONS 6.2. The space M under consideration is a cyclic Peano space which is strongly cyclic at the point p , but not two-cyclic at p , and for which every element U of the basis Σ such that $p \in U$ satisfies the following conditions:

- (a) The collection $\{X_i\}$ is empty.
- (b) For every component W of $E-\alpha$, $\bar{W} \cap F(M-E) \neq \emptyset$, where E and W are the sets of $\bar{U}$ as given in the above properties.

THEOREM 6.3. Let M be a Peano space satisfying the Restrictions 6.2. Then there exists a sequence $\{U_i\}$ of elements of the basis Σ such that $p \in U_i$ for every i , and corresponding sequences $\{E_i\}$, $\{N_i\}$, and $\{H_i\}$ as given by Properties 2 and 3, such that the following conditions hold:

- (i) $\{U_i\}$ is a null sequence, $U_i \supset \bar{U}_{i+1}$, and $\bar{U}_{i+1} \cap N_i = \emptyset$.
- (ii) $E_i \supset E_{i+1}$ and $H_i \supset \bar{H}_{i+1}$.
- (iii) If R_i is a component of H_i having a set of limit points in $\alpha - p$ which has p as a limit point, then there exists a sequence $\{R_j\}$, $j \geq i$, such that R_j is a component of H_j having a set of limit points in $\alpha - p$ which has p as a limit point, and $R_j \supset \bar{R}_{j+1}$.

Proof. The sequences $\{U_i\}$, $\{E_i\}$, $\{N_i\}$, and $\{H_i\}$ are easily seen to exist. To prove (iii), note that by Property 4 each set H_i contains at least one component having a set of limit points in $\alpha - p$ having p as a limit point. Let R_1 be such a component of H_1 . In the natural order from q to p on α let d be the last point of $\bar{N}_1 \cap \alpha$. There exists a sequence $\{r_n\}$ of points interior to the subarc dp of α such that $r_n \rightarrow p$, r_n is a limit point of R_1 , and $r_n \in U_2$, $n \in I$. Since H_2 has only a finite number of components, it is clear that there is a component R_2 of H_2 which has infinitely many of the points $\{r_n\}$ as limit points and $\bar{R}_2 \subset R_1$. Continuing this process inductively completes the proof of the theorem.

THEOREM 6.4. For each sequence $\{R_j\}$, $j \geq i$, of Theorem 6.3(iii) there exists an arc λ from a point $t_i \in F(R_i) - \alpha$ to p , and a sequence $\{\mu_j\}$ of arcs such that

- (a) $\lambda - p \subset \bar{R}_i - \alpha$; i.e., $\lambda \cap \alpha = p$.
- (b) $\lambda \cap R_j \neq \emptyset$ for every $j \geq i$.
- (c) $\mu_j = x_j y_j$ is an arc of $E_j - E_{j+1}$ such that $\mu_j \cap \alpha = x_j$, $\mu_j \cap \lambda = y_j$.
- (d) The sequences $\{x_j\}$ and $\{y_j\}$ converge monotonically to p on α and λ respectively.

Proof. An arc λ satisfying (a) and (b) is readily constructed. In each set E_i , let $t_i \in \alpha$, $z_i \in \lambda$. There exists an arc $p z_i t_i$ in E_i , and this arc contains a subarc $\mu_i = x_i y_i$ such that $\mu_i \cap \alpha = x_i$, $\mu_i \cap \lambda = y_i$. It is clear that no generality is lost in assuming that (c) and (d) hold.

THEOREM 6.5. Each set H_i , $i \in I$, of Theorem 6.3 contains exactly one component R_i having a set of limit points in $\alpha - p$ which has p as a limit point. Thus the sequence $\{R_i\}$, $i \in I$, of Theorem 6.3(iii) is unique, as is the sequence $\{G_i\}$ where G_i is the set of all points $x \in R_i$ which can be joined to $\alpha - p$ by an arc of E_i not meeting λ .

Proof. Assume that there is a set H_i which has two components having a set of limit points in $\alpha - p$ each having p as a limit point. It may be assumed that H_1 is such a set. By Theorem 6.3(iii) there exist two distinct sequences $\{R_i\}$ and $\{S_i\}$ as there described. As described in Theorem 6.4, there exist arcs λ and $\{\mu_i\}$ for the sequence $\{R_i\}$, and arcs λ^* and $\{\mu_i^*\}$ for the sequence $\{S_i\}$. It is easily seen that this makes M two-cyclic at p . This contradiction proves the theorem.

A review of Properties 1–5 and Theorems 6.3–6.5 reveals that there is no loss of generality in making the following assumptions.

ASSUMPTIONS 6.6. For the Peano space M of Properties 1–5 and Theorems 6.3–6.5 the following conditions hold:

(a) In the natural order from q to p on α , $x_i < p_i < x_{i+1}$, $i \in I$.

(b) $p_i \in E_i - E_{i+1}$, $i \in I$.

(c) $\mu_i \subset E_i - E_{i+1}$.

(d) λ is an arc from a point $t \in F(R_1) - \alpha$ to the point p such that λ satisfies Theorem 6.4 for the sequence $\{R_i\}$.

THEOREM 6.7. *Let M be a Peano space satisfying the restrictions 6.2 and subject to the Assumptions 6.6. Let $\{\alpha_i\}$ be the collection of subarcs of α defined by $\alpha_i = x_i p_i x_{i+1}$, $\{\lambda_i\}$ the collection of subarcs of λ defined by $\lambda_i = y_i y_{i+1}$. Let $\alpha'_i = \alpha_i - (x_i \cup x_{i+1})$, $\lambda'_i = \lambda_i - (y_i \cup y_{i+1})$. For each set G_j of the sequence $\{G_i\}$ of sets defined in Theorem 6.5, consider the set $G_j - \bigcup_{i \in I} \mu_i$, $j \in I$. Then there exists a positive integer k such that for $j \geq k$ the following hold:*

(i) *No component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in two distinct sets α'_m and α'_n where $|m - n| \geq 2$.*

(ii) *If a component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in a set α'_m and a set λ'_n , then $m = n$.*

(iii) *If a component of $G_j - \bigcup_{i \in I} \mu_i$ has limit points in a set α'_m , then any limit points of this component in $\bigcup_{i \in I} \mu'_i$, where $\mu'_i = \mu_i - (x_i \cup y_i)$, must lie in $\mu'_m \cup \mu'_{m+1}$.*

Proof. If no such integer k exists, it is easy to construct two arcs Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = p$ and each contains infinitely many of the points $\{p_n\}$.

Proof of Theorem III. Theorem 6.1 (iv) and property 5 give special cases in which Theorem III is true. It may thus be assumed that M satisfies the Restrictions 6.2 and is subject to the Assumptions 6.6. Moreover, it may be assumed in Theorem 6.7 that the integer $k = 1$. In the natural order from q to p on α , let d be the last point of $\bar{N}_1 \cap \alpha$. Then there exists an integer r such that for $i \geq r$, x_i is a point interior to the subarc dp of α . No generality is lost in assuming that $r = 1$. Define an arc Γ as follows:

$$\Gamma = p \cup \left(\bigcup_{i \in I} \alpha_{2i-1} \right) \cup \left(\bigcup_{i \in I} \mu_i \right) \cup \left(\bigcup_{i \in I} \lambda_{2i} \right).$$

Note that the arc Γ lies in the set E_1 and is disjoint from $\bar{N}_1$.

Next define a collection of sets $\{K_i\}$ as follows:

$K_i = \alpha'_{2i} \cup (\text{all components of } G_1 - \bigcup_{i \in I} \mu_i \text{ having limit points in } \alpha'_{2i})$.

Observe that each set K_i is a component of $E_1 - \Gamma$ by Theorem 6.7. Thus E_1 is separated into infinitely many components $\{K_i\}$ by the arc Γ . Moreover, the $\{K_i\}$ converge to p , for otherwise the hypothesis of Lemma 2.4 is satisfied thereby making E_1 , and hence M , two-cyclic at p . Thus Γ , and every subarc of Γ having p as an endpoint, separates M into infinitely many components converging to p , the collection of components in such cases being a

subcollection of the collection $\{K_i\}$. This completes the proof of Theorem III.

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