LATTICES WITH INVOLUTION(1)

BY
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Introduction. By a "lattice with involution," or "i-lattice," we shall mean a lattice L together with an involution [1, p. 4] $x \rightarrow x'$ in L. A distributive i-lattice in which $x \cap x' \leqq y \cup y'$ for all x and y will be called a "normal" i-lattice. The underlying lattice of an l-group becomes a normal i-lattice when x' is defined as the group inverse of x; also a Boolean algebra becomes a normal i-lattice when x' is defined as the complement of x. In this paper l-groups and Boolean algebras will always be understood to have the involutions defined above. §1 of the paper contains subdirect decomposition theorems for distributive and normal i-lattices, with applications; in §2, as a contribution to the study of nondistributive i-lattices, modular and nonmodular i-lattices are classified with respect to certain laws each of which, for distributive i-lattices, is equivalent to normality; and §§3 and 4 contain some extension and embedding theorems concerning normal i-lattices.

1. **Subdirect decomposition of distributive and normal i-lattices.** Every i-lattice is an algebra [1, p. vii] with operations $\cap$, $'$ which satisfy the identities $x \cap y = y \cap x$, $x \cap (y \cap z) = (x \cap y) \cap z$, $x' = x' \cap (x \cap y)'$, and $x'' = x$; it may be proved that these identities are independent postulates for i-lattices. We shall apply the usual terminology of abstract algebra (cf. [1, pp. viif.]) to i-lattices, except that we shall use the terms "i-sublattice," "i-homomorphism," and "i-isomorphism" instead of "subalgebra," "homomorphism," and "isomorphism."

Let L be a distributive i-lattice. For elements x, y, p of L we shall set $x \equiv y(C(p))$ if and only if $x \cap p = y \cap p$ and $x' \cap p = y' \cap p$. It is easily verified that this defines a congruence relation $C(p)$ on L, and that

$$C(p) \cap C(q) = C(p \cup q) \text{ for all } p \text{ and } q \text{ in } L. \tag{1}$$

Using [1, p. 28, Lemma 1(ii)] we see that if O is the zero congruence relation on L then

$$C(p) \cap C(p') = O \text{ for all } p \text{ in } L. \tag{2}$$

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Also, since $p\equiv p\cup p'(C(p))$ for all p, $p\geqq p'$ if $C(p)=O$, while, if $p\geqq p'$, then $C(p)=O$ by (1) and (2): thus

$$C(p) = O \text{ if and only if } p \geqq p'. \tag{3}$$

LEMMA 1. *Let L be a subdirectly irreducible distributive i-lattice. Then*

$$x \text{ is comparable with } x' \text{ for each } x \text{ in } L, \tag{4}$$

and, for elements x, y, z of L,

$$\text{if } x > x' \text{ and } y > y' \text{ then } x \cap y > (x \cap y)', \text{ and} \tag{5}$$

$$\text{if } y > y' \text{ and } z \leqq z' \text{ then } y > z. \tag{6}$$

Proof. Since L is subdirectly irreducible, (4) follows from (2) and (3). If $x>x'$ and $y>y'$ but $x\cap y \not> (x\cap y)'$ then $x\cap y\leqq(x\cap y)'$, by (4), and hence $C(x'\cup y') =O$, by (3); using (1), we deduce that $x'\geqq x$ or $y'\geqq y$, which contradicts the hypothesis. Suppose now that $y>y'$ and $z\leqq z'$; if $(y\cap z')'>y\cap z'$ then, by (5), $y\cap(y'\cup z)>y'\cup(y\cap z')=y\cap(y'\cup z')\geqq y\cap(y'\cup z)$, a contradiction; hence $(y\cap z')'\leqq y\cap z'$, and it follows that $y\geqq y\cap z'\geqq y'\cup z\geqq z$; but $y\neq z$, hence $y>z$.

For each element x of a given i-lattice we shall set $|x| =x\cup x'$. We shall call an element z of a given i-lattice a "zero" if $z=z'$. We shall denote the i-lattice with four elements and two zeros by $\mathfrak{D}$. The i-lattice whose underlying lattice is the finite chain with n elements will be denoted by $\mathfrak{C}_n$. If L and M are i-isomorphic i-lattices we shall write $L\cong M$.

LEMMA 2. *The i-lattices $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, and $\mathfrak{D}$ are subdirectly irreducible, and are to within i-isomorphism the only subdirectly irreducible distributive i-lattices.*

Proof. Each of the given i-lattices is obviously subdirectly irreducible. Conversely, let L be a subdirectly irreducible distributive i-lattice. For elements x, y of L we shall set $x\sim y$ if and only if one of the following statements is true: (a) $x>x'$ and $y>y'$, (b) $x<x'$ and $y<y'$, (c) $x=x'=y=y'$. Using Lemma 1 and trivial arguments we see that $\sim$ defines a congruence relation on L. Also, for each p in L, we may define congruence relations $D(p)$ and $E(p)$ on L by setting $x\equiv y(D(p))$ if and only if $x\sim y$ and $|x|\cap p=|y|\cap p$, and $x\equiv y(E(p))$ if and only if $x\sim y$ and $|x|\cup p=|y|\cup p$. Using [1, p. 28, Lemma 1(ii)], we see that $D(p)\cap E(p)=O$ for all p in L and hence, since L is subdirectly irreducible, that

$$\text{either } D(p) = O \text{ or } E(p) = O \text{ for each } p \text{ in } L. \tag{7}$$

If $x=x'$ for all x in L, then $L\cong\mathfrak{C}_1$; we may therefore assume in the rest of the proof that L has an element c such that $c>c'$. We shall prove that if L has an element x distinct from c and c' then x is a zero of L. First, L cannot have three distinct elements x such that $x > x'$; for, if L has three distinct such elements, it has a chain $x > y > w$ of such elements, and then $x \equiv y(D(y))$ and $w \equiv y(E(y))$, contradicting (7). It follows that L has at most two such elements, and that they must be comparable if they exist; but if $x > y > y' > x'$, and F is the congruence relation on L with congruence classes $\{x\}$, $\{x'\}$, and $\{w : y' \leqq w \leqq y\}$ (cf. (6)), then $F \neq O$, $D(y) \neq O$, and $F \cap D(y) = O$, contradicting the subdirect irreducibility of L. Thus, as asserted, every element of L distinct from c and c' is a zero of L; moreover, by [1, p. 28, Lemma 1(ii)], L has at most two zeros. Thus $L \cong \mathfrak{O}_2$, $\mathfrak{O}_3$, or $\mathfrak{D}$ according as L has 0, 1, or 2 zeros. This completes the proof.

It will be convenient to call *any* i-sublattice of a direct union of i-lattices L_γ a "subdirect union" of the L_γ (cf. [1, p. 91]). With this nomenclature we have

THEOREM 1. *Every distributive i-lattice is i-isomorphic with a subdirect union of i-isomorphic images of* $\mathfrak{D}$.

Proof. Every algebra A can be represented as a subdirect union of subdirectly irreducible homomorphic images of A (cf. [1, p. 92]). Observing that every i-homomorphic image of a distributive i-lattice is again a distributive i-lattice, and that each $\mathfrak{O}_i$ $(i = 1, 2, 3)$ is i-isomorphic with an i-sublattice of $\mathfrak{D}$, we deduce Theorem 1.

THEOREM 2. *Every normal i-lattice is i-isomorphic with a subdirect union of i-isomorphic images of* $\mathfrak{O}_3$.

This result is easily deduced from Theorem 1. The well known theorem that every Boolean algebra except $\mathfrak{O}_1$ is i-isomorphic with a subdirect union of i-isomorphic images of $\mathfrak{O}_2$ similarly follows from Theorem 2.

If L is an i-lattice, and P is a partly ordered set with involution $t \rightarrow t'$, the cardinal power L^P [1, pp. 8, 25] becomes an i-lattice if we define $f'(t) = (f(t'))'$ for each f in L^P and t in P.

THEOREM 3. *Every normal i-lattice is i-isomorphic with an i-sublattice of a vector lattice.*

Proof. Let L be a normal i-lattice. Then, by Theorem 2, there exists an i-isomorphism ρ of L into a cardinal power $(\mathfrak{O}_3)^\Gamma$, where Γ is an unordered set with involution given by $\gamma' = \gamma$ for each γ in Γ. If we take $\mathfrak{O}_3$ to be the chain $-1 < 0 < 1$ then ρ becomes an i-isomorphism of L into the vector lattice R^Γ of all real-valued functions on Γ.

An alternative proof of Theorem 3 may be based on [1, p. 140, Corollary] and the existence part of Theorem 7 below (cf. p. 98 of the author's thesis).

Let P be a class of equations for i-lattices; then an i-lattice L will be said to be "P-proper" if none of the equations in P holds in L, and to be "P-

complete" if it is P-proper and no P-proper i-lattice M is such that the set of identities of L is strictly included in the set of identities of M. Let P_0, P_1, and P_2 be the one-element classes whose elements are the equations $x=y$, $|x|=|y|$, and $|x|'\cap|y|=|x|'$, respectively; then P_0-completeness is precisely equational completeness [4].

THEOREM 4. *An i-lattice is P_0-complete if and only if it is a Boolean algebra with at least two elements*[2]. *An i-lattice is P_1-complete if and only if it is normal but is not a Boolean algebra. An i-lattice is P_2-complete if and only if it is distributive but is not normal.*

Proof. We observe that if L is a P_2-proper i-lattice then $\mathfrak{D}$ is an i-homomorphic image of an i-sublattice of L; indeed if a, b in L are such that $|a|'\cap|b| \neq |a|'$ then $M = \{t: |a|' \leqq t \leqq |a|\} \cup \{t: |b|' \leqq t \leqq |b|\} \cup \{t: t \leqq |a|\cap|b|\} \cup \{t: t \geqq |a|'\cup|b|'\}$ is an i-sublattice of L, and $\mathfrak{D}$ is an i-homomorphic image of M. Using this observation and Theorem 1 the reader may prove by the methods of [4] that an i-lattice is P_2-complete if and only if it is distributive but not normal. The proofs of the corresponding results for P_0- and P_1-completeness are similar but easier.

The results of §1 depend on [1, p. 92, Theorem 10], and hence on the axiom of choice. Although the results of §1 will be used later in the paper, all the results of §§2–4 can be proved without using the axiom of choice.

2. **A classification of i-lattices.** For elements x, y of any i-lattice set $x \triangle y = (x\cup y)\cap(x'\cup y')$. Then, using Theorems 1 and 2, we may verify that each of the following laws is a necessary and sufficient condition for a given distributive i-lattice to be normal: (A) $x \triangle (y \triangle z) = (x \triangle y) \triangle z$, (B) $x \triangle y = (x\cap y')\cup(x'\cap y)$, (C) $|x\cap y|\cap|x\cup y| = |x|\cap|y|$, (C^*) $|x \triangle y| = |x|\cap|y|$, (D) $|x\cap y|\cap|x\cup y| \leqq |x|\cap|y|$, (E) $|x\triangle y| \leqq |x|\cap|y|$. Similarly the laws (α) if $a \triangle c = b \triangle c$ for some c then $a=b$, and (β) $x\cap(y \triangle z) = (x\cap y) \triangle (x\cap z)$ are each necessary and sufficient for a given distributive i-lattice to be a Boolean algebra; using [2, Theorem 1] we may deduce that *any* i-lattice satisfying (α) is a Boolean algebra. For Boolean algebras, the laws (A), (B), (α), and (β) are well known (cf. [1, pp. 154 f.]), and the laws (C) and (C^*) are trivial; for 1-groups, the laws (A), (B), (C), and (C^*) are believed to be new. It may be proved that an arbitrary i-lattice satisfies (C^*) if and only if it satisfies (C).

Let ω be the set whose elements are the laws (A), (B), (C), (D), (E), (F) $|x|' \leqq |y|$, and (M) the modular law. For each $\xi\subseteq\omega$ let $\xi\mathfrak{c}$ be the set of all (Y) in ω such that every i-lattice which satisfies all (X) in ξ necessarily satisfies (Y); then $\mathfrak{c}: \xi\rightarrow\xi\mathfrak{c}$ is a closure operation on the subsets of ω. If $\xi\subseteq\omega$ is nonempty and has elements $(X), \cdots, (Y)$ we shall write $\xi = \{X \cdots Y\}$, and we shall denote the family of all i-lattices which satisfy all (Z) in ξ by

(2) Cf. [4, Theorem 3.3; 1, p. 189].

$[X \cdots Y]$. It may be proved that $\mathfrak{c}$ is the intersection (cf. [5, §1]) of all those closure operations $\mathfrak{d}$ on the subsets of ω which are such that $\{A\}\mathfrak{d} \supseteq \{BC\}$, $\{B\}\mathfrak{d} \supseteq \{E\}$, $\{C\}\mathfrak{d} \supseteq \{D\}$, $\{D\}\mathfrak{d} \supseteq \{E\}$, $\{E\}\mathfrak{d} \supseteq \{F\}$, $\{BM\}\mathfrak{d} \supseteq \{A\}$, and $\{EM\}\mathfrak{d} \supseteq \{D\}$; from this result we may deduce

THEOREM 5. *If $\{X \cdots Y\}$ is a nonempty subset of ω then $[X \cdots Y]$ is equal to exactly one of the families $[A]$, $[B]$, $[C]$, $[D]$, $[E]$, $[F]$, $[M]$, $[AM] = [BM]$, $[BC]$, $[BD]$, $[CM]$, $[DM] = [EM]$, and $[FM]$. In the presence of a set of axioms for i-lattices the laws indicated are in each case independent axioms for the corresponding family.*

3. **Normal extensions.** It is easily proved that an i-lattice satisfying the law (F) of §2 can have at most one zero, and that a modular i-lattice with a zero satisfies (F) if the zero is unique; thus a distributive i-lattice with a zero is normal if and only if it has no other zero. In an i-lattice with a unique zero 0 we shall set $x_+ = x \cup 0$ and $x_- = x' \cup 0$ for each x, and we shall say that x is "positive" if $x \geqq 0$; then $x_+ \cap x_- = x \,\Delta\, 0$ for all x, and the law $x_+ \cap x_- = 0$ (cf. [1, p. 220, Lemma 4]) is implied by (E) of §2, and implies (F). In this section we shall determine all the normal i-lattices with zero which have a given sublattice of positive elements.

If a is any element of a given distributive lattice P then the ordered pairs (x, y) of elements of P which are such that $x \cap y \leqq a \leqq x \cup y$ form a normal i-lattice $P(a)$ with operations given by $(x, y) \cap (z, w) = (x \cap z, y \cup w)$ and $(x, y)' = (y, x)$, and the elements (x, y) of $P(a)$ which are such that $a = x \leqq y$ or $a = y \leqq x$ form an i-sublattice $P\langle a\rangle$ of $P(a)$. If P has a least element O, and L is a normal i-lattice with zero whose sublattice L_+ of positive elements is lattice-isomorphic with P, we shall call L a "normal extension" of P; then, if λ is a lattice-isomorphism of L_+ onto P, $x \rightarrow (x_+\lambda, x_-\lambda)$ is an i-isomorphism of L into $P(O)$. Using this observation the reader may prove

THEOREM 6. *Let P be a distributive lattice with least element O. Then every i-sublattice L of $P(O)$ which is such that $P\langle O\rangle \subseteq L$ is a normal extension of P, and every normal extension of P is i-isomorphic with an i-sublattice L of $P(O)$ which is such that $P\langle O\rangle \subseteq L$. All normal extensions of P are i-isomorphic if and only if O is meet-irreducible in P.*

4. **Embedding theorems.** In §4 if L is a normal i-lattice and $x \in L$ then x^- will denote the principal ideal of all $y \leqq x$, $\vee$ will denote the join operation in the complete lattice $\bar{L}$ of all closed ideals [1, p. 59] of L, and Z will denote the set of all $|y|'$ for y in L. Then $Z \in \bar{L}$ and (cf. [3, p. 2]) the mapping $x \rightarrow x^- \vee Z$ is a lattice-homomorphism of L into $\bar{L}$. Hence $\{x^- \vee Z : x \in L\}$ is a distributive sublattice of $\bar{L}$; this lattice will be denoted by L^*.

THEOREM 7. *Let L be a normal i-lattice. Then the set L_Ω of all those ordered pairs $(x^- \vee Z, y^- \vee Z)$ of elements of L^* which are such that $x \leqq y'$ is an i-sub-*

lattice of $L^*(Z)$. L_Ω *is a normal i-lattice with zero* (Z, Z), *and* $\pi: x \rightarrow (x^- \vee Z, x'^- \vee Z)$ *is an i-isomorphism of L into* L_Ω; π *maps L onto* L_Ω *if and only if L has a zero. If* τ *is an i-isomorphism of L into a normal i-lattice M with zero* 0, *then* $\rho: (x^- \vee Z, y^- \vee Z) \rightarrow (x\tau)_+ \cap (y\tau)' (x \leqq y')$ *is an i-isomorphism of* L_Ω *onto the i-sublattice of M generated by* $L\tau$ *and* 0, *and* $\tau = \pi\rho$. *If N is a normal i-lattice with zero having the property*: *there exists an i-isomorphism* π_0 *of L into N such that every i-isomorphism* τ_0 *of L into a normal i-lattice P with zero O is of the form* $\tau_0 = \pi_0\rho_0$ *for some i-isomorphism* ρ_0 *of N into P, then* $N \cong L_\Omega$.

The proof depends on the fact that $x^- \vee Z = y^- \vee Z$ in L^* if and only if $x \cap x' = x \cap y'$ and $y \cap y' = y \cap x'$ in L; the details will be left to the reader.

If L is any i-lattice we may (and shall) identify the cardinal power $L^{\mathfrak{D}_2}$ (cf. §1) with the set of all ordered pairs (x, y) of elements of L which are such that $x \leqq y$.

THEOREM 8. *Let L be a normal i-lattice, and let* $M = L^{\mathfrak{D}_2}$. *Then* $\phi: (x, y) \rightarrow (x^- \vee Z, y'^- \vee Z)$ *is an i-homomorphism of M onto* L_Ω; ϕ *is an i-isomorphism if and only if L is a Boolean algebra. If* ψ *is an i-homomorphism of M onto a (distributive) i-lattice N, then N is normal if and only if* $\Psi \geqq \Phi$, *where* Φ *and* Ψ *are the congruence relations on M defined by* ϕ *and* ψ *respectively.*

The main step in the proof is to show that if N is normal, and (x, y), (u, v) in M are such that $(x, y)\phi = (u, v)\phi$, then $(x \cup u, y \cap v) \in M$ and $(x, y)\psi_\pm = (x \cup u, y \cap v)\psi_\pm = (u, v)\psi_\pm$.

A normal i-lattice L will be said to be "closed" if for every pair x, y of elements of L_Ω such that $x \cap y = 0$ (the zero of L_Ω) there exists z in L_Ω such that $z_+ = x$ and $z_- = y$. A distributive lattice with least element has, to within i-isomorphism, just one closed normal extension. Also, the following embedding theorem may be proved.

THEOREM 9. *Let L be a normal i-lattice. Then the set* L^C *of all those ordered pairs* $(x^- \vee Z, y^- \vee Z)$ *of elements of* L^* *which are such that* $x \Delta y' \in Z$ *is an i-sublattice of* $L^*(Z)$. L^C *is a closed normal i-lattice, and* $\pi: x \rightarrow (x^- \vee Z, x'^- \vee Z)$ *is an i-isomorphism of L into* L^C; π *maps L onto* L^C *if and only if L is closed. If* τ *is an i-isomorphism of L into a closed normal i-lattice M then there exists an i-isomorphism* σ *of* L^C *into M such that* $\tau = \pi\sigma$. *If N is a closed normal i-lattice having the property*: *there exists an i-isomorphism* π_0 *of L into N such that every i-isomorphism* τ_0 *of L into a closed normal i-lattice P is of the form* $\tau_0 = \pi_0\sigma_0$ *for some i-isomorphism* σ_0 *of N into P, then* $N \cong L^C$.

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