GENERALIZED POWERS OF THE DIFFERENCE OPERATOR

BY
D. B. SUMNER

1. The difference operator Δ with span h

$$\Delta f(x) = [f(x+h) - f(x)]/h \tag{1.1}$$

together with its integer powers

$$\Delta^N f(x) = \Delta\Delta^{N-1} f(x) = \sum_0^N (-)^{N-p} \binom{N}{p} f(x+ph)/h^N, \tag{1.2}$$

has a long history and an extensive literature, the most elegant and systematic accounts of it being those of Nörlund [4] and Milne-Thomson [3]. The study of the operator has taken several directions, the main ones being the Bernoulli numbers and polynomials associated with it, the inversion of the operator, and the solution of certain difference equations. Nörlund [4] broke fresh ground when he considered the role of the span h in the solution of the difference equation

$$f(x+h) - f(x) = h\phi(x), \tag{1.3}$$

and obtained [4, pp. 38–98] a *principal solution* of (1.3) which reduced to the integral of the function on taking the limit as $h \to 0$ [4, p. 59].

The main purpose of this paper is to give a definition of noninteger powers of the difference operator (1.1). Some of the more important properties of the Bernoulli numbers and polynomials associated with it will be obtained, thus giving a partial solution of the problem indicated by Nörlund [4, p. 146] and solved from another point of view by R. Lagrange [2, pp. 258–284]. The inverse of the operator Δ^λ will be obtained, and both $\Delta^{-\lambda}$ and Δ^λ will be shown when applied to functions for which they have a meaning, to have the properties of Nörlund's principal solution.

The key device used in generalizing (1.2) for real noninteger powers of Δ will be the fact that Δ (with span h) can be expressed in terms of a shift operator:

$$h\Delta f(x) = f(x+h) - f(x) = [\exp(hD) - 1] f(x). \tag{1.4}$$

It is then possible to define $\Delta^\lambda f(x)$ by obtaining representations of the function $[\exp(ht) - 1]^\lambda$ or suitable combinations of it. The method naturally im-

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poses restrictions on the functions to which the operator can be applied: we assume throughout that the functions to which the operator is applied are of finite exponential order, a condition which turns out to be sufficient.

2. **Notation and definitions.** The following definitions will be used throughout:

$$\alpha_0(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0; \end{cases} \tag{2.1}$$

$$\alpha_1(t) = \int_{t-1}^{t} \alpha_0(v)dv = \begin{cases} 0, & t \leqq 0, \\ t & 0 \leqq t \leqq 1, \\ 1, & t \geqq 1; \end{cases} \tag{2.2}$$

$$\alpha_p(t) = \int_{-\infty}^{\infty} \alpha_{p-1}(t-v)d\alpha_1(v) = \int_{t-1}^{t} \alpha_{p-1}(v)dv, \qquad (p = 1, 2, \cdots); \tag{2.3}$$

$$\beta_n(t) = \sum_{p=0}^{n} \binom{n}{p} (-)^{n-p}\alpha_p(t), \qquad (n = 0, 1, 2, \cdots); \tag{2.4}$$

$$\alpha_\lambda(t) = \sum_{n=0}^{\infty} \binom{\lambda}{n} \beta_n(t), \qquad \lambda \geqq 0. \tag{2.5}$$

When λ is a positive integer N, it is easily verified that $\alpha_\lambda(t) = \alpha_N(t)$, so that the notation $\alpha_\lambda(t)$ for the function (2.5) is justified.

$$\delta(t) = (e^t - 1)/t; \tag{2.6}$$

$$\delta^\lambda(t) = [(e^t - 1)/t]^\lambda, \qquad \arg [(e^t - 1)/t]^\lambda = 0 \text{ when } t > 0. \tag{2.7}$$

In the sequel, when the range of integration is not stated, it will be over $(-\infty, \infty)$. The number λ will be real.

We shall say that a function

$$\begin{gathered} f(z) \in E(h, \kappa) \text{ if it is entire and of exponential order } \kappa, \\ 0 < \kappa |h| < 2\pi, \text{ complex values being allowed for } h. \end{gathered} \tag{2.8}$$

3. **Bernoulli numbers and polynomials of general order.** In this section λ and μ may be positive or negative.

The Bernoulli numbers $B_n^{-\lambda}$ and the Bernoulli polynomials $B_n^{-\lambda}(x)$ of general order λ are defined by the generating relations:

$$\delta^\lambda(t) = \sum_0^\infty t^n B_n^{-\lambda}/n!, \tag{3.1}$$

$$\delta^\lambda(t) \exp(xt) = \sum_0^\infty t^n B_n^{-\lambda}(x)/n!. \tag{3.2}$$

The series on the right converge for all x and all t when $\lambda > 0$, and for all x

and for $|t| < 2\pi$ when $\lambda < 0$. On comparing coefficients in (3.2) we deduce

$$B_n^{-\lambda}(x) = \sum_0^n \binom{n}{p} x^{n-p} B_p^{-\lambda}, \tag{3.3}$$

$$B_n^{-\lambda}(0) = B_n^{-\lambda}. \tag{3.4}$$

The following recurrence relations hold:

$$B_{n+1}^{-\lambda}(x) = (x+\lambda)B_n^{-\lambda}(x) + \lambda \sum_0^n \binom{n}{p} B_{n-p}^{-\lambda}(x) B_{p+1}/(p+1), \quad (n \geqq 0); \tag{3.5}$$

$$B_{n+1}^{-\lambda} = \lambda \sum_0^n \binom{n}{p} B_{n-p}^{-\lambda} \gamma_p, \qquad (n \geqq 0), \tag{3.6}$$

where B_n are the Bernoulli numbers of order 1, and

$$\gamma_0 = 1 + B_1, \; \gamma_p = B_{p+1}/(p+1), \qquad (p \geqq 1).$$

The second is deduced by setting $x=0$ in the first, and to prove (3.5) we note that

$$B_n^{-\lambda}(x) = \left\{ \left(\frac{d}{dt} \right)^n [e^{xt} \delta^\lambda(t)] \right\}_{t=0}.$$

Write

$$F(t) = e^{xt} \delta^\lambda(t).$$

Then

$$F(0) = B_0^{-\lambda}(x) = 1,$$

and

$$F'(t) = xF(t) + \lambda F(t) g(t),$$

$$F^{(n+1)}(t) = xF^n(t) + \lambda \sum_0^n \binom{n}{p} F^{(n-p)}(t) g^p(t), \qquad (n \geqq 0), \tag{3.7}$$

where $g(t) = e^t/(e^t - 1) - 1/t$. Now

$$\sum_0^\infty \frac{g^{(p)}(0) t^p}{p!} = \frac{1}{t} \cdot \frac{t}{e^t - 1} + 1 - \frac{1}{t} = (1 + B_1) + \sum_1^\infty \frac{t^p B_{p+1}}{(p+1)!},$$

so that

$$g(0) = 1 + B_1, \qquad g^{(p)}(0) = B_{p+1}/(p+1), \qquad (p \geqq 1).$$

On setting $t=0$ in (3.7) we obtain (3.5).

When Cauchy's theorem on double power series is applied to the function $e^{t(x+y)}\delta^{\lambda+\mu}(t)$ we deduce the important identity

$$B_n^{-\lambda-\mu}(x+y) = \sum_0^n \binom{n}{p} B_{n-p}^{-\lambda}(x) B_p^{-\mu}(y), \qquad (\lambda, \mu \text{ real}). \tag{3.8}$$

We note also the trivial results

$$\Delta B_{n+1}^{-\lambda}(x) = (n+1) B_n^{-\lambda-1}(x), \qquad (\Delta f = f(x+1) - f(x)), \tag{3.9}$$

$$DB_n^{-\lambda}(x) = nB_{n-1}^{-\lambda}(x), \tag{3.10}$$

which are easily deduced from (3.2).

4. The functions $\alpha_n(t)$, $\beta_n(t)$, $\alpha_\lambda(t)$. We establish the following properties of these functions:

(4.1) *For $n = 1, 2, \cdots$, $\alpha_n(t)$ are continuous nondecreasing functions.*

(4.2) *For* $n = 1, 2, \cdots$, $\alpha_n(t) = \begin{cases} 0, & t \leqq 0, \\ 1, & t \geqq n. \end{cases}$

For $n-k<t<n-k+1$, $k=1, 2, \cdots n$, $n \geqq 1$,

$$\alpha_n(t) = 1 - \sum_{r=0}^{k-1} \frac{(-)^r (n-r-t)^n}{r!(n-r)!}, \tag{4.4}$$

$$\alpha_n(t) = \sum_{r=k}^{n} \frac{(-)^r (n-r-t)^n}{r!(n-r)!}. \tag{4.5}$$

(4.6) *For any fixed t, the sequence $\{\alpha_n(t)\}_0^\infty$ is nonincreasing.*

(4.7) *When m is an integer not less than unity*

$$\sum_0^{m-1} (-)^r \binom{\mu+1}{r} = (-)^{m-1} \binom{\mu}{m-1};$$

$$\int t^p d\alpha_n(t) = B_p^{-n}; \tag{4.8}$$

$$\int \alpha_n(t) dt = n/2; \tag{4.9}$$

$$\int_0^n t^p \alpha_n(t) dt = (n^{p+1} - B_{p+1}^{-n})/(p+1). \tag{4.10}$$

(4.11) *When* $n - k < t < n - k + 1$,

$$k = 1, 2, \cdots, n, \; n \geqq 1, \; \tau = n - k + 1 - t$$

$$|\beta_n(t)| \leqq \frac{(k-1+\tau)^n}{n!} \left[\underset{0 \leqq m \leqq n-1}{\text{Max}} \binom{n-1}{m} \right]^2.$$

(4.12) *The series* (2.5) *defining* $\alpha_\lambda(t)$ *converges absolutely for any* t *and for* $\lambda \geqq 0$.

$$(4.13)\qquad \alpha_\lambda(-\infty) = 0, \qquad \alpha_\lambda(\infty) = 1.$$

$$(4.14)\qquad \int t^p d\alpha_\lambda(t) = B_p^{-\lambda}$$

This relation is a generalization of (4.8).

(4.15) *The function* $\alpha_\lambda(t)$ *is continuous for* $\lambda > 0$.

$$(4.18)\qquad \int \alpha_\lambda(t-v)d\alpha_\mu(v) = \alpha_{\lambda+\mu}(t), \qquad (\lambda, \mu \geqq 0).$$

When $\lambda = 0 = \mu$, the integral is undefined at $t=0$: but since in this case the right and left hand limits of $\alpha_0(t)$ are finite, the integral can be defined by normalization.

$$(4.16)\qquad \int \beta_m(t-v)d\alpha_n(v) = \sum_{p=0}^{n}\binom{n}{p}\beta_{m+n-p}(t), \qquad (m, n \geqq 0).$$

$$(4.17)\qquad \int \beta_m(t-v)d\beta_n(v) = \beta_{m+n}(t), \qquad (m, n \geqq 0).$$

Proof of (4.1). That the functions $\alpha_n(t)$, $n=1, 2, \cdots$, are continuous follows from their definition (2.3) as Stieltjes convolutions of the continuous function $\alpha_1(t)$. Let $t_1 > t_2$, and suppose that $\alpha_n(t_1) \geqq \alpha_n(t_2)$. Then

$$\alpha_{n+1}(t_1) - \alpha_{n+1}(t_2) = \left\{\int_{t_1-1}^{t_1} - \int_{t_2-1}^{t_2}\right\}\alpha_n(v)dv = \int_{t_2}^{t_1}[\alpha_n(v) - \alpha_n(v-1)]dv,$$

which is non-negative by the inductive hypothesis. The conclusion follows from the observation that $\alpha_1(t)$ is nondecreasing.

Proof of (4.2) (4.3). That these propositions are true is an easy deduction from the definition (2.3).

Proof of (4.4) (4.5). It is easily verified that

$$\alpha_2(t) = \begin{cases} 0, & t \leqq 0, \\ t^2/2, & 0 \leqq t \leqq 1, \\ 2t - 1 - t^2/2, & 1 \leqq t \leqq 2, \\ 1, & t \geqq 2 \end{cases}$$

so that both propositions are satisfied for $n=2$. Details of the inductive proof are given for (4.4). Assume that (4.4) is satisfied for n and $1 \leqq k \leqq n$, and consider $\alpha_{n+1}(t)$ in the interval $n+1-k < t < n+2-k$. Then

$$\begin{aligned}
\alpha_{n+1}(t) &= \left\{\int_{t-1}^{n+1-k} + \int_{n+1-k}^{t}\right\}\alpha_n(v)dv \\
&= \int_{t-1}^{n+1-k}\left\{1 - \sum_{0}^{k-1}\frac{(-)^r(n-r-v)^n}{r!(n-r)!}\right\}dv \\
&\quad + \int_{n+1-k}^{t}\left\{1 - \sum_{0}^{k-2}\frac{(-)^r(n-r-v)^n}{r!(n-r)!}\right\}dv, \\
&= \int_{t-1}^{t}\left\{1 - \sum_{0}^{k-2}\frac{(-)^r(n-r-v)^n}{r!(n-r)!}\right\}dv \\
&\quad + (-)^k\int_{t-1}^{n+1-k}\frac{(n+1-k-v)^n dv}{(k-1)!(n+1-k)!}, \\
&= 1 + \sum_{0}^{k-2}\frac{(-)^r\{(n-r-t)^{n-1} - (n+1-r-t)^{n+1}\}}{r!(n-r)!(n+1)} \\
&\quad + \frac{(-)^k(n+2-k-t)^{n+1}}{(k-1)!(n+1-k)!(n+1)}, \\
&= 1 - \sum_{0}^{k-1}\frac{(-)^p(n+1-p-t)^{n+1}}{p!(n+1-p)!}
\end{aligned}$$

by an easy reduction.

Proof of (4.6). It is easily verified from (2.3) that

$$\alpha_1(t) - \alpha_2(t) = \begin{cases} 0, & \leqq 0, \\ t(1-t/2), & 0 \leqq t \leqq 1, \\ (t-2)^2/2, & 1 \leqq t \leqq 2, \\ 0, & t \geqq 2, \end{cases}$$

so that $\alpha_1(t) - \alpha_2(t) \geqq 0$. Then

$$\alpha_n(t) - \alpha_{n+1}(t) = \int_{t-1}^{t}[\alpha_{n-1}(v) - \alpha_n(v)]dv \geqq 0$$

by the inductive hypothesis.

Proof of (4.7). This identity is easily deduced by comparing coefficients of x^{m-1} in the expansion of $(1-x)^\mu = (1-x)^{\mu+1}\sum_{m=0}^{\infty} x^m$.

Proof of (4.8) (4.9) (4.10). Proposition (4.8) is a consequence of (2.8) and (5.2), while (4.10) follows when (4.8) is integrated by parts, and (4.9) is used. Proposition (4.9) is certainly true when $n=1$; and on making the inductive hypothesis $\int_0^{n-1}\alpha_{n-1}(t)d = (n-1)/2$, we have

$$\int_0^n \alpha_n(t)dt = \int_0^n [A(t) - A(t-1)]dt.$$

where $A(t)=\int_0^t\alpha_{n-1}(v)dv$. Thus

$$\int_0^n \alpha_n(t)dt = n[A(n) - A(n-1)] - \int_0^n t\{\alpha_{n-1}(t) - \alpha_{n-1}(t-1)\}dt,$$
$$= n - \int_0^n t\alpha_{n-1}(t)dt + \int_{-1}^{n-1}(1+t)\alpha_{n-1}(t)dt$$
$$= n - \int_0^{n-1} t\alpha_{n-1}(t)dt - \int_{n-1}^{n} tdt + \int_0^{n-1}(1+t)\alpha_{n-1}(t)dt$$
$$= n - (2n-1)/2 + (n-1)/2 = n/2$$

by the inductive hypothesis.

Proof of (4.11). When $t \geqq n-k$, $1=\alpha_0=\alpha_1=\cdots=\alpha_{n-k}$, and

$$\beta_n(t) = \sum_{p=0}^{n-k}(-)^{n-p}\binom{n}{p} + \sum_{n-k+1}^{n}(-)^{n-p}\binom{n}{p}\alpha_p(t),$$
$$= -\sum_0^{k-1}(-)^p\binom{n}{p}\{1-\alpha_{n-p}(t)\}.$$

The sequence $\{1-\alpha_{n-p}(t)\}_{p=0}^{k-1}$ is nonincreasing by (4.6), and by Abel's inequality

$$|\beta_n(t)| \leqq [1-\alpha_n(t)] \operatorname*{Max}_{0\leqq m\leqq k-1}\left|\sum_0^m(-)^p\binom{n}{p}\right|$$
$$= [1-\alpha_n(t)]\left[\operatorname*{Max}_{0\leqq m\leqq k-1}\binom{n-1}{m}\right]$$

by (4.7). Now by (4.4)

$$1-\alpha_n(t) = \frac{1}{n!}\sum_0^{k-1}(-)^r\binom{n}{r}(k-1-r+\tau)^n, \qquad (n-k<t<n-k+1)$$

where $\tau=n+1-k-t$ and $0<\tau<1$. The sequence $\{k-1-r+\tau\}_{r=0}^{k-1}$ is decreasing, and again by Abel's inequality and (4.7),

$$1-\alpha_n(t) \leqq \frac{(k-1+\tau)^n}{n!}\left[\operatorname*{Max}_{0\leqq m\leqq k-1}\binom{n-1}{m}\right].$$

Thus finally

$$|\beta_n(t)| \leqq \frac{(k-1+\tau)^n}{n!}\left\{\operatorname*{Max}_{0\leqq m\leqq k-1}\binom{n-1}{m!}\right\}^2$$
$$\leqq \frac{(k-1+\tau)^n}{n!}\left\{\operatorname*{Max}_{0\leqq m\leqq n-1}\binom{n-1}{m}\right\}^2.$$

Proof of (4.12). This follows easily from (4.11) and d'Alembert's test.

Proof of (4.13). Let t lie in the interval $(N, N+1)$; then

$$\alpha_\lambda(t) = \alpha_0(t) + \sum_{N+1}^{\infty} \binom{\lambda}{n} \beta_n(t).$$

By (4.12) the term $\sum_{N+1}^{\infty} = 0(1)(N\to\infty)$, and so $\alpha_\lambda(\infty) = \alpha_0(\infty) = 1$.

Proof of (4.14).

$$\int t^n d\alpha_\lambda(t) = \sum_{p=0}^{\infty} \binom{\lambda}{p} \sum_{q=0}^{p} \binom{p}{q} (-)^{p-q} \int t^n d\alpha_q(t)$$

$$= \sum_{p=0}^{\infty} \binom{\lambda}{p} \sum_{q=0}^{p} (-)^{p-q} B_n^{-q}$$

by (4.8). The inner sum is the coefficient of $t^n/n!$ in the expansion of

$$\sum_{0}^{p} \binom{p}{q} (-)^{p-q} \delta^q(t) = [\delta(t) - 1]^p;$$

and the whole sum the coefficient of $t^n/n!$ in

$$\sum_{0}^{\infty} \binom{\lambda}{p} \{\delta(t) - 1\}^p = \delta^\lambda(t).$$

Thus on taking note of (2.8) the conclusion follows.

Proof of (4.15). Since $0<\tau<1$, the estimate (4.11) may be written as

$$| \beta_n(t) | \leqq \frac{k^n}{n!} \left[\operatorname{Max}_{0 \leqq m \leqq n-1} \binom{n-1}{m} \right]^2, \qquad (n - k < t < n - k + 1);$$

and this is sufficient to guarantee the uniform convergence in any interval $(N, N+1)$ of the series (2.5) which defines $\alpha_\lambda(t)$. The additional fact that $\beta_n(t)$, which by (2.4) is a finite sum of functions continuous except at $t=0$, is then sufficient to establish the continuity of $\alpha_\lambda(t)$ for $t<0$ and for $t \geqq \epsilon > 0$.

To prove the continuity at $t=0$, we observe from (4.5) that in the interval $0<t\leqq 1$,

$$\alpha_n(t) = t^n/n!,$$

$$\beta_n(t) = \sum_{0}^{n} \binom{n}{p} (-)^{n-p} t^p/p! = (-)^n L_n(t)$$

by [4, p. 188], where $L_n(t)$ are Laguerre polynomials. Then by [4, p. 214],

$$\alpha_\lambda(t) = \sum_{0}^{\infty} \binom{\lambda}{n} (-)^n L_n(t) = t^\lambda/\Gamma(1+\lambda).$$

Since $\lambda > 0$, it follows that $\alpha_\lambda(0+) = 0 = \alpha_\lambda(0-)$.

Proof of (4.16) (4.17). The truth of these propositions follows in an obvious way from the definitions of $\alpha_n(t)$ and $\beta_n(t)$, and the easily established fact that

$$\alpha_n(t) = \sum_{p=0}^{n} \binom{n}{p} \beta_p(t).$$

Proof of (4.18). When λ, μ are positive, the series defining $\alpha_\lambda(t)$, $\alpha_\mu(t)$ converge absolutely by (4.12). Then

$$\begin{aligned}\int \alpha_\lambda(t-v)d\alpha_\mu(v) &= \sum_{m=0}^{\infty}\binom{\lambda}{m}\sum_{n=0}^{\infty}\binom{\mu}{n}\beta_{m+n}(t)\\ &= \sum_{p=0}^{\infty}\beta_p(t)\sum_{r=0}^{p}\binom{\lambda}{r}\binom{\mu}{p-r}\\ &= \sum_{p=0}^{\infty}\binom{\lambda+\mu}{p}\beta_p(t) = \alpha_{\lambda+\mu}(t).\end{aligned}$$

5. **The operator** $\delta^\lambda(D)$. Enough preliminary work has been done to make possible a definition of this operator, and to establish some of its properties. Since $|\delta(t)-1|$ is certainly less than unity when $|t|<6/5$, the expansion

$$\delta^\lambda(t) = [1+\{\delta(t)-1\}]^\lambda = \sum_{0}^{\infty}\binom{\lambda}{n}[\delta(t)-1]^n$$

has a positive radius of convergence. It is elementary that

(5.1) $$1 = \int e^{tu}d\alpha_0(u);$$

and by the product theorem for the Laplace-Stieltjes transform

(5.2) $$\delta^p(t) = \int e^{tu}d\alpha_p(u), \qquad (p = 1, 2, \cdots).$$

Hence

(5.3) $$\begin{aligned}[\delta(t)-1]^n &= \sum_{0}^{n}\binom{n}{p}(-)^{n-p}\delta^p(t) = \int e^{tu}d\beta_n(u),\\ \delta^\lambda(t) &= \int e^{tu}d\alpha_\lambda(u).\end{aligned}$$

We then *define*, using (5.3) and the shift operator exp (uD),

(5.4) $$\delta^\lambda(hD)f(x) = \int f(x+hu)d\alpha_\lambda(u),$$

and exhibit its effect in the following particular cases:

(5.5) $$\overset{\lambda}{\delta}(hD)x^{N} = h^{N}\overset{-\lambda}{B_N}(x/h);$$

(5.6) $$\overset{\lambda}{\delta}(hD)e^{\kappa x} = e^{\kappa x}\overset{\lambda}{\delta}(h\kappa), \qquad (0 < \kappa < 2\pi/h);$$

(5.7) $$\overset{\lambda}{\delta}(hD)\overset{-\mu}{B_n}(x/h) = \overset{-\lambda-\mu}{B_n}(x/h)$$

(5.8) *if* $f(x) = \sum_0^\infty c_n x^n/n!$ *belongs to the class* $E(h, \kappa)$, *then*

$\delta^\lambda(hD)f(x) = \sum_0^\infty c_n h^n B_n^{-\lambda}(x/h)/n!$, *and the series is convergent for all* x.

Proof of (5.5). By (5.4), (4.14) and (3.3)

$$\begin{aligned}\delta^\lambda(hD)x^N &= \sum_0^N \binom{N}{p} x^{N-p}h^p \int t^p d\alpha_\lambda(t) \\ &= h^N \sum_0^N \binom{N}{p} (x/h)^{N-p} \overset{-\lambda}{B_p} = h^N \overset{-\lambda}{B_N}(x/h).\end{aligned}$$

Proof of (5.6). From the definition (5.4)

$$\delta^\lambda(hD)e^{\kappa x} = \int e^{\kappa(x+hu)}d\alpha_\lambda(u) = e^{\kappa x}\delta^\lambda(h\kappa),$$

when the integral which represents $\delta^\lambda(h\kappa)$ converges. This integral converges when $0<\kappa<2\pi/h$.

Proof of (5.7). Using (5.4), (3.3) and (3.8) we have

$$\overset{\lambda}{\delta}(hD)\overset{-\mu}{B_n}(x/h) = \sum_{p=0}^n \binom{n}{p} \overset{-\mu}{B_p}\overset{-\lambda}{B_{n-p}}(x/h) = \overset{-\lambda-\mu}{B_n}(x/h).$$

Proof of (5.8). The series $\sum_0^\infty t^n B_n^{-\lambda}(x)/n!$ converges absolutely for all x and all $|t| < 2\pi$ since it is the product of the series obtained from the functions e^{xt} and $\delta^\lambda(t)$. It is well known that when $f(x) = \sum_0^\infty c_n x^n/n!$ is an entire function of exponential order κ, $\limsup_{n\to\infty} |C_n| = \kappa$. Accordingly the series $\sum_0^\infty c_n h^n B_n^{-\lambda}(x)/n!$ is dominated by

$$\sum_0^\infty [h(\kappa+\epsilon)]^n \,|\, \overset{-\lambda}{B_n}(x) \,|\, /n!$$

which is convergent when $h(\kappa+\epsilon)<2\pi$. It follows from (5.5) that

$$\overset{\lambda}{\delta}(hD)f(x) = \sum_0^\infty c_n h^n \overset{-\lambda}{B_n}(x)/n!.$$

Our next result establishes the exponential property of the operator $\delta^\lambda(hD)$. We prove

THEOREM I. *If* λ *and* μ *are positive, and* $f(x)\in E(h,\kappa)$ *then*

(5.9) $F(x) = \delta^{\mu}(hD)f(x)$ *is also of exponential order* κ;

(5.10) $$\delta^{\lambda}(hD)F(x) = \delta^{\lambda+\mu}(hD)f(x).$$

Proof of (5.9). Let ϵ be chosen so that $\kappa+\epsilon<2\pi/h$. Then by (5.8) and (3.3)

$$\begin{aligned}\delta^{\mu}(hD)f(x) &= \sum_{n=0}^{\infty}\frac{c_nh^n}{n!}\sum_{p=0}^{n}\binom{n}{p}B_{n-p}^{-\mu}\left(\frac{x}{h}\right)^p \\ &= \sum_{p=0}^{\infty}\frac{x^p}{p!}\sum_{r=0}^{\infty}c_{p+r}h^rB_r^{-\lambda}/r! \\ &\ll \sum_{p=0}^{\infty}\frac{[x(\kappa+\epsilon)]^p}{p!}\sum_{r=0}^{\infty}[h(\kappa+\epsilon)]^r\,|\,B_r^{-\lambda}\,|\,/r! \\ &= A\sum_{p=0}^{\infty}[x(\kappa+\epsilon)]^p/p!\end{aligned}$$

since $h(\kappa+\epsilon)<2\pi$. Thus $F(x)$ is of exponential order κ.

Proof of (5.10). By (5.8) and (5.7)

$$\delta^{\lambda}(hD)F(x) = \sum_{0}^{\infty}c_nh^nB_n^{-\lambda-\mu}(x/h)/n! = \delta^{\lambda+\mu}(hD)f(x).$$

6. **Real powers of** Δ. Let N be a non-negative integer, let $0\leqq\lambda<1$ and let $f(x)\in E(h,\kappa)$. The formal identities

$$\Delta = [\exp(hD)-1]/h, \qquad \Delta^{N+\lambda} = D^{N+\lambda}\delta^{N+\lambda}(hD)$$

lead to the definition

(6.1) $$\Delta^{N+\lambda}f(x) = G(x)$$

where

(5.4) $$F(x) = \delta^{N+\lambda}(hD)f(x) = \int f(x+ht)d\alpha_{N+\lambda}(t),$$

(6.2) $$\begin{aligned}G(x) &= D^{N+\lambda}F(x) = D^{N+1}\int_0^x(x-t)^{-\lambda}F(t)dt/\Gamma(1-\lambda) \\ &= \sum_{0}^{N}\frac{F^{(r)}(0)x^{r-\lambda-N}}{\Gamma(r+1-\lambda-N)}+\int_0^x\frac{(x-t)^{-\lambda}F^{(N+1)}(t)dt}{\Gamma(1-\lambda)}.\end{aligned}$$

The Riemann-Liouville definition adopted for $D^{N+\lambda}F(x)$ has a meaning since $0\leqq\lambda<1$. It is easily verified that $\lim_{h\to 0}\Delta^{N+\lambda}f(x)=D^{N+\lambda}f(x)$.

We exhibit the particular case where $f(x)=B_n^{\mu}(x/h)$, and verify that we then have a generalization of Nörlund's result [1, p. 131, (46)]:

$$\Delta^{N} B_n^m(x) = n! \quad B_{n-N}^{m-N}(x)/(n-N)! \qquad (\Delta \text{ with span unity}). \tag{6.3}$$

By (3.3), (3.8) and (5.4), with $n > N+\lambda$,

$$\delta^{N+\lambda}(hD)B_n^{\mu}(x/h) = B_n^{\mu-N-\lambda}(x/h);$$

$$\begin{aligned}\Delta^{N+\lambda} B_n^{\mu}(x/h) &= D^{N+1}\int_0^x (x-t)^{-\lambda} B_n^{\mu-N-\lambda}(t/h)dt/\Gamma(1-\lambda)\\ &= D^{N+1}\sum_0^n \binom{n}{p} B_p^{\mu-N-\lambda} x^{n-p+1-\lambda} h^{p-n}(n-p)!/\Gamma(1-\lambda)\\ &= h^{\lambda+N}\sum_0^n \frac{n!B_p^{\mu-N-\lambda}(x/h)^{n-p-\lambda-N}}{p!\Gamma(n-p-\lambda-N+1)}.\end{aligned}$$

On setting $h=1$, $\lambda=0$, $\mu=m$ and taking account of the terms in the summation which vanish, this reduces to (6.3).

It is clear that while the operator $\delta^{N+\lambda}(hD)$ does not affect the exponential order of $f(x)$, the operator $\Delta^{N+\lambda}$ does, on account of the fractional integration involved in $D^{N+\lambda}$. In fact it will be seen from the proof of (5.8) that when $f(x)$ is an entire function of exponential order κ, so also is the function $\delta^{N+\lambda}(hD)f(x)$; but when the further operation $D^{N+\lambda}$, as defined in (6.2) is applied, the resulting function has a branch point at the origin. This conclusion was to be expected in view of Boas' theorem, [5, p. 207, 11.2.3].

7. **The operator** $[D/2 \sinh (D/2)]^n$. We first consider some functions required to obtain a representation of this operator. Let

$$\phi_1(y) = e^{\pi y}/(e^{\pi y} + e^{-\pi y}), \tag{7.1}$$

$$\phi_{n+1}(y) = \int \phi_n(y-v)d\phi_1(v), \qquad (n = 1, 2, \cdots). \tag{7.2}$$

These functions have the following properties:

(7.3) $\phi_n(y) + \phi_n(-y) = 1,$ $(n = 1, 2, \cdots);$

(7.4) $\phi_n(y)$, *are continuous distribution functions*; $(n = 1, 2, \cdots).$

(7.5) $\phi_n'(y)$ *exist for all* y *and* $n = 1, 2, \cdots;$

(7.6) $\phi_n'(y) = O[\exp(-2\pi|y|)],$ $(|y| \to \infty).$

We prove (7.3) by induction. Clearly $\phi_1(y)+\phi_1(-y)=1$, $\phi_1'(y)$ exists and $\phi_1'(y)=\phi_1'(-y)$. Then

$$\begin{aligned}\phi_{n+1}(y) + \phi_{n+1}(-y) &= \int \phi_n(y-v)\phi_1'(v)dv + \int \phi_n(-y+v)\phi_1'(-v)dv\\ &= \int [\phi_n(y-v) + \phi_n(-y+v)]d\phi_1(v) = 1\end{aligned}$$

by the inductive hypothesis.

To prove (7.4) we observe that $\phi_1(y)$ is a continuous distribution function, and an easy induction shows that $\phi_n(y) \in \uparrow(-\infty, \infty)$. Another inductive argument shows that $0<\phi_n(y)<1$, so that $\phi_n(\infty)$ exist for $n=1, 2, \cdots$. It then follows from (7.2) that $\phi_n(\infty)=1$, $\phi_n(-\infty)=0$. That (7.5) is true follows by induction. To prove (7.6), we observe that $\phi_1'(y)=O[e^{-2\pi|y|}]$, $(|y|\to\infty)$ and we assume that $\phi_n'(y)=O[\exp(-2\pi|y|)]$, $(|y|\to\infty)$. Then

$$\phi_{n+1}'(y) = \int_0^\infty \phi_n'(u)\{\phi_1'(y-u) - \phi_1'(y+u)\}du.$$

By an easy calculation

$$\phi_1'(y-u) - \phi_1'(y+u)\begin{cases} >0, & y>0, \\ <0, & y<0. \end{cases} \tag{7.7}$$

When $y>0$ we use the inductive hypothesis and $\phi_n'(u)>0$, to obtain

$$\begin{aligned} \phi_{n+1}'(y) &\leqq A\int_0^\infty e^{-2\pi u}\phi_1'(y-u)du \\ &= A\left[\phi_1(y) - 2\pi\int_0^\infty \phi_1(y-u)e^{-2\pi u}du\right] \\ &= O(e^{-2\pi y}) \qquad\qquad (y\to\infty) \end{aligned}$$

as is seen by evaluting the integral. The order of $\phi_n'(y)$ for $y\to-\infty$, follows by taking the other term in (7.7).

It is well known that

$$\epsilon(t) = t/2\sinh(t/2) = \int e^{ity}d\phi_1(y); \tag{7.8}$$

and by the product theorem for Fourier-Stieltjes transforms

$$\epsilon^n(t) = \int e^{ity}d\phi_n(y). \tag{7.9}$$

Since $\delta^{-n}(t)=\epsilon^n(t)\exp(-nt/2)$, we then have the representation

$$\delta^{-n}(t) = \int \exp[t(iy - n/2)]d\phi_n(y). \tag{7.10}$$

We accordingly adopt the *definition*

$$\delta^{-n}(hD)f(x) = \int f[x + h(iy - n/2)]d\phi_n(y); \tag{7.11}$$

and prove the following propositions:

(7.12) *when* $f(x) \in E(h, \kappa)$, *the integral in* (7.10) *exists*;

$$\delta^{-n}(hD)x^{m} = h^{m}B_m^{n}(x/h); \tag{7.13}$$

$$\delta^{-n}(hD)B_m^{p}(x/h) = B_m^{p+n}(x/h); \quad (n\ a + ve,\ p \text{ any integer}). \tag{7.14}$$

The truth of (7.12) is an easy consequence of (7.6). The requirement that $f(x)$ be of exponential order on vertical lines would be sufficient for our purpose here, but the more restrictive assumption is made, since we shall have to apply $\delta^{\lambda}(hD)$ to $f(x)$ with $\lambda > 0$, for which exponential order of $f(x)$ along horizontal lines (see (5.4), (5.6)) is needed.

That (7.13) is true follows easily from the fact that

$$\delta^{-n}(hD)x^m = \int [x + h(iy - n/2)]^m d\phi_n(y)$$

is the coefficient of $t^m/m!$ in the expansion of $e^{xt}[ht/(e^{ht}-1)]^n$; while the truth of (7.14) follows from (3.3), (7.13) and (3.8).

An easy consequence of (7.14) is

$$\delta^{-n}(hD)B_m^{-n}(x/h) = (x/h)^m. \tag{7.15}$$

We now prove a proposition which is important in the inversion theorem:

(7.16) *if* $f(x) \in E(h, \kappa)$

n is a positive integer and $F(x) = \delta^n(hD)f(x)$, *then* $\delta^{-n}(hD)F(x) = f(x)$.

Let $f(x) = \sum_0^\infty c_r x^r$; then by (5.8)

$$F(x) = \sum_0^\infty c_r h^r B_r^{-n}(x/h), \tag{7.17}$$

the series converging uniformly in any x-interval. The conclusion then follows from (7.15), the termwise application of $\delta^{-n}(hD)$ to the series (7.17) being justified by the exponential character of $f(x)$.

8. **The inverse operator $\Delta^{-N-\lambda}$.** Making use of the formal identity

$$\Delta^{-N-\lambda} = \left[\frac{hD}{\exp(hD) - 1}\right]^{N+\lambda} = \delta^{-N-1}(hD)\delta^{1-\lambda}(hD)D^{-N-\lambda}$$

and bearing in mind the remarks at the end of paragraph 6, we *adopt the definition*:

(8.1) *when N is a non-negative integer,* $0 \leqq \lambda < 1$, *and the function $F(x)$ is such that*

$$H_1(x) = \int_0^x (x - t)^{N+\lambda-1}F(t)dt/\Gamma(N + \lambda) \in E(h, k), \tag{8.2}$$

then $\Delta^{-N-\lambda}F(x) = H(x)$ *where*

$$H_2(x) = \delta^{1-\lambda}(hD)H_1(x), \tag{8.3}$$

$$H(x) = \delta^{-N-1}(hD)H_2(x), \tag{8.4}$$

the operations (8.3) (8.4) *being defined by* (5.4) *and* (7.11) *respectively.*

This definition enables us to prove the inversion theorem:

THEOREM II. *Let N be a non-negative integer,* $0 \leqq \lambda < 1$ *and* $f(x) \in E(h, \kappa)$. *Then if* $F(x) = \Delta^{N+\lambda} f(x)$ *is defined by* (6.1)

$$\Delta^{-N-\lambda}F(x) = f(x).$$

The proof of the theorem involves showing that the successive operations

$$\delta^{-N-1}(hD)\cdot\delta^{1-\lambda}(hD)\cdot D^{-N-\lambda}\cdot D^{N+\lambda}\cdot\delta^{N+\lambda}(hD)\cdot f(x)$$

carried out *in this order* leave the function $f(x)$ unchanged. It is well known that the successive operations

$$D^{-N-\lambda}\, D^{N+\lambda}$$

defined by (6.2) and (8.2) respectively leave unchanged the function to which they are applied. It is then sufficient to show that

$$\delta^{-N-1}(hD)\cdot\delta^{1-\lambda}(hD)\cdot\delta^{N+\lambda}(hD)f(x) = f(x),$$

all functions resulting from the successive operations being by Theorem I and (7.16) entire functions of exponential order κ. Theorem I assures us that

$$\delta^{1-\lambda}(hD)\cdot\delta^{N+\lambda}(hD)f(x) = \delta^{N+1}(hD)f(x),$$

since $1-\lambda$ and $N+\lambda$ are non-negative. The final conclusion then follows from (7.16).

It is an easy conclusion from (8.2) (8.3) (8.4) that

$$\lim_{h\to 0} \Delta^{-N-\lambda}F(x) = \int_0^x (x-t)^{N+\lambda-1}F(t)dt/\Gamma(N+\lambda) \tag{8.5}$$

so that the solution given by the operator $\Delta^{-N-\lambda}$ has the same property as Nörlund's Hauptlösung [4, p. 59].

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McMASTER UNIVERSITY,
HAMILTON, ONT.