TRANSFORMATION GROUPS ON A HOMOLOGICAL MANIFOLD

BY

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1. **Introduction.** In a recent paper of Montgomery-Samelson-Zippin [1], the following theorem is proved. If a compact Lie group acts as a topological transformation group on an n-dimensional manifold such that the highest dimension of any orbit is r, then the union of all the orbits of dimension $\leqq k$, $0 \leqq k < r$, is a closed set of dimension $\leqq n-r+k-1$. Hence the singular set, i.e., the union of all the orbits not of the highest dimension, is a closed set of dimension $\leqq n-2$. The purpose of the present paper is to generalize this theorem to a homological manifold. This generalization together with some other results of this paper will be used in a further study [2] of transformation groups on a homological manifold.

Let X be an n-dimensional manifold and let G be a compact Lie group acting topologically on X. For each $x \in X$, G_x denotes the isotropic subgroup of G at x. It is known [3] that every point p of X has an open neighborhood U such that there is a subset Y of U containing p and an open cell Q in G containing the identity such that (i) whenever $g \in G$ and $x \in Y$, $g(x) \in Y$ if and only if $g \in G_p$ and (ii) $(g, x) \rightarrow g(x)$ defines a homeomorphism of $Q \times Y$ onto U. Y is called a slice at p and may be assumed to be connected. As one can easily see, G_p may be regarded as a transformation group on Y and its orbit space coincides with that of $G(Y)$ acted on by G. Moreover, an orbit in $G(Y)$ is of the same type as $G(p)$ if and only if it intersects Y at a single point. Therefore certain properties of the set of the orbits in $G(Y)$ which are of the same type as $G(p)$, can be obtained by studying the fixed point set of G_p in Y.

If G acts differentiably on X, we may have open cells as slices [4] and then the proof of the cited theorem can be greatly simplified by using the remark of the preceding paragraph. For the nondifferentiable case it is not known whether Y is a manifold, as it is still an open question whether a topological space is a k-dimensional manifold if the product of the space and the real line is known to be an $(n+1)$-dimensional manifold. However, Y can be shown to be a homological manifold. Therefore it seems more natural to begin with a homological manifold X and then to proceed to the proof just as in the differential case. Moreover, it is pointed out in [1] that the original proof depends only on the local homology property of a manifold, and this gives us another justification to deal with homological manifolds rather than ordinary manifolds.

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By a homological n-manifold we mean a connected, finite-dimensional, locally compact Hausdorff space which is of type P_n in the sense of Smith [5], where the dimension is defined to be the highest Lebesgue covering dimension of any compact subset of the space. But we can not use the group $\mathfrak{C}_q$ of integers modulo a fixed prime number q as the coefficient group as Smith did in [5], because we need the property P_n with respect to various $\mathfrak{C}_q$ at the same time. It will be seen later that we can meet this requirement by using the group $\mathfrak{P}$ of reals modulo 1 as coefficient group. Another reason for using $\mathfrak{P}$ as coefficient group is that it follows from [6] that the dimension of such a homological n-manifold is exactly equal to n. Note that the homological manifolds in our sense and the generalized manifolds defined in Wilder [7] are alike to a great extent.

To conclude the introduction the author wishes to express his gratitude to Professor Montgomery for his valuable suggestions and encouragement when this paper was prepared.

2. **A homology theory on $\mathfrak{A}_{LC}$.** We shall use a compact abelian group $\mathfrak{C}$ as coefficient group which of course can be the additive group $\mathfrak{P}$ of reals modulo 1 or the additive group $\mathfrak{C}_q$ of integers modulo a prime number q. Let $\mathfrak{A}_{LC}$ be the admissible category defined to consist of all the pairs (X, A) such that X is a locally compact Hausdorff space and A is a closed subset of X, and to consist of all the maps f of such pairs such that the inverse image of any compact set under f is compact. Let $\mathfrak{A}_C$ be the subcategory of $\mathfrak{A}_{LC}$ which consists of all the compact pairs and all the maps of such pairs. By the one-point-compactification process, we can use the Čech homology theory on $\mathfrak{A}_C$ to define a homology theory on $\mathfrak{A}_{LC}$ [8]. This homology theory on $\mathfrak{A}_{LC}$ satisfies the Eilenberg-Steenrod axioms and the continuity axiom [8] and will be used throughout this paper.

Whenever $(X, A)\in\mathfrak{A}_{LC}$, $H_k(X, A; \mathfrak{C})$ denotes the kth homology group of (X, A), i.e., the kth Čech homology group of $(X\cup\omega, A\cup\omega)$ with coefficients in $\mathfrak{C}$, where $X\cup\omega$ is the one-point-compactification of X. ∂ denotes the boundary operator of a homology sequence and Δ denotes the boundary operator of a Mayer-Vietoris sequence.

If f: $(X, A)\rightarrow(Y, B)$ is a map in $\mathfrak{A}_{LC}$, the notation

$$H_k(X, A; \mathfrak{C}) \xrightarrow{f_*} H_k(Y, B; \mathfrak{C})$$

means the homomorphism of $H_k(X, A; \mathfrak{C})$ into $H_k(Y, B; \mathfrak{C})$ induced by f. If in particular f is the inclusion map, we write only

$$H_k(X, A; \mathfrak{C}) \rightarrow H_k(Y, B; \mathfrak{C})$$

and call it the *natural* homomorphism of $H_k(X, A; \mathfrak{C})$ into $H_k(Y, B; \mathfrak{C})$. Whenever S is a subset or an element of $H_k(X, A; \mathfrak{C})$, $S|(Y, B)$ will denote the natural homomorphic image of S in $H_k(Y, B; \mathfrak{C})$. Any arrow to which a symbol $\approx$ is attached will indicate an isomorphism onto.

LEMMA 1. *Let* $(X, A) \in \mathfrak{A}_{LC}$ *and let* $e \in H_k(X, A; \mathfrak{C})$. *Let* Y *be a closed subset of* X. *Then* $e|(X, Y \cup A) = 0$ *if and only if* $e \in H_k(Y \cup A, A; \mathfrak{C})|(X, A)$. *Moreover, there exists a minimal* Y *satisfying these conditions, and such a set* Y *is contained in* $(X-A)^-$.

LEMMA 2. *Let* $(X, A) \in \mathfrak{A}_{LC}$ *and let* Y *be a closed t-cell with boundary* B. *Then there is a natural isomorphism* i *of* $H_k(X, A; \mathfrak{C})$ *onto* $H_{k+t}(X \times Y, (X \times B) \cup (A \times Y); \mathfrak{C})$. *If* $f: (X, A) \to (X', A')$ *is a map in* $\mathfrak{A}_{LC}$ *and* $j: (Y, B) \to (Y, B)$ *is the identity map, then the commutativity relation* $(f \times j)_* i = i f_*$ *holds in the diagram*

$$\begin{array}{ccc} H_k(X, A; \mathfrak{C}) & \xrightarrow{i} & H_{k+t}(X \times Y, (X \times B) \cup (A \times Y); \mathfrak{C}) \\ \uparrow f^* & & \uparrow (f \times j)_* \\ H_k(X', A'; \mathfrak{C}) & \xrightarrow{i} & H_{k+t}(X' \times Y, (X' \times B) \cup (A' \times Y); \mathfrak{C}). \end{array}$$

3. **The property** P_n. A subset of a locally compact Hausdorff space is called *bounded* if its closure is compact. If X is a locally compact Hausdorff space and x is a point of X, a *neighborhood* of x (in X) will always mean a *bounded open* subset of X containing x.

DEFINITION. Let X be a locally compact Hausdorff space and let n be a non-negative integer. A non-null bounded open subset U of X and a subgroup S of $H_n(X, X-U; \mathfrak{C})$ are said to form a *fundamental* $(\mathfrak{C}, n)$-*pair* (U, S) if the following conditions are satisfied.

(i) S is isomorphic to $\mathfrak{C}$.

(ii) Whenever W is a non-null open subset of U, the natural homomorphism of $H_n(X, X-U; \mathfrak{C})$ into $H_n(X, X-W; \mathfrak{C})$ maps S isomorphically onto $S|(X, X-W)$.

(iii) Whenever y is a point of U and V is a neighborhood of y, there is a neighborhood W of y contained in $U \cap V$ and such that

$$H_n(X, X - V; \mathfrak{C}) | (X, X - W) \subset S | (X, X - W)$$

and for $k \neq n$,

$$H_k(X, X - V; \mathfrak{C}) | (X, X - W) = 0.$$

REMARK 1. If (U, S) is a fundamental $(\mathfrak{C}, n)$-pair and Q is a non-null open subset of U, then $(Q, S|(X, X-Q))$ is also a fundamental $(\mathfrak{C}, n)$-pair.

DEFINITION. Let X be a locally compact Hausdorff space. At a point x of X, X possesses the *property* $P_n(\mathfrak{C})$ if there is a fundamental $(\mathfrak{C}, n)$-pair (U, S) with U containing x. X is of *type* $P(\mathfrak{C})$ if X possesses the property $P_n(\mathfrak{C})$ at every point x, n depending on x. If in particular n is a constant over X, X is of *type* $P_n(\mathfrak{C})$.

For $\mathfrak{C} = \mathfrak{C}_q$ these two definitions are due to Smith [5].

REMARK 2. If X is connected and of type $P(\mathfrak{C})$, then X is of type $P_n(\mathfrak{C})$ for some integer n.

As a consequence of these definitions and Lemma 2, we have

LEMMA 3. *Let X be a locally compact Hausdorff space and let R^t be the euclidean t-space. Let U be a non-null bounded open subset of X and let V be a bounded open t-cell in R^t. Let S be a subgroup of $H_n(X, X-U; \mathfrak{C})$ and let T be a natural isomorphic image of S in $H_{n+t}(X\times R^t, (X\times R^t)-(U\times V); \mathfrak{C})$; then (U, S) is a fundamental $(\mathfrak{C}, n)$-pair if and only if $(U\times V, T)$ is a fundamental $(\mathfrak{C}, n+t)$-pair. Hence X is of type $P_n(\mathfrak{C})$ if and only if $X\times R^t$ is of type $P_{n+t}(\mathfrak{C})$.*

DEFINITION. A locally compact Hausdorff space X is of *dimension* $\leqq k$ if the Lebesgue covering dimension of every compact subset of X is $\leqq k$.

A characterization of the dimension is [6]

LEMMA 4. *A finite-dimensional locally compact Hausdorff space X is of dimension k if and only if k is the largest integer such that for some compact subsets M, N of X, $M\supset N$ and $H_k(M, N; \mathfrak{P})\neq 0$.*

REMARK 3. Because of Lemma 1, the compact subsets M, N in Lemma 4 can be so chosen that for some nonzero element e of $H_k(M, N; \mathfrak{P})$, $e| (M, K\cup N) \neq 0$ whenever K is a proper compact subset of $(M-N)^-$.

COROLLARY. *In a finite-dimensional locally compact Hausdorff space X there is a point x such that every neighborhood of x is of the same dimension as X.*

Because of Lemma 4, we have

LEMMA 5. *Let X be a finite-dimensional locally compact Hausdorff space and let R^t be the euclidean t-space. Then*

$$\dim(X \times R^t) = (\dim X) + t.$$

LEMMA 6. *Let X be a finite-dimensional locally compact Hausdorff space and let Y be a closed subset of X. Then*

$$\dim X = \max (\dim Y, \dim (X - Y)).$$

Lemma 6 is an immediate consequence of some results of [9].

LEMMA 7. *Let A_α be a collection of open subsets of a non-null finite-dimensional locally compact Hausdorff space X indexed by a well-ordered set $\{\alpha\}$. If $\bigcup_\alpha A_\alpha = X$ and $\alpha' > \alpha$ implies $A_{\alpha'}\supset A_\alpha$, then there is some index β such that*

$$\dim X = \dim\left(A_\beta - \bigcup_{\alpha<\beta} A_\alpha\right).$$

Proof. Let K be a compact subset of X which is of the same dimension as X. Since $K\subset X=\bigcup_\alpha A_\alpha$, it follows that K is covered by a finite number of the sets A_α. But A_α is ordered by inclusion; we infer that there is some A_γ which contains K and then is of the same dimension as X. Since A_α is well-ordered, there is a minimal β such that A_β and X are of the same dimension. Hence the conclusion of our lemma follows from Lemma 6 if we can show that the dimension of $\bigcup_{\alpha<\beta} A_\alpha$ is less than that of X.

If there is an index γ preceding to β, then $\bigcup_{\alpha<\beta} A_\alpha = A_\gamma$ and hence our assertion follows from the minimality of β. Suppose now that there is no index preceding to β. If β is the first index, then our assertion is trivial. If β is not the first index, then as shown in the preceding paragraph, $\bigcup_{\alpha<\beta} A_\alpha$ and A_γ are of the same dimension for some $\gamma<\beta$. Hence again our assertion follows from the minimality of β.

LEMMA 8. *Let X be a finite-dimensional locally compact Hausdorff space of type $P_n(\mathfrak{G})$. Then for any bounded open subset Q of X and any integer $k>n$,*

$$H_k(X, X - Q; \mathfrak{G}) = 0.$$

If in particular $\mathfrak{G} = \mathfrak{P}$, X is of dimension n.

Proof. Since X is finite-dimensional, there is a greatest integer m such that for some bounded open subset Q of X

$$H_m(X, X - Q; \mathfrak{G}) \neq 0.$$

Let e be a nonzero element of $H_m(X, X-Q; \mathfrak{G})$. By Lemma 1 there exists a minimal closed set M contained in $\overline{Q}$ and such that for some $e' \in H_m(M \cup (X-Q), X-Q; \mathfrak{G})$,

$$e = e' \mid (X, X - Q).$$

Since $e \neq 0$, $M \cap Q \neq \varnothing$ and then we may take a point x of $M \cap Q$. Suppose that $m>n$; then there exists, by the definition of the property $P_n(\mathfrak{G})$, a neighborhood A of x contained in Q and such that

$$H_m(X, X - Q; \mathfrak{G}) \mid (X, X - A) = 0.$$

$$\begin{tikzcd}
H_m(X, (M-A)\cup(X-Q); \mathfrak{G}) \arrow[rr] & & H_m(X, M\cup(X-Q); \mathfrak{G}) & & H_m(X, X-A; \mathfrak{G}) \\
& H_m(X, X-Q; \mathfrak{G}) \arrow[ul] \arrow[ur] \arrow[urrr] & & & H_m(M\cup(X-A), X-A; \mathfrak{G}) \arrow[u] \\
& & & H_m(M\cup(X-Q), (M-A)\cup(X-Q); \mathfrak{G}) \arrow[ur, "\approx"'] & \\
H_m((M-A)\cup(X-Q), X-Q; \mathfrak{G}) \arrow[uur] \arrow[rr] & & H_m(M\cup(X-Q), X-Q; \mathfrak{G}) \arrow[uul] \arrow[ur] & & H_{m+1}(X, M\cup(X-A); \mathfrak{G}) \arrow[uu, "\partial"']
\end{tikzcd}$$

Because of our choice of M,

$$e' \notin H_m((M - A) \cup (X - Q), X - Q; \mathfrak{G}) \mid (M \cup (X - Q), X - Q);$$

therefore $e'|(M\cup(X-Q), (M-A)\cup(X-Q))\neq 0$ and hence

$$e' \mid (M \cup (X - A), X - A) \neq 0.$$

Since $e'|(X, X-A)=e|(X, X-A)=0$, it follows that

$$e' \mid (M \cup (X - A), X - A) \in \partial H_{m+1}(X, M \cup (X - A); \mathfrak{C}).$$

Hence $H_{m+1}(X, M\cup(X-A); \mathfrak{C})\neq 0$, contrary to our assumption on m. This proves that $m\leqq n$.

On the other hand, there is a fundamental $(\mathfrak{C}, n)$-pair (U, S). For such a bounded open set U, $H_n(X, X-U; \mathfrak{C})\neq 0$. Hence $m\geqq n$ and consequently $m=n$.

Suppose now that $\mathfrak{C}=\mathfrak{P}$. If (U, S) is a fundamental $(\mathfrak{P}, n)$-pair, then $H_n(\overline{U}, \overline{U}-U; \mathfrak{P})\neq 0$ as it is isomorphic to $H_n(X, X-U; \mathfrak{P})$. Hence by Lemma 4, X is of dimension $\geqq n$.

For any compact subsets M, N of X with $M\supset N$, and any integer $k>n$, $H_k(M, N; \mathfrak{P})=0$. In fact, let Q be a bounded open subset of X such that $M-Q=N$. Then the composition

$$H_k(M, N; \mathfrak{P}) \to H_k(M \cup (X - Q), N \cup (X - Q); \mathfrak{P}) \to H_k(X, X - Q; \mathfrak{P})$$

is an isomorphism into and hence our assertion follows. This proves that X is of dimension $\leqq n$.

Lemma 9. *Let X be a finite-dimensional locally compact Hausdorff space of type $P_n(\mathfrak{C})$. Let M, N be compact subsets of X with $M\supset N$. If there is a nonzero element e of $H_n(M, N; \mathfrak{C})$ such that for no proper closed subset K of $(M-N)$, $e|(M, K\cup N)=0$, then $M-N$ is open in X. Hence if $\mathfrak{C}=\mathfrak{P}$, a closed subset of X is of dimension n if and only if it is somewhere dense.*

Proof. Let x be a point of $M-N$ and let (U, S) be a fundamental $(\mathfrak{C}, n)$-pair with U containing x. Then x has a neighborhood V contained in $U-N$ and such that for some $u\in S$, $e|(X, X-V)=u|(X, X-V)$. By hypothesis, $e|(M, M-V)\neq 0$ and then $e|(M\cup(X-V), X-V)\neq 0$. Since

$$H_{n+1}(X, M \cup (X - V); \mathfrak{C}) = 0$$

(Lemma 8), it follows that $e|(X, X-V)\neq 0$. Hence $u\neq 0$.

Suppose that $V\not\subset M$. Then $W=V-M$ is a non-null open subset of U. Therefore

$$u \mid (X, X - W) \neq 0.$$

On the other hand,

$$u \mid (X, X - W) = e \mid (X, X - W) = 0.$$

Hence we have arrived at a contradiction. This proves that x is an inner point of $M-N$. Since x is arbitrary, it follows that $M-N$ is open.

Suppose now that $\mathfrak{C}=\mathfrak{P}$ and let X' be a closed subset of X. If X' is n-

dimensional, there exist, by Remark 3, compact subsets M, N of X' such that $M \supset N$ and for some nonzero element e of $H_n(M, N; \mathfrak{P})$, $e|(M, K \cup N) \neq 0$ whenever K is a proper compact subset of $(M-N)^-$. It follows that $M-N$ is open. Hence X' is somewhere dense. Conversely, if X' is somewhere dense, then the interior of X' is not null and therefore is n-dimensional. Hence X' is n-dimensional.

Lemma 10. *Let X be a finite-dimensional locally compact Hausdorff space of type $P_n(\mathfrak{C})$ and let Q be a connected non-null bounded open subset of X. Then for any non-null open subset Q' of Q, the natural homomorphism of*

$$H_n(X, X - Q; \mathfrak{C})$$

into $H_n(X, X-Q'; \mathfrak{C})$ is one-one. If in particular there is a fundamental $(\mathfrak{C}, n)$-pair (Q, S), then $S = H_n(X, X-Q; \mathfrak{C})$ and for any connected non-null open subset Q' of Q, the natural homomorphism of $H_n(X, X-Q; \mathfrak{C})$ into $H_n(X, X-Q'; \mathfrak{C})$ is one-one and onto.

Proof. Let e be a nonzero element of $H_n(X, X-Q; \mathfrak{C})$. By Lemma 1, there is a minimal compact subset M of $\overline{Q}$ such that for some

$$e' \in H_n(M \cup (X - Q), X - Q; \mathfrak{C}), e = e' | (X, X - Q).$$

Clearly there is no proper compact subset K of $(M \cap Q)^-$ such that

$$e' | (M \cup (X - Q), K \cup (X - Q)) = 0.$$

Since the natural homomorphism of $H_n(M, M-Q; \mathfrak{C})$ into

$$H_n(M \cup (X - Q), X - Q; \mathfrak{C})$$

is one-one and onto, we may apply Lemma 9. Therefore $M \cap Q$ is open in X. But $M \cap Q$ is clearly non-null and closed in Q. It follows from the connectedness of Q that $M \cap Q = Q$. Hence

$$M = \overline{Q}.$$

It is an immediate consequence of the minimality of M that whenever Q' is a non-null open subset of Q,

$$e | (X, X - Q') \neq 0.$$

Since e is an arbitrary nonzero element of $H_n(X, X-Q; \mathfrak{C})$, the natural homomorphism of $H_n(X, X-Q; \mathfrak{C})$ into $H_n(X, X-Q'; \mathfrak{C})$ is one-one.

Suppose that there is a fundamental $(\mathfrak{C}, n)$-pair (Q, S). For any connected non-null open subset Q' of Q we take a point x of Q'. Then there is a neighborhood A of x contained in Q' and such that

$$H_n(X, X - Q'; \mathfrak{C}) | (X, X - A) \subset S | (X, X - A).$$

Since the natural homomorphism of $H_n(X, X-Q; \mathfrak{C})$ into $H_n(X, X-A; \mathfrak{C})$

and that of $H_n(X, X-Q'; \mathfrak{C})$ into $H_n(X, X-A; \mathfrak{C})$ are both one-one, it follows that the natural homomorphism of $H_n(X, X-Q; \mathfrak{C})$ into $H_n(X, X-Q'; \mathfrak{C})$ maps S isomorphically onto $H_n(X, X-Q'; \mathfrak{C})$. Let $Q'=Q$; then we have

$$S = H_n(X, X - Q; \mathfrak{C}).$$

4. $(\mathfrak{C}, n)$-manifolds.

DEFINITION. A $(\mathfrak{C}, n)$-*manifold* is a connected, finite-dimensional, locally compact Hausdorff space of type $P_n(\mathfrak{C})$.

It is easily seen that every $(\mathfrak{C}, n)$-manifold is locally connected [7].

REMARK 4. An n-dimensional manifold in the ordinary sense is a $(\mathfrak{C}, n)$-manifold. The dimension of a $(\mathfrak{C}, n)$-manifold is not less than n and is equal to n if $\mathfrak{C}=\mathfrak{P}$ (Lemma 8).

On a $(\mathfrak{C}, n)$-manifold the fundamental $(\mathfrak{C}, n)$-pairs (U, S) with U connected are of particular importance. As shown in Lemma 10, the group S of such a fundamental $(\mathfrak{C}, n)$-pair (U, S) is equal to $H_n(X, X-U; \mathfrak{C})$ which is determined by U and is isomorphic to $\mathfrak{C}$. Therefore we may use $(U, [\mathfrak{C}])$ to denote such a fundamental $(\mathfrak{C}, n)$-pair.

LEMMA 11. *Let X be a $(\mathfrak{P}, n)$-manifold and let $\mathfrak{C}$ be an arbitrary compact abelian group. If $(U, [\mathfrak{P}])$ is a fundamental $(\mathfrak{P}, n)$-pair with U connected, then there is a fundamental $(\mathfrak{C}, n)$-pair $(U, [\mathfrak{C}])$. Hence every $(\mathfrak{P}, n)$-manifold is a $(\mathfrak{C}, n)$-manifold.*

Proof. Let X be a $(\mathfrak{P}, n)$-manifold. Whenever Q is a bounded open subset of X, we denote by $H^k(X, X-Q)$ the kth integral Čech cohomology group of $(X, X-Q)$. It is known [10] that

$$H_k(X, X - Q; \mathfrak{C}) = \mathrm{Hom}\,(H^k(X, X - Q), \mathfrak{C}) \oplus \mathrm{Ext}\,(\mathfrak{C}, H^{k+1}(X, X - Q)).$$

Let $(U, [\mathfrak{P}])$ be a fundamental $(\mathfrak{P}, n)$-pair with U connected. Since X is of dimension n (Lemma 8), $H^{n+1}(X, X-U)=0$. Therefore $H^n(X, X-U)$ is isomorphic to the additive group of integers and hence $H_n(X, X-U; \mathfrak{C})$ is isomorphic to $\mathfrak{C}$. Now we claim that $(U, H_n(X, X-U; \mathfrak{C}))$ is a fundamental $(\mathfrak{C}, n)$-pair.

For any connected non-null open subset W of U, the natural homomorphism of $H_n(X, X-U; \mathfrak{P})$ into $H_n(X, X-W; \mathfrak{P})$ is one-one and onto (Lemma 10) and then so is the natural homomorphism of $H^n(X, X-W)$ into $H^n(X, X-U)$. Therefore the natural homomorphism of $H_n(X, X-U; \mathfrak{C})$ into $H_n(X, X-W; \mathfrak{C})$ is one-one and onto. Hence for any non-null open subset W of U the natural homomorphism of $H_n(X, X-U; \mathfrak{C})$ into $H_n(X, X-W; \mathfrak{C})$ is one-one.

Let y be a point of U and let V be a neighborhood of y. Then there is a neighborhood W of y contained in $U\cap V$ and such that $H_n(X, X-V; \mathfrak{P}) \mid (X, X-W) \subset H_n(X, X-U; \mathfrak{P}) \mid (X, X-W)$ and for $k\neq n$, $H_k(X, X-V; \mathfrak{P}) \mid (X, X-W)=0$. Since X is locally connected, we may assume W connected.

Therefore $H_n(X, X-U; \mathfrak{C})\,|\,(X, X-W)=H_n(X, X-W; \mathfrak{C})$ (Lemma 10). Hence

$$H_n(X, X - V; \mathfrak{C})\,|\,(X, X - W) \subset H_n(X, X - U; \mathfrak{C})\,|\,(X, X - W).$$

Since $H^n(X, X-W)$ is isomorphic to the additive group of integers, $\mathrm{Ext}\,(\mathfrak{C}, H^n(X, X-W))=0$. Therefore the natural homomorphism of $\mathrm{Ext}\,(\mathfrak{C}, H^n(X, X-V))$ into $\mathrm{Ext}\,(\mathfrak{C}, H^n(X, X-W))$ is trivial. Let $k\neq n$. Since $H_k(X,X-V; \mathfrak{P})\,|\,(X,X-W)=0$, the natural homomorphism of $H^k(X,X-W)$ into $H^k(X, X-V)$ is trivial. Therefore the homomorphism of

$$\mathrm{Hom}\,(H^k(X, X - V), \mathfrak{C}) \text{ into } \mathrm{Hom}\,(H^k(X, X - W), \mathfrak{C})$$

and that of $\mathrm{Ext}\,(\mathfrak{C}, H^k(X, X-V))$ into $\mathrm{Ext}\,(\mathfrak{C}, H^k(X, X-W))$ are both trivial. Hence for $k\neq n$,

$$H_k(X, X - V; \mathfrak{C})\,|\,(X, X - W) = 0.$$

5. **$\mathfrak{C}$-orientability.** Let X be a $(\mathfrak{C}, n)$-manifold and let $\mathfrak{F}$ be the totality of fundamental $(\mathfrak{C}, n)$-pairs $(U, [\mathfrak{C}])$ with U connected. An ordered pair of elements $(U, [\mathfrak{C}])$, $(V, [\mathfrak{C}])$ of $\mathfrak{F}$ is called a *step* if either $U\subset V$ or $V\subset U$. A finite sequence of elements of $\mathfrak{F}$

$$(U_1, [\mathfrak{C}]), \cdots, (U_t, [\mathfrak{C}])$$

is called a *path* from $(U_1, [\mathfrak{C}])$ to $(U_t, [\mathfrak{C}])$ if every pair of adjacent terms is a step. For each $i=1, \cdots, t-1$, we have either $U_i\subset U_{i+1}$ or $U_i\supset U_{i+1}$; it follows from Lemma 10 that there is a natural isomorphism of $H_n(X, X-U_i; \mathfrak{C})$ onto $H_n(X, X-U_{i+1}; \mathfrak{C})$. The composition of these isomorphisms is an isomorphism of $H_n(X, X-U_1; \mathfrak{C})$ onto $H_n(X, X-U_t; \mathfrak{C})$, called the isomorphism induced by the path.

Definition. A $(\mathfrak{C}, n)$-manifold is *$\mathfrak{C}$-orientable* if there is a fundamental $(\mathfrak{C}, n)$-pair $(U, [\mathfrak{C}])$ with U connected and such that every path from $(U, [\mathfrak{C}])$ to itself induces the identity isomorphism on $H_n(X, X-U; \mathfrak{C})$.

Remark 5. The following are equivalent.

(i) X is $\mathfrak{C}$-orientable.

(ii) For every fundamental $(\mathfrak{C}, n)$-pair $(U, [\mathfrak{C}])$ with U connected, every path from $(U, [\mathfrak{C}])$ to itself induces the identity isomorphism on $H_n(X, X-U; \mathfrak{C})$.

(iii) There exist two fundamental $(\mathfrak{C}, n)$-pairs $(U, [\mathfrak{C}])$ and $(V, [\mathfrak{C}])$ such that U and V are connected and all the paths from $(U, [\mathfrak{C}])$ to $(V, [\mathfrak{C}])$ induces the same isomorphism of $H_n(X, X-U; \mathfrak{C})$ onto $H_n(X, X-V; \mathfrak{C})$.

(iv) The statement (iii) holds for any two fundamental $(\mathfrak{C}, n)$-pairs $(U, [\mathfrak{C}])$, $(V, [\mathfrak{C}])$ with U, V connected.

Remark 6. Every point of a $(\mathfrak{C}, n)$-manifold has a $\mathfrak{C}$-orientable neighborhood.

Remark 7. An orientable n-dimensional manifold in the ordinary sense is always $\mathfrak{C}$-orientable. A $(\mathfrak{C}_2, n)$-manifold is always $\mathfrak{C}_2$-orientable.

LEMMA 12. *Let X be a $\mathfrak{C}$-orientable $(\mathfrak{C}, n)$-manifold and let Q be a non-null bounded open subset of X. Then there is a subgroup S of $H_n(X, X-Q; \mathfrak{C})$ such that (Q, S) is a fundamental $(\mathfrak{C}, n)$-pair.*

Proof. We first prove that, if Q is connected, then $(Q, H_n(X, X-Q; \mathfrak{C}))$ is a fundamental $(\mathfrak{C}, n)$-pair.

For each point x of $\overline{Q}$ there is a fundamental $(\mathfrak{C}, n)$-pair $(U, [\mathfrak{C}])$ with U connected and containing x. Since $\overline{Q}$ is compact, there exist a finite number of fundamental $(\mathfrak{C}, n)$-pairs

$$(U_1, [\mathfrak{C}]), (U_2, [\mathfrak{C}]), \cdots, (U_t, [\mathfrak{C}])$$

such that each U_i is connected and $U_1 \cup \cdots \cup U_t \supset \overline{Q}$. Since $\overline{Q}$ is connected, we may assume that for each $i = 1, \cdots, t-1$,

$$(U_1 \cup \cdots \cup U_i) \cap U_{i+1} \neq 0.$$

Since X is $\mathfrak{C}$-orientable, all the paths from $(U_1, [\mathfrak{C}])$ to $(U_2, [\mathfrak{C}])$ induces the same isomorphism h of $H_n(X, X-U_1; \mathfrak{C})$ onto $H_n(X, X-U_2; \mathfrak{C})$. Therefore whenever $e \in H_n(X, X-U_1; \mathfrak{C})$,

$$e \mid (X, X - V) = h(e) \mid (X, X - V)$$

for all components V of $U_1 \cap U_2$ and so

$$e \mid (X, X - (U_1 \cap U_2)) - h(e) \mid (X, X - (U_1 \cap U_2)) = 0.$$

By the exactness of the Mayer-Vietoris sequence of the triad $(X; X-U_1, X-U_2)$, there is some $e' \in H_n(X, X-(U_1 \cup U_2); \mathfrak{C})$ such that

$$e' \mid (X, X - U_1) = e, \qquad e' \mid (X, X - U_2) = h(e).$$

Since $U_1 \cup U_2$ is connected, it follows from Lemma 10 that the natural homomorphism of $H_n(X, X-(U_1 \cup U_2); \mathfrak{C})$ into $H_n(X, X-U_1; \mathfrak{C})$ is one-one. Hence it is an isomorphism onto. This isomorphism followed by h is the natural homomorphism of $H_n(X, X-(U_1 \cup U_2); \mathfrak{C})$ into $H_n(X, X-U_2; \mathfrak{C})$ which is also an isomorphism onto. Then we can easily see that $(U_1 \cup U_2, H_n(X, X-(U_1 \cup U_2); \mathfrak{C}))$ is a fundamental $(\mathfrak{C}, n)$-pair.

Repeating this process, we can finally have a fundamental $(\mathfrak{C}, n)$-pair $(U_1 \cup \cdots \cup U_t, H_n(X, X-(U_1 \cup \cdots \cup U_t); \mathfrak{C}))$. Hence $(Q, H_n(X, X-Q; \mathfrak{C}))$ is a fundamental $(\mathfrak{C}, n)$-pair.

Now let Q be any non-null bounded open subset of X. Since X is connected, there is a connected bounded open subset Q' of X containing Q. We have just shown that $(Q', H_n(X, X-Q'; \mathfrak{C}))$ is a fundamental $(\mathfrak{C}, n)$-pair. It follows that $(Q, H_n(X, X-Q'; \mathfrak{C}) \mid (X, X-Q))$ is a fundamental $(\mathfrak{C}, n)$-pair.

COROLLARY. *For any connected non-null bounded open subset U of a $\mathfrak{C}$-orientable $(\mathfrak{C}, n)$-manifold there is a fundamental $(\mathfrak{C}, n)$-pair $(U, [\mathfrak{C}])$. Hence for any $\mathfrak{C}$-orientable $(\mathfrak{C}, n)$-manifold X, $H_n(X; \mathfrak{C})$ is isomorphic to $\mathfrak{C}$.*

6. Local separation.

LEMMA 13. *Let X be a $(\mathfrak{C}, n)$-manifold and let M, N be compact subsets of X with $M \supset N$. If there is a nonzero element e of $H_{n-1}(M, N; \mathfrak{C})$ such that for no proper compact subset K of $(M-N)^-$, $e|(M, K \cup N) = 0$, then every point x of $M-N$ has a neighborhood Q such that every neighborhood of x meets at least two components of $Q-M$.*

Proof. Let x be a point of $M-N$ and let $(U, [\mathfrak{C}])$ be a fundamental $(\mathfrak{C}, n)$-pair such that U is connected and $x \in U \subset X-N$. Let Q be a neighborhood of x contained in U and such that

$$H_{n-1}(X, X-U; \mathfrak{C}) \mid (X, X-Q) = 0.$$

The closure of $Q-M$ contains Q. Otherwise, $Q' = Q - (Q-M)^-$ is a nonnull open set contained in M. It follows from

$$\begin{array}{ccccc} H_{n-1}(M, N; \mathfrak{C}) & \rightarrow & H_{n-1}(X, X-U; \mathfrak{C}) & \rightarrow & H_{n-1}(X, X-Q; \mathfrak{C}) \\ \downarrow & & & & \downarrow \\ H_{n-1}(M, M-Q'; \mathfrak{C}) & & \xrightarrow{\approx} & & H_{n-1}(X, X-Q'; \mathfrak{C}) \end{array}$$

that $e|(M, M-Q') = 0$, contrary to our hypothesis.

Suppose that there is a neighborhood of x which meets only one component A of $Q-M$. Then

$$V = Q - [Q - (M \cup A)]^-$$

is a neighborhood of x. Both V and $V-M$ contain A and are contained in $\bar{A}$; therefore they are connected and hence the natural homomorphism of $H_n(X, X-V; \mathfrak{C})$ into $H_n(X, M \cup (X-V)\ \mathfrak{C})$; is one-one and onto.

$$\begin{array}{ccccc} H_{n-1}(X, X-V; \mathfrak{C}) & \leftarrow & H_{n-1}(M \cup (X-V), X-V; \mathfrak{C}) & \xleftarrow{\partial} & H_n(X, M \cup (X-V); \mathfrak{C}) \\ \uparrow & & \uparrow \approx & & \uparrow \approx \\ & & H_{n-1}(M, M-V; \mathfrak{C}) & & H_n(X, X-V; \mathfrak{C}) \\ & & \uparrow & & \\ H_{n-1}(X, X-U; \mathfrak{C}) & \longleftarrow & H_{n-1}(M, N; \mathfrak{C}) & & \end{array}$$

By hypothesis, $e|(M, M-V) \neq 0$. Therefore $e|(X, X-V) \neq 0$. On the other hand,

$$e \mid (X, X-V) = (e \mid (X, X-U)) \mid (X, X-V) = 0.$$

Hence we have arrived at a contradiction.

LEMMA 14. *Let X be a $(\mathfrak{P}, n)$-manifold and let Y be a closed subset of X. Then Y is $(n-1)$-dimensional if and only if Y is nowhere dense and there is a point x of Y and a neighborhood Q of x such that every neighborhood of x meets at least two components of $Q-Y$.*

Proof. Suppose that Y is $(n-1)$-dimensional; then it is nowhere dense by

Lemma 9. By Remark 3, there exist compact subsets M, N of Y such that $M\supset N$ and for some nonzero element e of $H_{n-1}(M, N; \mathfrak{P})$, $e|(M, K\cup N)\neq 0$ whenever K is a proper compact subset of $[M-N]^-$. It follows from Lemma 13 that there is a point x of $M-N$ and a neighborhood Q of x such that every neighborhood of x meets at least two components of $Q-M$. Since Y is nowhere dense, no open set can be contained in Y. Hence every neighborhood of x meets at least two components of $Q-Y$.

Conversely, suppose that Y is nowhere dense and that there is a point x of Y and a neighborhood Q of x such that every neighborhood of x meets at least two components of $Q-Y$. By Lemma 9, the dimension of Y is $\leqq n-1$. Since X is a $(\mathfrak{P}, n)$-manifold, there is a fundamental $(\mathfrak{P}, n)$-pair $(U, [\mathfrak{P}])$ with U connected and contained in Q. Clearly $U-Y$ contains at least two components. Let A be a component of $U-Y$. Let

$$M = [(\overline{A} - A)\cap U]^-, \qquad N = M - U.$$

We can easily see that M, N are compact subsets of Y and that the natural homomorphism of $H_{n-1}(M, N; \mathfrak{P})$ into $H_{n-1}(\overline{A}-A, \overline{A}-U; \mathfrak{P})$ is one-one and onto. Since $H_n(\overline{A}, \overline{A}-U; \mathfrak{P})=0$ and $H_n(\overline{A}, \overline{A}-A; \mathfrak{P})$ is isomorphic to $\mathfrak{P}$ (Lemma 10), it follows from

$$H_{n-1}(M, N; \mathfrak{P}) \xrightarrow{\approx} H_{n-1}(\overline{A} - A, \overline{A} - U; \mathfrak{P}) \xleftarrow{\partial} H_n(\overline{A}, \overline{A} - A; \mathfrak{P})$$
$$\longleftarrow H_n(\overline{A}, \overline{A} - U; \mathfrak{P})$$

that $H_{n-1}(M, N; \mathfrak{P})\neq 0$. Hence the dimension of Y is $\geqq n-1$ by Lemma 4. This proves that Y is $(n-1)$-dimensional.

LEMMA 15. *Let X be a $(\mathfrak{C}_q, n)$-manifold and let T be a periodical transformation on X of period q, where q is any prime number. Then the fixed point set L of T is closed (and hence locally compact), locally connected, nowhere dense and of type $P(\mathfrak{C}_q)$. Hence every component of L is open in L and is a $(\mathfrak{C}_q, k)$-manifold, where k depends on the component and is $\leqq n-1$. The equality holds only if $q=2$.*

REMARK 8. This statement, though slightly general, is essentially the same as the one given in [5]. Firstly, the conclusion is a local property; therefore the compactness of X required in [5] can be replaced by the local compactness. Secondly, the assumption that every bounded open set is a countable union of compact sets is not essential and then can be omitted. Finally, the property Q used in [5] can be removed, as we can prove that a locally compact Hausdorff space which possesses the property $P_n(\mathfrak{C})$ at point x possesses also the property $Q(\mathfrak{C})$ at x (see the Appendix).

LEMMA 16. *Let X be a $(\mathfrak{C}_2, n)$-manifold, let T be a periodical transformation on X of period 2 and let L be the fixed point set of T. If x is a point of L at which L possesses the property $P_{n-1}(\mathfrak{C}_2)$ (Lemma 15), then x has a connected neighborhood Q such that $Q=T(Q)=\operatorname{Int}\overline{Q}$ (i.e., the interior of $\overline{Q}$) and $Q-L$ has exactly two components which are mapped into each other by T.*

Proof. By hypothesis, there is a fundamental $(\mathfrak{S}_2, n-1)$-pair (U, S) of L with U containing x. Since the natural homomorphism of $H_{n-1}(\overline{U}, \overline{U}-U; \mathfrak{S}_2)$ into $H_{n-1}(L, L-U; \mathfrak{S}_2)$ is one-one and onto, there is an element e of $H_{n-1}(\overline{U}, \overline{U}-U; \mathfrak{S}_2)$ such that for no proper compact subset K of $\overline{U}$, $e|(\overline{U}, K\cup(\overline{U}-U))=0$. It follows from Lemma 13 that there is a neighborhood A of x such that every neighborhood of x meets at least two components of $A-\overline{U}$. Since L is nowhere dense (Lemma 15), every neighborhood of x meets at least two components of $A-L$. Let B be the component of $A\cap T(A)$ containing x and let

$$Q = \operatorname{Int} \overline{B}.$$

It is clear that Q is connected, $Q=T(Q)=\operatorname{Int}\overline{Q}$ and $Q-L$ has at least two components. We remain to prove that the components of $Q-L$ are two in number and are mapped into each other by T.

Let C be a component of $Q-L$. Then $T(C)\neq C$, since otherwise, the periodical transformation $T'\colon Q\to Q$ defined by

$$T'(x) = \begin{cases} T(x) & \text{if } x \in C, \\ x & \text{if } x \in Q - C \end{cases}$$

has a fixed point set which is not nowhere dense, contrary to Lemma 15.

Now we claim that every point of $Q\cap\overline{C}$ is contained in $\operatorname{Int}(\overline{C}\cup T(\overline{C}))$. Let y be a point of $Q\cap\overline{C}$ and let D be a neighborhood of y whose closure is contained in Q. In the diagram

$$\begin{tikzcd}
H_n(M, M_2\cup N; \mathfrak{S}_2) \arrow[rr, "\partial"] & & H_{n-1}(M_2\cup N, N; \mathfrak{S}_2) & & \\
H_n(M_1, N'\cup N_1; \mathfrak{S}_2) \arrow[u] \arrow[d, "\partial"] & & H_{n-1}(M_2, N_2; \mathfrak{S}_2) \arrow[u] & & \\
H_{n-1}(N'\cup N_1, N_1, \mathfrak{S}_2) & & H_{n-1}(M_2, N''; \mathfrak{S}_2) \arrow[u] & & \\
& H_{n-1}(N', N''; \mathfrak{S}_2) \arrow[ul, "\approx"'] \arrow[ur] \arrow[dd, "\approx"] & & H_{n-1}(N_2, N''; \mathfrak{S}_2) \arrow[ul] \arrow[uuul] \arrow[dd, "\approx"] & \\
& & H_n(M_2, N'\cup N_2; \mathfrak{S}_2) \arrow[uu, "\Delta"] \arrow[dl, "\partial"'] \arrow[dr, "\partial"] \arrow[dd, "\partial"] & & \\
& H_{n-1}(N'\cup N_2, N_2; \mathfrak{S}_2) & & H_{n-1}(N'\cup N_2, N'; \mathfrak{S}_2) & \\
& & H_{n-1}(N'\cup N_2; \mathfrak{S}_2) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

we let

$$\begin{aligned}
M_1 &= (C\cap D)^-, & M_2 &= T(M_1), & M &= M_1\cup M_2; \\
N_1 &= M_1 - D, & N_2 &= T(N_1), & N &= N_1\cup N_2; \\
N' &= M_1 - C = M_1\cap M_2; & & & & \\
N'' &= M_1 - (C\cup D) = N_1\cap N_2 = N'\cap N_1 = N'\cap N_2. & & & &
\end{aligned}$$

Since $H_n(X, X-Q; \mathfrak{C}_2)$ is isomorphic to $\mathfrak{C}_2$ (Lemmas 10 and 12), there is only one element u of $H_n(X, X-Q; \mathfrak{C}_2)$ different from 0. Under the natural homomorphisms

$$H_n(X, X - Q; \mathfrak{C}_2) \longrightarrow H_n(X, X - (M_i - (N' \cup N_i)); \mathfrak{C}_2)$$
$$\xleftarrow{\approx} H_n(M_i, N' \cup N_i; \mathfrak{C}_2)$$

$u\,|\,(X,\ X-(M_i-(N'\cup N_i)))=u_i\,|\,(X,\ X-(M_i-(N'\cup N_i)))$ determines a unique nonzero element u_i of $H_n(M_i, N'\cup N_i; \mathfrak{C}_2)$, $i=1, 2$.

Every group of the above diagram has an element which is corresponding to either u_1 or u_2 or both. The one in $H_{n-1}(M_2\cup N, N; \mathfrak{C}_2)$ is 0 as

$$H_{n-1}(N_2, N''; \mathfrak{C}_2)\,|\,(M_2 \cup N, N) = 0.$$

It follows that for some $e\in H_n(M, N; \mathfrak{C}_2)$,

$$u_1\,|\,(M, M_2 \cup N) = e\,|\,(M, M_2 \cup N).$$

Let K be a minimal compact subset of M such that for some $e'\in H_n(K\cup N, N; \mathfrak{C}_2)$,

$$e = e'\,|\,(M, N).$$

Let D' be any neighborhood of y not intersecting N. It follows from

$$\begin{aligned} e'\,|\,(X, X - (C \cap D')) &= e\,|\,(X, X - (C \cap D')) \\ &= u_1\,|\,(X, X - (C \cap D')) = u\,|\,(X, X - (C \cap D')) \neq 0 \end{aligned}$$

that $y\in K$. By Lemma 9, $K-N$ is a neighborhood of y contained in M. Hence y is an inner point of $\overline{C}\cup T(\overline{C})$.

As a consequence of this result, we have

$$Q = \operatorname{Int} (\overline{C} \cup T(\overline{C})).$$

In fact, it follows from $Q=\operatorname{Int}\overline{Q}$ that $Q\supset\operatorname{Int}(\overline{C}\cup T(\overline{C}))$. Suppose that our assertion is false. Then there exists, by the connectedness of Q, a point of $Q\cap(\overline{C}\cap T(\overline{C}))$ which does not belong to $\operatorname{Int}(\overline{C}\cup T(\overline{C}))$. But such a point clearly belongs to $Q\cap\overline{C}$ and then belongs to $\operatorname{Int}(\overline{C}\cup T(\overline{C}))$ by our result above. Hence we have arrived at a contradiction. This proves that $Q-L$ has exactly two components C and $T(C)$.

Corollary. *Let X be a $(\mathfrak{P}, n)$-manifold, let T be a periodical transformation on X of period 2 and let L be a component of the fixed point set of T. If L is a $(\mathfrak{C}_2, n-1)$-manifold (see Lemmas 11 and 15), then L is also a $(\mathfrak{P}, n-1)$-manifold.*

Proof. Let x be a point of L and let $(U, [\mathfrak{P}])$ be a fundamental $(\mathfrak{P}, n)$-pair with U connected and containing x. Since X is also a $(\mathfrak{C}_2, n)$-manifold (Lemma 11), it follows from Lemma 16 that x has a connected neighborhood

V such that $V \subset U$, $T(V)=V$ and $V-L$ has exactly two components C and $T(C)$. $(V, H_n(X, X-V; \mathfrak{P}))$ is clearly a fundamental $(\mathfrak{P}, n)$-pair.

For any point y of $L \cap V$ and any neighborhood Q of y there exists, by Lemma 16 and the definition of the property $P_n(\mathfrak{P})$, a connected neighborhood W of y such that (i) $W \subset V \cap Q$, (ii) $T(W)=W$, (iii) $W-L$ has exactly two components D and $T(D)$ with D contained in C, (iv) the natural homomorphism of $H_n(X, X-V; \mathfrak{P})$ into $H_n(X, X-W; \mathfrak{P})$ is one-one and onto, and (v) whenever $k \neq n$,

$$H_k(X, X-V; \mathfrak{P}) \mid (X, X-W) = 0.$$

Now let us observe the natural homomorphism of the Mayer-Vietoris sequence of $(X; X-C, X-T(C))$ into that of $(X; X-D, X-T(D))$.

$$\begin{tikzcd}
\vdots & & & \vdots \\
H_k(X, X-V; \mathfrak{P}) \arrow[u, "\Delta"'] \arrow[rrr] & & & H_k(X, X-W; \mathfrak{P}) \arrow[u, "\Delta"'] \\
H_k(X-C, X-V; \mathfrak{P}) \times H_k(X-T(C), X-V; \mathfrak{P}) \arrow[u] \arrow[rrr] & & & H_k(X-D, X-W; \mathfrak{P}) \times H_k(X-T(D), X-W; \mathfrak{P}) \arrow[u] \\
& H_k(L, L-V; \mathfrak{P}) \arrow[ul] \arrow[dl, "\approx"] \arrow[r] & H_k(L, L-W; \mathfrak{P}) \arrow[ur] \arrow[dr, "\approx"'] & \\
H_k(X-(C \cup T(C)), X-V; \mathfrak{P}) \arrow[uu] \arrow[rrr] & & & H_k(X-(D \cup T(D)), X-W; \mathfrak{P}) \arrow[uu] \\
H_{k+1}(X, X-V; \mathfrak{P}) \arrow[u, "\Delta"'] \arrow[rrr] & & & H_{k+1}(X, X-W; \mathfrak{P}) \arrow[u, "\Delta"'] \\
\vdots \arrow[u] & & & \vdots \arrow[u]
\end{tikzcd}$$

For $k<n$, the natural homomorphism

$$H_k(X-C, X-V; \mathfrak{P}) \times H_k(X-T(C), X-V; \mathfrak{P}) \\ \rightarrow H_k(X-D, X-W; \mathfrak{P}) \times H_k(X-T(D), X-W; \mathfrak{P})$$

is trivial. In fact, let e be any element of $H_k(X-C, X-V; \mathfrak{P})$. The image of $(e \mid (X-D, X-W), 0)$ in $H_k(X, X-W; \mathfrak{P})$ is 0; therefore $(e \mid (X-D, X-W), 0)$ is the image of some $e' \in H_k(X-(D \cup T(D)), X-W; \mathfrak{P})$. Since e' is invariant under the homomorphism induced by T, it follows that the isomorphism of $H_k(X-D, X-W; \mathfrak{P})$ onto $H_k(X-T(D), X-W; \mathfrak{P})$, induced by T, maps $e \mid (X-D, X-W)$ into 0. Hence $e \mid (X-D, X-W)=0$.

Let A be a neighborhood of y contained in W and such that whenever $k<n-1$,

$$H_{k+1}(X, X-W; \mathfrak{P}) \mid (X, X-A) = 0.$$

If $k<n-1$ and $e \in H_k(L, L-V; \mathfrak{P})$, then

$$e \mid (X - (D \cup T(D)), X - W) = \Delta e'$$

for some $e' \in H_{k+1}(X, X-W; \mathfrak{P})$. It follows from $e' \mid (X, X-A)=0$ that $e \mid (L, L-A)=0$. This proves that whenever $k<n-1$,

$$H_k(L, L - V; \mathfrak{P}) \mid (L, L - A) = 0.$$

Since L is of dimension $\leqq n-1$, it follows that whenever $k>n-1$, $H_k(L, L-V; \mathfrak{P})=0$ and hence

$$H_k(L, L - V; \mathfrak{P}) \mid (L, L - A) = 0.$$

It is clear that the natural homomorphism

$$H_n(X - C, X - V; \mathfrak{P}) \times H_n(X - T(C), X - V; \mathfrak{P}) \to H_n(X, X - V; \mathfrak{P})$$

is trivial. Therefore there is a subgroup S of $H_{n-1}(L, L-V; \mathfrak{P})$ such that Δ maps $H_n(X, X-V; \mathfrak{P})$ isomorphically onto $S \mid (X-(C \cup T(C)), X-V)$. Similarly there is a subgroup S' of $H_n(X, X-W; \mathfrak{P})$ isomorphically $S' \mid (X-(D \cup T(D)), X-W)$. Therefore it is clear that the natural homomorphism of $H_{n-1}(L, L-V; \mathfrak{P})$ into $H_{n-1}(L, L-W; \mathfrak{P})$ maps S isomorphically onto S'. For any $e \in H_{n-1}(L, L - V; \mathfrak{P})$, the image of $e \mid (L, L - W)$ in $H_{n-1}(X - D, X - W; \mathfrak{P}) \times H_{n-1}(X - T(D), X - W; \mathfrak{P})$ is 0; therefore $e \mid (X - (D \cup T(D)), X - W)$ belongs to $\Delta H_n(X, X - W; \mathfrak{P})$. Hence $e \mid (L, L-W) \in S'$.

From this result it follows that

$$H_{n-1}(L, L - V; \mathfrak{P}) \mid (L, L - A) = S \mid (L, L - A).$$

With W, A in place of V, Q we can find a neighborhood W' of y in W. Therefore the natural homomorphism of $H_{n-1}(L, L-W; \mathfrak{P})$ into $H_{n-1}(L, L-W'; \mathfrak{P})$ maps S' isomorphically into $H_{n-1}(L, L-W'; \mathfrak{P})$. Hence the natural homomorphism of $H_{n-1}(L, L-V; \mathfrak{P})$ into $H_{n-1}(L, L-A; \mathfrak{P})$ maps S isomorphically onto $S \mid (L, L-A)$.

Combining all these results, one can easily see that $(L \cap V, S)$ is a fundamental $(\mathfrak{P}, n-1)$-pair. This proves that L possesses that property $P_{n-1}(\mathfrak{P})$ at x. But x is an arbitrary point of L; hence L is a $(\mathfrak{P}, n-1)$-manifold.

REMARK 9. From this Corollary the following general question is raised. If X is a $(\mathfrak{P}, n)$-manifold and T is a periodical transformation on X of prime period, is every component of the fixed point set of T a homological manifold with respect to $\mathfrak{P}$?

7. **Theorem 1.**

THEOREM 1. *Let X be a $(\mathfrak{P}, n)$-manifold and let H be a finite group acting effectively on X. If H is not trivial, the dimension of the fixed point set of H is $\leqq n-1$. The equality holds only if the order of H is 2.*

Proof. Let T be an element of H of prime order q. Let L be the fixed point set of T and let L' be the fixed point set of H.

Since a $(\mathfrak{P}, n)$-manifold is of dimension n, L is of dimension $\leqq n$. By Lemma 11, X is also a $(\mathfrak{C}_q, n)$-manifold; it follows from Lemmas 15 and 9 that L is of dimension $\leqq n-1$.

Since $L' \subset L$, the dimension of L' is $\leqq n-1$. Now we assume that L' is of dimension $n-1$; then L is also of dimension $n-1$.

By Lemma 14, there is a point x of L' and a neighborhood Q of x such that every neighborhood of x meets at least two components of $Q-L$.

Let (U, S) be a fundamental $(\mathfrak{P}, n)$-pair with U containing x. Since both X and L are locally connected, there is a connected neighborhood V of x contained in $Q \cap U$ and such that $L \cap V$ is connected. It is clear that $(V, [\mathfrak{P}])$ is a fundamental $(\mathfrak{P}, n)$-pair and that $V-L$ has at least two components.

Let B be a component of $V-L$. Then $(B, [\mathfrak{P}])$ is a fundamental $(\mathfrak{P}, n)$-pair. By Lemma 11, $(B, [\mathfrak{C}_q])$ is a fundamental $(\mathfrak{C}_q, n)$-pair; therefore $H_n(\overline{B}, \overline{B}-B; \mathfrak{C}_q)$ has a nonzero element u. Let $M=\overline{B}-B$; then $V \cap M$ is not null and is contained in L. Let y be a point of $V \cap M$ and let $N=(V\cap M)^{-}$. For any neighborhood A of y contained in V there is a unique element e_A of $H_{n-1}(N, N-A; \mathfrak{C}_q)$ such that

$$e_A \mid (M, M-A) = \partial u,$$

where ∂ is the boundary operator of the homology sequence of the triple $(\overline{B}, M, M-A)$.

$$\begin{tikzcd}
& H_{n-1}(L, L-A; \mathfrak{C}_q) & & H_{n-1}(L, L-V; \mathfrak{C}_q) \arrow[ll] \\
H_n(L, N\cup(L-A); \mathfrak{C}_q) \arrow[r] & H_{n-1}(N\cup(L-A), L-A; \mathfrak{C}_q) \arrow[u] & & H_{n-1}(N\cup(L-V), L-V; \mathfrak{C}_q) \arrow[ll] \arrow[u] \\
& H_{n-1}(N, N-A; \mathfrak{C}_q) \arrow[u, "\approx"'] \arrow[d, "\approx"'] & & H_{n-1}(N, N-V; \mathfrak{C}_q) \arrow[ll] \arrow[u, "\approx"'] \arrow[d, "\approx"'] \\
& H_{n-1}(M, M-A; \mathfrak{C}_q) & & H_{n-1}(M, M-V; \mathfrak{C}_q) \arrow[ll] \\
& & H_n(B, M; \mathfrak{C}_q) \arrow[ul, "\partial"] \arrow[ur, "\partial"'] &
\end{tikzcd}$$

Since the dimension of L is $n-1$, $H_n(L, N\cup(L-A); \mathfrak{C}_q)=0$. Therefore

$$e'_A = e_A \mid (L, L-A) \neq 0.$$

Now e'_V is an element of $H_{n-1}(L, L-V; \mathfrak{C}_q)$ such that for every neighborhood A of y contained in V,

$$e'_V \mid (L, L-A) = e'_A \neq 0.$$

It follows that L possesses the property $P_{n-1}(\mathfrak{C}_q)$ at y (Lemma 15). Since $L \cap V$ is connected and contains both x and y, L possesses the property $P_{n-1}(\mathfrak{C}_q)$ at x (Lemma 15). Moreover, q must be equal to 2 (Lemma 15).

By Lemma 16, there is a connected neighborhood W of x contained in V and such that $W=T(W)$ and $W-L$ contains exactly two components C and $T(C)$.

Since W is contained in Q, $W-L'$ has at least two components. Therefore $W\cap L'=W\cap L$ and then $W-L'$ has exactly two components C and $T(C)$. Let Q' be a neighborhood of x invariant under H and contained in W. Then every element of H maps $Q'\cap C$ into $Q'\cap T(C)$ or itself.

Suppose that the order of H is >2. Then there is an element other than the identity maps $Q'\cap C$ into itself. Certain power of this element, denoted by T', is of prime order q. Just as we have seen above, $q=2$ and there is a neighborhood W' of x contained in Q' and such that $T'(W')=W'$ and $W'-L'$ has exactly two components C' and $T'(C')$. C' is contained in either C or $T(C)$; we may assume that $C'\subset C$. Since $T'(Q'\cap C)=Q'\cap C$, it follows that $T'(C')\subset C$. Therefore $C\cup L'$ contains W' and then contains x as an inner point. This is clearly impossible.

8. **Slices.**

Definition. Let X be a Hausdorff space and let G be a compact Lie group acting topologically on X. Let p be a point of X and G_p the isotropic subgroup of G at p. A *slice* at p is a subset Y of X containing p and satisfying the following conditions:

(i) Whenever $g\in G$ and $x\in Y$, $g(x)\in Y$ if and only if $g\in G_p$.

(ii) If Q is a small open cell which is a local cross-section to the cosets of G_p at the identity of G, then $(g, x)\to g(x)$ defines a homeomorphism of $Q\times Y$ onto $Q(Y)$. Moreover, $Q(Y)$ is open in X.

Lemma 17. *If X is completely regular Hausdorff space and G is a compact Lie group acting on X, then at every point of X there exists a slice.*

This Lemma was first proved by Montgomery-Yang [3] for complete metric spaces and then extended to completely regular spaces by Mostow [12].

9. **Theorem 2.**

Theorem 2. *Let X be a $(\mathfrak{P}, n)$-manifold and let G be a compact Lie group acting topologically on X such that the highest dimension of any orbit is r. Then for any integer k, $0\leqq k<r$, the union of all the orbits of dimension $\leqq k$ is a closed set of dimension $\leqq n-r+k-1$.*

Proof. Let X be a $(\mathfrak{P}, n)$-manifold and let G be a compact Lie group acting on X. For each point x of X, we denote by G_x the isotropic subgroup of G at x and by G_x^* the identity component of G_x. Then the order of the quotient group G_x/G_x^* is finite and will be denoted by $m(x)$.

For any integers $u \geqq 0$, $v \geqq 1$ we let

$$X_{u,v} = \{x : x \in X, \dim G(x) = u, m(x) = v\}, \qquad X_u = \bigcup_{v=1}^{\infty} X_{u,v}.$$

Since every point x of X has a neighborhood U such that whenever yU, G_y is conjugate to a subgroup of G_x [11], it follows that whenever $u \geqq 0$ and $v \geqq 0$,

$$Z_{u,v} = X_r \cup X_{r-1} \cup \cdots \cup X_{u+1} \cup X_{u,1} \cup X_{u,2} \cup \cdots \cup X_{u,v}$$

is open. Hence all $Z_{u,v}$ and consequently all $X_{u,v}$ are locally compact Hausdorff.

Each $X_{u,v}$ intersects $\overline{X}_r - X_r$ at a set of dimension $\leqq n-2$. In fact, let p be a point of $X_{u,v} \cap (\overline{X}_r - X_r)$ and let Y be a slice at p. Clearly Y may be assumed to be connected and then is a $(\mathfrak{P}, n-u)$-manifold (Lemma 3). Since $p \in \overline{X}_r$, there is a point x of Y belonging to X_r. It follows from $\dim G_p > \dim G_x$ that there is a finite subgroup H of G_p in which the index of $H \cap G_x$ is >2. Making use of Theorem 1, one can easily show that the fixed point set of H is of dimension $\leqq n-u-2$. Hence $X_{u,v} \cap Y$ is of dimension $\leqq n-u-2$, as it is contained in the fixed point set of H. Let Q be an open u-cell in G which contains the identity and is such that $(g, x) \rightarrow g(x)$ defines a homeomorphism of $Q \times Y$ onto $Q(Y)$ and that $Q(Y)$ is a neighborhood of p. Then $Q(X_{u,v} \cap Y)$ is a neighborhood of p in $X_{u,v}$ and is of dimension $\leqq n-2$ (Lemma 5). Since p is an arbitrary point of $X_{u,v} \cap (\overline{X}_r - X_r)$, the dimension of $X_{u,v} \cap (\overline{X}_r - X_r)$ is $\leqq n-2$ (Corollary to Lemma 4). From this result and Lemma 7, it follows that the dimension of $\overline{X}_r - X_r$ is $\leqq n-2$. Hence $\overline{X}_r = X$ and consequently $X - X_r$, i.e., the union of all the orbits not of the highest dimension, is a closed set of dimension $\leqq n-2$, since otherwise $\overline{X}_r - X_r$ would be of dimension $n-1$ by Lemma 9.

Let X^* be the orbit space of X and let f be the natural map of X onto X^*. Then all $f(Z_{u,v})$ are open in X^*. Hence all $f(Z_{u,v})$ and consequently all $f(X_{u,v})$ are locally compact Hausdorff. Let

$$\dim f(X_{u,v}) = k_{u,v}, \qquad 0 \leqq u < r.$$

Let p, Y, Q be as in the preceding paragraph. Then f maps $X_{u,v} \cap Y$ onto a neighborhood of $f(p)$ in $f(X_{u,v})$. Therefore $X_{u,v} \cap Y$ is of dimension $\leqq k_{u,v}$. Hence $Q(X_{u,v} \cap Y)$ is a neighborhood of p in $X_{u,v}$ whose dimension is $\leqq k_{u,v} + u$. Since p is an arbitrary point of $X_{u,v}$, it follows that the dimension of $X_{u,v}$ is $\leqq k_{u,v} + u$. Hence our conclusion that $X_0 \cup \cdots \cup X_k$, $0 \leqq k < r$, is a closed set of dimension $\leqq n-r+k-1$ follows from Lemma 7 if we can show that

$$k_{u,v} \leqq n - r - 1, \qquad 0 \leqq u < r.$$

Suppose that our assertion is false. Then there is some $X_{\alpha,\beta}$ such that (i) $0 \leqq \alpha < r$, (ii) $k = k_{\alpha,\beta} > n-r-1$ and (iii) if $u > \alpha$ or if $u = \alpha$ and $v < \beta$, then $k_{u,v} \leqq n-r-1$. By Remark 3, there exist compact subsets M, N of $X_{\alpha,\beta}$ such that $M \supset N$ and for some nonzero element e of $H_k(M, N; \mathfrak{P})$, $e \mid (M, K \cup N) \neq 0$ whenever K is a proper compact subset of $(M-N)^-$. Let p be a point of $f^{-1}(M-N)$ and let Y be a connected slice at p. Clearly G_p acts as a transformation group on Y with $f(Y)$ as the orbit space. Let A be a neighborhood of p in Y such that $G_p(A) = A$ and $f(A) \cap N = \emptyset$. Clearly for any neighborhood B of p in Y contained in A, $e_B = e \mid (M, M - f(B)) \neq 0$.

$$\begin{tikzcd}
H_k(f(Y)\cap M, (f(Y)\cap M) - f(A); \mathfrak{P}) \arrow[rr] \arrow[dd, "\approx"] & & H_k(f(Y)\cap M, (f(Y)\cap M) - f(B); \mathfrak{P}) \arrow[dd, "\approx"'] \arrow[ddddd, bend left] \\
& H_k(M, N; \mathfrak{P}) \arrow[dl] \arrow[dr] & \\
H_k(M, M - f(A); \mathfrak{P}) \arrow[rr] & & H_k(M, M - f(B); \mathfrak{P}) \\
H_k(f^{-1}(M), f^{-1}(M) - A; \mathfrak{P}) \arrow[rr] \arrow[u, "f_*", "\approx"'] \arrow[d] & & H_k(f^{-1}(M), f^{-1}(M) - B; \mathfrak{P}) \arrow[u, "f_*", "\approx"'] \arrow[d] \\
H_k(Y, Y - A; \mathfrak{P}) \arrow[rr] \arrow[d, "f_*"'] & & H_k(Y, Y - B; \mathfrak{P}) \arrow[d, "f_*"'] \\
H_n(f(Y), f(Y) - f(A); \mathfrak{P}) \arrow[rr] & & H_k(f(Y), f(Y) - f(B); \mathfrak{P})
\end{tikzcd}$$

Let $e'_B = f_*^{-1}(e_B) \mid (Y, Y-B)$; it is clear that $e'_B = e'_A \mid (Y, Y-B)$. k is $\leqq n-u-2$, as $X_{u,v}$ is of dimension $\leqq n-2$. Since Y is a $(\mathfrak{P}, n-u)$-manifold, B can be chosen such that $e'_B = 0$. Moreover, we may assume that $G_p(B) = B$. Let e''_B be the element of $H_k(f(Y)\cap M, (f(Y)\cap M) - f(B); \mathfrak{P})$ such that $e''_B \mid (M, M-f(B)) = e_B$. Then

$$e''_B \neq 0$$

and

$$e''_B \mid (f(Y), f(Y) - f(B)) = f_*(e'_B) = 0.$$

Since $f(Y)$ is contained in $f(Z_{\alpha,\beta})$ which is of dimension k by Lemma 7, it follows that the natural homomorphism of $H_k(f(Y)\cap M, (f(Y)\cap M) - f(B); \mathfrak{P})$ into $H_k(f(Y), f(Y) - f(B); \mathfrak{P})$ is one-one. Hence we have arrived at a contradiction. This completes the proof of Theorem 2.

10. **Appendix.**

Definition. Let X be a locally compact Hausdorff space and let $\mathfrak{C}$ be a compact abelian group. X is said to possess the *property* $Q(\mathfrak{C})$ at x if for every neighborhood A of x there is a neighborhood B of x contained in A and with the property that whenever y is a point of B and C is a neighborhood of y

contained in B, there is a neighborhood D of y contained in C and such that for every k,

$$H_k(X - C, X - A; \mathfrak{S}) \mid (X - D, X - B) = 0.$$

PROPOSITION. *Let X be a locally compact Hausdorff space and let x be a point of X. If X possesses the property $P_n(\mathfrak{S})$ at x, then X possesses the property $Q(\mathfrak{S})$ at x.*

Proof. By hypothesis, there is a fundamental $(\mathfrak{S}, n)$-pair (U, S) with U containing x. Let A be a fixed neighborhood of x. Then there is a neighborhood B of x contained in $A \cap U$ and such that

$$H_n(X, X - A; \mathfrak{S}) \mid (X, X - B) \subset S \mid (X, X - B) \tag{1}$$

and for every $k \neq n$,

$$H_k(X, X - A; \mathfrak{S}) \mid (X, X - B) = 0. \tag{2}$$

Let y be a point of B and let C be a neighborhood of y contained in B. Then there is a neighborhood D of y contained in C and such that

$$H_n(X, X - C; \mathfrak{S}) \mid (X, X - D) \subset S \mid (X, X - D) \tag{3}$$

and for every $k \neq n$,

$$H_k(X, X - C; \mathfrak{S}) \mid (X, X - D) = 0. \tag{4}$$

$$\begin{tikzcd}
 & & H_k(X, X - C; \mathfrak{S}) & & \\
 & H_k(X, X - A; \mathfrak{S}) \arrow[ur] \arrow[rr] & & H_k(X, X - B; \mathfrak{S}) \arrow[ul] & \\
 & H_k(X - C, X - A; \mathfrak{S}) \arrow[u] \arrow[rr] \arrow[dddd] & & H_k(X - C, X - B; \mathfrak{S}) \arrow[u] \arrow[dddd] & \\
 & & H_{k+1}(X, X - B; \mathfrak{S}) \arrow[d] & & \\
H_k(X - B, X - A; \mathfrak{S}) \arrow[uuur] \arrow[uur] \arrow[ddr] & & H_{k+1}(X, X - C; \mathfrak{S}) \arrow[d] \arrow[uul, "\partial"'] \arrow[uur, "\partial"] & & \\
 & & H_{k+1}(X, X - D; \mathfrak{S}) \arrow[dl, "\partial"] \arrow[dr, "\partial"'] & & \\
 & H_k(X - D, X - A; \mathfrak{S}) \arrow[rr] & & H_k(X - D, X - B; \mathfrak{S}) &
\end{tikzcd}$$

Let e be an element of $H_k(X-C, X-A; \mathfrak{S})$; then

$$e \mid (X, X - B) = 0.$$

In fact, if $k \neq n$, our assertion follows from (2). If $k = n$, it follows from (1) that for some $u \in S$,

$$e \mid (X, X - B) = u \mid (X, X - B).$$

Since $e \mid (X, X-C) = 0$, $u \mid (X, X-C) = 0$ and then $u = 0$. Hence $e \mid (X, X-B) = 0$.

From this result, it follows that for some $e' \in H_k(X-B, X-A; \mathfrak{S})$,

$$e \mid (X, X - A) = e' \mid (X, X - A).$$

Let

$$e'' = e - e' \mid (X - C, X - A).$$

Then

$$e'' \mid (X, X - A) = e \mid (X, X - A) - e' \mid (X, X - A) = 0.$$

Therefore for some $e''' \in H_{k+1}(X, X-C; \mathfrak{S})$,

$$\partial e''' = e''.$$

Now we claim that

$$\partial(e''' \mid (X, X - D)) = 0,$$

where ∂ is the boundary operator of the homology sequence of $(X, X-D, X-B)$. If $k \neq n-1$, then $e''' \mid (X, X-D) = 0$ by (4) and therefore our assertion holds. If $k = n-1$, then by (3) there is some $u \in S$ such that

$$\begin{aligned} e''' \mid (X, X - D) &= u \mid (X, X - D) \\ &= (u \mid (X, X - B)) \mid (X, X - D). \end{aligned}$$

Hence our assertion again holds.

From this result, we have

$$\begin{aligned} 0 &= \partial(e''' \mid (X, X - D)) \\ &= e'' \mid (X - D, X - B) \\ &= e \mid (X - D, X - B). \end{aligned}$$

This proves our proposition.

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