HOMOGENEOUS ALGEBRAS ON COMPACT ABELIAN GROUPS(1)

BY

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Introduction. Silov has studied in great detail (see [10] or [11]) a class of algebras of continuous functions on compact abelian groups. Another class of algebras, almost disjoint from Silov's, has been investigated in a recent work [1](2) by Arens and Singer. In this paper we initiate a study of the class of algebras obtained by retaining all of Silov's conditions except that the maximal ideal space of the algebra be precisely the group(3). In this manner we obtain a class of algebras that includes those of Silov and Arens-Singer as well as other interesting examples(4).

Our first task is the identification of the maximal ideal space for the class of algebras defined. This is accomplished in §2, the result being an extension of the corresponding identification in [1]. The proof makes essential use of a theorem relating polynomial and monomial convexity in the space of n complex variables, established for the present purpose in [4]. As a consequence of the identification we obtain also in §2 a classification of all possible maximal ideal spaces. §3 is devoted to a result on semisimplicity of completions that is needed later. It is observed that the result gives some information concerning a question raised by Kaplansky about norms on $C(X)$. In §4 we obtain a classification of the algebras under consideration by setting up a correspondence between algebras and norms. §5 is devoted to applications of the results obtained to a class of algebras of (perhaps analytic) almost periodic functions.

1. **Preliminary results.** Let G be a compact abelian group with $\hat{G}$ its discrete character group and S a subsemigroup of $\hat{G}$ that contains the unit and generates $\hat{G}$. Let Hom (S, C) be the set of all multiplicative maps of S into the complex numbers C that take the value one on the unit of S. Hom (S, C) is algebraically a semigroup if multiplication is defined by $(\psi\cdot\phi)(\chi)=\psi(\chi)\phi(\chi)$. Let $P(S)$ be all functions on G that are finite sums $\sum \alpha_\chi \chi$ of characters χ in S. $P(S)$ is an algebra with the ordinary function operations. It is clear that the pairing $\langle\cdot, \cdot\rangle$, defined between Hom (S, C) and $P(S)$ by

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(2) I wish to thank Arens and Singer for allowing me to see a manuscript of [1].

(3) In [9] Mirkil drops the same condition and obtains some results valid for noncommutative groups.

(4) The algebras studied in [5] are of the type considered here.

$$\langle\phi, \sum \alpha_\chi \chi\rangle = \sum \alpha_\chi \phi(\chi)$$

has the following properties: For each fixed ϕ in Hom (S, C), the map $f \rightarrow \langle\phi, f\rangle$ is an algebra homomorphism and for each fixed χ in S, the map $\phi \rightarrow \langle\phi, \chi\rangle$ is multiplicative.

Hom (S, C) shall be considered to be topologized with the weakest topology that makes the maps $\phi \rightarrow \langle\phi, \chi\rangle$ continuous for each χ in S. This is the same topology that Hom (S, C) receives as a subset of the space of all functions from S to C in the compact-open topology, and Hom (S, C) is a closed subset of this space. Thus, because of the Tychonoff theorem, if X is a closed subset of Hom (S, C) on which the mappings $\phi \rightarrow \langle\phi, \chi\rangle$ are bounded for each χ in S, X must be compact.

If for each element σ of G, the element ϕ_σ of Hom (S, C) is defined by $\phi_\sigma(\chi) = \chi(\sigma)$, the following statements are easily checked: The map $\sigma \rightarrow \phi_\sigma$ is an isomorphism and a homeomorphism of G into Hom (S, C) and $\langle\phi_\sigma \cdot \phi, f\rangle = \langle\phi, T_\sigma(f)\rangle$ where T_σ is defined by $[T_\sigma(f)](\tau) = f(\sigma\tau)$. A subset X of Hom (S, C) is called *G-circular* if $\phi_\sigma \cdot \phi$ is in X for every ϕ in X and every σ in G.

For any compact subset X of Hom (S, C) denote by $C_P(X)$ the set

$$\left\{\phi : |\langle\phi, f\rangle| \leqq \operatorname{Sup}_{\psi \in X} |\langle\psi, f\rangle|, \text{ all } f \text{ in } P(S)\right\}$$

and by $C_M(X)$ the set

$$\left\{\phi : |\langle\phi, \chi\rangle| \leqq \operatorname{Sup}_{\psi \in X} |\langle\psi, \chi\rangle|, \text{ all } \chi \text{ in } S\right\}.$$

Our basic preliminary result is the following:

THEOREM 1.1. *If X is a compact G-circular subset of* Hom (S, C), *then* $C_M(X) = C_P(X)$.

Thus will follow easily from

LEMMA 1.2. *If S has a finite number of generators, Theorem* 1.1 *is valid.*

Lemma 1.2 is simply a restatement of Theorem 1 of [4] which relates polynomial and monomial convexity for certain subsets of the space of n complex variables.

Proof of Theorem 1.1. Because of the definitions, $C_P(X) \subset C_M(X)$. Let θ be any element of Hom (S, C) not in $C_P(X)$. Then there is some f in $P(S)$ with

$$(1.1) \qquad |\langle\theta, f\rangle| > \operatorname{Sup}_{\psi \in X} |\langle\psi, f\rangle|.$$

To prove the theorem it suffices to find some χ in S with

$$|\langle\theta, \chi\rangle| > \sup_{\psi\in X} |\langle\psi, \chi\rangle|. \tag{1.2}$$

Let S' be the subsemigroup of S generated by the unit and the finite number of χ that occur in f. Let π: Hom $(S, C)\to$Hom (S', C) be the natural restriction mapping. Denote by K the annihalator of S', that is, the closed subgroup of G consisting of those σ with $\chi(\sigma)=1$ for all χ in S'. S' can be considered to be a subgroup of the dual of G/K and the duality theory of compact abelian groups shows that S' actually generates the dual. Thus S' bears the same relationship to G/K as S does to G, so that all that has been done previously for G, Hom (S, C) and $P(S)$ remains valid for G/K, Hom (S', C) and $P(S')$. In particular Lemma 1.2 is valid for G/K-circular subsets of Hom (S', C). $\pi(X)$ is G/K-circular since X is G-circular and $\phi_{\sigma+K}\cdot(\pi(\phi)) =\pi(\phi_\sigma\cdot\phi)$ for each ϕ in Hom (S, C) and each σ in G. Thus $C_M(\pi(X)) = C_P(\pi(X))$. Let f be the function in $P(S)$ that satisfies (1.1). Then f is also in $P(S')$ and (1.1) restated in terms of the pairing between Hom (S', C) and $P(S')$ is

$$|\langle\pi(\theta), f\rangle| > \sup_{\psi\in\pi(X)} |\langle\psi, f\rangle|.$$

But this means that $\pi(\theta)$ is not in $C_P(\pi(X))$ and since $C_P(\pi(X))=C_M(\pi(X))$ there must be some χ in S' with

$$|\langle\pi(\theta), \chi\rangle| > \sup_{\psi\in\pi(X)} |\langle\psi, \chi\rangle|.$$

But this restated in terms of the pairing between Hom (S, C) and $P(S)$ is just (1.2) and the proof is complete.

If A is any algebra of functions on G, a *G-invariant norm on A* is a positive real valued function on A that satisfies $n(f+g)\leqq n(f)+n(g)$, $n(fg)\leqq n(f)n(g)$, $n(\alpha f)=|\alpha|n(f)$, $n(1)=1$, $n(f)=0$ implies $f=0$ and $n(T_\sigma(f))=n(f)$ for all σ in G and f in A, where T_σ is defined by $[T_\sigma(f)](\tau)=f(\sigma\tau)$. Let n be a G-invariant norm on $P(S)$. It is easy to check that all multiplicative linear functionals of $P(S)$ are of the form $f\to\langle\phi, f\rangle$ for fixed ϕ in Hom (S, C) and the subset of those that are continuous with respect to the topology induced by n on $P(S)$ is

$$\{\phi: |\langle\phi, f\rangle| \leqq n(f), \text{ all } f \text{ in } P(s)\}.$$

Denote this subset of Hom (S, C) by V_n and let p_n be the function defined on $P(S)$ by

$$p_n(f) = \sup_{\psi\in V_n} |\langle\psi, f\rangle|. \tag{1.3}$$

$p_n(f)$ is the spectral radius of f in the algebra $P(S)$ supplied with norm n so that we have by the Gelfand formula for spectral radius (see [8, p. 75])

(1.3) $$p_n(f) = \lim_{r\to\infty} [n(f^r)]^{1/r}.$$

Let U_n be the subset

$$\{\phi: |\langle\phi, \chi\rangle| \leqq n(\chi), \text{ all } \chi \text{ in } S\}$$

of Hom (S, C), V_n is clearly a subset of U_n and we shall proceed to prove that they are equal. Take any ϕ in U_n. $|\langle\phi, \chi^r\rangle| \leqq n(\chi^r)$ for any χ in S and any positive integer r since ϕ is in U_n. Thus $|\langle\phi, \chi\rangle|^r \leqq n(\chi^r)$ and

$$|\langle\phi, \chi\rangle| \leqq [n(\chi^r)]^{1/r},$$

so by (1.3) we have $|\langle\phi, \chi\rangle| \leqq p_n(\chi)$. Because of the definition of p_n, this means

$$|\langle\phi, \chi\rangle| \leqq \operatorname{Sup}_{\psi\in V_n} |\langle\psi, \chi\rangle|$$

for each χ in S. But this is equivalent to ϕ being in $C_M(V_n)$. Since ϕ was an arbitrary element of U_n we have proven that U_n is a subset of $C_M(V_n)$. Because of its definition and the fact that n is G-invariant, V_n is a G-circular subset of Hom (S, C) and $C_P(V_n) = V_n$. Now finally using Theorem 1.1, $C_M(V_n) = V_n$. So we have proven that U_n is a subset of $C_M(V_n) = V_n$ while directly from the definitions we have that V_n is a subset of U_n. Thus $V_n = U_n$ and we have

THEOREM 1.3. *Let n be a G-invariant norm on $P(S)$. Then the two subsets $\{\phi: |\langle\phi, \chi\rangle| \leqq n(\chi)$, all χ in $S\}$ and $\{\phi: |\langle\phi, f\rangle| \leqq n(f)$, all f in $P(S)\}$ of Hom (S, C) are identical.*

Note that Theorem 1.1 is the special case of Theorem 1.3 for norms of the form

$$n(f) = \operatorname{Sup}_{\psi\in X} |\langle\psi, f\rangle|$$

for X a compact circular subset of Hom (S, C). Note also Theorem 1.3, which will be applied twice in what follows, has been derived from Lemma 1.2 using standard techniques of Banach algebra and group theory. The proof of Lemma 1.2 on the other hand (see [4]) depends on some rather involved convexity arguments in the space of n conplex variables.

2. **Identification of maximal ideal space.** Let G be a compact abelian group. A Banach algebra A of continuous functions on G will be called a *homogeneous algebra* on G if it satisfies the conditions:

A1. Addition and multiplication in A are pointwise addition and multiplication on G.

A2. A contains 1 and enough functions to separate points of G.

A3. The topology of A is stronger than pointwise convergence on G.

A4. A is translation invariant, that is, $T_\sigma(f)$ is in A for each f in A and σ in G (where $[T_\sigma(f)](\tau) = f(\sigma\tau)$).

A5. The map $\sigma \to T_\sigma(f)$ is continuous from G to A for each f in A.

The definition given by Silov (see [10] or [11]) differs from ours only in that he insists that the maximal ideal space of A be precisely G, that is, that every multiplicative linear functional of A be of the form $f \to f(\sigma)$ for some σ in G. By dropping this condition we include within our domain the algebras of generalized analytic functions studied in [1]. For an example of a homogeneous algebra that does not satisfy the Silov condition and also is not an algebra of generalized analytic functions one can take all functions on the circle $\{z: |z| = 1\}$ that can be extended to be continuous in the annulus $\{z: a \leqq |z| \leqq b\}$ and analytic at its interior points. Also almost all of the algebras in [5] will do.

Note that we do not take a fixed norm to be part of the structure of a homogeneous algebra. If A is a homogeneous algebra, n is a *homogeneous norm for* A if n is a G-invariant norm on A (defined in §1) that induces on A the correct topology. It follows from [11, Theorem 1.5], that homogeneous norms for A always exist.

Because of [11, 2.7 and Theorem 1.5] A5 in the presence of the remaining axioms is equivalent to the following: A5′. Linear combinations of those characters of G that are in A form a dense subspace of A. Since the maximal ideal space of A may be larger than G we may not conclude that A contains the full character group $\hat{G}$ of G as in [11]. Nevertheless, because of A2 and A5′, the subsemigroup of $\hat{G}$ that occurs in A contains the unit and generates $\hat{G}$. Let us denote this semigroup by S. Every nontrivial multiplicative linear functional of A when cut down to S becomes an element of Hom (S, C) so there is a natural mapping of the maximal ideal space of A into Hom (S, C). Because of A5′, the behavior of an (automatically continuous) multiplicative linear functional on A is determined by its behavior on S. Thus the natural mapping of the maximal ideal space into Hom (S, C) is one-one and it is easy to see from the manner in which the topology of the maximal ideal space is defined that this map is continuous and thus a homeomorphism. We shall therefore identify the maximal ideal space of A with its image in Hom (S, C) and denote that subset by $M(A)$.

Before proceeding with the identification of $M(A)$, note that if G is totally disconnected, A must automatically be a Silov homogeneous algebra. For every element of $\hat{G}$ is of finite order so S must be all of $\hat{G}$ and Hom (S, C) is G itself.

Now let n be any homogeneous norm for A. An element ϕ of Hom (S, C) will be called an *n-semicharacter of S* if $|\langle \phi, \chi \rangle| \leqq n(\chi)$ for each χ in S. (Note that if p_n is the spectral radius of A, because of (1.3) the collection of n-semicharacters is identical with the collection of p_n-semicharacters.) Our identification of $M(A)$ is

THEOREM 2.1. *$M(A)$ is identical with the set of all n-semicharacters of S, where n is any homogeneous norm for A and S is the semigroup of characters in A.*

First we shall prove

LEMMA 2.2. *$M(A)$ is* $\{\phi: |\langle\phi, f\rangle| \leqq n(f), \text{ all } f \text{ in } P(S)\}$.

Proof. Let ψ be any multiplicative linear functional on A and ϕ its restriction to S. Then ϕ is in $M(A)$ and $|\langle\phi, f\rangle| = |\psi(f)| \leqq n(f)$ for each f in $P(S)$. Conversely take any ϕ in Hom (S, C) that satisfies $|\langle\phi, f\rangle| \leqq n(f)$ for all f in $P(S)$. Then the mapping $f \rightarrow \langle\phi, f\rangle$ can be extended to a multiplicative linear functional of A since $P(S)$ is dense, so that ϕ is in $M(A)$.

Theorem 2.1 now follows from Theorem 1.3 and the fact that the set of n-semicharacters of S is

$$\{\phi: |\langle\phi, \chi\rangle| \leqq n(\chi), \text{ all } \chi \text{ in } S\}.$$

In the special case of A an algebra of generalized analytic functions (see [1]), n can be chosen so that $n(\chi) = 1$ for all χ in S and Theorem 2.1 reduces to the fact proven in [1] that $M(A)$ is

$$\{\phi: |\langle\phi, \chi\rangle| \leqq 1, \text{ all } \chi \text{ in } S\}.$$

We are now able to classify all maximal ideal spaces of homogeneous algebras on G. This will be accomplished by giving, for each subsemigroup S of $\hat{G}$ that contains the unit and generates $\hat{G}$, a classification of those subsets of Hom (S, C) that can be an $M(A)$ for some homogeneous algebra A on G that is generated by S.

THEOREM 2.3. *There is a natural one-one correspondence between the collection of all subsets of* Hom (S, C) *that can be an $M(A)$ and the set of all real valued functions p on S that satisfy*

1. $p(\chi) \geqq 1$ *for all χ in S*,
2. $p(\chi_1\chi_2) \leqq p(\chi_1)p(\chi_2)$,
3. $p(\chi^2) = p(\chi)^2$.

The correspondence is set up by associating to each subset X the function p_X defined by

$$p_X(\chi) = \operatorname{Sup}_{\psi \in X} |\langle\psi, \chi\rangle|$$

and to each p the subset

$$X_p = \{\phi: |\langle\phi, \chi\rangle| \leqq p(\chi), \textit{ all } \chi \text{ in } S\}$$

of Hom (S, C).

Proof. p_X for an arbitrary compact subset X of Hom (S, C) will satisfy conditions 2 and 3. If X is an $M(A)$, p_X will furthermore satisfy 1 since $p_X(\chi)$ is the spectral radius of χ and G is a subset of the maximal ideal space of A. Conversely, if p satisfies conditions 1, 2 and 3, it is easy to check that X_p is $M(A_p)$, where A_p is the algebra of all functions on G that are sums of absolutely convergent series $\sum \alpha_\chi \chi$, with χ in S and $\sum |\alpha_\chi| p(\chi)$ finite, normed with $\|f\| = \sum |\alpha_\chi| p(\chi)$ if $f = \sum \alpha_\chi \chi$. It remains to prove that the maps $p \to X_p$ and $X \to p_X$ are mutual inverses. Let p be any function on S satisfying 1, 2 and 3. Then $p(\chi)$ is the norm of χ in the algebra A_p constructed above, while $p_{(X_p)}(\chi)$ is the spectral radius of χ in A_p. But the spectral radius of χ and the norm of χ in A are identical because of condition 3 and the Gelfand formula for spectral radius. Thus $p_{(X_p)} = p$. Now let X be some $M(A)$. $X_{(p_X)}$ is simply $C_M(X)$. But Theorem 2.1 shows that $C_M(X) = X$ so that $X_{(p_X)} = X$. Thus we have shown that the maps $p \to X_p$ and $X \to p_X$ are mutual inverses and the theorem is proved.

3. **Semisimplicity of completions.** We preserve the notation and definitions of §1. The following result will be used in the establishment of the classification theorem of the next section.

THEOREM 3.1. *Let n be a G-invariant norm on $P(S)$ that induces a topology stronger than uniform convergence on G. Then the completion of $P(S)$ under n is a semisimple algebra.*

Proof. Let A^n be the completion of $P(S)$ under n. The natural injection mapping $P(S) \to C(G)$ is continuous if $P(S)$ is supplied with the topology given by n and $C(G)$ the topology of uniform convergence. Therefore it extends to a continuous linear transformation $I: A^n \to C(G)$. Somewhat more than the assertion of the theorem will be proven, namely that I is one-one. For each σ in G, the translation operator T_σ is an isometry of $P(S)$ and thus can be extended to be an isometry of A^n which will be denoted by the same symbol. For every χ in S, $T_\sigma(\chi) = \chi(\sigma)\chi$ so that the map $\sigma \to T_\sigma(\chi)$ of G into A^n is continuous. Thus for every f in $P(S)$, the map $\sigma \to T_\sigma(f)$ is continuous. If k is any element in A^n, let $\{f_i\}$ be a sequence in $P(S)$ converging to k. Then the map $\sigma \to T_\sigma(k)$ is a uniform limit of the continuous maps $\sigma \to T_\sigma(f_i)$ and is thus continuous. The proof will be completed by computing the Fourier coefficients of the vector valued function $\sigma \to T_\sigma(k)$, in particular by proving the validity of the formula

$$\int_G T_\sigma(k)\chi^{-1}(\sigma)d\sigma = \left[\int_G [I(k)](\sigma)\chi^{-1}(\sigma)d\sigma\right]\chi \tag{3.1}$$

for each k in A^n and each χ in S, where $d\sigma$ is Haar measure on G and the integral on the left is an A^n valued integral. The formula is correct if k is an element of S because of orthonormality of characters. Since both sides of the formula are linear in k, it is valid for k in $P(S)$. Finally both sides are continuous in k so the formula must be valid for all k in A^n as it is valid for $P(S)$ which is dense. Now let k be any element in A^n with $I(k) = 0$. Because of (3.1), all of the Fourier coefficients of the function $\sigma \to T_\sigma(k)$ are zero. It is known (see [11, 2.5]) that this can only happen if $T_\sigma(k) = 0$ for all σ in G and thus k must be zero. This completes the proof of Theorem 3.1.

The idea of the proof is essentially due to Silov who in [11, 4.6] uses it to obtain a more special result. The method is of even greater generality than needed here, for example G need not be abelian, the norm need not satisfy $n(fg) \leqq n(f)n(g)$ and what is more significant n can be assumed to be stronger than topologies different than uniform convergence, for example L_p or the topology of distributions.

Denote by A_n the subalgebra of $C(G)$ that is the isomorphic image of A^n under the map I. It is clear that A_n satisfies Axioms 1, 2, 3, 4 and 5′ of the previous section and so is a homogeneous algebra.

As a byproduct of Theorem 3.1 we obtain a meager bit of information concerning norms on $C(X)$. Kaplansky in [7] proved that any norm on a $C(X)$ must be larger than the uniform norm and has asked whether it must actually be equivalent.

THEOREM 3.2. *Let n be a G-invariant norm on $C(G)$ and which is such that for each f in $C(G)$, the map $\sigma \rightarrow T_\sigma(f)$ is continuous if $C(G)$ is given the topology induced by n. Then n is equivalent to the uniform norm.*

Proof. We shall denote by $C(G)_n$ the space $C(G)$ supplied with the topology induced by the norm n and by $C(G)_u$ the same space supplied with the uniform topology. Let A^n be the completion of $C(G)_n$. The identity map $C(G)_n \rightarrow C(G)_u$ is continuous because of [7, Theorem 6.2] and thus can be extended to a map $J: A^n \rightarrow C(G)_u$. Because of [11, 2.7] and the fact that the maps $\sigma \rightarrow T_\sigma(f)$ of G into $C(G)_n$ are assumed continuous, $P(\hat{G})$ is dense in $C(G)_n$. Thus A^n can be identified with the completion of $P(\hat{G})$ and the map J becomes the map I constructed in the proof of Theorem 3.1 and proved there to be one-one. J can only be one-one if $A^n = C(G)_n$ so that $C(G)_n$ is already complete. Then the open mapping theorem (see for example [8, p. 18]) applied to the identity map $C(G)_n \rightarrow C(G)_u$ shows that it is a homeomorphism and therefore n is equivalent to the uniform norm.

4. **Classification of homogeneous algebras.** In this section the results of §§2 and 3 are used to obtain a type of classification of all homogeneous algebras.

Let n be a G-invariant norm on $P(S)$ of the type considered in the previous section; that is, n induces a topology on $P(S)$ that is stronger than the topology of uniform convergence on G. Then for any σ in G, the map $f \rightarrow f(\sigma)$ is a continuous multiplicative linear functional of $P(S)$ so that $|f(\sigma)| \leqq n(f)$ for each f in $P(S)$ and in particular $1 \leqq n(\chi)$ for each χ in S.

The converse of this depends on Theorem 1.3.

LEMMA 4.1. *Let n be a G-invariant norm on $P(S)$ that satisfies $n(\chi) \geqq 1$ for all χ in S. Then the topology induced by n on $P(S)$ is stronger than that of uniform convergence on G.*

Proof. Recall that for each σ in G, ϕ_σ is the element in Hom (S, C) defined by $\langle \phi_\sigma, \chi \rangle = \chi(\sigma)$ so that $\langle \phi_\sigma, f \rangle = f(\sigma)$ for any f in $P(S)$. Because $n(\chi) \geqq 1$ for all χ, all of the ϕ_σ are in the subset $\{\phi: |\langle \phi, \chi \rangle| \leqq n(\chi), \text{all } \chi \text{ in } S\}$ of Hom (S, C).

By Theorem 1.3 this set is identical with $\{\phi: |\langle\phi, f\rangle| \leqq n(f), \text{ all } f \text{ in } P(S)\}$ so that $|f(\sigma)| = |\langle\phi_\sigma, f\rangle| \leqq n(f)$ for all σ in G and all f in $P(S)$. This completes the proof of the lemma.

Thus if n is a G-invariant norm on $P(S)$ with $n(\chi) \geqq 1$ for all χ in S, the considerations of the previous section can be applied and a homogeneous algebra A_n that is essentially the completion of $P(S)$ under n is obtained. Furthermore if n and m are two such norms, A_n is the same homogeneous algebra as A_m if and only if n and m are equivalent. Conversely, if A is any homogeneous algebra on G generated by S and n is a homogeneous norm for A, since $P(S)$ is dense in A, A_n will be identical with A. Thus we have the classification theorem.

THEOREM 4.2. *There is a natural one-one correspondence between the collection of all homogeneous algebras on G generated by S and the collection of all equivalence classes of G-invariant norms on P(S) that satisfy* $n(\chi) \geqq 1$ *for all* χ *in S.*

5. **Algebras of almost periodic functions.** In this section we shall apply the results obtained thus far to a particular class of algebras of almost periodic functions. The algebras will consist of functons analytic in a strip of the complex plane (which may degenerate to the real axis) and the strip will form a dense subset of the maximal ideal space.

We shall denote by $[\alpha, \beta]$ the closed interval $\{x: \alpha \leqq x \leqq \beta\}$ of the reals and by $[[\alpha, \beta]]$ the closed strip $\{s+it: \alpha \leqq t \leqq \beta\}$ of the complex plane. α and β will always satisfy $\alpha \leqq 0 \leqq \beta$ and we do not exclude the case of both α and β being 0 and $[[\alpha, \beta]]$ being the real axis. For each real λ, χ_λ is the function defined by $\chi_\lambda(z) = e^{i\lambda z}$. $P(\hat{R})$ will denote the class of all trigonometric polynomials, that is, linear combinations of the χ_λ.

A Banach algebra B, supplied with a norm n, and that consists of bounded continuous functions on $[[\alpha, \beta]]$ analytic at each interior point of $[[\alpha, \beta]]$ will be called an $[[\alpha, \beta]]$ *homogeneous algebra* if it satisfies the conditions:

B1. Addition and multiplication are pointwise addition and multiplication on $[[\alpha, \beta]]$.

B2. B contains all of the χ_λ and $n(\chi_\lambda)$ a measurable function of λ.

B3. The topology of B is stronger than that of pointwise convergence on $[[\alpha, \beta]]$.

B4. B and n are translation invariant, that is, $T_a(f)$ is in B for each f in B and real number a (where $[T_a(f)](b) = f(a+b)$) and $n(T_a(f)) = n(f)$.

B5. $\{T_a(f): \text{all real } a\}$ is a totally bounded subset of B (that is, it has compact closure) for each f in B.

Since each f in B is analytic at interior points of $[[\alpha, \beta]]$, it is determined by its restriction to the real axis R. Because of B3 and B4 the topology of B is stronger than that of uniform convergence on R (actually uniform convergence on all of $[[\alpha, \beta]]$ as an application of the closed graph theorem shows). Then, due to B5, the restriction of any f in B to R is an almost periodic function on R.

Let G be the almost periodic compactification of the group R. G can be described as the essentially unique compact group containing R as a dense subgroup and which is such that every continuous almost periodic function on R can be extended to be continuous on all of G. In what follows we shall use well known properties of G (all of which can be found in Chapter 7 of [12]) without further comment.

For any f in B, let $\bar{f}$ be the function on G obtained by first restricting f to R and then extending the almost periodic function thus obtained to all of G. Denote by $\bar{B}$ the set of functions $\{\bar{f}: f \text{ in } B\}$. $\bar{B}$ is an algebra of functions and the map $f \rightarrow \bar{f}$ is an algebra isomorphism and isometry of B onto $\bar{B}$ if $\bar{B}$ is supplied with the norm $\bar{n}$ defined by $\bar{n}(\bar{f}) = n(f)$. We prove next that $\bar{B}$ is a homogeneous algebra on G so that the previous results concerning homogeneous algebras can be applied to the isomorphic algebra B.

LEMMA 5.1. *$\bar{B}$ is a homogeneous algebra.*

Proof. Axiom A1 of §2 is satisfied because of known properties of the almost periodic compactification. A2 is satisfied since all of the $\bar{\chi}_\lambda$ are in $\bar{B}$ and they are all of the characters of G. Because n is stronger than uniform convergence on R, $\bar{n}$ is stronger than uniform convergence on G so that A3 is satisfied. If a is in R and $\bar{f}$ in $\bar{B}$, $T_a(\bar{f})$ is in $\bar{B}$ since it is the image of $T_a(f)$. If σ is any element in G, let $\{a_\gamma\}$ be any directed subset of R that converges to σ. Then the directed subset $\{T_{a_\gamma}(\bar{f})\}$ of $\bar{B}$ will have a cluster point in $\bar{B}$ because of B5 and the completeness of $\bar{B}$. Because $\bar{f}$ is continuous and $\bar{n}$ is stronger than uniform convergence on G, this cluster point must be $T_\sigma(\bar{f})$ and A4 is proved. Note that as a consequence of our argument, $\bar{n}(T_\sigma(\bar{f})) = \bar{n}(\bar{f})$ for all $\bar{f}$ in $\bar{B}$ and all σ in G so that $\bar{n}$ is a G-invariant norm. At this point we could appeal to the approximation theorem of [2] to verify A5′. Instead we proceed directly to verify A5. Let $\bar{f}$ be any function in $\bar{B}$. Since $\bar{n}$ is G-invariant, it suffices to find for each $\epsilon > 0$ a neighborhood U_ϵ of the identity in G which is such that $n(\bar{f} - T_\sigma(\bar{f})) \leqq \epsilon$ if σ is in U_ϵ. By B5 there is a finite set $\{a_1, \cdots, a_r\}$ of real numbers so that the set $\{T_{a_1}(\bar{f}), \cdots, T_{a_r}(\bar{f})\}$ is $\epsilon/2$ dense in $\{T_a(\bar{f}): a \text{ in } R\}$. Let V_i be the subset $\{a: \bar{n}(T_a(\bar{f}) - T_{a_1}(\bar{f})) \leqq \epsilon/2\}$ of R. Then R is the union of the V_i and since R is dense in G, G is the union of the closures of the V_i. If σ is some element in the closure of V_i, there is a directed subset $\{a_\gamma\}$ of V_i which converges to σ. We have seen that $T_\sigma(\bar{f})$ is a cluster point of the $T_{a_\gamma}(\bar{f})$ so we must have $n(T_\sigma(\bar{f}) - T_{a_i}(\bar{f})) \leqq \epsilon/2$ for each σ in the closure of V_i. Since G is the union of the closures of the V_i, some one must contain an open set W and if U_ϵ is any translate of W that contains the unit of G, $n(\bar{f} - T_\sigma(\bar{f})) \leqq \epsilon$ for each σ in U_ϵ as demanded.

$\bar{B}$ is now known to be a homogeneous algebra so we may apply our previous results to obtain information concerning B. Because of A5′, $P(\hat{G})$ is dense in $\overline{B}$. Since $P(\hat{G})$ is the image of $P(\hat{R})$ under the canonical isomorphism $B \rightarrow \overline{B}$ we have an approximation theorem for $[[\alpha, \beta]]$ homogeneous algebras

THEOREM 5.2. *$P(\hat{R})$ is dense in B; that is, every function in B can be approximated arbitrarily well in the topology of B by trigonometric polynomials.*

Note that thus far we have not really used the assumption that the functions in B must be analytic at interior points of $[[\alpha, \beta]]$, but only its weaker consequence that they are determined by their restriction to R. Theorem 5.2 shows that if we make only the weaker assumption the analyticity comes free as the functions in $P(\hat{R})$ are analytic and the topology of B is stronger than uniform convergence on $[[\alpha, \beta]]$.

Our next task is to identify the maximal ideal space of B. Let $\bar{p}$ be the spectral radius of $\overline{B}$. Since $\overline{B}$ is a homogeneous algebra its maximal ideal space can be identified with the subset $M(\overline{B})$ of Hom $(\hat{G}, C)$ that by Theorem 2.1 and the comment preceding consists of the $\bar{p}$-semicharacters of $\hat{G}$. $\hat{G}$ consists of the $\bar{\chi}_\lambda$ that multiply according to the rule $\bar{\chi}_\lambda \bar{\chi}_\mu = \bar{\chi}_{\lambda+\mu}$ so that the $\bar{p}$-semicharacters of $\hat{G}$ are in one-one correspondence with the collection of all complex valued functions on R that satisfy $\phi(\lambda+\mu) = \phi(\lambda)\phi(\mu)$, $\phi(0) = 1$ and $|\phi(\lambda)| \leqq q(\lambda)$ where q is defined by $q(\lambda) = \bar{p}(\bar{\chi}_\lambda)$. This collection of functions will be denoted by $M(B)$. It will not be hard to identify as q is of a very simple nature.

LEMMA 5.3. *If B is an $[[\alpha, \beta]]$ homogeneous algebra with norm n, $q(\lambda) = e^{-a\lambda}$ for $\lambda \geqq 0$ and $q(\lambda) = e^{-b\lambda}$ for $\lambda \leqq 0$ with*

$$a = -\lim_{r\rightarrow\infty} \frac{1}{r} \log n(\chi_r) \quad \text{and} \quad b = -\lim_{r\rightarrow -\infty} \frac{1}{r} \log n(\chi_r). \tag{5.1}$$

Also $a \leqq \alpha$ and $b \geqq \beta$.

Proof. Since $\bar{n}(\bar{\chi}_\lambda) = n(\chi)$ is measurable in λ,

$$q(\lambda) = \bar{p}(\chi_\lambda) = \lim_{r\rightarrow\infty} [\bar{n}(\chi_{r\lambda})]^{1/r} \tag{5.2}$$

will be also. Since $\bar{p}$ is a spectral radius, $q(\lambda+\mu) \leqq q(\lambda)q(\mu)$ and $q(2\lambda) = q(\lambda)^2$. Thus the function $f = \log(q)$ is subadditive, satisfies $f(2\lambda) = 2f(\lambda)$ and is measurable. It is known that for a measurable subadditive function

$$\lim_{\lambda\rightarrow\infty} \frac{f(\lambda)}{\lambda} \quad \text{and} \quad \lim_{\lambda\rightarrow -\infty} \frac{f(\lambda)}{\lambda}$$

must exist (see for example [6, Th. 6.11.1]). Since $f(2\lambda) = 2f(\lambda)$ this is possible only if f is linear on both half lines $(-\infty, 0]$ and $[0, +\infty)$. Thus q must have the form claimed, the formulas for a and b follow from (5.2) and it remains to verify that $a \leqq \alpha$ and $b \geqq \beta$. Since B is an $[[\alpha, \beta]]$ homogeneous algebra, the maps $f \rightarrow f(i\alpha)$ and $f \rightarrow f(i\beta)$ are continuous multiplicative linear functionals on B. If $\lambda>0$ $e^{-\alpha\lambda}=\chi_\lambda(i\alpha)\leqq \bar{p}(\bar{\chi}_\lambda)=q(\lambda)=e^{-a\lambda}$ so $a\leqq\alpha$. Similarly if $\lambda<0$, $e^{-\beta\lambda}=\chi_\lambda(i\beta)\leqq \bar{p}(\bar{\chi}_\lambda)=q(\lambda)=e^{-b\lambda}$ so $b\geqq\beta$.

Note that we have used for the first time the measurability of $n(\chi_\lambda)$.

Now if ϕ is a function in $M(B)$, $\phi(\lambda)\neq 0$ for all λ since $1=\phi(0)=\phi(\lambda-\lambda)=\phi(\lambda)\phi(-\lambda)$. Thus the factorization $\phi=|\phi|(\phi/|\phi|)$ makes sense. We shall identify both factors separately. If h is defined by $h=\log|\phi|$, $h(\lambda+\mu)=h(\lambda)+h(\mu)$, $h(\lambda)\leqq a\lambda$ if $\lambda\geqq 0$ and $h(\lambda)\leqq -b\lambda$ if $\lambda\leqq 0$. In particular h is bounded near 0 and it is well known that such an h must be linear. Then $h(\lambda)=-t\lambda$ for some t with $a\leqq t\leqq b$ and thus $|\phi(t)|=e^{-t\lambda}$. The other factor $\phi/|\phi|$ is a multiplicative map of R into the unit circle and thus a character of the discrete reals. It is known that every character of the discrete reals is of the form $\lambda\rightarrow\bar{\chi}_\lambda(\sigma)$ for some σ in G. Thus we have shown that every ϕ in $M(B)$ is of the form $\phi(\lambda)=e^{-t\lambda}\bar{\chi}_\lambda(\sigma)$ for some t satisfying $a\leqq t\leqq b$ and some σ in G. Conversely, any ϕ of this form will be in $M(B)$ so we have shown that $M(B)$ (and thus the maximal ideal space of B) can be identified with the product $G\times[a, b]$. By working back through the various canonical identifications it is seen that the point (σ, t) of $G\times[a, b]$ corresponds to the multiplicative linear functional $\phi_{\sigma,t}$ of B that satisfies $\phi_{\sigma,t}(\chi_\lambda)=\bar{\chi}_\lambda(\sigma)\chi_\lambda(it)$ (so that if σ is in the subgroup R of G, $\phi_{\sigma,t}(\chi_\lambda)=\chi_\lambda(\sigma+it)$). Using this it is easy to see that this correspondence between $G\times[a, b]$ and the maximal ideal space of B is a homeomorphism. Summarizing:

THEOREM 5.4. *Let B be an $[[\alpha, \beta]]$ homogeneous algebra with norm n. Then each multiplicative linear functional of B is obtained by choosing some σ in G and some t in $[a, b]$, where a and b are determined by* (5.1), *and extending the map $\chi_\lambda\rightarrow\bar{\chi}_\lambda(\sigma)\chi_\lambda(it)$ by linearity to $P(\hat{R})$ and then by continuity to B. This correspondence between multiplicative linear functionals and pairs (σ, t) is a homeomorphism between the maximal ideal space of B and the product space $G\times[a, b]$.*

We have observed above that if s is in R and t in $[a, b]$ the multiplicative linear functional $\phi_{s,t}$ that corresponds to the point (s, t) in $G\times[a, b]$ satisfies $\phi_{s,t}(f)=f(s+it)$ for f any χ_λ and thus for each f in $P(\hat{R})$. As a consequence, if $s+it$ is a point in the strip $[[\alpha, \beta]]$, $\phi_{s,t}(f)=f(s+it)$ for all f in B as $P(\hat{R})$ is dense in B and the topology of B is stronger than pointwise convergence on $[[\alpha, \beta]]$.

Recall now that $a\leqq\alpha$ and $b\geqq\beta$ so the strip $[[a, b]]$ is at least as large as the strip $[[\alpha, \beta]]$. It is now clear that every function f in B has a canonical extension to a function $\tilde{f}$ having domain $[[a, b]]$; simply define $\tilde{f}$ by $\tilde{f}(s+it)=\phi_{s,t}(f)$ for each $s+it$ in $[[a, b]]$. Denote by $\tilde{B}$ the class of $\tilde{f}$ so obtained from f in B. If $\tilde{B}$ is supplied with the norm $\tilde{n}$ defined by $\tilde{n}(\tilde{f})=n(f)$, it is easy to check that $\tilde{B}$ is an $[[a, b]]$ homogeneous algebra and that the map $f\rightarrow\tilde{f}$ is an algebra isomorphism and an isometry.

We shall call $\tilde{B}$ the *proper extension* of B and B will be called a *proper*

$[[\alpha, \beta]]$ homogeneous algebra if B and $\tilde{B}$ are identical, or equivalently if $a=\alpha$ and $b=\beta$.

THEOREM 5.5. *Let B be an $[[\alpha, \beta]]$ homogeneous algebra. Then the following are equivalent:*

1. *B is a proper $[[\alpha, \beta]]$ homogeneous algebra.*
2. *The strip $[[\alpha, \beta]]$ is dense in the maximal ideal space of B.*
3. *If f is in B and F is a function analytic in a neighborhood of the closure of the range of f, the composite function $F(f)$ is in B.*
4. *If f is in B and $|f(z)| \geqq K > 0$ for all z in $[[\alpha, \beta]]$, then $1/f$ is in B.*
5. *If f is analytic and periodic in some open strip containing $[[\alpha, \beta]]$ then f is in B.*
6. *If f is in B, the spectral radius of f is*

$$\operatorname*{Sup}_{z\in[[\alpha,\beta]]} |f(z)|.$$

Proof. 1 implies 2: Under our identification of the maximal ideal space of B with $G\times[a, b]$, we have seen that the maximal ideal corresponding to the point $s+it$ of the strip $[[\alpha, \beta]]$ is identified with the point (s, t) of $G\times[a, b]$. If B is proper, $[a, b]$ is $[\alpha, \beta]$ and since R is dense in G, these points (s, t) will be dense in $G\times[a, b]$. 2 implies 3: If the strip $[[\alpha, \beta]]$ is dense in the maximal ideal space of B, the closure of the range of f will be identical with the spectrum of f. Then 3 follows from the well known theorem concerning analytic functions of Banach algebra elements (see for example [8, p. 78]). 3 implies 4: 4 is the special case of 3 with $F(z)=1/z$. 3 implies 5: $f(z)=\sum a_n e^{in\lambda z}$ for some real λ and with the series converging absolutely in some strip containing $[[\alpha, \beta]]$. Then $f=F(\chi_\lambda)$ with $F(z)=\sum a_n z^n$ and thus 5 is also a special case of 3. 4 implies 1: Let w be any point in the complex plane not in $[[\alpha, \beta]]$. Then if f is defined by $f(z)=e^{iz}-e^{iw}$, f is in B and is bounded away from 0 in $[[\alpha, \beta]]$ so $1/f$ is in B. w cannot be in the strip $[[a, b]]$ since $1/f$ can be extended to be continuous and analytic at interior points of $[[a, b]]$. Since w was any point not in $[[\alpha, \beta]]$, $a=\alpha$ and $b=\beta$ so B is proper. 5 implies 1: Same proof using the fact that $1/f$ is analytic and periodic in a strip containing $[[\alpha, \beta]]$. 2 implies 6: The spectral radius of f is the sup of $|\phi(f)|$ over all multiplicative linear functionals of B, and those of the form $f\to f(z)$ for z in $[[\alpha, \beta]]$ are a dense subset. 6 implies 1: If $[[a, b]]$ is larger than $[[\alpha, \beta]]$, the spectral radius of some χ_λ will be larger than

$$\operatorname*{Sup}_{z\in[[\alpha,\beta]]} |\chi_\lambda(z)|.$$

Condition 4 should be compared with Corollary 2 of Theorem 7 of [3] which is at the same time more special and more precise. It states that if f is analytic almost periodic in an open strip and $f(z)\neq 0$ in that strip, then $1/f$ is analytic almost periodic in the same strip. Since we are dealing with closed strips we can not weaken Condition 4 to $f(z) \neq 0$ for z in $[[\alpha, \beta]]$. Condition 5 can be improved somewhat but it is not correct that every function analytic almost periodic in a strip containing $[[\alpha, \beta]]$ is in B. Bohr on p. 279 of [3] gives (for a different purpose) an example of a function which (if the complex plane is rotated 90°) is analytic almost periodic in a strip containing the real axis $[[0, 0]]$ but which is not in the proper $[[0, 0]]$ homogeneous algebra of all f of the form $f(x) = \sum a_n e^{i\lambda_n x}$ with $\sum |a_n| < \infty$.

Let us point out that if $n(\chi_\lambda)$ is not assumed to be measurable, all of the preceding analysis may (and actually does in cases) fail. In particular, if B is an $[[\alpha, \beta]]$ homogeneous algebra except for the condition that $n(\chi_\lambda)$ be measurable, it may be impossible to extend the functions in B to a larger strip while at the same time $[[\alpha, \beta]]$ will be far from dense in the maximal ideal space. Actually what is needed for the preceding is the boundedness of $n(\chi_\lambda)$ for small λ, and this is a consequence of measurability.

We proceed now to determine which norms on $P(\hat{R})$ come from $[[\alpha, \beta]]$ homogeneous algebras and derive an analog of Theorem 4.2. A norm m on $P(\hat{R})$ will be called *R-invariant* if $m(T_a(f)) = m(f)$ for all a in R and f in $P(\hat{R})$ and *measurable* if $m(\chi_\lambda)$ is a measurable function of λ. Let m be any R-invariant measurable norm on $P(\hat{R})$ that satisfies $m(\chi_\lambda) \geqq 1$ for all χ_λ. Recall that if f is an almost periodic function on R, $\bar{f}$ is its extension to G. Define $\bar{m}$ on $P(\hat{G})$ by $\bar{m}(\bar{f}) = m(f)$. $\bar{m}$ is a G-invariant norm on $P(\hat{G})$ with $\bar{m}(\bar{\chi}_\lambda) \geqq 1$ for each $\bar{\chi}_\lambda$ so that by Lemma 4.1, $\bar{m}$ is stronger than unform convergence on G (and thus m is stronger than uniform convergence on R). In §3 we have seen that the completion of $P(\hat{G})$ under the norm $\bar{m}$ can be identified with a homogeneous algebra on G. If B is the restriction to R of the functions of this algebra (or equivalently, the completion of $P(\hat{R})$ under m), it is easy to see that B is a $[[0, 0]]$ homogeneous algebra. If $\tilde{B}$ is the proper extension of B and $\tilde{m}$ its norm, $m(f) = \tilde{m}(\bar{f})$ for all f in $P(\hat{R})$. Thus we have shown that every R-invariant measurable norm on $P(\hat{R})$ with $m(\chi_\lambda) \geqq 1$ for all λ arises by restricting the norm of some proper $[[\alpha, \beta]]$ homogeneous algebra to $P(\hat{R})$. Conversely if m is the restriction of the norm of a proper $[[\alpha, \beta]]$ homogeneous algebra to $P(\hat{R})$, it is by definition R-invariant and measurable and satisfies $m(\chi_\lambda) \geqq 1$ for all λ since R is a subset of the maximal ideal space. It is clear that the association between proper algebras and norms is one-one. Summarizing:

THEOREM 5.6. *There is a natural one-one correspondence between proper* $[[\alpha, \beta]]$ *homogeneous algebras and measurable R-invariant norms on* $P(\hat{R})$ *that satisfy* $n(\chi_\lambda) \geqq 1$ *for all* λ. *The* $[[\alpha, \beta]]$ *corresponding to a given n is determined by* (5.1).

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