A FINITE RECURSION FORMULA FOR THE COEFFICIENTS IN ASYMPTOTIC EXPANSIONS[1]

BY

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1. **Introduction.** In recent years the asymptotic behavior in the neighborhood of infinity of entire functions, $G(z)$, which are defined by their power series developments

$$G(z) = \sum_{n=0}^{\infty} g(n)z^n,$$

have been studied by W. B. Ford [2], E. M. Wright [9; 10; 11], H. K. Hughes [3; 4], and J. H. B. Kemperman [6]. To apply their theory it is required that the coefficient $g(n)$ may be considered as a function $g(w)$ of a complex variable and as such admits an asymptotic factorial expansion in every right half-plane. From their investigations it follows that the constants occurring in this asymptotic expansion of $g(w)$ are precisely the constants which occur in the asymptotic expansion of $G(z)$.

It is known from a lemma which appears in [2; 4; 11] that the function

$$(1.1) \qquad {}_pg_q(w) = \prod_{i=1}^{p} \Gamma(w + \sigma_i) \Big/ \sum_{j=0}^{q} \Gamma(w + \rho_j)$$

admits the required factorial expansion. Here p and q are non-negative integers with $p \leqq q$; the σ_i and ρ_j denote arbitrary complex parameters. The complete determination of the asymptotic expansion for large $|z|$ of the important class of functions

$${}_pG_q(z) = \sum_{n=0}^{\infty} {}_pg_q(n)z^n$$

is thus dependent upon a procedure for computing the constants occurring in the expansion of ${}_pg_q(w)$. In an earlier paper [7] the fact that ${}_pg_q(w)$ satisfies a first order difference equation was used to obtain an inductive formula for the coefficients depending on all the previously calculated coefficients.

This paper deals with a simpler formula for calculating these coefficients. For this purpose we introduce a function $\phi(t)$ defined as an inverse transform

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of ${}_pg_q(w)$ and use the difference equation to obtain a differential equation of Fuchsian type satisfied by $\phi(t)$. The expansion of $\phi(t)$ about one of the regular singular points of the differential equation will be determined and shown to generate the constants in question. By this procedure we shall obtain a finite recursion formula of fixed length q for the coefficients.

2. **The factorial expansion.** The lemma cited in §1 is now stated as a theorem. The proof which is based on Stirling's formula is omitted.

THEOREM 2.1. *Let h be any real number and let M be any non-negative integer. Then in the half-plane* $\mathrm{Re}(w) > -h$ *with* $|w|$ *large,* ${}_pg_q(w)$ *admits the asymptotic development*

$$\begin{aligned} {}_pg_q(w) &= ((2\pi)^{1/2})^{1-\alpha}\alpha^{\alpha w+\beta-1/2} \\ &\quad\cdot\left[\sum_{m=0}^{M}\frac{c_m}{\Gamma(\alpha w+\beta+m)}+O\left(\frac{1}{\Gamma(\alpha w+\beta+M+1)}\right)\right], \end{aligned}\tag{2.1}$$

where $c_0 = 1$ *and*

$$\alpha = q+1-p; \qquad \beta = \sum_{j=0}^{q}\rho_j - \sum_{i=0}^{p}\sigma_i + (1-\alpha)/2.\tag{2.2}$$

In the subsequent investigation it is convenient to introduce the auxiliary function

$$H(x) = ((2\pi)^{1/2})^{\alpha-1}\alpha^{-x-q-5/2}\Gamma(x)\ {}_pg_q\left(\frac{x-\beta+q+3}{\alpha}\right),\tag{2.3}$$

where x is a complex argument. From the multiplication theorem of Gauss [8, p. 240] it follows that $H(x)$ may be written in the form

$$H(x) = \alpha^{-q-3}\ {}_pg_q\left(\frac{x-\beta+q+3}{\alpha}\right)\prod_{r=0}^{\alpha-1}\Gamma\left(\frac{x+r}{\alpha}\right).\tag{2.4}$$

We note that as long as x is confined to an arbitrary right half-plane,

$$\Gamma(x)/\Gamma(x+a) \sim x^{-a} \quad \text{for} \quad |x| \text{ large}.$$

This is true for any complex number a by Stirling's formula. It therefore follows from (2.4), (1.1) and (2.2) that if x is restricted to any right half-plane then

$$H(x) \sim x^{-q-3} \quad \text{for} \quad |x| \text{ large}.\tag{2.5}$$

Moreover, it follows from (2.3) and Theorem 2.1 that in any right half-plane $\mathrm{Re}(x) > -h$, $H(x)$ has the asymptotic development

$$H(x) = \sum_{m=0}^{M}\frac{c_m\Gamma(x)}{\Gamma(x+q+3+m)} + O\left(\frac{\Gamma(x)}{\Gamma(x+q+4+M)}\right).\tag{2.6}$$

3. Definition of $\phi(t)$. We introduce the notation

$$\lambda_0 = \max\left\{0, \max_i \operatorname{Re}(\beta - q - 3 - \alpha\sigma_i)\right\}, \tag{3.1}$$
$$\lambda = \text{any real number greater than } \lambda_0.$$

It follows from (2.3) and (1.1) that $H(x)$ is single-valued and regular in the half-plane $\operatorname{Re}(x) > \lambda_0$. From (2.5) we have $H(x) = O(x^{-q-3})$ for x in any right half-plane and $|x|$ large. Hence for real values ξ the inversion integral

$$U(\xi) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{\xi x}H(x)dx$$

converges. It follows from a well-known theorem [1, p. 126] that $H(x)$ is represented by the Laplacian integral

$$H(x) = \int_0^\infty e^{-\xi x}U(\xi)d\xi,$$

provided $\operatorname{Re}(x) > \lambda_0$. Making the transformation $\xi = \log t^{-1}$, $t = e^{-\xi}$, these two relations become

$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} t^{-x}H(x)dx, \qquad 0 < t \leqq 1, \tag{3.2}$$

and

$$H(x) = \int_0^1 t^{x-1}\phi(t)dt, \qquad \operatorname{Re}(x) > \lambda_0, \tag{3.3}$$

where we have replaced $U(\log t^{-1})$ by $\phi(t)$.

We now derive upper bounds for $\phi(t)$ at both $t=1$ and $t=0+$.

THEOREM 3.1. *The function $\phi(t)$ defined by* (3.2) *is $O(t^{-\lambda})$ as $t \to 0+$, where λ is defined by* (3.1). *Moreover, $\phi(t) = O((1-t)^{q+2})$ near $t=1$.*

Proof. Let v be real and set $x = \lambda + iv$, then

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-\lambda-iv}H(\lambda + iv)dv.$$

From (2.5) there exists a constant M such that

$$|H(\lambda + iv)| < M(\lambda^2 + v^2)^{-(q+3)/2} \tag{3.4}$$

for $|\lambda + iv|$ large enough, $\lambda > \lambda_0$. Since $H(\lambda + iv)$ has no poles for $\lambda > \lambda_0$ it follows that (3.4) holds, for $\lambda \geqq \lambda_0 + \epsilon$ (for any $\epsilon > 0$) with M independent of λ and v. Thus

$$|\phi(t)| < \frac{M}{\pi} t^{-\lambda} \int_0^\infty (\lambda^2 + v^2)^{-(q+3)/2} dv.$$

We set $v^2 = \lambda^2 u$ and obtain

$$|\phi(t)| < \frac{M}{2\pi} \lambda^{-q-2} t^{-\lambda} \int_0^\infty (u + 1)^{-(q+3)/2} u^{-1/2} du,$$

from which the first assertion of the theorem follows since the integral converges. If we choose $\lambda = (1-t)^{-1}$ in the latter inequality we see that near $t=1$

$$\phi(t) = O((1 - t)^{q+2})$$

since $t^{-\lambda}$ is bounded for this special choise of λ.

4. **A differential equation satisfied $\phi(t)$.** The reader will readily note that by applying the recurrence relation $\Gamma(x+1) = x\Gamma(x)$ to (2.4) we immediately obtain the difference equation

$$P(x)H(x) = Q(x + \alpha)H(x + \alpha), \tag{4.1}$$

where

$$\begin{aligned} P(x) &= \prod_{i=1}^{p} (x - \beta + q + 3 + \alpha\sigma_i) \prod_{r=0}^{\alpha-1} (x + r), \\ Q(x) &= \prod_{j=0}^{q} (x - \alpha - \beta + q + 3 + \alpha\rho_j). \end{aligned} \tag{4.2}$$

Equations (3.2) and (4.1) may be combined to yield a linear differential equation satisfied by $\phi(t)$.

THEOREM 4.1. *Let $\theta = t(d/dt)$, then the function $\phi(t)$ defined by* (3.2) *is an integral of the linear differential equation*

$$L(y) \equiv \{t^\alpha Q(-\theta) - P(-\theta)\} y(t) = 0, \tag{4.3}$$

where $P(x)$ and $Q(x)$ are defined by (4.2).

Proof. The integral (3.2) which defines $\phi(t)$ may be differentiated $m(m \leqq q+1)$ times under the integral sign. To show this it suffices to show that

$$\int_{\lambda-i\infty}^{\lambda+i\infty} (-\theta)^m t^{-x} H(x) dx = \int_{\lambda-i\infty}^{\lambda+i\infty} t^{-x} x^m H(x) dx$$

is uniformly convergent. This is indeed true for $m \leqq q+1$ as follows from (2.5). Observing that

$$P(-\theta)t^{-x} = P(x)t^{-x} \quad \text{and} \quad Q(-\theta)t^{-x} = Q(x)t^{-x},$$

we have

$$\begin{aligned} P(-\theta)\phi(t) &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} P(-\theta)t^{-x}H(x)dx \\ &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} P(x)t^{-x}H(x)dx \\ &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} Q(x+\alpha)t^{-x}H(x+\alpha)dx \\ &= \frac{t^\alpha}{2\pi i}\int_{\lambda+\alpha-i\infty}^{\lambda+\alpha+i\infty} Q(x)t^{-x}H(x)dx \\ &= t^\alpha Q(-\theta)\phi(t) \end{aligned}$$

which proves the assertion.

In order to write $L(y)=0$ in the usual linear form we shall need the following lemma.

LEMMA 4.1. *Let $\theta = td/dt$ and let B_m $(m=0, 1, \cdots, q)$ be arbitrary complex parameters. Then the constants A_k in the operator identity*

$$\prod_{m=0}^{q} (\theta + B_m) = \sum_{k=0}^{q+1} A_k t^k \frac{d^k}{dt^k} \tag{4.4}$$

are given recursively by the formula

$$A_k = \frac{1}{k!}\prod_{m=0}^{q} (k + B_m) - \sum_{j=0}^{k-1} \frac{A_j}{(k-j)!}\,. \tag{4.5}$$

In particular,

$$A_{q+1} = 1; \qquad A_q = \sum_{m=0}^{q} B_m + q(q+1)/2. \tag{4.6}$$

Proof. It is clear that such constants exist. Observing that

$$\prod_{m=0}^{q} (\theta + B_m)t^x = t^x \prod_{m=0}^{q} (x + B_m)$$

and

$$t^k \frac{d^k}{dt^k} t^x = \frac{\Gamma(x+1)}{\Gamma(x+1-k)} t^x,$$

we have the identity

$$\prod_{m=0}^{q} (x + B_m) = \sum_{k=0}^{q+1} A_k \frac{\Gamma(x+1)}{\Gamma(x+1-k)}\,,$$

or

$$\frac{1}{\Gamma(x+1)}\prod_{m=0}^{q}(x+B_m) = \sum_{j=0}^{q+1}\frac{A_j}{\Gamma(k+1-j)}. \tag{4.7}$$

We put $x=k$ in (4.7) and obtain

$$\frac{1}{k!}\prod_{m=0}^{q}(k+B_m) = \sum_{j=0}^{k}\frac{A_j}{\Gamma(x-j+1)},$$

from which (4.5) follows. Relations (4.6) follow from a direct operation of $\prod_{m=0}^{m=q}(\theta+B_m)$ on a function $y(t)$.

THEOREM 4.2. *The differential equation $L(y)=0$, of which $\phi(t)$ is an integral, may be written in the form*

$$L(y) \equiv \sum_{k=\alpha}^{q+1} t^{k-\alpha}(E_k t^\alpha - C_k)y^{(k)}(t) + \sum_{k=0}^{\alpha-1} E_k t^k y^{(k)}(t) = 0, \tag{4.8}$$

where

$$E_k = \frac{1}{k!}\prod_{m=0}^{q}(\beta+k-p-2-\alpha\rho_m) - \sum_{j=0}^{k-1}\frac{E_j}{(k-j)!} \quad (0 \leqq k \leqq q+1), \tag{4.9}$$

and

$$C_k = \frac{1}{k!}\prod_{m=0}^{p-1}(\beta+k-q-3-\alpha\sigma_{m+1})\prod_{m=p}^{q}(k+p-m) - \sum_{j=\alpha}^{k-1}\frac{C_j}{(k-j)!} \quad (\alpha \leqq k \leqq q+1). \tag{4.10}$$

In particular,

$$E_{q+1} = 1; \qquad E_q = (q+1)(\beta-p-2) - \alpha\sum_{m=0}^{q}\rho_m + q(q+1)/2, \tag{4.11}$$

and

$$C_{q+1} = 1; \quad C_q = p(\beta-q-3) - \alpha\sum_{m=1}^{p}\sigma_m + \alpha(1-\alpha)/2 + q(q+1)/2. \tag{4.12}$$

Here α and β are the quantities defined by (2.2).

Proof. The differential equation $L(y)=0$ may be written in the form

$$t^\alpha\prod_{m=0}^{q}(\theta+\beta-p-2-\alpha\rho_m)y(t) = \prod_{m=0}^{p-1}(\theta+\beta-q-3-\alpha\sigma_{m+1})\prod_{m=p}^{q}(\theta+p-m)y(t). \tag{4.13}$$

In Lemma 4.1 put $B_m=\beta-p-2-\alpha\rho_m$ and obtain the following representation for the left side of (4.13):

$$t^\alpha \sum_{k=0}^{q+1} E_k t^k \frac{d^k}{dt^k} y(t), \tag{4.14}$$

where E_k $(k=0, 1, \cdots, q+1)$ is given by (4.9). If we put $B_m=\beta-q-3-\alpha\sigma_{m+1}$ for $m=0, 1, \cdots, p-1$ and $B_m=p-m$ for $m=p, \cdots, q$, then from Lemma 4.1 the right side of (4.13) takes the form

$$\sum_{k=0}^{q+1} C_k t^k \frac{d^k}{dt^k} y(t), \tag{4.15}$$

where

$$C_k = \frac{1}{k!}\prod_{m=0}^{p-1}(\beta + k - q - 3 - \alpha\sigma_{m+1})\prod_{m=p}^{q}(k + p - m) - \sum_{j=0}^{k-1}\frac{C_j}{(k-j)!}$$
$$(0 \leqq k \leqq q + 1).$$

By recursion $C_k=0$ for $k=0, 1, \cdots, \alpha-1$ and hence this latter formula reduces to (4.10). From (4.14) and (4.15) we have

$$t^\alpha \sum_{k=0}^{q+1} E_k t^k y^{(k)}(t) = \sum_{k=\alpha}^{q+1} C_k t^k y^{(k)}(t)$$

which yields (4.8). Finally, relations (4.11) and (4.12) are a consequence of (4.6) with the above choices of B_m.

5. **Properties of the differential equation.** Let the variable t in the differential equation $L(y(t))=0$ be extended to the complex plane. From (4.8) we see that $L(y)=0$ is of Fuchsian type [5]. If $p=0$ the regular singular points of $L(y)=0$ are $t=\infty$ and the α roots of unity for which $t^\alpha=1$. If $p\geqq 1$ the regular singular points are $t=0$, $t=\infty$ and the α roots of unity for which $t^\alpha=1$. In any case, there are no singularities for t real with $0<t<1$.

The point $t=1$ is a regular singularity of $L(y)=0$ for each $p\leqq q$. From (4.8) we see that the indicial equation of the differential equation at this point is

$$\mu(\mu-1)\cdots(\mu-q) + \frac{E_q - C_q}{\alpha}\mu(\mu-1)\cdots(\mu-q+1) = 0$$

which has the solutions

$$\mu = 0, 1, \cdots, q-1 \quad \text{and} \quad \mu = q+2. \tag{5.1}$$

Here we have used the relation $C_q-E_q=2\alpha$.

When $p=0$ the point $t=0$ is an ordinary point of $L(y)=0$. If $p\geqq 1$ then $t=0$ is a regular singularity of the differential equation and from (4.3) it follows that the indicial equation at the origin is

$$\prod_{k=1}^{p}(\mu+\beta-q-3-\alpha\sigma_k)\prod_{r=0}^{\alpha-1}(\mu-r)=0$$

which has the solutions

$$(5.2)\qquad \mu=0,1,\cdots,\alpha-1;\qquad \mu=-\beta+q+3+\alpha\sigma_k\qquad (1\leqq k\leqq p).$$

The function $\phi(t)$ has been defined by (3.2) for t real with $0<t\leqq 1$. Since $\phi(t)$ solves $L(y)=0$ for $0<t\leqq 1$ it follows from the location of the singularities of the differential equation that $\phi(t)$ is analytic for $0<t<1$. Hence, $\phi(t)$ may be extended analytically and the extension is regular except for possible branch points at the regular singular points of $L(y)=0$.

If we let K denote the distance from $t=1$ to the nearest of the other singularities of $L(y)=0$, then it follows from the theory of Frobenius [5] and (5.1) that the functions

$$y_0(t)=(1-t)^{q+2}\psi_{0,0}(t)\quad y_k(t)$$
$$=(1-t)^{q-k}\sum_{m=0}^{k}[\log(t-1)]^m\psi_{k,m}(t)\qquad(1\leqq k\leqq q),$$

form a fundamental system of solutions of $L(y)=0$ in the region $|t-1|<K$. Here the functions $\psi_{k,m}(t)$ are regular for $|t-1|<K$ and $\psi_{k,0}(1)\neq 0$ for $k=0,1,\cdots,q$. Since $\phi(t)$ is an integral of $L(y)=0$ in the neighborhood of $t=1$

$$\phi(t)=\sum_{k=0}^{q}A_k y_k(t),\qquad |t-1|<K,$$

where the A_k are constants. From Theorem 3.1 we have that $\phi(t)=O((1-t)^{q+2})$ as $t\rightarrow 1$ through positive values of t. Since $\psi_{k,0}(1)\neq 0$ it follows that $A_k=0$ for $k=1,\cdots,q$ and hence $\phi(t)$ is a multiple of $y_0(t)$. Hence, by Taylor's theorem,

$$(5.3)\qquad \phi(t)=\sum_{m=0}^{\infty}\frac{\phi^{(q+2+m)}(1)}{\Gamma(q+3+m)}(t-1)^{q+2+m}$$

for values of t such that $|t-1|<K$.

The results of this section may be formulated in the following theorem.

THEOREM 5.1. *Let the variable t in the differential equation* (4.3) *be extended to the complex domain. Then $L(y(t))=0$ is of Fuchsian type. If $p=0$ the regular singular points are $t=\infty$ and the α roots of unity for which $t^\alpha=1$. If $p>0$ then $t=0$ is also a regular singular point of $L(y(t))=0$. If K denotes the distance from $t=1$ to the nearest of the other singularities of the differential equation, then $\phi(t)$ has the Taylor series expansion* (5.3) *for values of t such that $|t-1|<K$.*

6. **Identification of the constants.** In this section we shall see that $\phi(t)$ acts as a generating function for the constants c_m. To show this we shall first need the following result.

LEMMA 6.1. *Let m be any non-negative integer and let λ_0 be defined by* (3.1). *Then*

$$\lim_{t\to 0} t^{x+m}\phi^{(m)}(t) = 0,$$

provided Re $(x) > \lambda_0$.

Proof. $\phi(t)$ satisfies $L(y)=0$ which has $t=0$ as an ordinary point if $p=0$ and a regular singular point if $p>0$. The assertion is trivially true for $p=0$ since $\lambda_0 \geqq 0$. If $p>0$ it follows from the theory of Frobenius that $\phi(t)$ admits the expansion

$$\phi(t) = \sum_i (\log t)^{k_i} t^{\mu_i} \sum_{n=0}^{\infty} d_{i,n} t^n \qquad (d_{i,0} \neq 0),$$

for $|t| < 1$. From (5.2) and (3.1) we have that

$$\text{Re } (\mu_i) \geqq -\lambda_0 \text{ for all } i.$$

Consequently, for any non-negative integer m,

$$\phi^{(m)}(t) = O(t^{-\lambda-m}) \qquad (\lambda > \lambda_0)$$

as $t \to 0$, which proves the assertion.

THEOREM 6.1. *The constants c_m ($m=0, 1, \cdots$) occurring in the asymptotic factorial expansion of ${}_pg_q(w)$ are precisely the constants which occur in the Taylor series expansion of $\phi(t)$ about $t=1$. In fact,*

$$c_m = (-1)^{m+q}\phi^{(q+2+m)}(1) \qquad (m \geqq 0). \tag{6.1}$$

Proof. Integration of (3.3) by parts $N+1$ times yields

$$H(x) = \sum_{n=0}^{N} \frac{(-1)^n\Gamma(x)}{\Gamma(x+n+1)} \left[\phi^{(n)}(t)t^{x+n}\right]_0^1 + \frac{(-1)^{N+1}\Gamma(x)}{\Gamma(x+N+1)} \int_0^1 t^{x+N}\phi^{(N+1)}(t)dt,$$

provided $\text{Re}(x) > \lambda_0$. From Lemma 6.1 we have that

$$\left[\phi^{(n)}(t)t^{x+n}\right]_0^1 = \phi^{(n)}(1)$$

for $\text{Re}(x) > \lambda_0$. Since $\phi^{(n)}(1) = 0$ for $n=0, 1, \cdots, q+1$, by (5.3), we may set $n = m+q+2$ and obtain

$$H(x) = \sum_{m=0}^{M} \frac{(-1)^{q+2+m}\Gamma(x)}{\Gamma(x+q+3+m)} \phi^{(q+2+m)}(1) + R_M, \tag{6.2}$$

provided $\text{Re}(x) > \lambda_0$. Here M is any non-negative integer and

$$R_M = \frac{(-1)^{q+3+M}\Gamma(x)}{\Gamma(x+q+4+M)}\left\{\phi^{(q+3+M)}(1) - \int_0^1 t^{x+q+3+M}\phi^{(q+4+M)}(t)dt\right\}.$$

Let x tend to infinity along the positive real axis. Then the integral is of $O(1)$ and consequently

$$R_M = O(\Gamma(x)/\Gamma(x+q+4+M)).$$

The relation (6.1) now follows by comparing (6.2) with (2.6). For, the coefficients in (2.6) are already uniquely determined if we require (2.6) only for x real and positive.

7. **The recursion formula.** We substitute (6.1) into (5.3) and obtain

$$\phi(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+q}c_m}{\Gamma(q+3+m)}(t-1)^{q+2+m} \qquad |t-1| < K. \tag{7.1}$$

Since $\phi(t)$ is an integral of the differential equation $L(y)=0$ we may determine a recursion formula for the constants c_m by substituting (7.1) into (4.8). The formula provides a method of computing the constants since we know that $c_0=1$.

THEOREM 7.1. *Let the constants* C_k $(k=\alpha, \alpha+1, \cdots, q+1)$ *and* E_k $(k=0, 1, \cdots, q+1)$ *be as defined in the formulation of Theorem* 4.2. *The constants* c_m $(m=0, 1, \cdots)$, *defined by Theorem* 2.1, *satisfy the following recursion formula of length* q:

$$c_0 = 1$$

and

$$c_m = (m+2)!(\alpha m)^{-1}\sum_{i=1}^{q} a_{mi}c_{m-i}, \tag{7.2}$$

$(c_{-1}=c_{-2}= \cdots =0)$, *where*

$$a_{mi} = (-1)^{i-1}\sum_{k=q-i}^{q+1}\binom{k}{q-i}E_k/(m+q+2-i-k)! \tag{7.3}$$

if $p \leqq i \leqq q$, *while*

$$a_{mi} = (-1)^{i-1}\sum_{k=q-i}^{q+1}\left\{\binom{k}{q-i}E_k - \binom{k-\alpha}{q-i-\alpha}C_k\right\}\Big/ (m+q+2-i-k)! \tag{7.4}$$

if $1 \leqq i \leqq p-1$. *Note that the last formula and, hence, the* C_k *are of no importance if* $p=0$ *or* $p=1$.

Proof. Let us transform the independent variable in (4.8) by the relation $t=t'+1$ and then drop the prime. We obtain the differential equation

$$\sum_{k=\alpha}^{q+1}\{E_k(t+1)^k - C_k(t+1)^{k-\alpha}\}y^{(k)}(t) + \sum_{k=0}^{\alpha-1} E_k(t+1)^k y^{(k)}(t) = 0$$

which is satisfied by

$$y(t) \equiv \phi(t+1) = \sum_{m=0}^{\infty} \frac{(-1)^{m+q}c_m}{\Gamma(q+3+m)}\, t^{q+2+m} \tag{7.5}$$

in the neighborhood of $t=0$. Employing the binomial theorem, we may write the differential equation in the form

$$\sum_{j=0}^{\alpha-1}\sum_{k=j}^{q+1}\binom{k}{j}E_k t^{k-j}y^{(k)}(t) + \sum_{j=\alpha}^{q+1}\sum_{k=j}^{q+1}\left\{\binom{k}{j}E_k - \binom{k-\alpha}{j-\alpha}C_k\right\}t^{k-j}y^{(k)}(t) = 0.$$

By (7.5) the coefficient of t^{m+2} in $t^{k-j}y^{(k)}(t)$ is equal to

$$(-1)^{m+j}c_{m+j-q}/(m+2-j-k)!,$$

mplying that, for each integer m,

$$\sum_{j=0}^{\alpha-1}(-1)^{m+j}c_{m+j-q}\sum_{k=j}^{q+1}\binom{k}{j}E_k/(m+2+j-k)!$$
$$+\sum_{j=\alpha}^{q+1}(-1)^{m+j}c_{m+j-q}\sum_{k=j}^{q+1}\left\{\binom{k}{j}E_k - \binom{k-\alpha}{j-\alpha}C_k\right\}\Big/(m+2+j-k)! = 0,$$

provided we make the agreement that $c_m=0$ for $m<0$. From (4.11) and (4.12), the coefficients of c_{m+1} and c_m in the latter formula are 0 and

$$(-1)^{m+q}\alpha m/(m+2)!$$

respectively. Hence,

$$c_m = \frac{(m+2)!}{\alpha m}\left[\sum_{j=0}^{\alpha-1}(-1)^{j+q+1}c_{m+j-q}\sum_{k=j}^{q+1}\binom{k}{j}E_k/(m+2+j-k)!\right.$$
$$\left.+\sum_{j=\alpha}^{q-1}(-1)^{j+q+1}c_{m+j-q}\sum_{k=j}^{q+1}\left\{\binom{k}{j}E_k - \binom{k-\alpha}{j-\alpha}C_k\right\}\Big/(m+2+j-k)!\right].$$

This formula reduces to (7.2) if we set $i=q-j$ and introduce the notations (7.3) and (7.4).

8. **Several examples.** As a first illustration let us consider the coefficient function ${}_0g_1(w)$. In the special case where $\rho_0=1$ ${}_0g_1(w)$ reduces to the coefficient function of the Bessel function. Here $p=0$, $q=1$, $\alpha=2$ and $\beta=\rho_0+\rho_1-1/2$; (7.2) reduces to

$$c_m = c_{m-1}(m+2)!\,(2m)^{-1}\sum_{k=0}^{2} E_k/(m+2-k)!, \qquad (m \geqq 1).$$

From (4.11) we recursively calculate

$$E_0 = (\rho_1 - \rho_0 - 5/2)(\rho_0 - \rho_1 - 5/2); \quad E_1 = -4; \quad E_2 = 1.$$

Hence,

$$\begin{aligned} c_m &= \frac{c_{m-1}}{2m}\left[(m+1)(m+2) - 4(m+2) - (\rho_1 - \rho_0)^2 + 25/4\right] \\ &= \frac{c_{m-1}}{8m}\left[(2m-1)^2 - 4(\rho_1 - \rho_0)^2\right], \qquad (m \geqq 1). \end{aligned}$$

Since $c_0 = 1$ it follows that

$$c_m = \frac{1}{2^{3m} m!}\prod_{k=1}^{m}\left[(2k-1)^2 - 4(\rho_1 - \rho_0)^2\right] \qquad (m \geqq 0). \tag{8.1}$$

When $\rho_0 = 1$ this reduces to the familiar expression for the coefficients in the asymptotic expansion of the Bessel function [8, p. 368].

Theorem 7.1 also yields the explicit expression for the constants in the factorial expansion of ${}_1g_1(w)$. In this case one obtains

$$c_m = \frac{1}{m!}\sum_{k=0}^{m-1}(\rho_0 - \sigma_1 + k)(\rho_1 - \sigma_1 + k), \qquad (m \geqq 0). \tag{8.2}$$

If we set $\rho_0 = 1$ these reduce to the coefficients which occur in the asymptotic expansion of the confluent hypergeometric function [8, p. 342].

As a final illustration let us consider the coefficient function ${}_2g_2(w)$. In this case $p = q = 2$, $\alpha = 1$ and $\beta = \rho_0 + \rho_1 + \rho_2 - \sigma_1 - \sigma_2$; (7.2) reduces to

$$\begin{aligned} c_m = \frac{(m+2)!}{m}\Bigg[& c_{m-1}\sum_{k=1}^{3}\{kE_k - C_k\}/(m+3-k)! \\ & - c_{m-2}\sum_{k=0}^{3} E_k/(m+2-k!)\Bigg], \qquad (m \geqq 1). \end{aligned} \tag{8.3}$$

Here the constants E_k and C_k are given by Theorem 4.2. The recursion formula for c_m thus depends on the last two previously calculated coefficients since $q = 2$. Consequently, we cannot write the general explicit formula for c_m. For specified values of ρ_j and σ_i, however, there is no difficulty as may be illustrated by a numerical example.

Suppose $\rho_0 = 1$, $\rho_1 = 2.2 - 0.5i$, $\rho_2 = 2.2 + 0.5i$, $\sigma_1 = 0.6$ and $\sigma_2 = -0.2$. Then $\beta = 5$ and Theorem 4.2 yields $E_0 = 1$, $E_1 = 0.29$, $E_2 = 0.6$, $E_3 = 1$, $C_1 = 0.48$, $C_2 = 2.6$ and $C_3 = 1$. From (8.3) we calculate

$$c_0 = 1,$$
$$c_1 = 7.61\, c_0 = 7.61,$$
$$c_2 = 9.105\, c_1 - 8.98\, c_0 = 60.30905,$$
$$c_3 = 10.9367\, c_2 - 24.4833\, c_1 = 473.2816, \text{ etc.}$$

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