INVARIANT MEANS AND CONES WITH VECTOR INTERIORS

BY

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1. **Introduction.** The Hahn-Banach theorem on the extension of distributive functionals bounded from above by a positive-homogeneous subadditive functional and the theorem on the extension of monotone distributive functionals can be generalized by taking an ordered linear space with certain special properties as the range space instead of the real number field and requiring that the extension be invariant with respect to certain collections of distributive operators on the domain space. Theorems of this nature have been proved in [1; 6; 7; 8]. It will be shown that a semigroup of operators G on the domain space, and the range space V permit invariant Hahn-Banach and monotone extensions if G has an invariant mean and V is a boundedly complete vector lattice whose cone of positive elements is sharp and has a vector interior point.

2. **Preliminaries.** In this section needed background material is presented. The linear spaces are presumed to have the real numbers as the scalar field. A linear space V is an *ordered linear space* if there exists a transitive binary relationship $\geqq$ in V such that if x, y, z, are in V and t is a nonnegative real number, then if $x \geqq y$, $tx \geqq ty$, and $x+z \geqq y+z$. The ordering is *sharp* if $x \geqq y \geqq x$ imply $x=y$. A nonempty subset C of a linear space V is a *cone*, if x, y are in C and t is a nonnegative real number then $x+y$ and tx are in C. A cone C is *sharp* if x and $-x$ are in C imply $x=0$. A cone C in a linear space V determines an ordering which makes V an ordered linear space. $x \geqq y$ if and only if $x-y$ is in C. The set of elements greater than or equal 0 in an ordered linear space is a cone and the ordering determined from this cone is the original ordering. A cone is sharp if and only if its associated ordering is sharp. The definitions of upper bound, lower bound, least upper bound, greatest lower bound and other terms associated with partially ordered systems are assumed.

A function f from an ordered linear space X to an ordered linear space Y is *monotone* if $x \geqq x'$ implies $f(x) \geqq f(x')$ and f is *nonnegative* if $x \geqq 0$ implies $f(x) \geqq 0$. If f is a distributive function, the monotone property of f is equivalent to the nonnegative property of f. The following theorem is proved in [2].

A. Consider an ordered linear space V, then statements 1, 2, and 3 are equivalent and they imply 4.

1. V is a *boundedly complete vector lattice*. That is, every set of elements in V with an upper bound has a least upper bound.

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2. Given subsets A and B of V such that $A \geqq B$ (i.e. $a \geqq b$, $a \in A$ and $b \in B$), there exists a v in V such that $A \geqq v \geqq B$.

3. V has the *monotone extension property*. That is, if (1) X is an ordered subspace of an ordered linear space Y with cone C such that the cone in X is $X \cap C$, (2) for all y in Y, $(y+X)\cap C \neq \phi$ if and only if $(-y+X)\cap C \neq \phi$, then every monotone distributive function from X to V has a monotone distributive extension from Y to V.

4. V has the *Hahn-Banach extension property*. That is, if (1) X is a subspace of a linear space Y, (2) p is a positive-homogeneous subadditive function from Y to V, (3) f is a distributive function from X to V such that $f(x) \leqq p(x)$, $x \in X$, then there exists a distributive extension F of f from Y to V such that $F(y) \leqq p(y)$ for every y in Y.

A *representation* of a semigroup $\overline{G}$ is defined to be a homomorphism or anti-homomorphism of $\overline{G}$ into the space of distributive operators on a linear space (or the image of $\overline{G}$ under such a map).

DEFINITION 1. The pair $[\overline{G}, V]$ where

(a) $\overline{G}$ is an abstract semigroup,

(b) V is a boundedly complete vector lattice whose positive cone K is sharp, has

(α) the *Hahn-Banach extension property*, if and only if for every collection $[Y, X, G, p, f]$, where (i) Y is a linear space, (ii) X is a subspace of Y, (iii) G is a representation of $\overline{G}$ on Y such that $gx \in X$ for $x \in X$ and $g \in G$, (iv) p is a positive-homogeneous subadditive function from Y to V such that $p(gy) \leqq p(y)$ for $g \in G$ and $y \in Y$, and (v) f is a distributive function from X to V such that $f(x) \leqq p(x)$ and $f(gx) = f(x)$ for $y \in Y$ and $g \in G$, there exists a distributive extension F of f to all of Y such that $F(y) \leqq p(y)$ and $F(gy) = F(y)$ for every $y \in Y$ and $g \in G$;

(β) the *monotone extension property* if and only if for every collection $[Y, C, X, G, f]$, where (i) Y is an ordered linear space with positive cone C, (ii) X is an ordered linear subspace of Y (with order induced by C) such that $(y+X)\cap C \neq \phi$, $y \in Y$, (iii) G is a representation of $\overline{G}$ on Y such that gz is in C and gx is in X for g in G, z in C, and x in X, and (iv) f is a monotone distributive function from X to V such that $f(gx) = f(x)$ for each x in X and g in G, there exists a monotone distributive extension F of f to all of Y such that $F(gy) = F(y)$, for y in Y and g in G.

The following theorem is proved in [7; 8].

B. The pair $[\overline{G}, V]$ has the monotone extension property if and only if it has the Hahn-Banach extension property.

A semigroup $\overline{G}$ has an *invariant mean* if there exists a positive linear functional m of unit norm defined over the Banach space of bounded real-valued functions over $\overline{G}$, $M(\overline{G})$, which is invariant with respect to the right and left regular representations of $\overline{G}$ on $M(\overline{G})$.

For a discussion of semigroups with invariant means see [3].

A positive cone C in an ordered linear space W is *reproducing* if every element in W is the difference of two elements in C. If U is a subspace of the space of distributive functionals, $W^{\#}$ of an ordered linear space W with cone C, then the *induced cone* in U, $C(U)=\{u\in U \mid u(w)\geqq 0,\ w\in C\}$ is a cone and determines an ordering so that U is an ordered linear space. The next theorem is proved in [8].

C. Let W be an ordered linear space with reproducing cone. Let V be an ordered linear space with cone K, such that V satisfies properties (b) of Definition 1, and in addition is a subspace of the space of distributive functionals of W such that K is the cone induced in V from W. Let $\overline{G}$ be a semigroup with an invariant mean. Then $[\overline{G}, V]$ has the monotone extension property and the Hahn-Banach extension property.

3. **Cones with vector interiors.** A set N in a linear space V is a *vector neighborhood* of v_0 in V if it contains v_0 and contains a segment of every line through v_0 subject to the restriction that v_0 is not an end point of any of these segments. A set in a linear space has a *vector interior* if it contains a point along with a vector neighborhood of that point. Every neighborhood of a linear topological space is a vector neighborhood. Further, if v_0 is a vector interior point of a set A and v_0 is in a subspace X of V then every translate of X by an element of V meets A. The principle theorem of this paper can now be proved.

THEOREM. *If V is a boundedly complete vector lattice whose positive cone K is sharp and has a vector interior point v_0, and if $\overline{G}$ is a semigroup with an invariant mean, then $[\overline{G}, V]$ has the monotone extension property and the Hahn-Banach extension property.*

Proof. Consider $V^{\#}$, the space of distributive functionals on V. Let $K_1=\{f \text{ in } V^{\#} \mid f(v)\geqq 0,\ v \text{ in } K\}$. Then clearly, V_1, the linear hull of K_1, is an ordered linear space whose positive cone K_1 is reproducing. Further, the set K_1 distinguishes positive elements in V from nonpositive elements in V. This will be proved in the following lemma.

LEMMA. *For any $v_1\neq 0$ in V, there exists a function $f\in K_1$ such that $f(v_1)\neq 0$ and if v_1 is not in K, $f(v_1)<0$.*

Proof. Let X be the subspace of V generated by v_1 and v_0. Then for any v in V, $v+X$ meets K since v_0 is an interior point of K. There are three cases to consider.

CASE 1. $v_1\geqq 0$, $v_1\neq 0$. Let $k=\sup\{t\geqq 0 \mid -tv_1+v_0\geqq 0\}$. The set of elements $\{-tv_1+v_0\geqq 0\}$ is not empty since v_0 is a vector interior point of K. The number k exists, for otherwise $-v_1+v_0/t\geqq 0$ for every positive number t and therefore it follows that $-v_1=\inf_t(-v_1+v_0/t)\geqq 0$, a contradiction of the sharpness of K.

Define the distributive functional h on X: $h(av_0+bv_1)=ak+b$. If av_0+bv_1

$\geqq 0$, then, by sharpness, a, b cannot be both less than zero. If $a>0$ and $b>0$, then $ak+b>0$. If $a>0$ and $b<0$, then $-b/a \leqq k$ and hence $ak+b \geqq 0$. If $a<0$ and $b>0$, then $v_0+bv_1/a \leqq 0$. Therefore, $-b/a \geqq k$ and $ak+b \geqq 0$.

CASE 2. $v_1 \leqq 0, v_1 \neq 0$. Consider $-v_1$ and use Case 1. Note that Cases 1 and 2 include $v_1 = sv_0$.

CASE 3. $v_1 \not\leqq 0$, $v_1 \not\geqq 0$. Let $k = \sup \{ |t| \mid tv_1 + v_0 \geqq 0 \}$ the number k exists. If for all positive t, $tv_1+v_0 \geqq 0$, then $v_1 = \inf_t (v_1 + v_0/t) \geqq 0$. Therefore, t is bounded from above. Similarly, it follows that t is bounded from below. Thus k is defined. Define the distributive functional h on X: $h(av_0+bv_1) = ak-b$. Therefore $h(v_0)=k$ and $h(v_1)=-1$. Consider $av_0+bv_1 \geqq 0$. First, $a \geqq 0$; for otherwise $bv_1 \geqq -av_0 \geqq 0$ and v_1 is comparable with zero. Since $a \geqq 0$, $v_0+bv_1/a \geqq 0$ and $|b|/a \leqq k$. Therefore, $ak-b \geqq ak-|b| \geqq 0$.

Therefore, since the real numbers form a boundedly complete vector lattice, V, X, K, h satisfy the conditions for Y, X, C, and f of Theorem A.3, there exists a monotone distributive extension of h to all of V, and the lemma is proved.

Consider now V_2 the linear space of distributive functionals of V_1. Consider the cone induced in V_2 from V_1: $K_2 = \{ w \text{ in } V_2 \mid w(f) \geqq 0, f \text{ in } K_1 \}$. Thus V_2 is an ordered linear space relative to K_2. Consider the natural mapping Q from V into V_2: $Qv(f) = f(v)$, v in V, f in V_1. The mapping Q is distributive and monotone clearly. Since V_1 has sufficiently many functionals Q is 1-1 onto $QV \subset V_2$. The mapping Q^{-1} from QV to V is thus defined and in addition, distributive. Further Q^{-1} is monotone, for if $Qv=w$ is in K_2 then $w(f) \geqq 0$, for f in K_1. By virtue of the lemma this implies that v is in K.

The image of V under Q, QV, is a boundedly complete vector lattice. This follows from Theorem A.2 and the fact that Q and Q^{-1} are monotone mappings.

Consider now $[Y, C, X, G, f]$ as in Definition $1(\beta)$, where G is a representation of $\overline{G}$. Define f_1 from X to QV: $Qf=f_1$, then the collection $[Y, C, X, G, f_1]$ satisfies the conditions in Definition $1(\beta)$. It is now noted that the pair $[\overline{G}, QV]$, satisfy the conditions of Theorem C. Hence this pair has the monotone extension property. Thus there exists a monotone distributive extension F_1 of f_1 to all of Y with values of QV, such that $F_1(gy) = F_1(y)$ for y in Y and g in G. Define $F = Q^{-1}F_1$. This is clearly an invariant, monotone, distributive extension of f. Hence $[\overline{G}, V]$ has the monotone and the Hahn-Banach extension properties.

A theorem is proved in [8] which implies the converse of the previous theorem and is quoted. If V is a boundedly complete vector lattice with a sharp positive cone with a vector interior point, then, if (1) Y is an ordered linear space with cone C, (2) X is a subspace of Y with order induced by C, and such that $y+X$ meets C for all y in Y, (3) f is a monotone distributive function from X to V, (4) G is a semigroup of operators on Y such that gx is in X, gz is in C, and $f(gx)=f(x)$, for all g in G, x in X, and z in C, implies that there exists a monotone, distributive extension F of f to all of Y such that $F(gy)=F(y)$, for g in G and y in Y, it follows that the semigroup of operators G, considered as an abstract semigroup, has an invariant mean.

Bibliography

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

2. M. M. Day, *Notes on ordered linear spaces*, University of Illinois, 1950, unpublished.

3. ———, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 89–108.

4. S. Kakutani, *Concrete representations of abstract* (*M*)-*spaces*, Ann. of Math. vol. 42 (1941) pp. 523–537.

5. M. G. Kreĭn and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space* [Uspehi Matematicheskii Nauk (N.S.) vol. 3 (23) (1948) pp. 3–95] Amer. Math. Soc. Translations, no. 26, 1950.

6. A. Morse and R. Agnew, *Extensions of linear functionals, with applications to limits, integrals, measures, and densities*, Ann. of Math. vol. 39 (1938) pp. 20–30.

7. R. J. Silverman, *Invariant linear functions*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 411–424.

8. ———, *Means on semigroups and the Hahn-Banach extension property*, Trans. Amer. Math. Soc. vol. 83 (1956) pp. 222–237.

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