THE STRAIN-ENERGY FUNCTION FOR ANISOTROPIC ELASTIC MATERIALS

BY

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1. **Introduction.** If we consider a body of perfectly elastic material to undergo deformation in which a point initially at X_i' in the rectangular Cartesian coordinate system x_i' moves to x_i' in the same coordinate system, then the strain-energy function W is a single-valued function of the quantities g'_{ij} defined by

$$g'_{ij} = \frac{\partial x'_k}{\partial X'_i}\frac{\partial x'_k}{\partial X'_j} - \delta_{ij}. \tag{1.1}$$

We assume (see Appendix) that W is expressible as a polynomial in the quantities g'_{ij} and denote this by

$$W = W(g'_{ij}). \tag{1.2}$$

We define the displacement components u_i' by

$$u'_i = x'_i - X'_i. \tag{1.3}$$

Then, if the displacement gradients $\partial u_i'/\partial X_j'$ are sufficiently small compared with unity, the strain-energy function W, considered as a polynomial in $\partial u_i'/\partial X_j'$, may be approximated by the leading terms of first and second degree in $\partial u_i'/\partial X_j'$. If we assume that the stress in the material is zero when $u_i' = 0$, then the coefficients of the first degree terms are zero and W is a homogeneous quadratic expression of the form

$$W = c_{ijlm}\left(\frac{\partial u'_i}{\partial X'_j} + \frac{\partial u'_j}{\partial X'_i}\right)\left(\frac{\partial u'_l}{\partial X'_m} + \frac{\partial u'_m}{\partial X'_l}\right), \tag{1.4}$$

where c_{ijlm} are constants.

If the material possesses symmetry, certain restrictions, depending on the type of symmetry, are imposed on these constants. These restrictions were fully investigated by Voigt (1910) for the various classes of crystal symmetry.

A higher order approximation to W, in the case when $\partial u_i'/\partial X_j' \ll 1$, is

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obtained by retaining in the expression for W as a polynomial in $\partial u_i'/\partial X_j'$ all terms of third and lower degree in $\partial u_i'/\partial X_j'$. Alternatively, in the expression for W as a polynomial in g_{ij}' all terms of third or lower degree in g_{ij}' may be retained. In both cases, the coefficients of the first degree terms vanish if the stress is assumed zero when $u_i' = 0$. Considering W to be a polynomial of third degree in g_{ij}', Birch (1947) has investigated the restrictions imposed on the coefficients in this polynomial if the material considered has cubic symmetry of the hexoctahedral, gyroidal, hextetrahedral, diploidal or tetartoidal classes. Murnaghan (1951) has considered the restrictions imposed on these coefficients by certain basic types of symmetry, e.g. rotational symmetry about an axis. Sheng (1955) has investigated these restrictions for the various crystal classes[2].

In the present paper, we discuss the restrictions imposed by symmetry on the form of the strain-energy function, for elastic materials belonging to the various crystal classes, without in any way restricting the degree of the expression for the strain-energy function as a polynomial in g_{ij}'. The results are obtained by consistent use of theorems in classical invariant theory, although in certain of the simple cases, e.g. materials having rhombic symmetry, the conclusions can be readily reached by inspection (Green and Wilkes (1954)). It is seen that the 32 crystal classes have strain-energy functions of eleven different types, which coalesce into nine types if the approximations of classical elasticity theory are made.

Throughout this paper, the nomenclature for the various crystal classes is that employed by Dana and Hurlbut (1952).

For each of the crystal classes, a preferred rectangular Cartesian coordinate system x_i is chosen and the symmetry of the material is described by the group of transformations which transforms this coordinate system into its equivalent positions. The strain-energy function is assumed to be a polynomial in the six quantities g_{ij}, which are the quantities $g_j^{i\prime}$ given in (1.1) but defined in the reference system x_i. This polynomial must, of course, be form-invariant under the transformations describing the particular crystal class considered. Hence, the problem of determining the limitations imposed on the strain-energy function reduces to the determination of a polynomial basis for polynomials in g_{ij} which are form-invariant under the group of transformations associated with the crystal class considered.

Such a polynomial basis $I_1, I_2, \cdots, I_n$ (say) is determined for each of the crystal classes. It has the following properties:

(i) each of the quantities $I_1, I_2, \cdots, I_n$ is a polynomial in g_{ij} which is form-invariant under the group of transformations associated with the crystal class considered and hence any polynomial in $I_1, I_2, \cdots, I_n$ is form invariant under this group of transformations;

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(2) Sheng (1955) considers the restrictions imposed on the expressions for the stress by the symmetry of the material. The restrictions on the strain-energy function follow readily from these results.

(ii) any polynomial in g_{ij} which is form-invariant under the group of transformations, and hence the strain-energy function W, is expressible as a polynomial in $I_1, I_2, \cdots, I_n$.

From this knowledge of W, we can determine the stress components τ_{ij} in the coordinate system x_i by employing the formula (see, for example, Truesdell (1952) or Green and Zerna (1954))

$$\tau_{ij} = \frac{1}{\det \dfrac{\partial x_p}{\partial X_q}} \left[\frac{\partial x_i}{\partial X_k} \frac{\partial W}{\partial(\partial x_j/\partial X_k)} \right]. \tag{1.5}$$

In this formula x_i and X_i are the positions in the coordinate system x_i of a particle of the material in the deformed and undeformed states respectively.

From Equation (1.5) and (ii) it follows immediately that

$$\tau_{ij} = \frac{1}{\det \dfrac{\partial x_p}{\partial X_q}} \sum_{r=1}^{n} \frac{\partial W}{\partial I_r} \frac{\partial x_i}{\partial X_k} \frac{\partial I_r}{\partial(\partial x_j/\partial X_k)}. \tag{1.6}$$

2. **Symmetry transformations.** For each of the materials considered in this paper, there exist in the undeformed state three preferred directions in the material which may be defined by the unit vectors $\boldsymbol{h}_i$ $(i=1, 2, 3)$. Except for the pedial class of the triclinic system, there are a number of equivalent ways in which the triad of vectors $\boldsymbol{h}_i$ may be chosen. The symmetry properties of the material may be defined by the group of orthogonal transformations which transform any of these triads into its equivalent positions.

The following transformations are sufficient for the description of the symmetry properties in the various crystal systems:

$$\boldsymbol{I} = (1, 1, 1), \qquad \boldsymbol{C} = (-1, -1, -1),$$

$$\boldsymbol{R}_1 = (-1, 1, 1), \qquad \boldsymbol{R}_2 = (1, -1, 1), \qquad \boldsymbol{R}_3 = (1, 1, -1),$$

$$\boldsymbol{D}_1 = (1, -1, -1), \qquad \boldsymbol{D}_2 = (-1, 1, -1), \qquad \boldsymbol{D}_3 = (-1, -1, 1),$$

$$\boldsymbol{T}_1 = \begin{Vmatrix} 1, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & 0 \end{Vmatrix}, \qquad \boldsymbol{T}_2 = \begin{Vmatrix} 0, & 0, & 1 \\ 0, & 1, & 0 \\ 1, & 0, & 0 \end{Vmatrix}, \qquad \boldsymbol{T}_3 = \begin{Vmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 1 \end{Vmatrix},$$

$$\boldsymbol{M}_1 = \begin{Vmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ 1, & 0, & 0 \end{Vmatrix}, \qquad \boldsymbol{M}_2 = \begin{Vmatrix} 0, & 0, & 1 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{Vmatrix}, \tag{2.1}$$

$$\boldsymbol{S}_1 = \begin{Vmatrix} -1/2, & 3^{1/2}/2, & 0 \\ -3^{1/2}/2, & -1/2, & 0 \\ 0, & 0, & 1 \end{Vmatrix} \quad \text{and} \quad \boldsymbol{S}_2 = \begin{Vmatrix} -1/2, & -3^{1/2}/2, & 0 \\ 3^{1/2}/2, & -1/2, & 0 \\ 0, & 0, & 1 \end{Vmatrix}.$$

In Equation (2.1), the notation

$$(2.2) \qquad (a, b, c) = \begin{Vmatrix} a, & 0, & 0 \\ 0, & b, & 0 \\ 0, & 0, & c \end{Vmatrix}$$

is used. $\boldsymbol{I}$ is the identity transformation. $\boldsymbol{C}$ is the central inversion transformation. $\boldsymbol{R}_j$ is the reflection transformation which transforms a rectangular Cartesian coordinate system x_i, the x_j axis of which lies in the direction of the unit vector $\boldsymbol{h}_j$, into its image in the plane normal to $\boldsymbol{h}_j$. The transformation $\boldsymbol{D}_j$ transforms this coordinate system into that obtained by rotating it through 180° about the x_j axis. The transformation $\boldsymbol{T}_j$ transforms a rectangular Cartesian coordinate system x_i, the axes of which lie in the directions of the unit vectors $\boldsymbol{h}_i$, into the coordinate system obtained by reflecting the system x_i in a plane passing through the axis x_j and bisecting the angle between the remaining two axes. The transformations $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ transform a rectangular Cartesian coordinate system x_i, the axes of which lie in the directions of $\boldsymbol{h}_i$, into the systems obtained by rotating the system x_i through 120° and 240° respectively about a line passing through the origin and the point (1, 1, 1). The transformations $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ transform a rectangular Cartesian coordinate system x_i, the x_3 axis of which lies in the direction of $\boldsymbol{h}_3$ into the systems obtained by rotation of this coordinate system through 120° and 240° respectively about the x_3 axis.

3. **Form-invariance of the strain-energy function.** For each of the symmetry classes that we consider, we shall choose as reference system a rectangular Cartesian coordinate system x_i, so related to the preferred directions $\boldsymbol{h}_i$ in the material under consideration that the symmetry of the material may be described by one or more of the transformations (2.1). Let X_i and x_i denote the positions of a particle of the material in this coordinate system in the undeformed and deformed states respectively. Then, from (1.1), it is seen that the strain-energy function W is expressible as a polynomial in the quantities g_{ij} defined by

$$(3.1) \qquad g_{ij} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j} - \delta_{ij}.$$

We shall express this as

$$(3.2) \qquad W = W(g_{ij}).$$

Let $\bar{x}_i$ be the coordinate system into which the system x_i is transformed by a transformation $\|t_{ij}\|$ of the group describing the symmetry properties of the material under consideration. Then, if $\overline{X}_i$ and $\bar{x}_i$ are the coordinates in the system $\bar{x}_i$ corresponding to the coordinates X_i and x_i in the system x_i, we have

(3.3) $$\overline{X}_i = t_{ij}X_j \quad \text{and} \quad \bar{x}_i = t_{ij}x_j.$$

Defining $\bar{g}_{ij}$ by

(3.4) $$\bar{g}_{ij} = \frac{\partial \bar{x}_k}{\partial \overline{X}_i}\frac{\partial \bar{x}_k}{\partial \overline{X}_j} - \delta_{ij},$$

we have

(3.5) $$\bar{g}_{ij} = t_{ip}t_{jq}g_{pq}.$$

Now, since the coordinate systems x_i and $\bar{x}_i$ are equivalent, we must have

(3.6) $$W(g_{ij}) = W(\bar{g}_{ij}),$$

i.e. the strain-energy function must be form-invariant under any transformation of the group defining the symmetry properties of the material. It is the object of this paper to determine the limitations imposed on W by the condition (3.6), with (3.5), for each of the classes of the crystal systems. For each crystal class, the condition (3.6) must, of course, be satisfied for every transformation $\|t_{ij}\|$ of the group describing the symmetry of the material. In Tables 1 and 2, the values of $\bar{g}_{ij}$, obtained from (3.5), are given in terms of

TABLE 1

Line No.	$\|t_{ij}\|$	$\bar{g}_{11}$	$\bar{g}_{22}$	$\bar{g}_{33}$	$\bar{g}_{23}$	$\bar{g}_{31}$	$\bar{g}_{12}$
1	I, C	g_{11}	g_{22}	g_{33}	g_{23}	g_{31}	g_{12}
2	R_1, D_1	g_{11}	g_{22}	g_{33}	g_{23}	$-g_{31}$	$-g_{12}$
3	R_2, D_2	g_{11}	g_{22}	g_{33}	$-g_{23}$	g_{31}	$-g_{12}$
4	R_3, D_3	g_{11}	g_{22}	g_{33}	$-g_{23}$	$-g_{31}$	g_{12}
5	T_1, CT_1	g_{11}	g_{33}	g_{22}	g_{23}	g_{12}	g_{31}
6	R_1T_1, D_1T_1	g_{11}	g_{33}	g_{22}	g_{23}	$-g_{12}$	$-g_{31}$
7	R_2T_1, D_2T_1	g_{11}	g_{33}	g_{22}	$-g_{23}$	g_{12}	$-g_{31}$
8	R_3T_1, D_3T_1	g_{11}	g_{33}	g_{22}	$-g_{23}$	$-g_{12}$	g_{31}
9	T_2, CT_2	g_{33}	g_{22}	g_{11}	g_{12}	g_{31}	g_{23}
10	R_1T_2, D_1T_2	g_{33}	g_{22}	g_{11}	g_{12}	$-g_{31}$	$-g_{23}$
11	R_2T_2, D_2T_2	g_{33}	g_{22}	g_{11}	$-g_{12}$	g_{31}	$-g_{23}$
12	R_3T_2, D_3T_2	g_{33}	g_{22}	g_{11}	$-g_{12}$	$-g_{31}$	g_{23}
13	T_3, CT_3	g_{22}	g_{11}	g_{33}	g_{31}	g_{23}	g_{12}
14	R_1T_3, D_1T_3	g_{22}	g_{11}	g_{33}	g_{31}	$-g_{23}$	$-g_{12}$
15	R_2T_3, D_2T_3	g_{22}	g_{11}	g_{33}	$-g_{31}$	g_{23}	$-g_{12}$
16	R_3T_3, D_3T_3	g_{22}	g_{11}	g_{33}	$-g_{31}$	$-g_{23}$	g_{12}
17	M_1, CM_1	g_{22}	g_{33}	g_{11}	g_{31}	g_{12}	g_{23}
18	R_1M_1, D_1M_1	g_{22}	g_{33}	g_{11}	g_{31}	$-g_{12}$	$-g_{23}$
19	R_2M_1, D_2M_1	g_{22}	g_{33}	g_{11}	$-g_{31}$	g_{12}	$-g_{23}$
20	R_3M_1, D_3M_1	g_{22}	g_{33}	g_{11}	$-g_{31}$	$-g_{12}$	g_{23}
21	M_2, CM_2	g_{33}	g_{11}	g_{22}	g_{12}	g_{23}	g_{31}
22	R_1M_2, D_1M_2	g_{33}	g_{11}	g_{22}	g_{12}	$-g_{23}$	$-g_{31}$
23	R_2M_2, D_2M_2	g_{33}	g_{11}	g_{22}	$-g_{12}$	g_{23}	$-g_{31}$
24	R_3M_2, D_3M_2	g_{33}	g_{11}	g_{22}	$-g_{12}$	$-g_{23}$	g_{31}

g_{pq} for each of the transformations entering into the description of the symmetry properties of the various crystal classes. These transformations are either those given in (2.1) or their products. In Table 2 the notation

$$
\begin{aligned}
&y_1 = g_{11}, \quad y_2 = \frac{1}{4} g_{11} - \frac{3^{1/2}}{2} g_{12} + \frac{3}{4} g_{22}, \quad y_3 = \frac{1}{4} g_{11} + \frac{3^{1/2}}{2} g_{12} + \frac{3}{4} g_{22}, \\
(3.7)\quad & \\
&z_1 = g_{31}, \quad z_2 = -\frac{1}{2} g_{31} + \frac{3^{1/2}}{2} g_{23}, \qquad z_3 = -\frac{1}{2} g_{31} - \frac{3^{1/2}}{2} g_{23}
\end{aligned}
$$

is used. It is, of course, evident that

$$(3.8) \qquad z_1 + z_2 + z_3 = 0.$$

The limitation imposed on W by the requirement that it be form-invariant under one of the transformations is obtained from (3.6) by substituting for $\bar{g}_{ij}$ the values appropriate to that transformation given in the tables.

TABLE 2

Line No.	$\|\|t_{ij}\|\|$	$\bar{g}_{11}$	$\bar{g}_{22}$	$\bar{g}_{33}$	$\bar{g}_{23}$	$\bar{g}_{31}$	$\bar{g}_{12}$
1	S_1, CS_1	y_2	$\frac{1}{3}(2y_1-y_2+2y_3)$	g_{33}	$-\frac{1}{3^{1/2}}(z_1-z_3)$	z_2	$\frac{1}{3^{1/2}}(y_1-y_3)$
2	S_2, CS_2	y_3	$\frac{1}{3}(2y_1+2y_2-y_3)$	g_{33}	$\frac{1}{3^{1/2}}(z_1-z_2)$	z_3	$\frac{1}{3^{1/2}}(y_2-y_1)$
3	R_1S_1, D_1S_1	y_2	$\frac{1}{3}(2y_1-y_2+2y_3)$	g_{33}	$-\frac{1}{3^{1/2}}(z_1-z_3)$	$-z_2$	$\frac{1}{3^{1/2}}(y_3-y_1)$
4	R_1S_2, D_1S_2	y_3	$\frac{1}{3}(2y_1+2y_2-y_3)$	g_{33}	$\frac{1}{3^{1/2}}(z_1-z_2)$	$-z_3$	$\frac{1}{3^{1/2}}(y_1-y_2)$
5	R_2S_1, D_2S_1	y_2	$\frac{1}{3}(2y_1-y_2+2y_3)$	g_{33}	$\frac{1}{3^{1/2}}(z_1-z_3)$	z_2	$\frac{1}{3^{1/2}}(y_3-y_1)$
6	R_2S_2, D_2S_2	y_3	$\frac{1}{3}(2y_1+2y_2-y_3)$	g_{33}	$-\frac{1}{3^{1/2}}(z_1-z_2)$	z_3	$\frac{1}{3^{1/2}}(y_1-y_2)$
7	R_3S_1, D_3S_1	y_2	$\frac{1}{3}(2y_1-y_2+2y_3)$	g_{33}	$\frac{1}{3^{1/2}}(z_1-z_3)$	$-z_2$	$\frac{1}{3^{1/2}}(y_1-y_3)$
8	R_3S_2, D_3S_2	y_3	$\frac{1}{3}(2y_1+2y_2-y_3)$	g_{33}	$-\frac{1}{3^{1/2}}(z_1-z_2)$	$-z_3$	$\frac{1}{3^{1/2}}(y_2-y_1)$

4. **Some theorems on polynomial bases.** It has been seen that the requirement that the strain-energy function for a particular crystal class, regarded as a polynomial in the quantities g_{ij} defined by (3.1), be form-invariant under the transformations associated with that class imposes on W a number of restrictions of the type expressed by (3.6) and (3.5). It is well-known that polynomials in a number of variables which are form-invariant under a finite

group of transformations possess a finite polynomial basis, such that each of the polynomials can be expressed as a polynomial in the elements of the basis and any polynomial in the elements of the basis is form-invariant under the group of transformations. Thus, in order to determine explicitly the limitations imposed on the form of W by the requirement of form-invariance under the group of transformations characterizing a crystal class, we have to determine a polynomial basis for W regarded as a function of the quantities g_{ij}. This will be done for each of the crystal classes in §§5 to 10. It will be seen there that for each of the crystal classes the restrictions imposed on W are equivalent to the restriction that it be form-invariant under some sub-group of the full symmetric group of permutations of a number of linear combinations of the six quantities g_{ij}. In this section we shall give some theorems regarding the determination of a polynomial basis for polynomials in a number of variables which are form-invariant under certain sub-groups of the full symmetric group of permutations.

THEOREM 1. *A polynomial basis for polynomials which are symmetric in the two sets of variables* $(y_1, y_2, \cdots, y_n)$ *and* $(z_1, z_2, \cdots, z_n)$ *is formed by the quantities*

$$\begin{aligned} I_j &= (y_j + z_j)/2 \qquad (j = 1, ,2 \cdots, n), \\ I_{jk} &= (y_j z_k + y_k z_j)/2 \qquad (j, k = 1, 2, \cdots, n). \end{aligned} \tag{4.1}$$

THEOREM 2. *A polynomial basis for polynomials which are symmetric in the three pairs of variables* (y_1, z_1), (y_2, z_2) *and* (y_3, z_3) *is formed by the quantities,*

$$\begin{aligned} &J_1 = y_1 + y_2 + y_3, \quad J_2 = y_2 y_3 + y_3 y_1 + y_1 y_2, \quad J_3 = y_1 y_2 y_3, \\ &J_4 = z_1 + z_2 + z_3, \quad J_5 = z_2 z_3 + z_3 z_1 + z_1 z_2, \quad J_6 = z_1 z_2 z_3, \\ &J_7 = y_2 z_3 + y_3 z_2 + y_3 z_1 + y_1 z_3 + y_1 z_2 + y_2 z_1, \\ &J_8 = y_1 z_2 z_3 + y_2 z_3 z_1 + y_3 z_1 z_2, \quad J_9 = z_1 y_2 y_3 + z_2 y_3 y_1 + z_3 y_1 y_2. \end{aligned} \tag{4.2}$$

Theorems 1 and 2 follow immediately from a well-known theorem in the theory of invariants (see, for example, H. Weyl, 1946, p. 36 et seq.).

THEOREM 3. *A polynomial basis for polynomials in the variables* $y_1, y_2, y_3, z_1, z_2, z_3$ *which are form-invariant under cyclic rotation of the subscripts* 1, 2, 3 *is formed by the quantities*

$$\begin{aligned} &K_1 = y_1 + y_2 + y_3, \quad K_2 = y_2 y_3 + y_3 y_1 + y_1 y_2, \quad K_3 = y_1 y_2 y_3, \\ &K_4 = z_1 + z_2 + z_3, \quad K_5 = z_2 z_3 + z_3 z_1 + z_1 z_2, \quad K_6 = z_1 z_2 z_3, \\ &K_7 = y_2 z_3 + y_3 z_1 + y_1 z_2, \quad K_8 = z_2 y_3 + z_3 y_1 + z_1 y_2, \\ &K_9 = y_3 y_2^2 + y_1 y_3^2 + y_2 y_1^2, \quad K_{10} = z_3 z_2^2 + z_1 z_3^2 + z_2 z_1^2, \\ &K_{11} = y_1 z_2 z_3 + y_2 z_3 z_1 + y_3 z_1 z_2, \quad K_{12} = z_1 y_2 y_3 + z_2 y_3 y_1 + z_3 y_1 y_2, \\ &K_{13} = y_1 y_2 z_2 + y_2 y_3 z_3 + y_3 y_1 z_1, \quad K_{14} = z_1 z_2 y_2 + z_2 z_3 y_3 + z_3 z_1 y_1. \end{aligned} \tag{4.3}$$

This theorem follows almost immediately from the fact that the degree of the elements of a polynomial basis for polynomials which are form-invariant under a finite group of transformations is less than or equal to the number of transformations in that group (see, for example, H. Weyl, 1946, p. 276, et seq.). In this case, there are three transformations in the transformation group, and hence the elements of the polynomial basis are of degree three or lower. With this fact, it can readily be shown that the given set of invariants forms a basis.

THEOREM 4. *A polynomial basis for polynomials in the variables* $y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n, I_1, I_2, \cdots, I_r$, *which are form-invariant under a group of transformations for which* $I_1, I_2, \cdots, I_r$ *are invariants, is formed by adjoining to the quantities* $I_1, I_2, \cdots, I_r$ *a polynomial basis for polynomials in the variables* $y_1, y_2, \cdots, y_n, z_1, z_2, \cdots, z_n$ *which are form-invariant under the given group of transformations.*

The validity of this result is immediately obvious.

5. **The triclinic system.** For a material having triclinic symmetry, there is no restriction on the orientation of the unit vectors $\boldsymbol{h}_1$, $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ defining the preferred directions. We may therefore choose any rectangular Cartesian coordinate system x_i as a reference system.

The triclinic system contains two crystal classes. These are the *pedial* class, the strain-energy function for which must be form-invariant under the identity transformation $\boldsymbol{I}$, and the *pinacoidal* class, the strain-energy function for which must be form-invariant under the transformation $\boldsymbol{I}$ and the central inversion transformation $\boldsymbol{C}$.

It is seen from Table 1 that neither of the transformations $\boldsymbol{I}$ and $\boldsymbol{C}$ imposes any restriction on the form of W. Hence, the six quantities g_{ij} form a polynomial basis for the triclinic system.

6. **The monoclinic system.** For a material having monoclinic symmetry, the unit vectors $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ are not at right-angles and the unit vector $\boldsymbol{h}_1$ is perpendicular to the plane defined by $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$. We take as our reference system a rectangular Cartesian coordinate system x_i, the x_1 axis of which is parallel to $\boldsymbol{h}_1$. The axes x_2 and x_3 may be in arbitrary perpendicular directions in the $\boldsymbol{h}_2\boldsymbol{h}_3$ plane.

The monoclinic system contains three classes which, together with the symmetry transformations defining them, are: *domatic* ($\boldsymbol{I}$, $\boldsymbol{R}_1$), *sphenoidal* ($\boldsymbol{I}$, $\boldsymbol{D}_1$), *prismatic*, ($\boldsymbol{I}$, $\boldsymbol{C}$, $\boldsymbol{R}_1$, $\boldsymbol{D}_1$).

It is seen from Table 1 that, for each of the three crystal classes of the monoclinic system, the limitation imposed on the form of W is given by

$$W(g_{11}, g_{22}, g_{33}, g_{23}, g_{31}, g_{12}) = W(g_{11}, g_{22}, g_{33}, g_{23}, -g_{31}, -g_{12}).$$

Taking

$$(y_1, y_2, \cdots, y_n) = (g_{31}, g_{12}),$$
$$(z_1, z_2, \cdots, z_n) = (-g_{31}, -g_{12})$$

and

$$(I_1, I_2, \cdots, I_r) = (g_{11}, g_{22}, g_{33}, g_{23}),$$

in Theorems 1 and 4, it follows that the quantities

$$g_{11},\ g_{22},\ g_{33},\ g_{23},\ g_{31}^2,\ g_{12}^2 \quad \text{and} \quad g_{12}g_{31}$$

form a polynomial basis for W.

7. **The rhombic system.** For a material having rhombic symmetry, the unit vectors $\boldsymbol{h}_i$ are mutually perpendicular. We take as our reference system a rectangular Cartesian coordinate system x_i, the axes of which are parallel to the vectors $\boldsymbol{h}_i$.

The rhombic system contains three classes which, together with the symmetry transformations characterizing them, are: rhombic-pyramidal $(\boldsymbol{I}, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1)$, rhombic-disphenoidal $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)$, rhombic-dipyramidal $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3)$.

It is seen that for each of these classes there are three restrictions on the form of W obtained from (3.6) by giving $\bar{g}_{ij}$ the values in lines 2, 3 and 4 respectively of Table 1. We have already seen in §6 that the first of these implies that

$$W = W'(g_{11}, g_{22}, g_{33}, g_{23}, g_{31}^2, g_{12}^2, g_{12}g_{31}), \tag{7.1}$$

where W' denotes a polynomial in the indicated variables. Now,

$$W'(g_{ij}) = W'(\bar{g}_{ij}), \tag{7.2}$$

where $\bar{g}_{ij}$ is given by the quantities in line 3 of Table 1. Thus,

$$\begin{aligned} &W'(g_{11}, g_{22}, g_{33}, g_{23}, g_{31}^2, g_{12}^2, g_{12}g_{31}) \\ &\quad = W'(g_{11}, g_{22}, g_{33}, -g_{23}, g_{31}^2, g_{12}^2, -g_{12}g_{31}). \end{aligned} \tag{7.3}$$

Taking, in Theorems 1 and 4,

$$\begin{aligned} (y_1, y_2, \cdots, y_n) &= (g_{23}, g_{12}g_{31}), \\ (z_1, z_2, \cdots, z_n) &= (-g_{23}, -g_{12}g_{31}), \\ (I_1, I_2, \cdots, I_r) &= (g_{11}, g_{22}, g_{33}, g_{31}^2, g_{12}^2), \end{aligned} \tag{7.4}$$

we see that W must be expressible as a polynomial in

$$g_{11},\ g_{22},\ g_{33},\ g_{23}^2,\ g_{31}^2,\ g_{12}^2 \text{ and } g_{23}g_{31}g_{12}. \tag{7.5}$$

It is readily seen that any such polynomial satisfies the restriction obtained from (3.6) by giving $\bar{g}_{ij}$ the values in line 4 of Table 1. Hence, the quantities (7.5) form a polynomial basis for W.

8. **The tetragonal system.** For a material having tetragonal symmetry, the unit vectors $\boldsymbol{h}_i$ are mutually perpendicular. As in the case of the rhombic system, we take as our reference system a rectangular Cartesian coordinate system x_i, the axes of which are parallel to the vectors $\boldsymbol{h}_i$. We shall take the x_3 axis to be the principal axis of symmetry.

The tetragonal system contains seven crystal classes which, together with the transformation groups characterizing them, are:

tetragonal-disphenoidal	$(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3)$,
tetragonal-pyramidal	$(\boldsymbol{I}, \boldsymbol{D}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3)$,
tetragonal-dipyramidal	$(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_3, \boldsymbol{D}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3)$,
tetragonal-scalenohedral	$(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3)$,
ditetragonal-pyramidal	$(\boldsymbol{I}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{D}_3, \boldsymbol{T}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3)$,
tetragonal-trapezohedral	$(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{C}\boldsymbol{T}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{R}_3\boldsymbol{T}_3)$,
ditetragonal-dipyramidal	$(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{T}_3, \boldsymbol{C}\boldsymbol{T}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{R}_3\boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3)$.

It is seen that for each of the first three classes of the tetragonal system listed above (tetragonal-disphenoidal, tetragonal-pyramidal, tetragonal-dipyramidal), the limitations on W are those obtained by substituting for $\bar{g}_{ij}$ in (3.6) from Lines 4, 14 and 15 respectively of Table 1. It can be shown in a manner analogous to that adopted in discussing the monoclinic system that the first of these limitations implies that

$$W(g_{ij}) = W'(g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}, g_{23}g_{31}), \tag{8.1}$$

where W' is a polynomial in the indicated variables. With (8.1), the restriction derived from line 14 of Table 1 yields

$$\begin{aligned} &W'(g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}, g_{23}g_{31}) \\ &\qquad = W'(g_{22}, g_{11}, g_{33}, g_{31}^2, g_{23}^2, -g_{12}, -g_{23}g_{31}). \end{aligned} \tag{8.2}$$

Employing Theorems 1 and 4 as before and omitting redundant elements, we see that W' and hence W is expressible as a polynomial in

$$\begin{gathered} g_{11} + g_{22},\ g_{33},\ g_{23}^2 + g_{31}^2,\ g_{12}^2,\ g_{11}g_{22},\ g_{12}(g_{11} - g_{22}), \\ g_{23}g_{31}(g_{11} - g_{22}),\ g_{23}g_{31}g_{12},\ g_{12}(g_{31}^2 - g_{23}^2), \\ g_{11}g_{23}^2 + g_{22}g_{31}^2,\ g_{23}g_{31}(g_{31}^2 - g_{23}^2) \text{ and } g_{23}^2g_{31}^2. \end{gathered} \tag{8.3}$$

Since such a polynomial satisfies the limitation derived from line 15 of Table 1, we see that the quantities (8.3) form a polynomial basis for the first three classes of the tetragonal system.

For the remaining four classes of the tetragonal system, the restrictions imposed on W are obtained by substituting in (3.6) from lines 2, 3, 4, 13, 14, 15 and 16 respectively of Table 1. In the case of the rhombic system it was seen that the first three of these restrictions implies that W is a polynomial W' (say) in the quantities (7.5). Introducing the limitation derived from line 13 of Table 1 and employing Theorems 1 and 4 as before, we see that W' and hence W must be expressible as a polynomial in

$$(8.4)\qquad g_{11}+g_{22},\ g_{33},\ g_{23}^2+g_{31}^2,\ g_{12}^2,\ g_{11}g_{22},\ g_{23}g_{31}g_{12},\ g_{11}g_{23}^2+g_{22}g_{31}^2 \text{ and } g_{23}^2g_{31}^2.$$

Such a polynomial automatically satisfies the limitations derived from lines 14, 15 and 16 of Table 1 and hence forms a polynomial basis for the tetragonal-scalenohedral, ditetragonal-pyramidal, tetragonal-trapezohedral and ditetragonal-dipyramidal classes.

9. **The cubic system.** For a material having cubic symmetry the unit vectors $\boldsymbol{h}_1$, $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ are mutually perpendicular. We take as our reference system a rectangular Cartesian coordinate system x_i, the axes of which are parallel to the vectors $\boldsymbol{h}_i$.

The cubic system contains five classes which, together with the symmetry transformations characterizing them, are:

tetartoidal $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{M}_1, \boldsymbol{D}_1\boldsymbol{M}_1, \boldsymbol{D}_2\boldsymbol{M}_1, \boldsymbol{D}_3\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{D}_1\boldsymbol{M}_2, \boldsymbol{D}_2\boldsymbol{M}_2, \boldsymbol{D}_3\boldsymbol{M}_2)$,

diploidal $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{M}_1, \boldsymbol{C}\boldsymbol{M}_1, \boldsymbol{R}_1\boldsymbol{M}_1, \boldsymbol{R}_2\boldsymbol{M}_1, \boldsymbol{R}_3\boldsymbol{M}_1, \boldsymbol{D}_1\boldsymbol{M}_1, \boldsymbol{D}_2\boldsymbol{M}_1, \boldsymbol{D}_3\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{C}\boldsymbol{M}_2, \boldsymbol{R}_1\boldsymbol{M}_2, \boldsymbol{R}_2\boldsymbol{M}_2, \boldsymbol{R}_3\boldsymbol{M}_2, \boldsymbol{D}_1\boldsymbol{M}_2, \boldsymbol{D}_2\boldsymbol{M}_2, \boldsymbol{D}_3\boldsymbol{M}_2)$,

hextetrahedral $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{T}_1, \boldsymbol{D}_1\boldsymbol{T}_1, \boldsymbol{D}_2\boldsymbol{T}_1, \boldsymbol{D}_3\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{D}_1\boldsymbol{T}_2, \boldsymbol{D}_2\boldsymbol{T}_2, \boldsymbol{D}_3\boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3, \boldsymbol{M}_1, \boldsymbol{D}_1\boldsymbol{M}_1, \boldsymbol{D}_2\boldsymbol{M}_1, \boldsymbol{D}_3\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{D}_1\boldsymbol{M}_2, \boldsymbol{D}_2\boldsymbol{M}_2, \boldsymbol{D}_3\boldsymbol{M}_2)$,

gyroidal $(\boldsymbol{I}, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{C}\boldsymbol{T}_1, \boldsymbol{R}_1\boldsymbol{T}_1, \boldsymbol{R}_2\boldsymbol{T}_1, \boldsymbol{R}_3\boldsymbol{T}_1, \boldsymbol{C}\boldsymbol{T}_2, \boldsymbol{R}_1\boldsymbol{T}_2, \boldsymbol{R}_2\boldsymbol{T}_2, \boldsymbol{R}_3\boldsymbol{T}_2, \boldsymbol{C}\boldsymbol{T}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{R}_3\boldsymbol{T}_3, \boldsymbol{M}_1, \boldsymbol{D}_1\boldsymbol{M}_1, \boldsymbol{D}_2\boldsymbol{M}_1, \boldsymbol{D}_3\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{D}_1\boldsymbol{M}_2, \boldsymbol{D}_2\boldsymbol{M}_2, \boldsymbol{D}_3\boldsymbol{M}_2)$,

hexoctahedral $(\boldsymbol{I}, \boldsymbol{C}, \boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{T}_1, \boldsymbol{C}\boldsymbol{T}_1, \boldsymbol{R}_1\boldsymbol{T}_1, \boldsymbol{R}_2\boldsymbol{T}_1, \boldsymbol{R}_3\boldsymbol{T}_1, \boldsymbol{D}_1\boldsymbol{T}_1, \boldsymbol{D}_2\boldsymbol{T}_1, \boldsymbol{D}_3\boldsymbol{T}_1, \boldsymbol{T}_2, \boldsymbol{C}\boldsymbol{T}_2, \boldsymbol{R}_1\boldsymbol{T}_2, \boldsymbol{R}_2\boldsymbol{T}_2, \boldsymbol{R}_3\boldsymbol{T}_2, \boldsymbol{D}_1\boldsymbol{T}_2, \boldsymbol{D}_2\boldsymbol{T}_2, \boldsymbol{D}_3\boldsymbol{T}_2, \boldsymbol{T}_3, \boldsymbol{C}\boldsymbol{T}_3, \boldsymbol{R}_1\boldsymbol{T}_3, \boldsymbol{R}_2\boldsymbol{T}_3, \boldsymbol{R}_3\boldsymbol{T}_3, \boldsymbol{D}_1\boldsymbol{T}_3, \boldsymbol{D}_2\boldsymbol{T}_3, \boldsymbol{D}_3\boldsymbol{T}_3, \boldsymbol{M}_1, \boldsymbol{C}\boldsymbol{M}_1, \boldsymbol{R}_1\boldsymbol{M}_1, \boldsymbol{R}_2\boldsymbol{M}_1, \boldsymbol{R}_3\boldsymbol{M}_1, \boldsymbol{D}_1\boldsymbol{M}_1, \boldsymbol{D}_2\boldsymbol{M}_1, \boldsymbol{D}_3\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{C}\boldsymbol{M}_2, \boldsymbol{R}_1\boldsymbol{M}_2, \boldsymbol{R}_2\boldsymbol{M}_2, \boldsymbol{R}_3\boldsymbol{M}_2, \boldsymbol{D}_1\boldsymbol{M}_2, \boldsymbol{D}_2\boldsymbol{M}_2, \boldsymbol{D}_3\boldsymbol{M}_2)$.

(i) *Tetartoidal and diploidal classes.*

It is seen that for each of these crystal classes the limitations imposed on W are obtained by substituting in (3.6) for $\bar{g}_{ij}$ from lines 2, 3, 4 and lines 17 to 24 respectively of Table 1. As in the case of the rhombic system, it is seen that the restrictions obtained from lines 2, 3 and 4 imply that W is a polynomial W' (say) in the quantities (7.5). It is readily seen that the restrictions obtained from lines 17 to 24 imply that W' must satisfy the relations

$$W'(g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}^2, g_{23}g_{31}g_{12})$$
$$(9.1) \qquad = W'(g_{22}, g_{33}, g_{11}, g_{31}^2, g_{12}^2, g_{23}^2, g_{23}g_{31}g_{12})$$
$$= W'(g_{33}, g_{11}, g_{22}, g_{12}^2, g_{23}^2, g_{31}^2, g_{23}g_{31}g_{12}).$$

Taking

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}^2)$$

and

$$(I_1, I_2, \cdots, I_r) = g_{23}g_{31}g_{12}$$

in Theorems 3 and 4, we see that W' and hence W must be expressible as a polynomial in $K_1, K_2, \cdots, K_{14}$ and $g_{23}g_{31}g_{12}$, where

$$K_1 = g_{11} + g_{22} + g_{33}, \qquad K_2 = g_{22}g_{33} + g_{33}g_{11} + g_{11}g_{22},$$
$$K_3 = g_{11}g_{22}g_{33}, \qquad K_4 = g_{23}^2 + g_{31}^2 + g_{12}^2,$$
$$K_5 = g_{31}^2g_{12}^2 + g_{12}^2g_{23}^2 + g_{23}^2g_{31}^2, \qquad K_6 = g_{23}^2g_{31}^2g_{12}^2,$$
$$K_7 = g_{22}g_{12}^2 + g_{33}g_{23}^2 + g_{11}g_{31}^2, \qquad K_8 = g_{31}^2g_{33} + g_{12}^2g_{11} + g_{23}^2g_{22},$$
$$(9.2) \qquad K_9 = g_{33}g_{22}^2 + g_{11}g_{33}^2 + g_{22}g_{11}^2, \qquad K_{10} = g_{12}^2g_{31}^4 + g_{23}^2g_{12}^4 + g_{31}^2g_{23}^4,$$
$$K_{11} = g_{11}g_{31}^2g_{12}^2 + g_{22}g_{12}^2g_{23}^2 + g_{33}g_{23}^2g_{31}^2,$$
$$K_{12} = g_{23}^2g_{22}g_{33} + g_{31}^2g_{33}g_{11} + g_{12}^2g_{11}g_{22},$$
$$K_{13} = g_{11}g_{22}g_{31}^2 + g_{22}g_{33}g_{12}^2 + g_{33}g_{11}g_{23}^2,$$
$$K_{14} = g_{23}^2g_{31}^2g_{22} + g_{31}^2g_{12}^2g_{33} + g_{12}^2g_{23}^2g_{11}.$$

It is evident that K_6 is a redundant element. Hence a polynomial basis for these two classes is formed by the quantities K_1 to K_5, K_7 to K_{14} and $g_{23}g_{31}g_{12}$.

(ii) *Hextetrahedral, gyroidal and hexoctahedral classes.*

For each of these classes the restrictions imposed on W are obtained by substituting in (3.6) for $\bar{g}_{ij}$ from lines 2 to 24 respectively of Table 1. The restrictions derived from lines 2, 3 and 4 imply that W must be expressible in the form

$$(9.3) \qquad W = W'(g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}^2, g_{23}g_{31}g_{12}),$$

where W' is a polynomial in the indicated variables. The remaining restrictions leave $g_{23}g_{31}g_{12}$ unaltered and imply that W' is unaltered by interchange of the pairs of variables (g_{11}, g_{23}^2), (g_{22}, g_{31}^2) and (g_{33}, g_{12}^2). Taking

$$(9.4)\qquad \begin{aligned} &(y_1, z_1) = (g_{11}, g_{23}^2), && (y_2, z_2) = (g_{22}, g_{31}^2),\\ &(y_3, z_3) = (g_{33}, g_{12}^2) \quad \text{and} && (I_1, I_2, \cdots, I_r) = g_{23}g_{31}g_{12} \end{aligned}$$

in Theorems 2 and 4, we see that W' and hence W must be expressible as a polynomial in $g_{23}g_{31}g_{12}$ and $J_1, J_2, \cdots, J_9$ where

$$(9.5)\qquad \begin{aligned} &J_1 = g_{11} + g_{22} + g_{33}, && J_2 = g_{22}g_{33} + g_{33}g_{11} + g_{11}g_{22},\\ &J_3 = g_{11}g_{22}g_{33}, && J_4 = g_{23}^2 + g_{31}^2 + g_{12}^2,\\ &J_5 = g_{31}^2g_{12}^2 + g_{12}^2g_{23}^2 + g_{23}^2g_{31}^2, && J_6 = g_{23}^2g_{31}^2g_{12}^2,\\ &J_7 = g_{22}g_{22}^2 + g_{33}g_{31}^2 + g_{33}g_{23}^2 + g_{11}g_{12}^2 + g_{11}g_{31}^2 + g_{22}g_{23}^2, &&\\ &J_8 = g_{11}g_{31}^2g_{12}^2 + g_{22}g_{12}^2g_{23}^2 + g_{33}g_{23}^2g_{31}^2, &&\\ &J_9 = g_{23}^2g_{22}g_{33} + g_{31}^2g_{33}g_{11} + g_{12}^2g_{11}g_{22}. && \end{aligned}$$

Since J_6 is a redundant element, we see that the quantities J_1 to J_5, J_7 to J_9 and $g_{23}g_{31}g_{12}$ form a polynomial basis for the hextetrahedral, gyroidal and hexoctahedral classes.

We note that the elements of the polynomial bases of first and second degree in g_{ij} are the same for all the classes of the cubic system. Consequently, if the displacement gradients, and hence the quantities g_{ij}, are sufficiently small so that terms of higher degree than the second in the expression for W as a polynomial in g_{ij} can be neglected, then W takes the same form for all the classes of the cubic system.

10. **The hexagonal system.** For a material having hexagonal symmetry, the unit vectors $\boldsymbol{h}_i$ are situated so that $\boldsymbol{h}_3$ is perpendicular to the plane defined by $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, and so that $\boldsymbol{h}_1$ can be made to coincide with $\boldsymbol{h}_2$ by a rotation of 120° about the direction of $\boldsymbol{h}_3$. We take as our reference system a right-handed rectangular Cartesian coordinate system x_i, the x_1 and x_3 axes of which are parallel respectively to $\boldsymbol{h}_1$ and $\boldsymbol{h}_3$.

The hexagonal system contains twelve crystal classes which, together with the transformation groups characterizing them, are:

trigonal-pyramidal	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2)$,
rhombohedral	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{C}, \boldsymbol{CS}_1, \boldsymbol{CS}_2)$,
ditrigonal-pyramidal	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{R}_1, \boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_1\boldsymbol{S}_2)$,
trigonal-trapezohedral	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2)$,
hexagonal-scalenohedral	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{C}, \boldsymbol{CS}_1, \boldsymbol{CS}_2, \boldsymbol{R}_1, \boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_1\boldsymbol{S}_2, \boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2)$,
trigonal-dipyramidal	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{R}_3, \boldsymbol{R}_3\boldsymbol{S}_1, \boldsymbol{R}_3\boldsymbol{S}_2)$,
hexagonal-pyramidal	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{D}_3, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_2)$,
hexagonal-dipyramidal	$(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{C}, \boldsymbol{CS}_1, \boldsymbol{CS}_2, \boldsymbol{R}_3, \boldsymbol{R}_3\boldsymbol{S}_1, \boldsymbol{R}_3\boldsymbol{S}_2, \boldsymbol{D}_3, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_2)$,

ditrigonal-dipyramidal $(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{R}_1, \boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_1\boldsymbol{S}_2, \boldsymbol{R}_3, \boldsymbol{R}_3\boldsymbol{S}_1, \boldsymbol{R}_3\boldsymbol{S}_2, \boldsymbol{D}_2, \boldsymbol{D}_2\boldsymbol{S}_1, \boldsymbol{D}_2\boldsymbol{S}_2)$,

dihexagonal-pyramidal $(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{R}_1, \boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_1\boldsymbol{S}_2, \boldsymbol{R}_2, \boldsymbol{R}_2\boldsymbol{S}_1, \boldsymbol{R}_2\boldsymbol{S}_2, \boldsymbol{D}_3, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_2)$,

hexagonal-trapezohedral $(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2, \boldsymbol{D}_2, \boldsymbol{D}_2\boldsymbol{S}_1, \boldsymbol{D}_2\boldsymbol{S}_2, \boldsymbol{D}_3, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_2)$,

dihexagonal-dipyramidal $(\boldsymbol{I}, \boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{C}, \boldsymbol{C}\boldsymbol{S}_1, \boldsymbol{C}\boldsymbol{S}_2, \boldsymbol{R}_1, \boldsymbol{R}_1\boldsymbol{S}_1, \boldsymbol{R}_1\boldsymbol{S}_2, \boldsymbol{R}_2, \boldsymbol{R}_2\boldsymbol{S}_1, \boldsymbol{R}_2\boldsymbol{S}_2, \boldsymbol{R}_3, \boldsymbol{R}_3\boldsymbol{S}_1, \boldsymbol{R}_3\boldsymbol{S}_2, \boldsymbol{D}_1, \boldsymbol{D}_1\boldsymbol{S}_1, \boldsymbol{D}_1\boldsymbol{S}_2, \boldsymbol{D}_2, \boldsymbol{D}_2\boldsymbol{S}_1, \boldsymbol{D}_2\boldsymbol{S}_2, \boldsymbol{D}_3, \boldsymbol{D}_3\boldsymbol{S}_1, \boldsymbol{D}_3\boldsymbol{S}_2)$.

(i) *Trigonal-pyramidal and rhombohedral classes.*

For each of these classes, the restrictions on W are obtained from (3.6) by assigning to $\bar{g}_{ij}$ the values given in lines 1 and 2 respectively of Table 2. They may be written as

$$
\begin{aligned}
&W(g_{11}, g_{22}, g_{33}, g_{23}, g_{31}, g_{12}) \\
&= W\left[y_2, \frac{1}{3}(2y_1 - y_2 + 2y_3), g_{33}, -\frac{1}{3^{1/2}}(z_1 - z_3), z_2, \frac{1}{3^{1/2}}(y_1 - y_3)\right] \\
&= W\left[y_3, \frac{1}{3}(2y_1 + 2y_2 - y_3), g_{33}, \frac{1}{3^{1/2}}(z_1 - z_2), z_3, \frac{1}{3^{1/2}}(y_2 - y_1)\right],
\end{aligned}
\tag{10.1}
$$

where the notation of Equations (3.7) is used. From (3.7), we see that

$$
\begin{aligned}
g_{11} = y_1,\ g_{22} &= \frac{1}{3}(2y_2 + 2y_3 - y_1), \\
g_{12} &= \frac{1}{3^{1/2}}(y_3 - y_2), \\
g_{31} = z_1 \quad \text{and} \quad g_{23} &= \frac{1}{3^{1/2}}(z_2 - z_3).
\end{aligned}
\tag{10.2}
$$

Introducing (10.2) into (10.1), and noting that we may regard W as a polynomial W' (say) in $y_1, y_2, y_3, z_1, z_2, z_3$ and g_{33}, we see that (10.1) may be rewritten as

$$
\begin{aligned}
&W'(y_1, y_2, y_3, z_1, z_2, z_3, g_{33}) \\
&= W'(y_2, y_3, y_1, z_2, z_3, z_1, g_{33}) \\
&= W'(y_3, y_1, y_2, z_3, z_1, z_2, g_{33}).
\end{aligned}
\tag{10.3}
$$

Employing Theorems 3 and 4, we see that a polynomial basis for W' and hence for W is formed by g_{33} and $K_1, K_2, \cdots, K_{14}$, where $K_1, K_2, \cdots, K_{14}$ are obtained from Equations (4.3) by substituting for $y_1, y_2, y_3, z_1, z_2, z_3$ from (3.7). It can be shown that if W' is expressible as a polynomial in $g_{33}, K_1, K_2, \cdots, K_{14}$ it is also expressible as a polynomial in

$$
\begin{aligned}
&g_{33},\ g_{11}+g_{22},\ g_{11}g_{22}-g_{12}^2,\ g_{11}[(g_{11}+3g_{22})^2-12g_{12}^2],\\
&g_{31}^2+g_{23}^2,\ g_{31}(g_{31}^2-3g_{23}^2),\ (g_{11}-g_{22})g_{31}-2g_{12}g_{23},\\
&(g_{22}-g_{11})g_{23}-2g_{12}g_{31},\ 3g_{12}(g_{11}-g_{22})^2-4g_{12}^3,\\
&g_{23}(g_{23}^2-3g_{31}^2),\ g_{22}g_{31}^2+g_{11}g_{23}^2-2g_{23}g_{31}g_{12},\\
&g_{31}[(g_{11}+g_{22})^2+4(g_{12}^2-g_{22}^2)]-8g_{11}g_{12}g_{23}.\\
&g_{23}[(g_{11}+g_{22})^2+4(g_{12}^2-g_{22}^2)]+8g_{11}g_{12}g_{31},\\
&(g_{11}-g_{22})g_{23}g_{31}+g_{12}(g_{23}^2-g_{31}^2)
\end{aligned}
\tag{10.4}
$$

and that any polynomial in the quantities (10.4) may be expressed as a polynomial in g_{33} and $K_1, K_2, \cdots, K_{14}$. These results can be readily obtained by inspection if we express each of the quantities $g_{33}, K_1, K_2, \cdots, K_{14}$ as a polynomial in the quantities (10.4). Thus, the expressions (10.4) form a polynomial basis for the trigonal-pyramidal and rhombohedral classes.

(ii) *Ditrigonal-pyramidal, trigonal-trapezohedral and hexagonal-scalenohedral classes.*

For each of these classes, the restrictions imposed on W are obtained from (3.6) by assigning to $\bar{g}_{ij}$ the values given in line 2 of Table 1 and lines 1, 2, 3 and 4 respectively of Table 2. As in the case of the trigonal-pyramidal and rhombohedral classes, we regard W as a polynomial W' in $y_1, y_2, y_3, z_1, z_2, z_3$ and g_{33} defined by (3.7) and obtain the limitations imposed on W'. We then employ the notation

$$
\begin{aligned}
&Y_1=y_1,\quad Y_2=y_2,\quad Y_3=y_3,\quad Z_1=\frac{1}{3^{1/2}}(z_2-z_3),\\
&Z_2=\frac{1}{3^{1/2}}(z_3-z_1),\quad Z_3=\frac{1}{3^{1/2}}(z_1-z_2)
\end{aligned}
\tag{10.5}
$$

and consider W' and hence W as a polynomial W'' (say) in $Y_1, Y_2, Y_3, Z_1, Z_2, Z_3$ and g_{33}. The restrictions on the form of W'' then take the form that it is unaltered by interchange of any pair of the variables (Y_1, Z_1), (Y_2, Z_2), (Y_3, Z_3) while the dependence on g_{33} involves no restrictions; i.e.,

$$
\begin{aligned}
&W''(Y_1, Y_2, Y_3, Z_1, Z_2, Z_3, g_{33})\\
&\quad = W''(Y_1, Y_3, Y_2, Z_1, Z_3, Z_2, g_{33})\\
&\quad = \text{etc.}
\end{aligned}
\tag{10.6}
$$

Then we see from Theorems 2 and 4 that the quantities $g_{33}, J_1, J_2, \cdots, J_9$ form a polynomial basis for W'', and hence for W, if we replace $y_1, y_2, \cdots, z_3$ by $Y_1, Y_2, \cdots, Z_3$ in the expressions (4.2) for $J_1, J_2, \cdots, J_9$. By expressing each of the quantities g_{33}, J_1, J_2, $\cdots$, J_9 so obtained as a polynomial in g_{ij} and then as a polynomial in the quantities

$$(10.7)\qquad \begin{aligned} &g_{33},\ g_{11}+g_{22},\ g_{11}g_{22}-g_{12}^2,\ g_{11}[(g_{11}+3g_{22})^2-12g_{12}^2],\\ &g_{31}^2+g_{23}^2,\ g_{23}(g_{23}^2-3g_{31}^2),\ (g_{11}-g_{22})g_{23}+2g_{12}g_{31},\\ &g_{11}g_{31}^2+g_{22}g_{23}^2+2g_{23}g_{31}g_{12},\\ &g_{23}[(g_{11}+g_{22})^2-4(g_{22}^2-g_{12}^2)]+8g_{11}g_{12}g_{31}, \end{aligned}$$

it can be readily seen that any polynomial in g_{33}, J_1, J_2, $\cdots$, J_9 can be expressed as a polynomial in the quantities (10.7) and conversely any polynomial in the quantities (10.7) can be expressed as a polynomial in g_{33}, J_1, J_2, $\cdots$, J_9. Thus, the expressions (10.7) form a polynomial basis for the ditrigonal-pyramidal, trigonal-trapezohedral and hexagonal-scalenohedral classes.

(iii) *Trigonal-dipyramidal, hexagonal-pyramidal and hexagonal-dipyramidal classes.*

For each of these classes, the restrictions imposed on W are obtained from (3.6) by assigning to $\bar{g}_{ij}$ the values given in line 4 of Table 1 and lines 1, 2, 7 and 8 respectively of Table 2. It can be shown in a manner analogous to that adopted in discussing the monoclinic system, that if W satisfies the restriction derived from line 4 of Table 1, it must be expressible in the form

$$(10.8)\qquad W = W'(g_{11}, g_{22}, g_{33}, g_{23}^2, g_{31}^2, g_{12}, g_{23}g_{31})$$

where W' is a polynomial. We now employ the notation

$$(10.9)\qquad Y_1 = y_1,\ Y_2 = y_2,\ Y_3 = y_3,\ Z_1 = z_1^2,\ Z_2 = z_2^2\ \text{ and }\ Z_3 = z_3^2$$

in (10.8), where y_1, y_2, $\cdots$, z_3 are defined by (3.7). It is then seen that W' may be considered as a polynomial W'' (say) in Y_1, Y_2, $\cdots$, Z_3 and g_{33}. Introducing into W'' the restrictions derived from lines 1, 2, 7 and 8 of Table 2, we obtain

$$(10.10)\qquad \begin{aligned} &W''(Y_1, Y_2, Y_3, Z_1, Z_2, Z_3, g_{33})\\ &\quad = W''(Y_2, Y_3, Y_1, Z_2, Z_3, Z_1, g_{33})\\ &\quad = W''(Y_3, Y_1, Y_2, Z_3, Z_1, Z_2, g_{33}). \end{aligned}$$

Employing Theorems 3 and 4, we see that g_{33} and the quantities K_1, K_2, $\cdots$, K_{14} defined by (4.3), with Y_1, Y_2, $\cdots$, Z_3 substituted for y_1, y_2, $\cdots$, z_3, form a polynomial basis for W'' and hence for W. Each of the quantities g_{33}, K_1, K_2, $\cdots$, K_{14} may be expressed as a polynomial in the quantities g_{ij} by substitution from (10.9) and (3.7). Then, by expressing each of the quanties g_{33}, K_1, K_2, $\cdots$, K_{14} as a polynomial in the quantities

$$
\begin{aligned}
&g_{33},\ g_{11}+g_{22},\ g_{11}g_{22}-g_{12}^2,\ g_{11}[(g_{11}+3g_{22})^2-12g_{12}^2],\\
&g_{31}^2+g_{23}^2,\ g_{31}^2(g_{31}^2-3g_{23}^2)^2,\ g_{11}g_{23}^2+g_{22}g_{31}^2-2g_{23}g_{31}g_{12},\\
&g_{12}(g_{31}^2-g_{23}^2)+(g_{22}-g_{11})g_{31}g_{23},\ 3g_{12}(g_{11}-g_{22})^2-4g_{12}^3,\\
&g_{31}g_{23}[3(g_{31}^2-g_{23}^2)^2-4g_{31}^2g_{23}^2],\\
&g_{11}(g_{31}^4+3g_{23}^4)+2g_{22}g_{31}^2(g_{31}^2+3g_{23}^2)-8g_{12}g_{23}g_{31}^3,\\
&g_{31}^2[(g_{11}+g_{22})^2-4(g_{22}^2-g_{12}^2)]-2g_{11}[(g_{11}+3g_{22})(g_{31}^2+g_{23}^2)-4g_{23}g_{31}g_{12}],\\
&g_{23}g_{31}[(g_{11}+g_{22})^2-4(g_{22}^2-g_{12}^2)]+4g_{11}g_{12}(g_{23}^2-g_{31}^2),\\
&g_{12}[(g_{31}^2+g_{23}^2)^2+4g_{23}^2(g_{31}^2-g_{23}^2)]-4g_{31}^3g_{23}(g_{11}-g_{22}),
\end{aligned}
\tag{10.11}
$$

we see that the quantities (10.11) form a polynomial basis for W for the trigonal-dipyramidal, hexagonal-pyramidal and hexagonal-dipyramidal classes.

(iv) *Ditrigonal-dipyramidal, dihexagonal-pyramidal, hexagonal-trapezohedral and dihexagonal-dipyramidal classes.*

For each of these classes the restrictions imposed on W are obtained from (3.6) by assigning to $\bar{g}_{ij}$ the values given in lines 2, 3 and 4 of Table 1 and lines 1, 2, $\cdots$, 8 of Table 2 respectively. It can be shown in the manner adopted in discussing the preceding three classes of the hexagonal system that if W satisfies the restrictions derived from line 4 of Table 1 and lines 1, 2, 7 and 8 of Table 2, it must be expressible as a polynomial W'' in g_{33} and the quantities Y_1, Y_2, Y_3, Z_1, Z_2, Z_3, defined by (10.9) and (3.7), which satisfies the relations (10.10). In a similar manner it can be shown that the restrictions on W derived from lines 2 and 3 of Table 1 and lines 3, 4, 5 and 6 of Table 2 impose three further restrictions on W''. These are

$$
\begin{aligned}
&W''(Y_1, Y_2, Y_3, Z_1, Z_2, Z_3, g_{33})\\
&\quad = W''(Y_1, Y_3, Y_2, Z_1, Z_3, Z_2, g_{33})\\
&\quad = W''(Y_2, Y_1, Y_3, Z_2, Z_1, Z_3, g_{33})\\
&\quad = W''(Y_3, Y_2, Y_1, Z_3, Z_2, Z_1, g_{33}).
\end{aligned}
\tag{10.12}
$$

The relations (10.10) and (10.12) together imply that W'' must be symmetric in the three pairs of variables (Y_1, Z_1), (Y_2, Z_2) and (Y_3, Z_3). We see from Theorems 2 and 4 that g_{33} and J_1, J_2, $\cdots$, J_9, defined by (4.2) with y_1, y_2, $\cdots$, z_3 replaced by Y_1, Y_2, $\cdots$, Z_3, form a polynomial basis for W'' and hence for W. Expressing g_{33}, J_1, J_2, $\cdots$, J_9 as polynomials in g_{ij} and then in the quantities

$$
\begin{aligned}
(10.13)\quad & g_{33},\ g_{11}+g_{22},\ g_{11}g_{22}-g_{12}^2,\ g_{11}[(g_{11}+3g_{22})^2-12g_{12}^2], \\
& g_{31}^2+g_{23}^2,\ g_{31}^2(g_{31}^2-3g_{23}^2)^2,\ g_{11}g_{23}^2+g_{22}g_{31}^2-2g_{23}g_{31}g_{12}, \\
& g_{11}(g_{31}^4+3g_{23}^4)+2g_{22}g_{31}^2(g_{31}^2+3g_{23}^2)-8g_{12}g_{23}g_{31}^3, \\
& g_{31}^2[(g_{11}+g_{22})^2-4(g_{22}^2-g_{12}^2)]-2g_{11}[(g_{11}+3g_{22})(g_{31}^2+g_{23}^2)-4g_{23}g_{31}g_{12}]
\end{aligned}
$$

it is readily seen analogously with the previous cases, that the quantities (10.13) form a polynomial basis for the ditrigonal-dipyramidal, dihexagonal-pyramidal, hexagonal-trapezohedral and dihexagonal-dipyramidal classes.

We note that the elements of first and second degree in g_{ij} of the polynomial bases for the crystal classes discussed in (iii) and (iv) are the same. Consequently, if the displacement gradients, and hence the quantities g_{ij}, are sufficiently small so that terms of higher degree than the second in the expression for W as a polynomial in g_{ij} can be neglected, then W takes the same form for all the crystal classes discussed in (iii) and (iv).

11. **Appendix.** In the introduction it is assumed that if a body of the material considered is deformed so that a particle initially at X_i' in the rectangular Cartesian coordinate system x_i' moves to x_i' in the same coordinate system, then the strain-energy function W, defined as the strain-energy per unit volume measured in the undeformed state, may be expressed as a polynomial in the quantities g_{ij}' defined by equation (1.1). It is the purpose of this section to discuss this assumption further.

If we assume initially that W is expressible as a polynomial in the gradients $\partial x_i'/\partial X_j'$, then W is unaltered if the deformed body is subjected to an arbitrary rigid-body rotation in which the particle at x_i' moves to x_i'', where

$$(11.1)\qquad x_i'' = a_{ik}x_k',$$

with

$$(11.2)\qquad a_{ij}a_{ik} = \delta_{jk} \quad \text{and} \quad \det a_{ij} = 1.$$

We therefore have

$$(11.3)\qquad W(\partial x_i'/\partial X_j') = W(\partial x_i''/\partial X_j') = W(a_{ik}\partial x_k'/\partial X_j'),$$

where W is a polynomial function of the indicated variables, for all a_{ij} satisfying the relations (11.2). It follows from a well-known theorem in the theory of invariants (see, for example, H. Weyl, 1946, p. 53 et seq.) that W must be expressible as a polynomial in the quantities $g_{ij}'+\delta_{ij}$ defined by (1.1) and $\det \partial x_i'/\partial X_j'$. Now, it is easily seen that

$$(11.4)\qquad \det (g_{ij}'+\delta_{ij}) = (\det \partial x_i'/\partial X_j')^2$$

and, since for any deformation which is possible in a real material det $\partial x_i'/\partial X_j'$ is positive, we have

$$\det \partial x_i'/\partial X_j' = [\det (g_{ij}' + \delta_{ij})]^{1/2}. \tag{11.5}$$

Also, since for any deformation which is possible in a real material det $(g_{ij}'+\delta_{ij})$ cannot be zero, $[\det (g_{ij}'+\delta_{ij})]^{1/2}$ may be approximated to any desired accuracy by a polynomial in det $(g_{ij}'+\delta_{ij})$. Thus, W may be approximated to any desired accuracy by a polynomial in the quantities $g_{ij}'+\delta_{ij}$ and hence in the quantities g_{ij}'.

We have already noted that it follows from our initial assumption that W must be expressible precisely as a polynomial in the quantities $g_{ij}'+\delta_{ij}$ and det $\partial x_i'/\partial X_j'$. Hence, it must be expressible precisely as a polynomial in the quantities g_{ij}' and det $\partial x_i'/\partial X_j'$. If such an expression is made the starting point for the discussion of the various crystal classes, then we have only to add to the basic invariants in terms of which W is expressed in each case the expression det $\partial x_i'/\partial X_j'$, since the latter is invariant under each of the transformations considered.

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