LIE ALGEBRAS OF TYPES A, B, C, D, AND F

BY

MARVIN L. TOMBER[1]

Let Φ be an arbitrary field of characteristic 0. Lie algebras of types A, B, C, and D over Φ have been described in terms of associative algebras over Φ [5; 6; 10][2]. Using these results, Jacobson showed in [8] that these algebras are derivation algebras of central simple Jordan algebras over Φ. By showing that any automorphism of the derivation algebra of a Jordan algebra is induced by an automorphism of the Jordan algebra, we give a direct proof that Lie algebras of types A, B, C, D, and F are derivation algebras of central simple Jordan algebras of degree $n \geqq 3$. The methods will be similar to those in [7] and [11].

1. Let Ω be an algebraically closed field of characteristic 0. The simple Jordan algebras over Ω are divided into classes A, B, C, D, and E, [4]. The degrees and dimensions of the algebras in these classes are respectively: A; n, n^2: B; n, $n(n+1)/2$: C; n, $2n^2-n$: D; 2, $\geqq 3$: E; 3, 27.

For simple Jordan algebras over Ω of degree at least three we may write $\mathfrak{J} = \sum \Omega e_i + \sum \mathfrak{A}_{jk}$, $i=1, \cdots, n=$degree of $\mathfrak{J}$, $j<k\leqq n$. The elements of the subspaces $\mathfrak{A}_{jk}$ may be taken to be in a simple alternative algebra $\mathfrak{B}$ over Ω. The complete multiplication table of $\mathfrak{J}$ in terms of this decomposition is given in [1, §18]. The properties that we will use most are: $e_i^2=e_i$, $e_ie_j=0$, $2e_ia_{ij} = a_{ij}$, $a_{ij}^2 \in \Omega(e_i+e_j)$, $\mathfrak{A}_{ij}\mathfrak{A}_{jk} \subseteq \mathfrak{A}_{ik}$, $e_i\mathfrak{A}_{jk}=0$, and $\mathfrak{A}_{ij}\mathfrak{A}_{kh}=0$, $1\leqq i<j<k<h\leqq n$ and $a_{ij}\in\mathfrak{A}_{ij}$. If $x\in\mathfrak{J}$, then $x=\sum \alpha_i e_i + \sum a_{jk}$, $\alpha_i \in \Omega$, and $a_{jk}\in\mathfrak{A}_{jk}$. The reduced trace of x, Sp x, is defined as Sp $x = \sum \alpha_i$. The bilinear form Sp xy is known to be nondegenerate. If we let $\mathfrak{J}_0 = \{x;\ x\in\mathfrak{J},\ \mathrm{Sp}\, x=0\}$, then the restriction of Sp xy to $\mathfrak{J}_0 \times \mathfrak{J}_0$ is also nondegenerate. Since Sp xy is nondegenerate on $\mathfrak{J}(\mathfrak{J}_0)$ we may define the adjoint A^* of a linear transformation A on $\mathfrak{J}(\mathfrak{J}_0)$ by

$$\mathrm{Sp}\,(A^*x)y = \mathrm{Sp}\, x(Ay), \qquad \text{for all } x, y \in \mathfrak{J}(\mathfrak{J}_0).$$

Let $\mathfrak{J}$ be a Jordan algebra over a field Φ of characteristic 0. We shall use the notation $R(x)$, $x\in\mathfrak{J}$ to stand for the linear transformation $R(x)y=yx$, for all $y\in\mathfrak{J}$. The linear transformations $\sum[R(x_i), R(y_i)]$ are derivations of $\mathfrak{J}$. If $\mathfrak{J}$ is semi-simple, then all derivations of $\mathfrak{J}$ have this form. Let $\mathfrak{D}(\mathfrak{J})$ denote the derivation algebra of $\mathfrak{J}$. The subspace $\mathfrak{J}_0$ is invariant with respect to $\mathfrak{J}$ and we shall use $\mathfrak{D}(\mathfrak{J}_0)$ to denote the restriction of $\mathfrak{D}(\mathfrak{J})$ to $\mathfrak{J}_0$. In [3] and [8] the derivation algebras of the simple Jordan algebras are computed.

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(2) Numbers in brackets refer to references at the end of the paper.

Over an algebraically closed field of characteristic 0 they are: $\mathfrak{J}$ of class A and degree n, $\mathfrak{D}(\mathfrak{J})=A_n$; $\mathfrak{J}$ of class B, and degree n, $\mathfrak{D}(\mathfrak{J})=B_p$ for $n=2p+1$ and $\mathfrak{D}(\mathfrak{J})=D_p$ for $n=2p$; $\mathfrak{J}$ of class C and degree n, $\mathfrak{D}(\mathfrak{J})=C_p$ where $n=2p$; $\mathfrak{J}$ of class D, $\mathfrak{D}(\mathfrak{J})=B_p$ or D_p; $\mathfrak{J}$ of class E, $\mathfrak{D}(\mathfrak{J})=F_4$.

If $\mathfrak{J}$ is simple over Ω of degree at least three, the subspace $\mathfrak{S}$ generated by e_1-e_2 and 1_{12}, 1 the identity of $\mathfrak{B}$, is irreducible with respect to the derivations $[R(e_1), R(1_{12})]$ and $[R(e_2), R(1_{12})]$. Thus some subspace $\mathfrak{M}$ of $\mathfrak{J}_0$ which is invariant with respect to $\mathfrak{D}(\mathfrak{J}_0)$ must contain $\mathfrak{S}$. If $\mathfrak{M}$ contains $\mathfrak{S}$, then it will contain $[R(e_1), R(a_{1j})](e_1-e_2)=a_{1j}/4$, $a_{1j}\in\mathfrak{A}_{1j}$. Thus $\mathfrak{M}$ also contains $\mathfrak{A}_{1j}$, $j=2, \cdots, n$. Similarly $\mathfrak{M}$ contains $\mathfrak{A}_{2j}$. Using the derivation $[R(e_j), R(a_{ij})]$, we have that e_1-e_j is in $\mathfrak{M}$. Continuing in this manner, it may be seen that $\mathfrak{M}=\mathfrak{J}_0$ and $\mathfrak{J}_0$ is an irreducible representation space for $\mathfrak{D}(\mathfrak{J}_0)$. With slight changes in the proof the same is true for algebras of degree two.

Since $\mathfrak{J}_0$ is an irreducible representation space for $\mathfrak{D}(\mathfrak{J}_0)$, by Schur's Lemma it follows that any linear transformation which commutes with every element of $\mathfrak{D}(\mathfrak{J}_0)$ is of the form σI, $\sigma\in\Omega$ and I the identity transformation of $\mathfrak{J}_0$. As in [11] it can be seen that if S is an automorphism of $\mathfrak{J}$, then $S\mathfrak{J}_0=\mathfrak{J}_0$ and if $\mathfrak{J}$ is a simple Jordan algebra over Φ and Σ an extension of Φ, then $\mathfrak{J}_{0\Sigma}=(\mathfrak{J}_\Sigma)_0$.

2. If $\mathfrak{L}$ is a Lie algebra over an arbitrary algebraically closed field of characteristic 0, then it is known that the irreducible representations of L are determined by the highest weight of the representation [2; 12]. In this section we shall determine the number of nonsimilar representations of Lie algebras as derivation algebras of Jordan algebras. For the Lie algebra F_4 this has been done in [11].

A. For a Lie algebra A_n, $n\geqq 2$, of class A the degree of the irreducible representation as $\mathfrak{D}(\mathfrak{J}_0)$ is n^2-1. If $\Lambda=m_1\lambda_1+m_2\lambda_2+\cdots+m_n\lambda_n$ is a weight of a representation of A_n, then $m_1+m_2+\cdots+m_n=0$. If the form Λ is a weight, then so are $\Lambda-(\lambda_i-\lambda_j), \cdots, \Lambda-(m_i-m_j)(\lambda_i-\lambda_j)$. The quantities m_i-m_j are integers and the m's are rational numbers, [2, p. 68]. The last form in the above string implies that if Λ is a weight, then so is any form obtained from Λ by a permutation of the m's. If Λ is the highest weight of an irreducible representation of A_n, then $m_1\geqq m_2\geqq\cdots\geqq m_n$.

Let Λ be the highest weight of an irreducible representation of A_n of degree n^2-1. We will show that $\Lambda=\lambda_1-\lambda_n$ and thus all irreducible representations of A_n are similar. First we consider Λ with $m_1\geqq 1$ which implies $m_1-m_n\geqq 2$. The weight $\Lambda'=\Lambda-(\lambda_1-\lambda_2)$ is distinct from Λ. If $m_1-1=m_2=\cdots=m_{n-1}=m_n=1$, then $\Lambda=\lambda_1-\lambda_n$. If $\Lambda\neq\lambda_1-\lambda_n$, then Λ or Λ' must have at least three distinct m's and the other two. Counting the weights obtained from Λ and Λ' we must have at least n^2 weights which is impossible for an irreducible representation of degree n^2-1 [12, p. 76]. For $m_1<1$ the possible highest weights and the degrees of the corresponding irreducible representations are given in [2, pp. 68–69]. The degrees are $C_{n,i}\neq n^2-1$.

Since there is a representation of degree n^2-1 its highest weight is $\lambda_1-\lambda_n$.

B. The Lie algebra B_p, $n=2p+1>5$, of class B has two irreducible representations as a derivation algebra of Jordan algebras. These have degree n and $2p^2+3p$. If $\Lambda=m_1\lambda_1+\cdots+m_p\lambda_p$ is a weight of an irreducible representation of B_p, then so are the forms: $\Lambda-\lambda_i, \cdots, \Lambda-2m_i\lambda_i; \Lambda-(\lambda_i+\lambda_j), \cdots, \Lambda-(m_i+m_j)(\lambda_i+\lambda_j); \Lambda-(\lambda_i-\lambda_j), \cdots, \Lambda-(m_i-m_j)(\lambda_i-\lambda_j)$. In particular, if Λ is a weight, so is $-\Lambda$ and any form obtained from Λ by a permutation of the m's.

If Λ is a highest weight of an irreducible representation, then $m_1\geqq m_2 \geqq\cdots\geqq m_p\geqq 0$. First let $m_1>1$ and $m_p\neq 0$. The 2^p forms $\pm m_1\lambda_1\pm m_2\lambda_2 \pm\cdots\pm m_p\lambda_p$ are weights. If all the m_i are not equal, using permutations, we get at least $p2^p$ distinct weights. If all m_i are equal, a similar set of weights generated from $\Lambda-\lambda_1$ will give at least $p2^p$ distinct weights. For $p\neq 3$, $p2^p>2p^2+3p$. For $p=3$ it is easy to enumerate more than 27 distinct weights. Thus there is no representation of B_p as $\mathfrak{D}(\mathfrak{J}_0)$ where the highest weight satisfies $m_1>1$ and $m_p\neq 0$. Next we consider the case $m_1=m_p=1$. From the first string of weights listed above we see that the forms $\sum s_i\lambda_i$, $s_i=\pm 1, 0$, are 3^p distinct weights. For $p\neq 3$, $3^p>2p^2+3p$ which is impossible. The degree of the irreducible representation corresponding to $\Lambda=\lambda_1+\lambda_2+\lambda_3$ is given in Cartan [2, p. 87]. This degree is 35 which is greater than 27. Let $m_p=0$ and $m_1>1$. If $\Lambda\neq 2\lambda_1$, we may obtain from Λ the weights $\pm 2\lambda_1\pm\lambda_2$ and $\pm\lambda_1\pm\lambda_2$. Taking the permutations of these weights we get $6p(p-1)$ distinct weights which is impossible. We shall see that $2\lambda_1$ is the highest weight of the representation of degree $2p^2+3p$ since all other weights will be eliminated. Finally, for $m_1<1$, or $m_1=1$ and $m_p=0$ the possible highest weights are listed in [2]. In Cartan's notation Π_1 corresponds to an irreducible representation of degree $2^p\neq 2p+1$, $2p^2+3p$; Π_2 is the highest weight of the representation of degree $2p+1$; Π_3 is the highest weight of the adjoint representation of degree $2p^2+p+1$. The system of weights for Π_i, $i\geqq 4$, contains the system of weights for Π_4 which clearly has more than $2p^2+3p$ distinct weights.

C. The Lie algebra C_p with $n=2p\geqq 4$ of class C has an irreducible representation as $\mathfrak{D}(\mathfrak{J}_0)$ of degree $2p^2-p-1$. If $\Lambda=m_1\lambda_1+\cdots+m_p\lambda_p$ is a weight of a representation of C_p, the forms $\Lambda-2\lambda_i, \cdots, \Lambda-2m_i\lambda_i$; $\Lambda-(\lambda_i+\lambda_j), \cdots, \Lambda-(m_i+m_j)(\lambda_i+\lambda_j); \Lambda-(\lambda_i-\lambda_j), \cdots, \Lambda-(m_i-m_j)(\lambda_i-\lambda_j)$ are also weights and the m_i are integers.

If $m_1>1$, use of transformations of the form $\Lambda'=\Lambda-m_j(\lambda_i+\lambda_j)$ shows that the system of weights includes $m_1\lambda_1$ or $m_1\lambda_1+m_2'\lambda_2$. For $\Lambda'=m_1\lambda_1+m_2'\lambda_2$, changes of sign and permutations give $2p(p-1)$ distinct weights and permutations of $\Lambda'-(\lambda_1+\lambda_2)$ give at least p more which is greater than $2p^2-p-1$. From $\Lambda'=m_1\lambda_1$ we obtain $(m_1-1)\lambda_1+\lambda_2$. Again changing signs and permuting the coefficients we find more than $2p^2-p-1$ distinct weights. For $m_1=1$ we know Π_1 is the highest weight of the representation of degree $2p$. It is clear that the weights Π_i, $i=3, \cdots, p$, may be eliminated. Thus $\Pi_2=\lambda_1+\lambda_2$ is the highest weight for a representation of C_p as $\mathfrak{D}(\mathfrak{J}_0)$.

D. The Lie algebra D_p, $n=2p\geqq 6$, has representations as $\mathfrak{D}(\mathfrak{J}_0)$ of degree $2p$ and $2p^2+p-1$. It will be convenient to make the restriction $p>4$. If $\Lambda=m_1\lambda_1+\cdots+m_p\lambda_p$ is a weight of an irreducible representation of D_p, we may obtain from Λ the weights $\Lambda-(\lambda_i+\lambda_j), \cdots, \Lambda-(m_i+m_j)(\lambda_i+\lambda_j)$; $\Lambda-(\lambda_i-\lambda_j), \cdots, \Lambda-(m_i-m_j)(\lambda_i-\lambda_j)$. The rational numbers m_i are either all integers or all fractions with denominator 2 and odd numerator.

Let Λ be a highest weight of an irreducible representation. We consider several cases. Let Λ be such that $m_p=0$ and $m_3\neq 0$. Permutations of the coefficients of Λ, $\Lambda-(m_1+m_2)(\lambda_1+\lambda_2)$, and $\Lambda-m_3(\lambda_3+\lambda_p)$ yield at least p, $p(p-1)(p-2)/2$, and $p(p-1)$ distinct weights. This is more than $2p^2+p-1$ weights, for $p>4$. If the m_i are all integers and $m_p\neq 0$, the transformation $\Lambda\rightarrow\Lambda-|m_p|(\lambda_{p-1}-\lambda_p)$ will give us a weight in the group just considered. Now consider those weights with $m_1>1$ and all m_i fractions. The system of weights obtainable in this case clearly contains a weight with $m_1\geqq m_2'\geqq m_3' = \cdots = m_{p-1}'=|m_p'|=1/2$ and it is easy to see that this will generate more than $2p^2+p-1$ distinct weights. The weights Π_1 and Π_2, $m_i=1/2$, for class D are the highest weights of the spin representations of D_p of degree $2^{p-1} \neq 2p^2+p-1$. Finally if $m_3=0$ and $m_1\neq 0$, $\Lambda=2\lambda_1$, $\lambda_1+\lambda_2$, λ_1, or Λ can be transformed into a weight of the first form we considered. The weight $\lambda_1+\lambda_2$ is the highest weight of the adjoint representation of D_p of degree $2p^2-3p+2$. Thus the only possible highest weights are λ_1 and $2\lambda_1$. The weight λ_1 belongs to the representation of degree $2p$ and $2\lambda_1$ is the highest weight of the representation of degree $2p^2+p-1$.

Thus we have proved:

THEOREM 1. *Over an algebraically closed field of characteristic* 0, *all irreducible representations of the Lie algebras of class* A_n, $n\geqq 2$, B_p, $p\geqq 3$, C_p, $p\geqq 2$, D_p, $p\geqq 5$, *and* F_4 *as* $\mathfrak{D}(\mathfrak{J}_0)$ *of the same degree are similar.*

3. This section will be devoted to the proof of the following theorem.

THEOREM 2. *If* $D\rightarrow\tilde{D}$ *is an automorphism of* $\mathfrak{D}(\mathfrak{J})$, $\mathfrak{J}$ *a simple Jordan algebra of degree at least* 3 *over an algebraically closed field* Ω *of characteristic* 0, *and* $\mathfrak{D}(\mathfrak{J})$ *is not the Lie algebra* D_4, *then there is a unique automorpism* S *of* $\mathfrak{J}$ *such that* $\tilde{D}=SDS^{-1}$.

Using Theorem 1, we begin as in [7] and [11] by defining a linear transformation S on $\mathfrak{J}$ such that $\tilde{D}=SDS^{-1}$, $S^*S=I$ the identity transformation on $\mathfrak{J}$, $Se=e$ the identity of $\mathfrak{J}$, Sp $(Sx)(Sy)=$ Sp xy, and Sp $Sx=$ Sp x, $x, y\in\mathfrak{J}$.

We consider Jordan algebras of degree $n\geqq 3$. All subscripts shall be understood to have the values $1, \cdots, n$, g, i, j, k, p, and q will be used as distinct subscripts, and x_{uv}, a_{iuv}, etc. will be used as elements of $\mathfrak{A}_{uv}$, $u<v$ and $u, v =g, i, j, k, p, q$. The derivation $D=[R(x_{ij}), R(x_{ik})]$ acts as follows: De_i

$=De_g=Da_{pq}=0$, $Da_{pj}\in\mathfrak{A}_{pk}$, $Da_{pk}\in\mathfrak{A}_{pj}$, $Da_{ij}\in\mathfrak{A}_{ik}$, $Da_{ik}\in\mathfrak{A}_{ij}$, De_j, $De_k\in\mathfrak{A}_{jk}$, and $Da_{jk}\in\Omega e_i+\Omega e_j+\Omega e_k$.

Let $f_i=Se_i$. Since S is nonsingular we may write

$$f_i^2 = \sum \alpha_{ih}f_h + \sum Sa_{ist}, \qquad s < t \text{ and } \alpha_{ih} \in \Omega.$$

To this equation we apply the derivation SDS^{-1} and obtain

$$2f_iSDe_i = 0 = \sum \alpha_{ih}SDe_h + \sum SDa_{ist}.$$

This implies, because of the linear independence of the subspaces involved, that (i) $\alpha_{ij}De_j+\alpha_{ik}e_k=0$, (ii) $Da_{ipj}=Da_{ipk}=0$, (iii) $Da_{iij}=Da_{iik}=0$, and (iv) $Da_{ijk}=0$. If $x_{ij}=1_{ij}$ and $x_{ik}=1_{ik}$, 1 the identity of $\mathfrak{B}$, then from (i) we have $\alpha_{ij}=\alpha_{ik}$. In the same manner (ii) implies that $a_{ipj}=a_{ipk}=0$. From [9, Lemma 1, p. 31] we see that $Da_{iij}=(x_{ik}x_{ij})a_{iij}$ and $Da_{iik}=-(x_{ij}x_{ik})a_{iik}$. The same choice of x_{ij} and x_{ik} shows that (iii) implies $a_{iij}=a_{iik}=0$. The element Da_{ijk} is in the subspace $\Omega e_i+\Omega e_j+\Omega e_k$. From (iv) we know that the component of Da_{ijk} in Ωe_k is 0. Suppose $a_{ijk}\neq 0$, then we may choose x_{ij} so that the product $a_{ijk}x_{ij}\in\mathfrak{A}_{ik}$ is different from 0, otherwise B would not be simple. Furthermore, since $\Omega e_i+\Omega e_k+\mathfrak{A}_{ik}$ is a simple Jordan algebra, we may choose x_{ik} such that $(a_{ijk}x_{ij})x_{ik}\neq 0$. Since $(a_{ijk}x_{ij})x_{ik}\in\Omega(e_i+e_k)$, we have a contradiction and $a_{ijk}=0$. If this is done for each pair j, k we get all $Sa_{ijk}=0$ and $\alpha_{ip}=\alpha_{iq}$. Thus f_i^2 may be written as

$$f_i^2 = \alpha_i f_i + \beta_i e$$

since $\sum f_h=e$. $\text{Sp}\, f_i^2=\text{Sp}\, e_i^2=\text{Sp}\, e_i=1$, $\text{Sp}\, f_i=\text{Sp}\, e_i=1$, and $\text{Sp}\, e=n$, thus

$$f_i^2 = (1 - n\beta_i)f_i + \beta_i e. \tag{1}$$

To equation (1) we apply the derivation $S[R(e_i),\, R(a_{ig})]S^{-1}$ and obtain

$$2f_iSa_{ig} = (1 - n\beta_i)Sa_{ig}. \tag{2}$$

Applying the same derivation to (2) and to (2) with i and g interchanged we have

$$2\theta f_i(f_g - f_i) + (Sa_{ig})^2 = \theta(1 - n\beta_i)(f_g - f_i) \tag{3}$$

and

$$2\theta f_g(f_g - f_i) - (Sa_{ig})^2 = \theta(1 - n\beta_g)(f_g - f_i) \tag{3'}$$

where $a_{ig}^2=\theta(e_i+e_g)$. The quantity a_{ig} may be chosen so that $\theta\neq 0$. Now we add (3) and (3′). Since e, f_i, and f_g are linearly independent we may equate the coefficients of e on each side of the equation and obtain $\beta_i=\beta_j$. Thus (1) and (2) may be written as

$$f_i^2 = (1 - n\beta)f_i + \beta e \tag{4}$$

and

$$2f_iSa_{ig} = (1 - n\beta)Sa_{ig}. \tag{5}$$

For $n \geqq 4$, let $f_if_g = \sum \alpha_{igh}f_h + \sum Sa_{igst}$. If we apply the derivations $S[R(x_{ij}), R(x_{ik})]S^{-1}$ and $S[R(x_{gj}), R(x_{gk})]S^{-1}$ to f_if_g, then in a manner similar to that for f_i^2 we obtain $f_if_g = Sa_{igig} + \sum \alpha_{igh}f_h$, $\alpha_{igp} = \alpha_{igq}$. Now, use of the derivation $S[R(e_i), R(x_{ig})]S^{-1}$ gives $\alpha_{igi} = \alpha_{igg}$ and $Sa_{igig} = 0$. Thus, the product f_if_g may be written as $f_if_g = \gamma_{ig}(f_i + f_g) + \delta_{ig}e$. For $n = 3$, the equation $(f_i + f_g)^2 = (e - f_k)^2$ gives the same result. Since $\text{Sp}\, f_if_g = \text{Sp}\, e_ie_g = 0$, $2\gamma_{ig} + n\delta_{ig} = 0$ and $f_if_g = -n\delta_{ig}(f_i + f_g)/2 + \delta_{ig}e$.

Since $\sum f_h = e$, $f_i = f_i(\sum f_h) = f_i^2 + \sum' f_if_h$ or

$$f_i = (1 - n\beta)f_i + \beta e + (\textstyle\sum' \delta_{ih})e - n(\textstyle\sum' \delta_{ih})(f_i + f_h)/2.$$

Comparing coefficients of f_j we have $0 = \beta + \sum' \delta_{ih} - n\delta_{ij}/2$, which implies that all δ_{ij} are equal and $\delta_{ij} = 2\beta/(2 - n)$. Thus

$$(n - 2)f_if_g = n\beta(f_i + f_g) - 2\beta e. \tag{6}$$

For any $a_{gk} \in \mathfrak{A}_{gk}$ the derivation $S[R(e_g), R(a_{gk})]S^{-1}$ applied to (6) yields

$$(n - 2)f_iSa_{gk} = n\beta Sa_{gk}. \tag{7}$$

For arbitrary $a_{ig} \in \mathfrak{A}_{ig}$ and $a_{ij} \in \mathfrak{A}_{ij}$ the derivations $S[R(e_i), R(a_{ig})]S^{-1}$ and $S[R(e_i), R(a_{ij})]S^{-1}$ applied to (7) give

$$(n - 2)(Sa_{ig})(Sa_{gk}) = (n - 2 - n^2\beta)Sa_{ik}, \tag{8}$$

where $a_{ik} = a_{ig}a_{gk}$, and

$$(Sa_{ij})(Sa_{gk}) = 0. \tag{9}$$

The relations (3) through (9) allow us to compute both sides of $(f_iSa_{ig}) \cdot (Sa_{ig})^2 = (f_i(Sa_{ig})^2)Sa_{ig}$. If a_{ig} is chosen so that $a_{ig}^2 \neq 0$, then this straightforward computation will give a cubic equation for β. The roots of this equation, the only possible values for β, are $\beta = 0$, $(n-2)/n^2$, and $2(n-2)/n^2$. For $\beta = (n - 2)/n^2$ we have; $f_i^2 = (2nf_i + (n - 2)e)/n^2$, $f_iSa_{ig} = Sa_{ig}/n$, $f_if_g = (nf_i + nf_g - 2e)/n^2$, $f_iSa_{gk} = Sa_{gk}/n$, $(Sa_{ig})(Sa_{gk}) = (Sa_{ij})(Sa_{gk}) = 0$, and $(Sa_{ig})^2 = 4\theta e/n$. Since every element of $\mathfrak{J}$ can be written in the form $\sum \alpha_hf_h + \sum Sa_{st}$ it is easy to see that every element of $\mathfrak{J}$ satisfies a quadratic equation when $\beta = (n-2)/n^2$. This is impossible for a Jordan algebra of degree greater than 2. Hence, $\beta = 0$ or $2(n-2)/n^2$.

Sx was defined as $\gamma e + \sigma^{-1/2}S_1x_0$ for $x = \gamma e + x_0$. Let $S'x = \gamma e - \sigma^{-1/2}S_1x_0$. Then S' has all the properties we have derived for S. $\text{Sp}\, x = \text{Sp}\, (Sx) = n\gamma$ and $Sx + S'x = 2\gamma e$ imply $Sx + S'x = 2(\text{Sp}\, x)e/n$. Calling $S'e_i = f_i'$, we have $f_i' = (2/n)e - f_i$ and

$$(f_i')^2 = \beta' e + (1 - n\beta')f_i' = (-\beta' + 2/n)e + (-1 + n\beta')f_i$$
$$= ((2/n)e - f_i)^2 = (\beta + 4/n^2)e + (1 - n\beta - 4/n)f_i.$$

Comparing coefficients we get $\beta' = (2n-4)/n^2 - \beta$. Thus if $\beta = (2n-4)/n^2$, $\beta' = 0$. Therefore (by replacing S by S' if necessary) we may assume that $\beta = 0$.

By placing $\beta = 0$ in (3)–(9) we see that S is an automorphism of $\mathfrak{J}$.

To see that S is unique, let R be an automorphism of $\mathfrak{J}$ such that $D \rightarrow \tilde{D} = SDS^{-1} = RDR^{-1}$. In §1 we saw that $R\mathfrak{J}_0 = \mathfrak{J}_0$. Thus on $\mathfrak{J}_0$, $R^{-1}S$ commutes with all D. Hence $R^{-1}S = \sigma I$, $\sigma \neq 0$ in Ω, on $\mathfrak{J}_0$; that is, $S = \sigma R$ on $\mathfrak{J}_0$. Choose $a_{12} \in \mathfrak{A}_{12}$, $a_{13} \in \mathfrak{A}_{13}$ such that $a_{12}a_{13} = a_{23} \neq 0$ in $\mathfrak{A}_{23}$. $Sa_{12} = \sigma Ra_{12}$, $Sa_{13} = \sigma Ra_{13}$, and $Sa_{23} = (Sa_{12})(Sa_{13}) = \sigma^2(Ra_{12})(Ra_{13}) = \sigma^2 Ra_{23}$. Hence $\sigma = 1$. Since $Re = Se$, $R = S$ on $\mathfrak{J}$.

4. In this section we assume merely that the base field Φ is of characteristic 0. A Lie algebra $\mathfrak{L}$ is said to be of *type* A, B, C, D, or F if $\mathfrak{L}_\Omega$ is the Lie algebra A, B, C, D, or F_4 over Ω where Ω is the algebraic closure of Φ. Our determination of the Lie algebras of types A, B, C, D, and F is given in terms of the central simple Jordan algebras over Φ of degree at least three. The restriction to Jordan algebras of degree at least three is made because the automorphism of the preceding section is not unique for algebras of degree two.

In [7] Jacobson characterizes Lie algebras of type G as the derivation algebras of Cayley algebras over Φ. The first three theorems in this section are restatements of analogous theorems for algebras of type G. The proofs of these theorems shall be omitted here. The statements made in §1 together with known results about Jordan algebras allow us to use Jacobson's proofs.

THEOREM 3. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be central simple Jordan algebras of degree at least three over a field Φ of characteristic* 0 *such that $\mathfrak{D}(\mathfrak{J}_1) \cong \mathfrak{D}(\mathfrak{J}_2)$ where $\mathfrak{D}(\mathfrak{J}_1)$ is not of type* D_4. *Then there exists a unique isomorphism S between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ such that the given isomorphism between $\mathfrak{D}(\mathfrak{J}_1)$ and $\mathfrak{D}(\mathfrak{J}_2)$ has the form $D \rightarrow E = SDS^{-1}$.*

The requirement $\mathfrak{D}(\mathfrak{J}_1) \cong \mathfrak{D}(\mathfrak{J}_2)$ ensures that J_1 and $\mathfrak{J}_2$ have the same split algebra and the proof in [7] goes through.

THEOREM 4. *If $\mathfrak{J}$ is a central simple Jordan algebra of degree at least three over a field Φ of characteristic* 0 *and $\mathfrak{D}(\mathfrak{J})$ is not of type* D_4, *then the group of automorphisms of $\mathfrak{D}(\mathfrak{J})$ is isomorphic to the group of automorphisms of $\mathfrak{J}$.*

THEOREM 5. *A necessary and sufficient condition that a Lie algebra $\mathfrak{L}$ over a field Φ of characteristic* 0 *be of type* A, B, C, D, *or* F *with the exception of type* D_4 *is that $\mathfrak{L} \cong \mathfrak{D}(\mathfrak{J})$, $\mathfrak{J}$ a central simple Jordan algebra of degree at least three over Φ.*

From [4] it can be seen that the Lie algebras of type $\mathrm{A_I}$ are the derivation algebras of the Jordan algebras of class A_I and that the Lie algebras of type A_{II} are the derivation algebras of the Jordan algebras of class A_{II}.

THEOREM 6. *A Lie algebra* $\mathfrak{L}$ *over a field* Φ *of characteristic* 0 *is simple with multiplication center* P *and of type* A, B, C, D, *or* F *with the exception of type* D_4 *over* P *if and only if* $\mathfrak{L} \cong \mathfrak{D}(\mathfrak{A})$ *for some simple Jordan algebra* $\mathfrak{A}$ *of degree at least three with center* P.

The proof of this theorem is similar to the proof of Theorem 5 of [11].

REFERENCES

1. A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

2. E. Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane*, Bull. Soc. Math. France vol. 41 (1913) pp. 53–96.

3. C. Chevalley and R. D. Schafer, *The exceptional simple Lie algebras* F_4 *and* E_6, Proc. Nat. Acad. Sci. U. S. A. vol. 36 (1950) pp. 137–141.

4. F. O. Jacobson and N. Jacobson, *Classification and representation of semi-simple Jordan algebras*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 141–169.

5. N. Jacobson, *A class of normal simple Lie algebras of characteristic* 0, Ann. of Math. vol. 36 (1937) pp. 508–517.

6. ———, *Simple Lie algebras of type* A, Ann. of Math. vol. 39 (1938) pp. 181–188.

7. ———, *Cayley numbers and normal simple Lie algebras of type* G, Duke Math. J. vol. 5 (1939) pp. 775–783.

8. ———, *Derivation algebras and multiplication algebras of semi-simple Jordan algebras*, Ann. of Math. vol. 50 (1949) pp. 866–874.

9. ———, *Structure of alternative and Jordan bimodules*, Osaka Math. J. vol. 6 (1954) pp. 1–71.

10. W. Landherr, *Über einfache Liesche Ringe*, Abh. Math. Sem. Univ. Hamburg vol. 11 (1935) pp. 41–64.

11. M. L. Tomber, *Lie algebras of type* F, Proc. Amer. Math. Soc.. vol. 4 (1953) pp. 759–768.

12. H. Weyl, *The structure and representation of continuous groups*, The Institute for Advanced Study, 1935.

MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICH.