# A HOMOTOPY THEOREM FOR MATROIDS, I

BY

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1. **Introduction.** By a *matroid* on a finite set $M$ we understand a class $\boldsymbol{M}$ of non-null subsets of $M$ which satisfies the following axioms.

AXIOM I. *No member of* $\boldsymbol{M}$ *contains another as a proper subset.*

AXIOM II. *If* $(X, Y) \in \boldsymbol{M}$, $a \in X \cap Y$ *and* $b \in X - (X \cap Y)$, *then there exists* $Z \in \boldsymbol{M}$ *such that* $b \in Z \subseteq (X \cup Y) - \{a\}$.

Such systems were introduced by Hassler Whitney [1].

As an example let $\boldsymbol{L}$ be any class of subsets of $M$ forming a group under mod 2 addition, and let $\boldsymbol{M}$ be the class of all minimal non-null members of $\boldsymbol{L}$. Then it is easily verified that $\boldsymbol{L}$ satisfies Axiom II and that each non-null member of $\boldsymbol{L}$ is a sum of non-null members of $\boldsymbol{M}$. It follows that $\boldsymbol{M}$ satisfies both axioms and is thus a matroid. Such a matroid we call *binary*.

In particular $\boldsymbol{M}$ may be the set of edges of a finite graph $G$ and $\boldsymbol{L}$ may be the class of 1-cycles mod 2 of $G$. Then it is found that the members of $\boldsymbol{M}$ are those sets of edges of $G$ which define circuits. In this case we call $\boldsymbol{M}$ the *circuit-matroid* of $G$.

Given a matroid $\boldsymbol{M}$ let $\boldsymbol{Q}$ be the class of all unions of members of $\boldsymbol{M}$. Then each element of $\boldsymbol{Q}$ is a subset of $M$. We partition $\boldsymbol{Q}$ into disjoint classes $\boldsymbol{Q}_{-1}, \boldsymbol{Q}_0, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \cdots$ according to the following rules.

(i) *The null subset* $\emptyset$ *of* $M$, *considered as an empty union, is the only member of* $\boldsymbol{Q}_{-1}$.

(ii) *When* $\boldsymbol{Q}_r$ *has been determined for* $-1 \leqq r \leqq k$ *we define* $\boldsymbol{Q}_{k+1}$ *as the class of all minimal members of*

$$\boldsymbol{P}_k = \boldsymbol{Q} - \bigcup_{r=-1}^{k} \boldsymbol{Q}_r.$$

That is $\boldsymbol{Q}_{k+1}$ consists of all members of $\boldsymbol{P}_k$ which have no other members of $\boldsymbol{P}_k$ as a subset. The members of $\boldsymbol{Q}$ are the *flats* of $\boldsymbol{M}$. Those belonging to $\boldsymbol{Q}_d$ are the flats of *dimension* $d$, or $d$*-flats*.

At the end of §2 of this paper we interpret the dimensions of the flats of a circuit-matroid in terms of graph theory.

We shall see that the flats of a matroid $\boldsymbol{M}$ on a set $M$ have some properties resembling those of the elements of a projective geometry. Because of this analogy we refer to the 0-flats, 1-flats and 2-flats of $\boldsymbol{M}$ as its *points*, *lines* and *planes* respectively. The points are simply the members of the class $\boldsymbol{M}$.

We have to recognize one distinction which has no analogue in projective

Received by the editors October 26, 1956.

geometry. A flat $F$ is *disconnected* if it can be represented as the union of two disjoint non-null subsets $F'$ and $F''$ of $M$ such that each point $X$ of $\boldsymbol{M}$ satisfying $X\subseteq F$ satisfies also either $X\subseteq F'$ or $X\subseteq F''$. If no such representation is possible then $F$ is *connected*. Thus the points of $\boldsymbol{M}$ and its $(-1)$-flat are connected.

A *path* in $\boldsymbol{M}$ is a finite sequence $P=(X_1, \cdots, X_k)$ of one or more points of $\boldsymbol{M}$, not necessarily all distinct, such that any two consecutive terms are distinct points of $\boldsymbol{M}$ which are subsets of the same connected line. The first and last terms of $P$ are its *origin* and *terminus* respectively. If they are the same point we call $P$ *re-entrant*. If $P$ has only one term we call it *degenerate*.

If $P=(X_1, \cdots, X_k)$ and $P'=(X_k, \cdots, X_m)$ are paths of $\boldsymbol{M}$ such that the origin of $P'$ is the terminus of $P$ then we define their *product* $PP'$ as the path $(X_1, \cdots, X_k, \cdots, X_m)$. Multiplication of paths is clearly associative. It is therefore permissible to write a path $(PQ)R$ or $P(QR)$ simply as $PQR$.

Suppose we have two paths $PR$ and $PQR$ where $Q$ is either (i) of the form $(X, Y, X)$ or (ii) of the form $(X, Y, Z, X)$ with $X$, $Y$ and $Z$ subsets of the same plane. Then we say that each of $PR$ and $PQR$ can be derived from the other by an *elementary deformation*. Two paths $P_1$ and $P_2$ are *homotopic* if they are identical or if one can be derived from the other by a finite sequence of elementary deformations. Homotopy is clearly an equivalence relation. A path homotopic to a degenerate path is said to be *null-homotopic*.

In this paper we show that every re-entrant path in a matroid is null-homotopic. Actually we prove a more general theorem, as the result just stated is not sufficient for the purposes of Paper II. We first agree to call a subclass $\boldsymbol{C}$ of $\boldsymbol{M}$ *convex* if it has the following property: if two distinct members $X$ and $Y$ of $\boldsymbol{C}$ are subsets of the same line $L$ then every point of $\boldsymbol{M}$ which is a subset of $L$ is a member of $\boldsymbol{C}$. Given a convex subclass $\boldsymbol{C}$ of $\boldsymbol{M}$ we say that a path $P$ is *off* $\boldsymbol{C}$ if no term of $P$ is a point of $\boldsymbol{C}$. We then enquire into the condition that a path $P$ off $\boldsymbol{C}$ can be transformed into a degenerate path by a finite sequence of elementary deformations so that all the intermediate paths are off $\boldsymbol{C}$. In this paper we show how the idea of an elementary deformation must be generalized so as to make this transformation possible for every re-entrant path $P$ off $\boldsymbol{C}$.

It is hoped that the technique here developed for the study of matroids will be found useful in graph theory when applied to the circuit-matroids of graphs.

2. **Flats.** Let $\boldsymbol{M}$ be a matroid on a set $M$. We refer to the elements of $M$ as the *cells* of the matroid. If $S$ and $T$ are subsets of $M$ we use the symbol $S\subset T$ to denote that $S$ is a proper subset of $T$. We write $\langle S\rangle$ for the union of all the points of $\boldsymbol{M}$ which are subsets of $S$. If $S$ is a flat of $\boldsymbol{M}$ we denote its dimension by $dS$.

(2.1) *If $S$ is a flat of $\boldsymbol{M}$ and $k$ is an integer satisfying $-1\leqq k<dS$ then there exists a flat $T$ of $\boldsymbol{M}$ such that $dT=k$ and $T\subset S$.*

**Proof.** If the theorem fails let $k$ be the greatest integer satisfying $-1$

$\leqq k < dS$ such that no $k$-flat $T$ of $\boldsymbol{M}$ satisfies $T \subset S$. Clearly $k > -1$ and therefore $dS \geqq 1$.

By the definition of $k$ there exists a $(k+1)$-flat $T'$ of $\boldsymbol{M}$ such that $T' \subseteq S$. By the definition of the classes $\boldsymbol{Q}_r$ we have

$$T' \in \boldsymbol{Q} - \bigcup_{r=-1}^{k} \boldsymbol{Q}_r \subset \boldsymbol{Q} - \bigcup_{r=-1}^{k-1} \boldsymbol{Q}_r = \boldsymbol{P}_{k-1}.$$

But $T'$ is not a minimal member of $\boldsymbol{P}_{k-1}$, since $dT'$ is not $k$. Hence there exists a minimal member $T$ of $\boldsymbol{P}_{k-1}$ such that $T \subset T'$. But then $dT = k$. Since $T' \subseteq S$ this contradicts the definition of $k$. The theorem follows.

(2.2) *If $S$ and $T$ are flats of $\boldsymbol{M}$ such that $S \subset T$, then $dS < dT$.*

**Proof.** Since $T$ is non-null we have $dT > -1$. If $dS \geqq dT$ there is a flat $U$ of $\boldsymbol{M}$ such that $U \subseteq S$ and $dU = dT$, by (2.1). But then $T$ is not a minimal member of $\boldsymbol{Q}_{dT}$, contrary to the definition of this class.

It follows from (2.2) that $\langle M \rangle$ has a greater dimension than any other flat of $\boldsymbol{M}$.

It is convenient to say that a flat $S$ is *on* a flat $T$ if either $S \subseteq T$ or $T \subseteq S$. If $S$ and $T$ are distinct we can distinguish between the two cases by comparing dimensions.

(2.3) *If $S$ is a flat of $\boldsymbol{M}$ and $a \in S$, then $d\langle S - \{a\} \rangle = dS - 1$.*

**Proof.** If possible choose $S$ and $a$ so that $d\langle S - \{a\} \rangle \neq dS - 1$ and so that $dS$ has the least value consistent with this.

By (2.1) there is a flat $T$ of $\boldsymbol{M}$ such that $T \subseteq S$ and $dT = dS - 1$. Choose $b \in S - T$. Then $T \subseteq \langle S - \{b\} \rangle \subset S$. Hence $d\langle S - \{b\} \rangle = dS - 1$, by (2.2).

Suppose $a \notin \langle S - \{b\} \rangle$. Then $\langle S - \{b\} \rangle \subseteq \langle S - \{a\} \rangle \subset S$. Hence $d\langle S - \{a\} \rangle = dS - 1$, by (2.2). But this is contrary to the choice of $S$ and $a$.

We deduce that $a \in \langle S - \{b\} \rangle$. Hence there exists $X \in \boldsymbol{M}$ such that $X \subseteq S$, $a \in X$ and $b \notin X$. Since $b \in S$ there exists $Y \in \boldsymbol{M}$ such that $Y \subseteq S$ and $b \in Y$. It follows by Axiom II that there exists $Z \in \boldsymbol{M}$ such that $Z \subseteq S$, $a \notin Z$ and $b \in Z$. ($Z = Y$ if $a \notin Y$). These results imply

$$\langle \langle S - \{a\} \rangle - \{b\} \rangle \subset \langle S - \{a\} \rangle \subset S, \tag{2.3a}$$

$$\langle \langle S - \{b\} \rangle - \{a\} \rangle \subset \langle S - \{b\} \rangle \subset S. \tag{2.3b}$$

We also have

$$\langle \langle S - \{a\} \rangle - \{b\} \rangle = \langle \langle S - \{b\} \rangle - \{a\} \rangle, \tag{2.3c}$$

since each side of this equation represents the union of those points of $\boldsymbol{M}$ which include neither $a$ nor $b$.

Since $d\langle S - \{b\} \rangle = dS - 1$ it follows from (2.3b) and the choice of $S$ and $a$ that $d\langle \langle S - \{b\} \rangle - \{a\} \rangle = dS - 2$. Hence $d\langle S - \{a\} \rangle = dS - 1$, by (2.2), (2.3a) and (2.3c). This contradiction establishes the theorem.

(2.4) *Let $S$ and $T$ be flats of $\boldsymbol{M}$ such that $S \subseteq T$. Then there exists a flat $U$ of $\boldsymbol{M}$ such that $U \subseteq T$, $\langle U \cap S \rangle = \emptyset$ and $dU = dT - dS - 1$.*

**Proof.** Write $S_0=S$, $T_0=T$. If possible choose $a_0\in S_0$ and write $S_1=\langle S_0-\{a_0\}\rangle$, $T_1=\langle T_0-\{a_0\}\rangle$. Observe that $S_1\subseteq T_1$. If possible choose $a_1\in S_1$ and write $S_2=\langle S_1-\{a_1\}\rangle$, $T_2=\langle T_1-\{a_1\}\rangle$. Then $S_2\subseteq T_2$. Continue this process until it terminates. By (2.3) this will be with $S_k$ and $T_k$, where $k=dS+1$ and $S_k=\emptyset$. Applying (2.3) to the sequence of the $T_i$ we find that $dT_k=dT-k=dT-dS-1$. We note that $\langle T_k\cap S\rangle\subseteq\langle S-\{a_0,\cdots,a_{k-1}\}\rangle=S_k=\emptyset$. Hence the theorem is satisfied with $U=T_k$.

(2.5) *If $S$ and $T$ are any flats of $\boldsymbol{M}$ then $d(S\cup T)+d\langle S\cap T\rangle\geqq dS+dT$.*

**Proof.** Write $S_0=S$. If possible choose $a_0\in S_0-(S_0\cap T)$ and write $S_1=\langle S_0-\{a_0\}\rangle$. If possible choose $a_1\in S_1-(S_1\cap T)$ and write $S_2=\langle S_1-\{a_1\}\rangle$, and so on. By (2.3) the process terminates with $S_k=\langle S\cap T\rangle$, where $k=dS-d\langle S\cap T\rangle$. We now have $T=S_k\cup T\subset S_{k-1}\cup T\subset\cdots\subset S_0\cup T=S\cup T$. Hence $d(S\cup T)-dT\geqq k=dS-d\langle S\cap T\rangle$, by (2.2).

Many "geometrical" results can be deduced from (2.2) and (2.5). For example any two distinct lines $L_1$ and $L_2$ on a plane $P$ have a unique common point. To prove this we first use (2.2) to show that $d\langle L_1\cap L_2\rangle<dL_1=1$ and $L_1\cup L_2=P$. Then $d\langle L_1\cap L_2\rangle\geqq 0$, by (2.5). Hence $d\langle L_1\cap L_2\rangle=0$ and $\langle L_1\cap L_2\rangle$ is a single point of $\boldsymbol{M}$. We can prove in the same way that if $P_1$ and $P_2$ are distinct planes on the same 3-flat $E$ of $\boldsymbol{M}$, then $\langle P_1\cap P_2\rangle$ is a line on $E$. Similarly if $P$ is a plane and $L$ a line on the same 3-flat $E$, and $L$ is not on $P$, then $\langle P\cap L\rangle$ is a point on $E$.

Not all the axioms of projective geometry are valid for matroids. For example two points are not necessarily on a common line. In general matroids are like geometrical figures but not like complete geometries.

Suppose $\boldsymbol{M}$ is the circuit-matroid of a graph $G$. If $S\in\boldsymbol{Q}$ we write $G\cdot S$ for the subgraph of $G$ made up of the edges of $S$ and their incident vertices. We see that the flats $S$ of $\boldsymbol{M}$ correspond to those subgraphs $G\cdot S$ in which each edge belongs to some circuit of the subgraph. In virtue of (2.3) $dS+1$ is the least number of edges which must be removed from $G\cdot S$ in order to destroy all its circuits, that is $dS+1$ is the rank or first Betti number of $G\cdot S$. The subgraph $G\cdot S$ is nonseparable if and only if the flat $S$ is connected.

3. **Connected flats.** We begin this section with a study of the line.

(3.1) *Any line $L$ of $\boldsymbol{M}$ is on at least two points. If $X$ and $Y$ are distinct points on $L$ then $L=X\cup Y$. Moreover $X\cap Y$ is non-null if and only if $L$ is connected.*

**Proof.** Choose $a\in L$. Then $\langle L-\{a\}\rangle$ is a point on $L$, by (2.3). Choose $b\in\langle L-\{a\}\rangle$. Then $\langle L-\{b\}\rangle$ is a point on $L$, by (2.3), which is distinct from $\langle L-\{a\}\rangle$.

Let $X$ and $Y$ be distinct points on $L$. Then $X\subset X\cup Y\subseteq L$. Hence $X\cup Y=L$, by (2.2). If $X\cap Y$ is non-null then $L$ is clearly connected. If $X\cap Y$ is null then either $L$ is disconnected or there exists $Z\in\boldsymbol{M}$ such that $Z\subseteq X\cup Y$ and $Z$ meets both $X$ and $Y$. In the latter case $X\subset X\cup Z\subset X\cup Y=L$, by Axiom I. This is impossible, by (2.2).

(3.2) *A disconnected line is on just two points, and a connected line is on at least three points.*

**Proof.** By (3.1) any two distinct points on a disconnected line $L$ are disjoint and have $L$ as their union. Hence $L$ has at most two points, and therefore just two by (3.1).

By (3.1) any connected line $L$ has two distinct points $X$ and $Y$, and we can find $a \in X \cap Y$. By (2.3) $\langle L - \{a\} \rangle$ is a point on $L$ distinct from $X$ and $Y$.

We shall need the following general theorems on connected flats.

(3.3) *Let $S$ and $T$ be connected flats of $\boldsymbol{M}$ such that $S \subset T$. Then there exists a connected $(dS+1)$-flat $U$ of $\boldsymbol{M}$ which is on both $S$ and $T$.*

**Proof.** Since $T$ is connected we can find $X \in \boldsymbol{M}$ such that $X \subseteq T$ and $X$ meets both $S$ and $T-S$. Choose such an $X$ so that $S \cup X$ has the least possible number of cells. Clearly $S \cup X$ is a connected flat of $\boldsymbol{M}$. Its dimension exceeds $dS$, by (2.2).

Suppose $d(S \cup X) > dS+1$. Choose $a \in (S \cup X) - S$. Then $d\langle (S \cup X) - \{a\} \rangle \geq dS+1$, by (2.3). Hence there exists $Y \in \boldsymbol{M}$ such that $Y \subseteq (S \cup X) - \{a\}$ and $Y$ meets $(S \cup X) - S$. But $Y \cap S$ is null, by the choice of $X$. Hence $Y \subset X$, which is impossible by Axiom I. We deduce that $d(S \cup X) = dS+1$. Hence the theorem is true with $U = S \cup X$.

(3.4) *Let $S$ be a connected $d$-flat on a connected $(d+2)$-flat $T$ of $\boldsymbol{M}$. Then there exist distinct connected $(d+1)$-flats $U$ and $V$ of $\boldsymbol{M}$ such that $S = \langle U \cap V \rangle$ and $T = U \cup V$.*

**Proof.** By (3.3) there is a connected $(d+1)$-flat $U$ which is on both $S$ and $T$. Choose $a \in U - S$ and write $W = \langle T - \{a\} \rangle$. By (2.3) $W$ is another $(d+1)$-flat on $S$ and $T$. By (2.4) there is a line $L$ on $T$ having no point in common with $S$. It meets $U$ and $W$ in points $X$ and $Z$ respectively, by (2.5). (See Figure I.) By (2.2) we have $S \cup X = U$ and $S \cup Z = W$. Hence $Z$ is not on $U$ and therefore $U \cup Z = T$, by (2.2).

Assume $W$ is not connected. Then $S \cap Z = \emptyset$.

Suppose $U \cap Z = \emptyset$. By the connection of $T$ there exists $Z' \in \boldsymbol{M}$ such that $Z' \subseteq T$ and $Z'$ meets both $U$ and $Z$. Then $U \subset U \cup Z' \subset U \cup Z = T$, by Axiom I. This is impossible by (2.2). We deduce that $U \cap Z \neq \emptyset$. A similar argument in which $X$, $S$ and $U$ replace $Z$, $U$ and $T$ respectively shows that $X \cap S \neq \emptyset$. Choose $b \in Z \cap U$ and $c \in X \cap S$.

Write $V = \langle T - \{b\} \rangle$. By (2.3) $V$ is a $(d+1)$-flat. It is on $S$ since $b \in Z$ and $S \cap Z = \emptyset$. By (2.5) it has a common point $Y$ with $L$. Clearly $V$ is distinct from $U$ and $W$ and therefore $Y$ is distinct from $X$ and $Z$, since $V = S \cup Y$ by (2.2).

Now $c \in S \cap X \subseteq S \cap L = S \cap (Y \cup Z) = S \cap Y$, by (3.1). Hence $V$ is connected.

If instead $W$ is connected we write $V = W$.

We now have two connected $(d+1)$-flats $U$ and $V$ of $\boldsymbol{M}$ each of which is on both $S$ and $T$. Hence $S \subseteq \langle U \cap V \rangle \subset U \subset U \cup V \subseteq T$, since $U$ and $V$ are

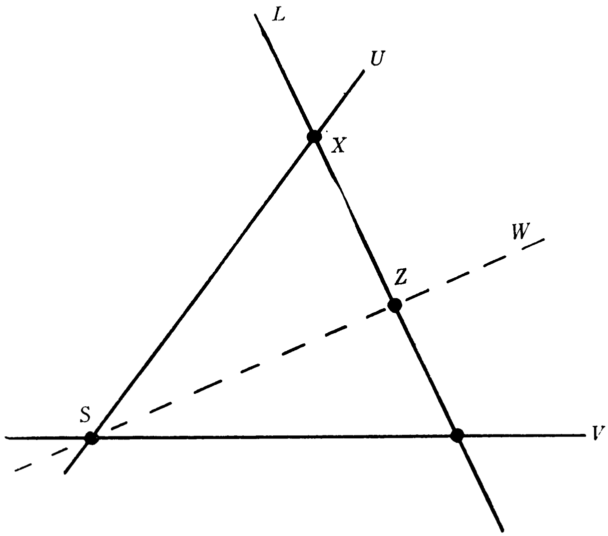


FIG. I

distinct and, by (2.2), neither is a subset of the other. In view of (2.2) this is possible only if $S=\langle U\cap V\rangle$ and $T=U\cup V$.

(3.5) *Let $S$, $T$ and $U$ be flats of $\boldsymbol{M}$ such that $S$ and $T$ are connected, $S\cup U\subseteq T$ and $\langle S\cap U\rangle=\varnothing$. Then there exists a connected flat $R$ of $\boldsymbol{M}$ such that $S\subseteq R\subseteq T$, $\langle R\cap U\rangle=\varnothing$ and $dR=dT-dU-1$.*

**Proof.** If possible choose $S$, $T$ and $U$ so that the theorem fails and $dU$ has the least value consistent with this. Then $dU>-1$ since otherwise the theorem holds with $T=R$. Let $W$ be a connected flat of $\boldsymbol{M}$ of greatest possible dimension such that $S\subseteq W\subseteq T$ and $W$ does not contain $U$. Then $dW=dT-1$ since otherwise, by (3.3) and (3.4), there exist distinct connected $(dW+1)$-flats $K$ and $L$ of $\boldsymbol{M}$ on $T$ such that $\langle K\cap L\rangle=W$, and these cannot both contain $U$. By the choice of $S$, $T$ and $U$ there is a connected flat $R$ of $\boldsymbol{M}$ such that $S\subseteq R\subseteq W\subset T$, $\langle R\cap U\rangle=\varnothing$ and $dR=dW-d\langle U\cap W\rangle-1$. But then $dR\geqq dT-dU-1$, by (2.2), and therefore $dR=dT-dU-1$, by (2.2) and (2.5). This contradiction establishes the theorem.

The foregoing results can be applied to circuit-matroids to obtain rather simple theorems about graphs. Thus from (3.5) with $S=\varnothing$ we find that if a nonseparable graph $G$ has rank $r$ and a subgraph $G\cdot U$ has a rank $s$ then there is a nonseparable subgraph $G\cdot R$ of $G$ of rank $r-s$ having no circuit in common with $G\cdot U$.

4. **The disconnected line.** By a *separation* $\{S_1, S_2\}$ of a disconnected flat $S$ of $\boldsymbol{M}$ we mean a pair of complementary non-null subsets of $S$ such that each $X\in\boldsymbol{M}$ satisfying $X\subseteq S$ satisfies either $X\subseteq S_1$ or $X\subseteq S_2$.

(4.1) *If $\{S_1, S_2\}$ is a separation of a flat $S$ of $\boldsymbol{M}$ and $X_1$ and $X_2$ are* *points of* $\mathbf{M}$ *such that* $X_1 \subseteq S_1$ *and* $X_2 \subseteq S_2$, *then* $X_1 \cup X_2$ *is a disconnected line of* $\mathbf{M}$.

**Proof.** Suppose $Y$ is a point on $X_1 \cup X_2$ distinct from $X_1$ and $X_2$. Since $Y \subseteq S$ we have $Y \subseteq S_1$ or $Y \subseteq S_2$. Hence $Y \subset X_1$ or $Y \subset X_2$, contrary to Axiom I. Thus the only subsets of $X_1 \cup X_2$ which are flats of $\mathbf{M}$ are $\emptyset$, $X_1$, $X_2$ and $X_1 \cup X_2$. Hence, by the definition of dimension, $X_1 \cup X_2$ is a line of $M$ having $\{X_1, X_2\}$ as a separation.

(4.2) *Let* $L$ *be a disconnected line on a connected* $d$-*flat* $S$ *of* $\mathbf{M}$, *where* $dS > 1$. *Then there exists a connected plane* $P$ *of* $M$ *such that* $L \subset P \subseteq S$.

**Proof.** Let the two points on $L$ be $X$ and $Y$. Let $P$ be a connected flat of $\mathbf{M}$ of least possible dimension such that $L \subset P \subseteq S$. Assume $dP > 2$.

Suppose first that there is a disconnected line $L'$, distinct from $L$, on $X$ and $P$. Let its point other than $X$ be $Z$. By (3.5) there is a connected $(dP-2)$-flat $U$ on $Y$ and $P$ having no point in common with $L'$. By (3.4) there are distinct connected $(dP-1)$-flats $V$ and $W$ of $\mathbf{M}$ on $P$ such that $\langle V \cap W \rangle = U$. By (2.2) and (2.5) $V$ and $W$ meet $L'$ in distinct points. Since there are only two points on $L'$ we may suppose $X$ is on $V$. But then $L$ is on $V$ and the definition of $P$ is contradicted. A similar argument applies if there is a disconnected line distinct from $L$ on $Y$ and $P$.

In the remaining case we choose $a \in P - L$ and write $R = \langle P - \{a\} \rangle$. Then $L \subseteq R$. Moreover $dR = dP - 1$, by (2.3). By the definition of $P$ the flat $R$ is disconnected. But there is no disconnected line on $R$, other than $L$, which is on either $X$ or $Y$. Hence, by (4.1), the only possible separation of $R$ is $\{X, Y\}$. Accordingly $R = L$ and $dP = 2$, contrary to assumption. From this contradiction we deduce that $P$ is a plane.

(4.3) *Let* $L$ *be a disconnected line on a connected plane* $P$ *of* $\mathbf{M}$. *Let* $X$ *and* $Y$ *be the two points of* $L$ *and let* $Z$ *be any other point on* $P$. *Then* $X \cup Z$ *and* $Y \cup Z$ *are connected lines. Moreover they are the only lines of* $\mathbf{M}$ *which are on both* $Z$ *and* $P$.

**Proof.** Any line on $Z$ and $P$ has a common point with the line $X \cup Y$. Hence, by (3.1), the only flats on $Z$ and $P$ which can be lines are $X \cup Z$ and $Y \cup Z$. By (3.4) both these flats must be connected lines.

(4.4) *Let* $L$ *be a disconnected line on a connected plane* $P$ *of* $\mathbf{M}$. *Then every line on* $P$ *other than* $L$ *is connected.*

**Proof.** Let $L'$ be any such line. By (3.1) it is on a point $Z$ distinct from $X$ and $Y$. Hence, by (4.3) it is one of the connected lines $X \cup Z$ and $Y \cup Z$.

5. **Convex subclasses.** Convex subclasses of a matroid $\mathbf{M}$ were defined in the Introduction. As an example we may take the class of all points of $\mathbf{M}$ on a given flat. The convexity of this class follows from (3.1).

Consider any path $P = (X_1, X_2, \cdots, X_k)$ of $\mathbf{M}$. We say $P$ is a path *from* $X_1$ *to* $X_k$. Any two consecutive terms of $P$ have a non-null intersection, by (3.1). Hence the flat $X_1 \cup X_2 \cup \cdots \cup X_k$ is connected. We denote this flat by $F(P)$. If $S$ is any flat of $\mathbf{M}$ such that $F(P) \subseteq S$ we say that $P$ is a path *on* $S$.

(5.1) *Let* $\mathbf{C}$ *be any convex subclass of* $\mathbf{M}$. *Let* $S$ *be a non-null connected flat* *of* $\mathbf{M}$ *and let* $X$ *and* $Y$ *be points on* $S$ *such that* $Y \notin \mathbf{C}$. *Then there exists a path* $P$ *from* $X$ *to* $Y$ *on* $S$ *such that no term of* $P$ *other than the first is a point of* $\mathbf{C}$.

**Proof.** If possible choose $S$, $X$ and $Y$ so that the theorem fails and $dS$ has the least value consistent with this. Clearly $dS>1$. By (3.3) and (3.4) there is a connected $(dS-2)$-flat $U$ and two distinct connected $(dS-1)$-flats $V$ and $W$ on $S$ such that $X\subseteq U=\langle V\cap W\rangle$. Now $Y$ is not on $V$ or $W$, for otherwise there would be a path from $X$ to $Y$ on $V$ or $W$ of the kind required. By (3.5) there is a connected line $L$ on $S$ and $Y$ such that $\langle L\cap U\rangle=\varnothing$. This meets $V$ and $W$ in distinct points $Z(V)$ and $Z(W)$ respectively, by (2.2) and (2.5). At least one of these, say $Z(V)$, belongs to $\mathbf{M}-\mathbf{C}$ since $\mathbf{C}$ is convex. By the choice of $S$, $X$ and $Y$ there is a path $Q$ from $X$ to $Z(V)$ on $V$ such that no term of $Q$ other than the first is a point of $\mathbf{C}$. Adjoining $Y$ to $Q$ we obtain a path $P$ from $X$ to $Y$ on $S$ of the kind required. This contradiction establishes the theorem.

We now distinguish four kinds of re-entrant paths of $\mathbf{M}$ as *elementary* with respect to a given convex subclass $\mathbf{C}$ of $\mathbf{M}$. The first kind consists of all paths off $\mathbf{C}$ of the form $(X, Y, X)$. The second consists of all paths off $\mathbf{C}$ of the form $(X, Y, Z, X)$ such that $d(X\cup Y\cup Z)\leqq 2$.

Suppose $P$ is a plane of $\mathbf{M}$ on which there are two distinct points $A$ and $B$ of $\mathbf{C}$ such that each connected line on $P$ is on either $A$ or $B$. Then any path off $\mathbf{C}$ on $P$ of the form $(X, Y, Z, T, X)$ such that $X$, $Y$, $Z$ and $T$ are distinct, the lines $X\cup Y$ and $Z\cup T$ are on $A$, and the lines $Y\cup Z$ and $T\cup X$ are on $B$ is an elementary re-entrant path of the third kind with respect to $\mathbf{C}$.

Suppose $E$ is a 3-flat of $\mathbf{M}$ on which there are three points $A$, $B$ and $C$ such that $A\cup B$, $B\cup C$ and $C\cup A$ are disconnected lines. Let there be just six connected planes on $E$, two on each of these disconnected lines. Suppose $A$, $B$ and $C$ are all in $\mathbf{M}-\mathbf{C}$ but there are two distinct members of $\mathbf{C}$ on each of the six connected planes. Then any path off $\mathbf{C}$ of the form $(A, X, B, Y, A)$, where $X$ and $Y$ are on distinct connected planes on $A\cup B$ and $E$, is an elementary re-entrant path of the fourth kind with respect to $\mathbf{C}$.

In studying the preceding case it is convenient to use the following notation. We write $Z_1$, $Z_2$ and $Z_3$ for $A$, $B$ and $C$. We enumerate the six connected planes as $P_1, \cdots, P_6$ in such a way that $\langle P_i\cap P_{i+3}\rangle=Z_j\cup Z_k$, where $1\leqq i\leqq 3$ and $(i, j, k)$ is a permutation of $(1, 2, 3)$. In general we write $\langle P_i\cap P_j\rangle=L_{ij}$ for $1\leqq i<j\leqq 6$. If $j=i+3$ then $L_{ij}$ is the disconnected line $(Z_1\cup Z_2\cup Z_3)-Z_i$. If $j\neq i+3$ let $k$ be that integer 1, 2 or 3 which is not congruent to $i$ or $j$ mod 3. Then $L_{ij}$ is on $Z_k$ and it meets $P_k$ and $P_{k+3}$ in two distinct points. It is therefore connected, by (3.2). Clearly it is on no connected plane on $E$ other than $P_i$ and $P_j$. The 12 lines $L_{ij}$, $j\neq i+3$, are the only connected lines on $E$, for by (3.4) any connected line on $E$ is on two distinct connected planes on $E$.

We write $\langle P_i\cup P_j\cup P_k\rangle=X_{ijk}$ for $1\leqq i<j<k\leqq 6$. Then $X_{ijk}$ is a point of $\mathbf{M}$, being identical with $\langle L_{ij}\cap P_k\rangle$. If two of the suffices $i$, $j$ and $k$ are congruent mod 3 then $X_{ijk}$ is one of the points $Z_1$, $Z_2$ and $Z_3$. The remaining eight points $X_{ijk}$ are all distinct, for on any one of them there can be only three

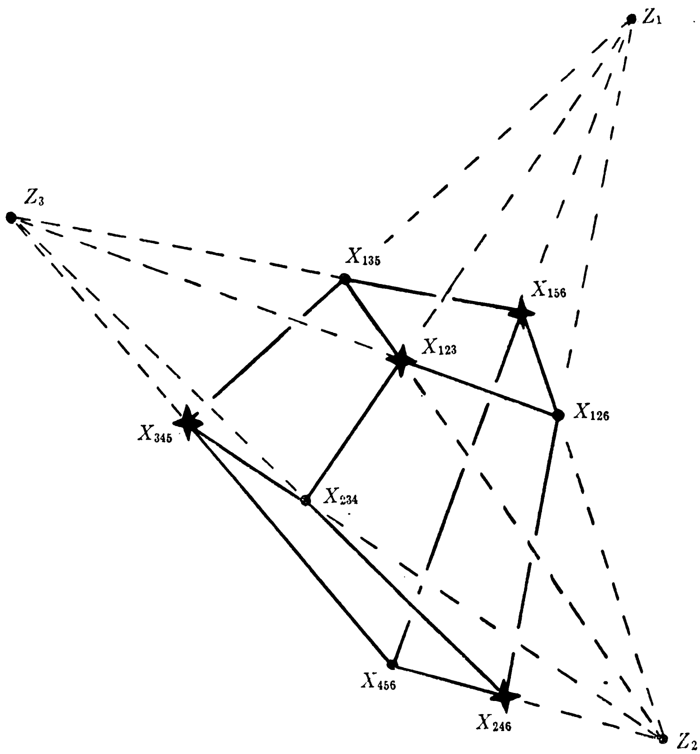


FIG. II

distinct planes such that each is on one of the lines $Z_1 \cup Z_2$, $Z_2 \cup Z_3$ and $Z_3 \cup Z_1$. These eight points, together with $Z_1$, $Z_2$ and $Z_3$, are the only points on $E$. For any point on $E$ is on three distinct connected planes on $E$, by two applications of (3.4). (See Figure II.)

Consider the plane $P_1$. The only points on it are $Z_2$, $Z_3$, $X_{123}$, $X_{126}$, $X_{135}$ and $X_{156}$. We may adjust the notation so that $X_{123} \in \mathbf{C}$. The other point of $\mathbf{C}$ on $P_1$ can have no common connected line with $X_{123}$ and must therefore be $X_{156}$. We now find that $X_{246} \in \mathbf{C}$ since this is the only point on $P_2$ having no common connected line with $X_{123}$. Proceeding in this way we find that $X_{ijk} \in \mathbf{C}$ if and only if no two of the suffices are congruent mod 3 and the number of suffices less than 4 is odd. In Figures II, III and IV we represent points of $\mathbf{C}$ by four-pointed stars.

To construct a matroid having the structure just described we may use a method based on (2.3). We take $M$ to be a set of six cells in 1-1 correspondence with the planes $P_i$. Any point $X_{ijk}$ is represented by the set of those cells not corresponding to planes on $X_{ijk}$.

Suppose we have two paths $PQR$ and $PR$ off $\mathbf{C}$, where $Q$ is an elementary re-entrant path of the $k$th kind with respect to $\mathbf{C}$. Then we call the process of deriving one of the paths $PQR$ and $PR$ from the other an *elementary deformation* of the $k$th kind with respect to $\mathbf{C}$. We say that two given paths $P'$ and $P''$ off $\mathbf{C}$ are *homotopic* with respect to $\mathbf{C}$ (written $P' \sim P''$ ($\mathbf{C}$)) if they are identical or if one can be derived from the other by a finite sequence of elementary deformations with respect to $\mathbf{C}$. Homotopy with respect to $\mathbf{C}$ is an equivalence relation. A path $P$ homotopic to a degenerate path with respect to $\mathbf{C}$ is said to be *null-homotopic* with respect to $\mathbf{C}$ (written $P \sim 0$ ($\mathbf{C}$)).

The null subset of $\mathbf{M}$ is clearly convex. If $\mathbf{C}$ is null we have only elementary deformations of the first and second kind to consider, and homotopy with respect to $\mathbf{C}$ becomes identical with the homotopy defined in the Introduction.

If $P$ is any path of $\mathbf{M}$ we write $P^{-1}$ for the path obtained by taking the terms of $P$ in reverse order.

(5.2) *If $P$ is any path off $\mathbf{C}$ then $PP^{-1} \sim 0$ ($\mathbf{C}$).*

**Proof.** If possible choose $P$ so that the theorem fails and $P$ has the least number $s$ of terms consistent with this. If $s > 1$ we can write $P = QR$, where $Q$ and $R$ have each fewer than $s$ terms. Since $RR^{-1}$ and $QQ^{-1}$ can be converted into degenerate paths by elementary deformations we have $PP^{-1} = QRR^{-1}Q^{-1} \sim QQ^{-1} \sim 0$ ($\mathbf{C}$). If $s = 1$ then $PP^{-1}$ is an elementary re-entrant path of the first kind, and so $PP^{-1} \sim 0$ ($\mathbf{C}$). The theorem follows.

(5.3) *If $PUR$ and $PVR$ are paths off $\mathbf{C}$ such that $UV^{-1} \sim 0$ ($\mathbf{C}$), then $PUR \sim PVR$ ($\mathbf{C}$).*

**Proof.** By (5.2) we have $V^{-1}V = V^{-1}(V^{-1})^{-1} \sim 0$ ($\mathbf{C}$). Hence $PUR \sim PUV^{-1}VR \sim PVR$ ($\mathbf{C}$).

(5.4) *Let $\mathbf{C}$ be any convex subclass of a matroid $\mathbf{M}$. Let $S$ be a $d$-flat of $\mathbf{M}$ on a $(d+1)$-flat $T$ of $\mathbf{M}$. Suppose all the points on $S$ and at least one other point on $T$ are members of $\mathbf{C}$. Then all the points on $T$ are members of $\mathbf{C}$.*

**Proof.** Suppose the theorem false. Then we can find points $X \in \mathbf{C}$ and $Y \notin \mathbf{C}$, both on $T$ but not on $S$. The flat $X \cup Y$ is connected since otherwise $S \subset S \cup X \subset T$, contrary to (2.2). By (5.1) there is a path from $X$ to $Y$ on $X \cup Y$ whose second term, $X'$ say, is not a member of $\mathbf{C}$. By (2.5) the line $X \cup X'$ has a point $X''$ in common with $S$. But $X'' \notin \mathbf{C}$, by the definition of a convex subclass. This is contrary to hypothesis.

## 6. Proof of the main theorem.

(6.1) *Let $\mathbf{C}$ be any convex subclass of a matroid $\mathbf{M}$ and let $P$ be any re-entrant path of $\mathbf{M}$ off $\mathbf{C}$. Then $P \sim 0$ ($\mathbf{C}$).*

**Proof.** Assume the theorem false. Let $P$ be any re-entrant path off $\mathbf{C}$ which is not null-homotopic with respect to $\mathbf{C}$, and for which $dF(P)$ has the least value, $n$ say, consistent with this condition. For an arbitrary path $Q$ of $\mathbf{M}$ we call $dF(Q)$ the *dimension* of $Q$.

By far the most difficult part of the proof is that covered by the following lemma.

LEMMA. *Suppose $n \geqq 3$. Let $Q = (W, X, Y, Z, W)$ be a path off $\mathbf{C}$ of dimension*

*$n$ such that $W \cup X \cup Y$ and $Y \cup Z \cup W$ are connected planes and $W \cup Y$ is a disconnected line. Then $Q \sim 0$ (**C**).*

**Proof.** Write $F_1 = W \cup X \cup Y$ and $F_2 = Y \cup Z \cup W$.

We note that if $Q' = (W, X', Y, Z', W)$ is a path off **C** such that $X'$ is on $F_1$ and $Y'$ on $F_2$, then

(6.1a) $$Q' \sim Q \text{ } (\mathbf{C}).$$

For

$$Q' \sim (W, X', Y)(Y, X, W)(W, X, Y)(Y, Z, W)(W, Z, Y)(Y, Z', W) \text{ } (\mathbf{C})$$

by (5.2). But $(W, X', Y)(Y, X, W)$ and $(W, Z, Y)(Y, Z', W)$ are re-entrant paths off **C** of dimension $<n$ and are therefore null-homotopic with respect to **C**. Hence $Q' \sim (W, X, Y)(Y, Z, W) = Q(\mathbf{C})$.

A *transversal of dimension* $n-1$ is a connected $(n-1)$-flat of $\boldsymbol{M}$ which is on $F(Q)$ but not on both $W$ and $Y$. By (2.2) and (2.5) such a transversal meets each of $F_1$ and $F_2$ in a line. These two lines are connected, by (4.4).

A *transversal of dimension* $n-2$ is a connected $(n-2)$-flat of $\boldsymbol{M}$ which is on $F(Q)$ but not on $W$ or $Y$. By (2.2) and (2.5) the transversal has just one point in common with each of $F_1$ and $F_2$. We call these two points the *poles* of the transversal.

Let $B$ be any transversal of dimension $n-2$, with poles $X'$ on $F_1$ and $Z'$ on $F_2$. Then $B$ is on two distinct connected $(n-1)$-flats of $\boldsymbol{M}$ on $F(Q)$, by (3.4). Using (2.5) we find that each of these is on one, but not both, of $W$ and $Y$. Hence, by (2.2) they are $B \cup W$ and $B \cup Y$. They are transversals of dimension $n-1$. The flats $X' \cup W$, $X' \cup Y$, $Z' \cup W$ and $Z' \cup Y$ are their connected lines of intersection with $F_1$ and $F_2$. We note that a path $(W, X', Y, Z', W)$ exists.

Assume that $Q$ is not null-homotopic with respect to **C**.

Suppose $B$ is a transversal of dimension $n-2$ with poles $X'$ on $F_1$ and $Z'$ on $F_2$. Suppose further that neither $X'$ nor $Z'$ belongs to **C**. Then, by (5.1) there is a path $R$ off **C** from $X'$ to $Z'$ on $B$. Now $(W, X')R(Z', W)$ and $(X', Y, Z')R^{-1}$ are paths on the $(n-1)$-flats $B \cup W$ and $B \cup Y$ respectively. Hence their dimensions are less than $n$ and so they are null-homotopic with respect to **C**. Using (6.1a), (5.3) and (5.2) we find $Q \sim (W, X', Y, Z', W) = (W, X')(X', Y, Z')(Z', W) \sim (W, Z')R^{-1}R(Z', W) \sim 0$ (**C**). This is contrary to assumption. We deduce that each transversal of dimension $n-2$ has at least one pole in **C**.

By (3.5) there is a transversal $A$ of dimension $n-1$ which is not on $Y$. Let its lines of intersection with $F_1$ and $F_2$ be $L_1$ and $L_2$ respectively. They are connected lines on $W$. By (3.2) there is a point $X'$ of $\boldsymbol{M} - \mathbf{C}$ other than $W$ on $L_1$. By (3.5) there is a connected $(n-2)$-flat $B$ of $\boldsymbol{M}$ which is on $A$ and $X'$ but not on $W$. Now $B$ is a transversal of dimension $n-2$. Let its pole on $L_2$ be $U_2$. Then $U_2 \in \mathbf{C}$. Similarly there is a transversal $B'$ of dimension $n-2$ on $A$ having a point $Z'$ of $\boldsymbol{M} - \mathbf{C}$ as its pole on $L_2$ and a point $U_1$ of **C**

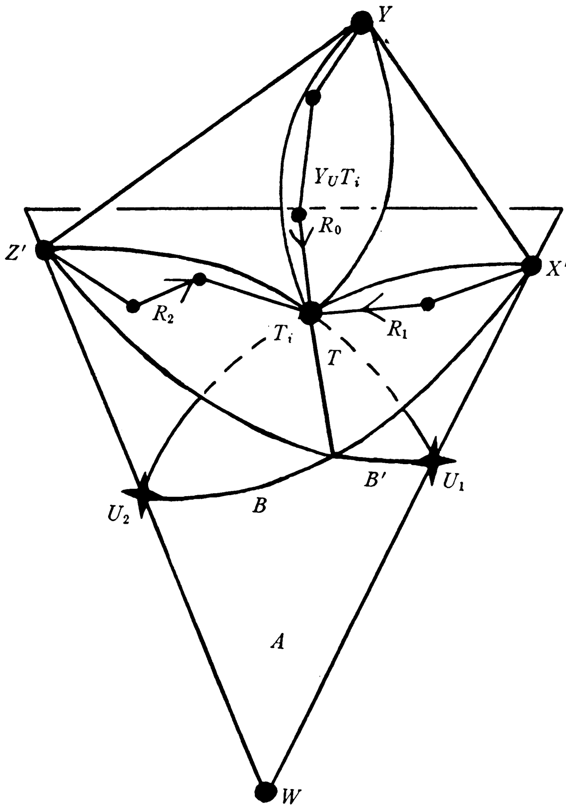


FIG. III

as its pole on $L_1$. (See Figure III.) We write $T=\langle B\cap B'\rangle$. By (2.2) and (2.5) $T$ is an $(n-3)$-flat of $\boldsymbol{M}$.

Let $\boldsymbol{S}$ be the class of all members of $\boldsymbol{M}-\boldsymbol{C}$ on $T$. Since $T\subset B$, $X'\in\boldsymbol{M}-\boldsymbol{C}$ and $U_2\in\boldsymbol{C}$ it follows by (5.4) that $\boldsymbol{S}$ is non-null.

Let $T_i$ be any point of $\boldsymbol{S}$. Suppose that the flat $Y\cup T_i$ is connected. Then there is a path $R_0$ from $Y$ to $T_i$ on $Y\cup T_i$ which is off $\boldsymbol{C}$, by (5.1). Similarly there is a path $R_1$ from $X'$ to $T_i$ on $B$ and a path $R_2$ from $Z'$ to $T_i$ on $B'$, both $R_1$ and $R_2$ being off $\boldsymbol{C}$. Now $(X', Y)R_0R_1^{-1}$ is a re-entrant path on the transversal $B\cup Y$ of dimension $n-1$, and $(Y, Z')R_2R_0^{-1}$ is a re-entrant path on the transversal $B'\cup Y$ of dimension $n-1$. Hence both these paths are null-homotopic with respect to $\boldsymbol{C}$. Applying (6.1a), (5.3) and (5.2) we find $Q\sim(W, X', Y, Z', W)=(W, X')(X', Y)(Y, Z')(Z', W)\sim(W, X')R_1R_0^{-1}R_0R_2^{-1}\cdot(Z', W)\sim(W, X')R_1R_2^{-1}(Z', W)$ $(\boldsymbol{C})$. But the last path is on the $(n-1)$-flat $A$ and is therefore null-homotopic with respect to $\boldsymbol{C}$. Hence $Q\sim 0$ $(\boldsymbol{C})$, contrary to assumption. We deduce that $Y\cup T_i$ has a separation $\{Y, T_i\}$. That is $Y\cup T_i$ is a disconnected line, and $Y$ and $T_i$ are the two points on it.

We can repeat the above argument with $B\cup Y$ replacing $A$. Instead of

$B'$ we then obtain a transversal $B''$ of dimension $n-2$ on $B\cup Y$ with its pole on $U_2\cup Y$ a member of $\boldsymbol{M}-\boldsymbol{C}$ and its pole on $X'\cup Y$ a member of $\boldsymbol{C}$. We denote the $(n-3)$-flat $\langle B\cap B''\rangle$ by $T'$. We find that any point $T_j'$ of $\boldsymbol{M}-\boldsymbol{C}$ on $T'$ is such that $W\cup T_j'$ is a disconnected line. But, by (2.5), $B''$ has a point in common with the disconnected line $Y\cup T_i$, and this point can only be $T_i$. Hence $T_i$ is one of the points of $\boldsymbol{M}-\boldsymbol{C}$ on $T'$.

We conclude that any point $T_i$ of $\boldsymbol{S}$ is such that $W\cup T_i$ and $Y\cup T_i$ are disconnected lines.

Let $E$ be a connected flat of $\boldsymbol{M}$ on $F_1$ and $F(Q)$ which is on some point of $\boldsymbol{S}$ and has the least dimension consistent with this property. It is clear that either $F(Q)$ or one of its subsets satisfies these conditions. We have

$$n = dF(Q) \geqq dE \geqq 3, \tag{6.1b}$$

$$d\langle E\cap T\rangle \geqq dE - 3, \tag{6.1c}$$

by (2.2) and (2.5). Choose a point $N$ on $\langle E\cap T\rangle$, taking $N\in\boldsymbol{C}$ if this is possible. By (3.5) there is a connected $(dE-1)$-flat $E'$ of $\boldsymbol{M}$ on $F_1$ and $E$ but not on $N$. By (2.2) and (2.5) $\langle E'\cap T\rangle$ is a $(d\langle E\cap T\rangle-1)$-flat on $\langle E\cap T\rangle$. All the points of $\boldsymbol{M}$ which are subsets of $\langle E'\cap T\rangle$ belong to $\boldsymbol{C}$, by the definition of $E$. By the choice of $N$ this implies that either $d\langle E'\cap T\rangle=-1$ or $N\in\boldsymbol{C}$. But in the latter case all the points on $\langle E\cap T\rangle$ belong to $\boldsymbol{C}$, by (5.4), contrary to the definition of $E$. Hence $d\langle E'\cap T\rangle=-1$ and therefore $d\langle E\cap T\rangle=0$. Hence $dE=3$, by (6.1b) and (6.1c). Henceforth we use the symbol $T_i$ to denote the single point $\langle E\cap T\rangle$, which must be in $\boldsymbol{S}$.

Suppose $n\geqq 4$. Then $F_2$ is not on $E$. By (3.5) there is a connected $(n-1)$-flat $E''$ of $\boldsymbol{M}$ on $F_2$ and $F(Q)$ but not on $T_i$. Write $F_3=\langle E''\cap E\rangle$. Then $F_3$ is a plane on $E$ and $W\cup Y$, by (2.2) and (2.5). By (3.5) there is a connected line $L$ on $E$ and $T_i$ having no common point with $W\cup Y$. Let its common points with $F_1$ and $F_3$ be $X_1$ and $X_3$ respectively. Neither of these is $T_i$. We note that $F_3=W\cup Y\cup X_3$, by (2.2). Now $X_1\cup W$ and $X_1\cup Y$ are connected lines by (4.3). Hence $X_1\cap W$ and $X_1\cap Y$ are both non-null, by (3.1). But we have shown that $W\cup T_i$ and $Y\cup T_i$ are disconnected lines. Hence $T_i\cap W$ and $T_i\cap Y$ are both null. But $L=X_1\cup T_i=X_3\cup T_i$, by (3.1). Hence $X_3\cap W$ and $X_3\cap Y$ are both non-null and therefore $F_3$ is connected.

By (5.1) there is a path $R$ from $Y$ to $W$ on $F_3$ which is off $\boldsymbol{C}$. The re-entrant paths $(W, X, Y)R$ and $(Y, Z, W)R^{-1}$ are on $E$ and $E''$ respectively and so have dimensions $<n$. Hence they are null-homotopic with respect to $\boldsymbol{C}$. Using (5.3) and (5.2) we find $Q=(W, X, Y)(Y, Z, W)\sim R^{-1}R\sim 0$ ($\boldsymbol{C}$), contrary to assumption.

We deduce, using (6.1b), that $n=3$. This implies $dT=0$. Hence $T$ is a point of $\boldsymbol{M}$, identical with $T_i$ and therefore a member of $\boldsymbol{M}-\boldsymbol{C}$. The three flats $W\cup Y$, $Y\cup T$ and $T\cup W$ are disconnected lines. The flat $W\cup Y\cup T$ is not a connected plane of $\boldsymbol{M}$ by (4.3).

Any plane on $F(Q)$ has a point in common with each of the disconnected lines $W \cup Y$, $Y \cup T$ and $T \cup W$. It is therefore on one of these lines. Each line of $F(Q)$ is on a plane of $F(Q)$, by (2.2) and (2.3). It follows that each line on $F(Q)$ is on one of the points $W$, $Y$ and $T$.

Let $F$ be any transversal of dimension 2. It meets $F_1$ and $F_2$ in connected lines $L_1$ and $L_2$ respectively. Let the points on $L_1$ other than $W$ or $Y$ be $X_1, \cdots, X_k$. By (3.4) there is a transversal $B_i$ of dimension 1 on $F$ and $X_i$ for each $i$. The line $B_i$ must be on $T$. Hence $B_i = X_i \cup T$, and $B_i$ is uniquely determined for each $i$. Let $X_i'$ denote the point of intersection of $B_i$ and $L_2$. Since $B_i = T \cup X_i'$ for each $i$ the $k$ points $X_1', \cdots, X_k'$ are all distinct. But at most one point on each of the lines $L_1$ and $L_2$ belongs to $\mathbf{C}$, and no transversal of dimension 1 has both its poles in $\mathbf{M} - \mathbf{C}$. Applying (3.2) we deduce that $k=2$. Moreover we can adjust the notation so that $(X_1, X_2') \in \mathbf{C}$ and $(X_2, X_1') \in \mathbf{M} - \mathbf{C}$.

Distinct lines on $T$ and $F$ meet $L_1$ in distinct points, by (3.1). Hence the only connected lines on $T$ and $F$ are $B_1$ and $B_2$. But each point of $F$ is on two connected lines on $F$, and one of these is on $T$. Hence $X_1$ and $X_2'$ are the only points of $\mathbf{C}$ on $F$. We thus prove that each connected plane on $F(Q)$ not on $W \cup Y$ is on just two points of $\mathbf{C}$.

Any connected line on $F_1$ is on a transversal of dimension 2, by (3.4). Hence it is on just three points, one of which is in $\mathbf{C}$. Distinct lines on $F_1$ and $W$ (or $Y$) meet a given connected line on $F_1$ and $Y$ (or $W$) in distinct points, by (3.1). It follows that on $F_1$ there are just two connected lines on each of the points $W$ and $Y$. As each point on $F_1$ is on two connected lines on $F_1$ we deduce that $F_1$ is on just two points of $\mathbf{C}$. Analogous results hold for $F_2$. Two distinct transversals of dimension 2 are both on $T$ and therefore meet $F_1$ in distinct lines, by (2.2). Accordingly there are just two connected planes on $F(Q)$ and $T \cup W$, and just two on $F(Q)$ and $Y \cup T$ (since $W \cup Y \cup T$ is not a connected plane).

It follows from these results that either $Q$ is an elementary re-entrant path of the fourth kind with respect to $\mathbf{C}$ or there is a third connected plane $F_3$ on $F(Q)$ and $W \cup Y$. The first alternative must be rejected since it implies $Q \sim 0$ ($\mathbf{C}$).

Let the points of intersection with $F_3$ of $B_1$ and $B_2$ be $X_1''$ and $X_2''$ respectively. These are both in $\mathbf{M} - \mathbf{C}$. Each is on two connected lines on $F_3$, one on $W$ and the other on $Y$, by (4.3). We have $(W, X, Y, X_1'', W) = (W, X, Y) \cdot (Y, X_1'', W) \sim 0$ ($\mathbf{C}$). For otherwise we can repeat the first parts of the preceding proof with $(W, X, Y, X_1'', W)$ replacing $Q$ and obtain a contradiction, for the transversal $B_2$ then has both poles in $\mathbf{M} - \mathbf{C}$. Similarly $(Y, Z, W, X_1'', Y) = (Y, Z, W)(W, X_1'', Y) \sim 0$ ($\mathbf{C}$). Applying (5.3) and (5.2) we find $Q = (W, X, Y)(Y, Z, W) \sim (W, X_1'', Y)(Y, X_1'', W) \sim 0$ ($\mathbf{C}$), contrary to assumption. The lemma follows.

We return to the path $P$ defined at the beginning of this proof. We note that $n = dF(P) \geqq 1$ since otherwise $P$ would be trivially null-homotopic with respect to $\mathbf{C}$. We choose a connected $(n-1)$-flat $E$ of $\mathbf{M}$ which is on $F(P)$

and the origin $X_0$ of $P$. This choice is possible, by (3.3).

Let $R=(X_0, \cdots, X_m, X_0)$ be any re-entrant path with the same origin as $P$ on $F(P)$. We write $u(R)$ for the number of terms of $R$, counting repetitions, which are not on $E$. If $u(R)>0$ we write $X_i$ for the first term of $R$ which is not on $E$. We then write $v(R)=d(X_{i-1}\cup X_i\cup X_{i+1})$, taking $X_{m+1}=X_0$ if $i=m$. If $u(R)=0$ we write $v(R)=0$.

Henceforth we suppose $R$ chosen so as to satisfy the following conditions:

(i) $R\sim P$ ($\mathbf{C}$),

(ii) $u(R)$ has the least value consistent with (i),

(iii) $v(R)$ has the least value consistent with (i) and (ii).

We consider first the case $u(R)>0$. Then $v(R)>0$. We may conveniently write $R$ in the form $R_1(X_{i-1}, X_i, X_{i+1})R_2$, noting that $R_1$ is a path on . We write also $F=X_{i-1}\cup X_i\cup X_{i+1}$.

Suppose $v(R)=1$. Then $F$ is a connected line. If $X_{i+1}=X_{i-1}$ we have $R\sim R_1R_2$ ($\mathbf{C}$), by an elementary deformation of the first kind. This is impossible since $u(R_1R_2)<u(R)$. If $X_{i+1}\neq X_{i-1}$ then $(X_{i-1}, X_i, X_{i+1}, X_{i-1})$ is an elementary re-entrant path of the second kind with respect to $\mathbf{C}$. Applying (5.3) we find $R\sim R_1(X_{i-1}, X_{i+1})R_2$ ($\mathbf{C}$). This is impossible since

$$u(R_1(X_{i-1}, X_{i+1})R_2) < u(R).$$

Suppose $v(R)=2$. Then $F$ is a connected plane on $F(Q)$. It meets $E$ in a line $L$, by (2.2) and (2.5). Let $Z$ be the point of intersection of the lines $L$ and $X_i\cup X_{i+1}$ on $F$. We discuss first the case $Z\in \boldsymbol{M}-\mathbf{C}$. In this case we define $Q$ as the degenerate path $(Z)$ if $Z=X_{i+1}$ and as the path $(Z, X_{i+1})$ otherwise. Then $(X_i, X_{i+1})Q^{-1}(Z, X_i)$ is an elementary re-entrant path of the first or second kind. Hence $R\sim R_1(X_{i-1}, X_i, Z)QR_2$ ($\mathbf{C}$), by (5.3). If $L$ is connected we have $(X_{i-1}, X_i, Z, X_{i-1})\sim 0$ ($\mathbf{C}$) and therefore $R\sim R_1(X_{i-1}, Z)QR_2$ ($\mathbf{C}$), by (5.3). If $L$ is not connected it is on a connected plane $F'$ of $\boldsymbol{M}$ on $E$, by (4.2). We can find a connected line $L'$ on $X_{i-1}$ and $F'$, and a point $T$ of $\boldsymbol{M}-\mathbf{C}$ distinct from $X_{i-1}$ on $L'$. Then $T\cup X_{i-1}$ and $T\cup Z$ are connected lines, by (4.3). Using the lemma and the definition of $n$ we find $(X_{i-1}, X_i, Z, T, X_{i-1})\sim 0$ ($\mathbf{C}$). Hence $R\sim R_1(X_{i-1}, T, Z)QR_2$ ($\mathbf{C}$), by (5.3). So whether $L$ is connected or not we have $R\sim R_3QR_2$ ($\mathbf{C}$), where $R_3$ is on $E$. This is impossible since $u(R_3QR_2)<u(R)$.

We go on to the case $Z\in\mathbf{C}$, illustrated in Figure IV. By (3.4) there is a connected line $L'$ other than $X_i\cup X_{i+1}$ on $X_{i+1}$ and $F$. If $L'$ is on $X_{i-1}$ we have $R\sim R_1(X_{i-1}, X_{i+1})R_2$ ($\mathbf{C}$), using (5.3) with the elementary re-entrant path $(X_{i-1}, X_i, X_{i+1}, X_{i-1})$ of the second kind. This is impossible since $u(R_1(X_{i-1}, X_{i+1})R_2)<u(R)$. Hence $L'$ must meet the lines $X_{i-1}\cup X_i$ and $L$ in distinct points $U$ and $V$ respectively. Since $Z\in\mathbf{C}$ and $X_{i-1}\in\boldsymbol{M}-\mathbf{C}$ we have $V\in\boldsymbol{M}-\mathbf{C}$.

Suppose $U\in\boldsymbol{M}-\mathbf{C}$. Using (5.3) with elementary re-entrant paths of the first and second kinds we find

$R \sim R_1(X_{i-1}, U, X_i, X_{i+1})R_2 \sim R_1(X_{i-1}, V, U, X_{i+1})R_2 \sim R_1(X_{i-1}, V, X_{i+1})R_2$ (**C**).

This is impossible since $u(R_1(X_{i-1}, V, X_{i+1})R) < u(R)$.

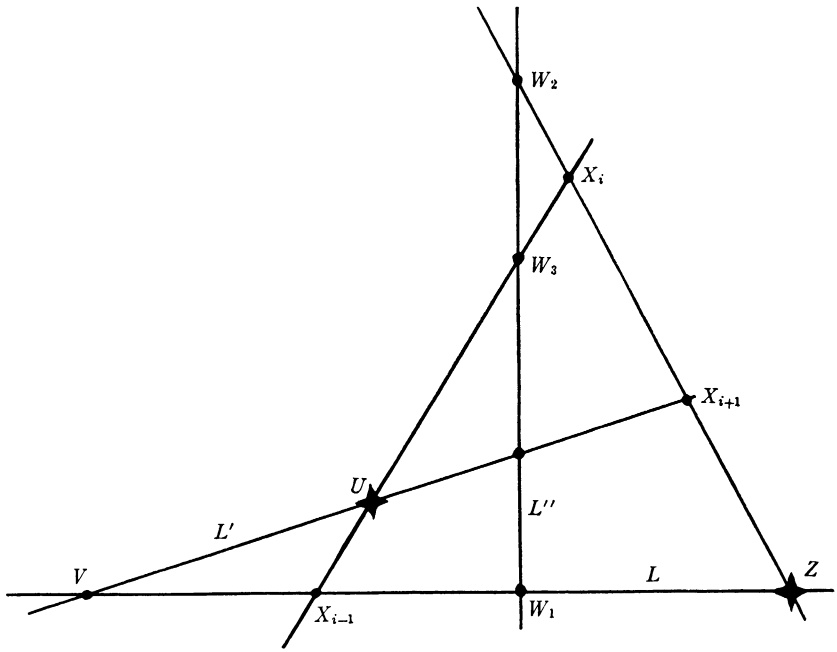


FIG. IV

Suppose $U \in$ **C**. It may happen that each connected line on $F$ is on either $U$ or $Z$. Then $(X_{i-1}, X_i, X_{i+1}, V, X_{i-1})$ is an elementary re-entrant path of the third kind with respect to **C**. Using (5.3) we have $R \sim R_1(X_{i-1}, V, X_{i+1})R_2$ (**C**), which is impossible, as before. Hence there is a connected line $L''$ on $F$ which is not on $U$ or $Z$. If $L''$ is on $X_{i+1}$ we can substitute it for $L'$ in the preceding argument and so reduce to the case $U \in \boldsymbol{M} - \mathbf{C}$. We may therefore suppose $L''$ is not on $X_{i+1}$.

If $L''$ is on $X_i$ it meets $L$ in a point $W_1$ distinct from $X_{i-1}$ and $Z$. Writing $R' = R_1(X_{i-1}, W_1, X_i, X_{i+1})R_2$ we have $R' \sim R$ (**C**), by (5.3). If $L''$ is not on $X_i$ it meets $X_i \cup X_{i+1}$ in a point $W_2$ distinct from $X_i$, $X_{i+1}$ and $Z$. If $L''$ is then on $X_{i-1}$ we write $R' = R_1(X_{i-1}, W_2, X_{i+1})R_2$ and have

$$R' \sim R_1(X_{i-1}, W_2, X_i, X_{i+1})R_2 \sim R\ (\mathbf{C}),$$

by (5.3). If instead $L''$ is not on $X_{i-1}$ it meets the lines $L$, $X_i \cup X_{i+1}$ and $X_{i-1} \cup X_i$ in distinct points $W_1$, $W_2$ and $W_3$ respectively of $\boldsymbol{M} - \mathbf{C}$. We then write $R' = R_1(X_{i-1}, W_1, W_2, X_{i+1})R_2$ and have $R' \sim R_1(X_{i-1}, W_1, W_3, W_2, X_i, X_{i+1})R_2$ $\sim R_1(X_{i-1}, W_3, X_i, X_{i+1})R_2 \sim R$ (**C**), by (5.3). For each of these three possibilities we have $R' \sim R$ (**C**), $u(R') = u(R)$ and $v(R') = v(R) = 2$. Hence we may replace $R$ by $R'$ in the preceding argument. This reduces the problem to the case $U \in \mathbf{M} - \mathbf{C}$, which we have found to lead to a contradiction.

We now consider the case $v(R) > 2$. By (3.3) there is a connected plane $K$ on $X_{i-1} \cup X_i$ and the connected $v(R)$-flat $F$. This plane meets $E$ in a line $L$. Choose a point $T$ distinct from $X_{i-1}$ on $L$ and if possible in $\mathbf{C}$. By (3.5) there is a connected $(v(R)-1)$-flat $F'$ on $X_i \cup X_{i+1}$ and $F$ but not on $T$. Now $F'$ is not on $X_{i-1}$, for otherwise we would have $F \subseteq F'$, contrary to (2.2). Hence $F'$ meets $L$ in a point $T'$ distinct from $X_{i-1}$ and $T$. It follows that $L$ is connected, by (3.2), and that $T' \in \mathbf{M} - \mathbf{C}$. The flats $K$ and $F'$ intersect in a line $L'$ on $X_i$ and $T'$. If $L'$ is connected we write

$$R' = R_1(X_{i-1}, T', X_i, X_{i+1})R_2$$

and have $R' \sim R$ ($\mathbf{C}$), by (5.3). If $L'$ is not connected it is on a connected plane $K'$ on $F'$, by (4.2). $K'$ meets $E$ in a connected line $L''$ on $T'$, by (4.4). We can find a point $U$ on $L''$ distinct from $T'$ and in $\mathbf{M} - \mathbf{C}$. The flat $U \cup X_i$ is a connected line, by (4.3). Using the lemma and the definition of $n$ we find $(T', U, X_i, X_{i-1}, T') \sim 0$ ($\mathbf{C}$). In this case we write

$$R' = R_1(X_{i-1}, T', U, X_i, X_{i+1})R_2.$$

Then by (5.2) and (5.3) we have

$$R' \sim R_1(X_{i-1}, T', U, X_i, X_{i-1}, T', X_{i-1}, X_i, X_{i+1})R_2$$
$$\sim R_1(X_{i-1}, T', X_{i-1}, X_i, X_{i+1})R_2 \sim R \ (\mathbf{C}).$$

So whether $L'$ is connected or not we have $R' \sim R$ ($\mathbf{C}$), $u(R') = u(R)$ and $v(R') < v(R)$, which is contrary to the definition of $R$.

From the above analysis we deduce that $u(R) = 0$. Hence $R$ is on $E$ and has dimension $< n$. Hence $P \sim R \sim 0$ ($\mathbf{C}$), contrary to assumption. The theorem follows.

**7. Special cases.** With $\mathbf{C}$ null in (6.1) we find that every re-entrant path in a matroid $\mathbf{M}$ is null-homotopic, as stated in the Introduction.

In applying this result to the circuit-matroid $\mathbf{M}$ of a graph $G$ we must remember that a path in $\mathbf{M}$ corresponds to a sequence of circuits of $G$ such that any two consecutive circuits form a nonseparable subgraph of rank 2. It can be shown that such a subgraph is made up of three arcs such that any two have both ends but no other edge or vertex in common. Each of the elementary deformations by which a re-entrant sequence of circuits can be transformed into a sequence with only one member operates within some nonseparable subgraph of rank $\leq 3$.

## Reference

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